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Research article

Analytical findings on bilinear fractional Hardy operators in weighted central Morrey spaces with variable exponents

Muhammad Asim[1,*] **and Ghada AlNemer**[2,*]

[1] Department of NUSASH, National University of Technology (NUTECH), Islamabad 44000, Pakistan

[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, Riyadh 11671, Saudi Arabia

* **Correspondence:** Email: masim@math.qau.edu.pk, gnnemer@pnu.edu.sa.

Abstract: This paper demonstrates the boundedness of the fractional bilinear Hardy operator and its adjoint on weighted λ-central Morrey spaces with a variable exponent. Analogous outcomes for their commutators were derived when the symbol functions are elements of the weighted λ-central bounded mean oscillation (λ-central BMO) spaces.

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1. Introduction

Through the progressive evolution of the domains encompassing partial differential equations, nonlinear analysis, and other cognate disciplines, modern harmonic analysis, by virtue of its idiosyncratic perspectives and methodologies, has established itself as an indispensable apparatus for the intricate classification and stratification of functions. This analytical ingenuity has, in turn, precipitated the formulation of multifarious specialized theories concerning function spaces, including, but not limited to: Hardy spaces, Herz spaces, and Morrey spaces. As early as 1931, Orlicz, in the seminal work [1], initiated the articulation and preliminary investigation of the theoretical framework underpinning variable exponent L^p spaces. Nevertheless, it was not until the year 1991, after the dissemination of the pivotal contributions by Kováčik and Rákosník (refer to [2]), that the paradigm of variable function spaces garnered significant and widespread scholarly attention. In [2, 3], the authors undertook the generalization of classical Sobolev and Lebesgue spaces, thereby extending them to the domains of variable Sobolev and Lebesgue spaces, while rigorously establishing the foundational properties inherent to these variable function spaces. Furthermore, such variable function spaces have found extensive applications in the analysis of fluid mechanics and differential equations characterized by non-standard growth conditions (see [4]). This has incited a substantial body of scholarly inquiry, wherein numerous researchers have dedicated considerable effort to the study of variable exponent spaces, yielding a proliferation of results. Additionally, this exploration has precipitated the development and investigation of further variable exponent spaces, including, but not limited to, variable Triebel-Lizorkin spaces, variable Besov spaces, variable Herz spaces, and variable Hardy spaces (see [5–10]).

In recent years, significant progress has been made in establishing the boundedness of numerous pivotal operators within the framework of harmonic analysis on variable exponent function spaces. For instance, in [5, 9–12], the respective authors have undertaken comprehensive investigations into the boundedness of various integral operators on such variable function spaces, employing the intricate properties inherent to variable L^p spaces as a foundational analytical tool. Conversely, since the seminal contributions of Peetre (see [13]), the advancement of Morrey spaces has progressively emerged as a dominant paradigm within the corpus of modern harmonic analysis. Numerous scholars have undertaken rigorous inquiries into the structural and functional properties of central Morrey spaces and central BMO spaces (refer to [10,14]), thereby rendering the theory of operator boundedness on central Morrey spaces increasingly comprehensive and refined. Motivated by the burgeoning development of variable function spaces, Fu et al., in their influential work [15], extended the classical central Morrey spaces from a constant exponent framework to the realm of variable exponents in the year 2019. They introduced the conceptual framework of central Morrey spaces and central BMO spaces with variable exponents, wherein they established comprehensive estimates for singular integral operators and their associated commutators. This endeavor significantly advanced the theoretical development and analytical sophistication of central Morrey spaces. Leveraging the conceptual underpinnings and intrinsic properties of variable function spaces, it is a natural intellectual progression for scholars to investigate the boundedness of operators within this framework. Between the years 2019 and 2022, various researchers engaged in the rigorous examination of the boundedness of numerous classes of operators and their commutators within the context of variable central Morrey spaces. For instance, Wang et al., in [16, 17], undertook an extensive exploration of the boundedness properties of multilinear singular integral operators and multilinear fractional integral operators, thereby extending and enriching the foundational study presented in [15]. Moreover, in 2022, Hussain et al., in [18], addressed the boundedness of the Hardy operator within the framework of variable central Morrey spaces, further advancing the theoretical discourse in this domain.

In the year 1920, Hardy initially introduced the notion of the one-dimensional Hardy operator, as delineated in [19]. Subsequently, an increasing number of researchers engaged in the study and refinement of the definition, alongside the exploration of various generalized forms of Hardy-type operators. In 1995, Christ and Grafakos, in their seminal work [20], extended the conceptual framework of the Hardy operator from the one-dimensional case to the n-dimensional setting and rigorously established its boundedness within the context of L^p spaces. Fu et al., in [21], advanced the generalization of Hardy operators by introducing the notion of n-dimensional fractional Hardy operators. Furthermore, they meticulously established the boundedness of the commutators associated with these operators within the analytical frameworks of Lebesgue spaces and homogeneous Herz spaces. Subsequently, the boundedness properties of Hardy operators garnered significant scholarly interest, motivating numerous researchers to delve into this intricate subject. Notably, Fu et al., in [22], undertook a profound investigation into the boundedness of n-dimensional rough Hardy operators and their associated commutators, thereby contributing substantially to the theoretical advancement of this field. Hussain et al., in [18], derived rigorous estimates pertaining to fractional Hardy operators and their commutators within the analytical framework of variable λ-central Morrey spaces, thereby augmenting the theoretical understanding of operator behavior in such variable exponent settings. These two scholarly contributions serve as a profound source of intellectual inspiration. With the ever-expanding impact of variable exponent function spaces on the advancement of disciplines such as information science, and related fields, the investigation into the boundedness of multilinear operators within the framework of variable exponent function spaces has, in recent years, ascended to the forefront of contemporary research endeavors. Accordingly, this paper endeavors to achieve a substantive breakthrough in the examination of the boundedness properties of n-dimensional bilinear fractional Hardy operators and their associated commutators within the framework of weighted central Morrey spaces characterized by variable exponents. Building upon the foundational analysis of the boundedness of bilinear fractional Hardy operators, this study further delves into the boundedness of adjoint bilinear fractional Hardy operators and their corresponding commutators. Subsequently, it undertakes a comparative assessment of the methodologies and results pertaining to the boundedness of bilinear fractional operators, thereby deriving pertinent conclusions regarding the broader class of multilinear operators.

Let ζ_1 and ζ_2 be functions residing in $\mathbb{R}^n$ that are locally integrable, and let the parameter τ satisfy $0 \leq \tau < mn$. The n-dimensional multilinear fractional Hardy operator, in conjunction with its adjoint counterpart, is formally defined as follows:

$$H_\tau(\zeta_1, ..., \zeta_m) = \frac{1}{|t|^{nm-\tau}} \int_{|(\xi_1,...,\xi_m)|<|t|} \prod_{i=1}^{m} \zeta_i(\xi_i) d\xi_1, ..., d\xi_m.$$

$$H_\tau^*(\zeta_1, ..., \zeta_m) = \int_{|(\xi_1,...,\xi_m)|>|t|} \frac{1}{|\xi|^{nm-\tau}} \prod_{i=1}^{m} \zeta_i(\xi_i) d\xi_1, ..., d\xi_m.$$

Moreover, the commutators associated with the n-dimensional multilinear fractional Hardy operator, as well as its adjoint counterpart, are rigorously characterized by the following expressions:

$$[b, H_\tau](\zeta_1, ..., \zeta_m)(\xi) = \sum_{i=1}^{m} [b_i, H_\tau^i](\zeta_1, ..., \zeta_m)(\xi)$$

$$[b, H_\tau^*](\zeta_1, ..., \zeta_m)(\xi) = \sum_{i=1}^{m} [b_i, H_\tau^{*i}](\zeta_1, ..., \zeta_m)(\xi)$$

$$[b_i, H_\tau^i](\zeta_1, ..., \zeta_m)(\xi) = b_i(\xi) H_\tau(\zeta_1, ..., \zeta_m)(\xi) - H_\tau(\zeta_1, ..., \zeta_{i-1}, \zeta_i b_i, \zeta_{i+1}, ..., \zeta_m)(\xi).$$

$$[b_i, H_\tau^{*i}](\zeta_1, ..., \zeta_m)(\xi) = b_i(\xi) H_\tau^*(\zeta_1, ..., \zeta_m)(\xi) - H_\tau^*(\zeta_1, ..., \zeta_{i-1}, \zeta_i b_i, \zeta_{i+1}, ..., \zeta_m)(\xi).$$

Subsequently, we shall elucidate the structural framework of this treatise. In Section 2, we commence by succinctly recapitulating certain foundational notations and pivotal lemmas within the theory of variable Lebesgue spaces, concomitantly introducing the formal definitions of central bounded mean oscillation (BMO) spaces and weighted central Morrey spaces characterized by variable exponents. Thereafter, in Section 3, we shall rigorously establish the boundedness properties of the n-dimensional bilinear fractional Hardy operator and its adjoint operator when acting upon weighted central Morrey spaces endowed with variable exponents. Finally, in Section 4, we shall expound upon the boundedness of the commutators associated with the n-dimensional bilinear fractional Hardy operator and its adjoint operator within the framework of central Morrey spaces exhibiting variable exponents. To govern the continuity prerequisites of the m-linear fractional Hardy operator, we shall invoke the boundedness properties of the fractional integral, formally delineated by the expression:

$$I_\tau(h)(\xi) = \int_{\mathbb{R}^n} \frac{h(\eta)}{|\xi - \eta|^{n-\tau}} d\eta.$$

2. Symbols and descriptions

In the subsequent exposition, we shall delineate certain foundational attributes of variable Lebesgue spaces alongside pivotal definitions pertaining to variable exponent function spaces. Throughout the entirety of this treatise, we employ the notations $|B|$, C, and χ_B to signify, respectively, the Lebesgue measure, a generic constant, and the characteristic function corresponding to a measurable subset $B \subset \mathbb{R}^n$.

Let $E \subset \mathbb{R}^n$ be an open set and $q(\cdot) : E \to [1, \infty)$ a measurable function. The space $L^{q(\cdot)}(E)$ is defined as the collection of measurable functions ζ on E for which there exists a constant λ such that the integral

$$\int_E \left(\frac{|\zeta(x)|}{\lambda}\right)^{q(x)} dx < \infty.$$

This set is endowed with the structure of a Banach function space upon the imposition of the Luxemburg-Nakano norm, articulated as

$$\|\zeta\|_{L^{q(\cdot)}(E)} = \inf\left\{\lambda > 0 : \int_E \left(\frac{|\zeta(x)|}{\lambda}\right)^{q(x)} dx \leq 1\right\}.$$

Such spaces are denominated as variable Lebsgue spaces $L^{q(\cdot)}$, insofar as they extend and generalize the classical framework of standard L^q spaces.

The space $L^{q(\cdot)}_{loc}(F)$ is rigorously characterized as

$$L^{q(\cdot)}_{loc}(F) = \{\zeta : \zeta \in L^{q(\cdot)}(E) \; for \; all \; compact \; subsets \; E \subset F\}.$$

We designate $\mathfrak{P}(F)$ as the collection of all measurable functions $q(\cdot) : F \to (1, \infty)$ satisfying the condition where

$$q_- := \operatorname*{essinf}_{x \in F} q(x), \; q_+ := \operatorname*{esssup}_{x \in F} q(x).$$

Furthermore, $q'(\cdot)$ signifies the conjugate exponent corresponding to $q(\cdot)$, defined implicitly through the functional relationship

$$\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1,$$

thereby establishing the duality condition inherent in their reciprocal interaction. Let $\mathfrak{D}(F)$ denote the subset of $\mathfrak{P}(F)$ for which the Hardy-Littlewood maximal operator $\mathcal{M}$, defined by

$$\mathcal{M}\zeta(x) = \sup_r \frac{1}{|B_r|} \int_{B_r \cap F} |\zeta(y)| dy$$

exhibits boundedness on the space $L^{p(\cdot)}(F)$. Here, $B_r = \{y \in \mathbb{R}^n : |x - y| < r\}$ represents the ball of radius r centered at x.

Within the framework of variable $L^{p(\cdot)}$ spaces, several pivotal lemmas are established as follows.

Definition 2.1. Let $q(\cdot)$ be a real-valued function defined on $\mathbb{R}^n$. The following designations and properties are articulated:
(i) $C^{\log}_{\text{loc}}(\mathbb{R}^n)$ represents the set of all locally log-Hölder continuous functions $q(\cdot)$ satisfying

$$|q(x) - q(y)| \leq \frac{-C}{\log(|x - y|)}, \quad |y - x| < \frac{1}{2}, \quad x, y \in \mathbb{R}^n.$$

(ii) For $q(\cdot) \in C^{\log}_0(\mathbb{R}^n)$, the following condition holds at the origin:

$$|q(x) - q(0)| \leq \frac{C}{\log(\frac{1}{|x|} + e)}, \qquad x \in \mathbb{R}^n.$$

(iii) For $q(\cdot) \in C^{\log}_\infty(\mathbb{R}^n)$, the following inequality is satisfied at infinity:

$$|q(x) - q_\infty| \leq \frac{C_\infty}{\log(|x| + e)}, \quad x \in \mathbb{R}^n.$$

(iv) The space $C^{\log} = C^{log}_{\text{loc}} \bigcap C^{log}_\infty$ denotes the set of all globally log-Hölder continuous functions $q(\cdot)$.

It has been demonstrated in [24] that if $q(\cdot) \in \mathfrak{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$.

Suppose $w(x)$ is a weight function on $\mathbb{R}^n$ that is nonnegative and locally integrable. The space $L^{q(\cdot)}(w)$ comprises all complex-valued functions ζ on $\mathbb{R}^n$ such that $\zeta w^{\frac{1}{q(\cdot)}} \in L^{q(\cdot)}(\mathbb{R}^n)$. This space forms a Banach function space with the norm

$$\|\zeta\|_{L^{q(\cdot)}(w)} = \|\zeta w^{\frac{1}{q(\cdot)}}\|_{L^{q(\cdot)}}.$$

In [23], Benjamin Muckenhoupt introduced the A_p-weight theory for $(1 < p < \infty)$ on $\mathbb{R}^n$. Subsequently, Noi and Izuki extended the Muckenhoupt A_p-class by allowing p to vary, as described in [25, 26].

Definition 2.2. Assume that $q(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. A weight w is deemed an $A_{q(\cdot)}$-weight if it satisfies the condition

$$\sup_B |B|^{-1} \|w^{-1/q(\cdot)} \chi_B\|_{L^{q'(\cdot)}} \|w^{1/q(\cdot)} \chi_B\|_{L^{q(\cdot)}} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. It was established in [27] that $w \in A_{q(\cdot)}$ if and only if the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on the variable exponent Lebesgue space $L^{q(\cdot)}$.

Remark 2.3. [25] Let $q(\cdot), p(\cdot) \in \mathfrak{P}(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ and suppose that $q(\cdot) \leq p(\cdot)$ holds pointwise on $\mathbb{R}^n$. Under these stipulations, it is deducible that

$$A_1 \subset A_{q(\cdot)} \subset A_{p(\cdot)}.$$

Definition 2.4. Let $\tau \in (0, n)$ and suppose that $p_1(\cdot), p_2(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ such that the relation

$$\frac{1}{p_1(x)} = \frac{1}{p_2(x)} + \frac{\tau}{n}$$

holds pointwise for all $x \in \mathbb{R}^n$. A weight w is designated as an $A(p_1(\cdot), p_2(\cdot))$-weight if it satisfies the inequality

$$|B|^{\frac{\tau}{n}-1} \|\chi_B\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C,$$

where the constant C is independent of the choice of the ball $B \subset \mathbb{R}^n$.

Definition 2.5. [25] Assume that $\tau \in (0, n)$, $p_1(\cdot), p_2(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$, and that the relationship

$$\frac{1}{p_1(x)} = \frac{1}{p_2(x)} + \frac{\tau}{n}$$

holds pointwise on $\mathbb{R}^n$. Under these conditions, it is both necessary and sufficient for $w \in A_{(p_1(\cdot), p_2(\cdot))}$ that $w^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)}$.

Definition 2.6. If $p(\cdot) \in \mathfrak{P}$ and $\lambda \in \mathbb{R}$, the weighted central Morrey space with a variable exponent, denoted as $\dot{B}^{p(\cdot),\lambda}(w^{p(\cdot)})$, is defined by:

$$\dot{B}^{p(\cdot),\lambda}(w) = \{\zeta \in L^{p(\cdot)}_{loc}(w) : \|\zeta\|_{\dot{B}^{p(\cdot),\lambda}(w)} < \infty\},$$

where the norm is given by:

$$\|\zeta\|_{\dot{B}^{p(\cdot),\lambda}(w)} = \sup_{R>0} \frac{\|\zeta\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}{|B(0,R)|^{\lambda} \|\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}.$$

Definition 2.7. If $p(\cdot) \in \mathfrak{P}$ and $\lambda < \frac{1}{n}$, the weighted λ−BMO space with a variable exponent, denoted by $CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})$, is defined as:

$$CBMO^{p(\cdot),\lambda}(w) = \{\zeta \in L^{p(\cdot)}_{loc}(w) : \|\zeta\|_{CBMO^{p(\cdot),\lambda}(w)} < \infty\},$$

where the norm is expressed by:

$$\|\zeta\|_{CBMO^{p(\cdot),\lambda}(w)} = \sup_{R>0} \frac{\|(\zeta - \zeta_{B(0,R)})\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}{|B(0,R)|^{\lambda} \|\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}.$$

Lemma 2.8. *[28] Let X be a Banach function space. Then the following properties hold:*

(1) The associated space X', defined as the Kothe dual of X, is necessarily also a Banach function space.

(2) The norms $\|\cdot\|_X$ and $\|\cdot\|_{(X')'}$ are equivalent, preserving the parallel structure of these dual spaces.

(3) (Generalized Hölder inequality) For every $\zeta \in X'$ and $\psi \in X$, the inequality

$$\int_{\mathbb{R}^n} |\psi(x)\zeta(x)| \leq \|\zeta\|_{X'} \|\psi\|_X$$

holds, characterizing the duality between X and X'.

Proposition 2.9. *[29] Let E be an open set and let $p(\cdot) \in \mathfrak{P}(E)$ satisfy the following conditions:*

$$|p(t) - p(z)| \leq \frac{-c}{log(|t-z|)}, \frac{1}{2} \geq |t-z| \tag{2.1}$$

$$|p(t) - p(z)| \leq \frac{-c}{log(|t|+e)}, |t| \leq |z|. \tag{2.2}$$

Then $p(\cdot) \in \mathfrak{D}(\mathbb{R}^n)$, where C is a positive constant independent of t and z.

Lemma 2.10. *[30] Let X be a Banach function space. Suppose that the Hardy-Littlewood maximal operator $\mathcal{M}$ is weakly bounded on X, satisfying the condition*

$$\|\chi_{\{\mathcal{M}\zeta > \rho\}}\|_X \lesssim \rho^{-1}\|\zeta\|_X$$

for all $\rho > 0$ and every $\zeta \in X$. Under this assumption, it follows that

$$\sup_{B:ball} \frac{1}{|B|}\|\chi_B\|_X\|\chi_B\|_{X'} < \infty.$$

Lemma 2.11. *[25] For any ball $B \subset \mathbb{R}^n$ and any Banach function space X, it holds that*

$$1 \leq \frac{1}{|B|}\|\chi_B\|_X\|\chi_B\|_{X'}.$$

Lemma 2.12. *[25] Let X be a Banach function space, and suppose the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on the associate space X'. Then, for any measurable sets $E \subset \mathbb{R}^n$ and $S \subset E$, there exists a constant $\sigma \in (0,1)$, such that the inequality*

$$\frac{\|\chi_S\|_X}{\|\chi_E\|_X} \lesssim \left(\frac{|S|}{|E|}\right)^{\sigma}$$

is satisfied, where the implicit constant is independent of S and E.

Lemma 2.13. *[31]*

(1) The space $X(\mathbb{R}^n, w)$, defined as

$$\|\zeta\|_{X(\mathbb{R}^n,w)} = \|\zeta w\|_X,$$

is a Banach function space under the norm

$$X(\mathbb{R}^n, w) = \{\zeta \in \mathcal{M} : \zeta w \in X\}.$$

(2) The associated space $X'(\mathbb{R}^n, w^{-1})$, defined analogously, is likewise a Banach function space.

Remark 2.14. Let $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. Upon comparing the variable exponent Lebesgue spaces $L^{p(\cdot)}(w^{p(\cdot)})$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ with the generalized function space $X(\mathbb{R}^n, W)$, the following observations arise:

1: If $w = W$ and $X = L^{p(\cdot)}(\mathbb{R}^n)$, then

$$L^{p(\cdot)}(w^{p(\cdot)}) = L^{p(\cdot)}(\mathbb{R}^n, w).$$

2: If $w^{-1} = W$ and $X = L^{p'(\cdot)}(\mathbb{R}^n)$, then

$$L^{p'(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{p'(\cdot)}(w^{-p'(\cdot)}).$$

By invoking the result of Lemma 2.6, we deduce that

$L^{p'(\cdot)}(\mathbb{R}^n, w^{-1}) = (L^{p(\cdot)}(\mathbb{R}^n, w))' = L^{p'(\cdot)}(w^{-p'(\cdot)}) = (L^{p(\cdot)}(w^{p(\cdot)}))'.$

Lemma 2.15. *[32] Assume $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ and that the weight $w^{p_2(\cdot)} \in A_{p_2(\cdot)}$ ensures $w^{-p_2'(\cdot)} \in A_{p_2'(\cdot)}$. Under these conditions, there exist constants $\sigma_{33}, \sigma_{44} \in (0,1)$ such that the following inequalities hold:*

$$\frac{\|\chi_S\|_{(L^{p_2(\cdot)}w^{p_2(\cdot)})'}}{\|\chi_E\|_{(L^{p_2(\cdot)}w^{p_2(\cdot)})'}} \lesssim \left(\frac{|S|}{|E|}\right)^{\sigma_{33}}, \quad \frac{\|\chi_S\|_{(L^{p_1(\cdot)}w^{p_1(\cdot)})'}}{\|\chi_E\|_{(L^{p_1(\cdot)}w^{p_1(\cdot)})'}} \lesssim \left(\frac{|S|}{|E|}\right)^{\sigma_{44}} \tag{2.3}$$

for every ball $E \subset \mathbb{R}^n$ and for all measurable subsets $S \subset E$.

In an analogous manner, should the conditions $p_1(\cdot), p_2(\cdot) \in \mathfrak{D}(\mathbb{R}^n)$ be satisfied, it follows by virtue of Lemma 2.15 that there exist constants $\sigma_{11} \in (0, \frac{1}{(q_1)_+})$ and $\sigma_{22} \in (0, \frac{1}{(q_2)_+})$ such that the following inequalities hold:

$$\frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_E\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|E|}\right)^{\sigma_{11}} \tag{2.4}$$

$$\frac{\|\chi_S\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_E\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|E|}\right)^{\sigma_{22}} \tag{2.5}$$

for all balls $E \subset \mathbb{R}^n$ and $S \subset E$.

Lemma 2.16. *[25] Let $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$, $0 < \tau < \frac{n}{p_+}$, and suppose the relationship $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\tau}{n}$ holds. Under these conditions, the fractional integral operator I_τ is bounded from the variable exponent Lebesgue space $L^{p(\cdot)}(w^{P(\cdot)})$ to $L^{q(\cdot)}(w^{q(\cdot)})$ provided that the weight w satisfies the Muckenhoupt-type condition $w \in A(p(\cdot), q(\cdot))$.*

Lemma 2.17. *[33] Assume that $q(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. Then, for any $b \in BMO$ and for all integers all $j, i \in Z$ with $j > i$, the following inequalities hold:*

1. The characterization of the BMO via variable exponent Lebesgue norms is:

$$C^{-1}\|b\|_{BMO} \leq \sup_{B:Ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}}}\|(b - b_B)\chi_B\|_{L^{q(\cdot)}} \leq C\|b\|_{BMO}, \tag{2.6}$$

where $b_B = \frac{1}{|B|}\int_B b(x)dx$, and C is a constant independent of b and B.

2. A decay estimate for the difference of b over disjoint balls is:

$$\|(b - b_{B_i})\chi_{B_j}\|_{L^{q(\cdot)}} \leq C(j - i)\|b\|_{BMO}\|\chi_{B_j}\|_{L^{q(\cdot)}}, \tag{2.7}$$

where B_i and B_j are nested balls, and C is a constant depending only on $q(\cdot)$.

3. Boundedness properties of the n-dimensional bilinear fractional Hardy operator and its adjoint operator

Theorem 3.1. *Presume that $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ complies with the stipulations delineated in conditions (2.1) and (2.2) as articulated in Proposition 2.9. Let us prescribe the variable exponent $q(\cdot)$ such that $\frac{1}{q(\cdot)} + \frac{\tau}{n} = \frac{1}{p(\cdot)}$, where $\frac{1}{p(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$. Additionally, assume that $w^{q_1(\cdot)}, w^{q_2(\cdot)} \in A_1$. Define the parameter $\lambda = \lambda_1 + \lambda_2 + \frac{\tau}{n}$ and impose the restriction $\lambda > (\sigma_{33} + \sigma_{44} + \sigma)$, where the constants σ_{33}, σ_{44}, and σ are identical to those introduced in Lemmas (2.15) and (2.12). Under these assumptions, the following inequality holds:*

$$\|H_\tau(\zeta_1, \zeta_2)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})} \leq C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}.$$

Proof. If we formulate the identities $\zeta_1 = \zeta_1 \cdot \chi_i = \zeta_1 \cdot \chi_{B_i}$ and $\zeta_2 = \zeta_2 \cdot \chi_i = \zeta_2 \cdot \chi_{B_i}$ for an arbitrary $i \in \mathbb{Z}$, it follows that the functions ζ_1 and ζ_2 can be decomposed in the following manner:

$$\zeta_1(x) = \sum_{i=-\infty}^{\infty} \zeta_1(x) \cdot \chi_i(x) = \sum_{i=-\infty}^{\infty} \zeta_1 \chi(x)_{B_i}.$$

$$\zeta_2(x) = \sum_{i=-\infty}^{\infty} \zeta_2(x) \cdot \chi_i(x) = \sum_{i=-\infty}^{\infty} \zeta_2 \chi(x)_{B_i}.$$

Leveraging the extended Hölder inequality, we deduce the following bound:

$$\begin{aligned}
|H_\tau(\zeta_1, \zeta_2)(x) \cdot \chi_j(x)| &\leq \frac{1}{|x|^{2n-\tau}} \int_{B_j} \int_{B_j} |\zeta_1(y_1)||\zeta_2(y_2)| dy_1 dy_2 \cdot \chi_j(x) \\
&= \frac{1}{|x|^{2n-\tau}} \int_{B_j} |\zeta_1(y_1)| dy_1 \int_{B_j} |\zeta_2(y_2)| dy_2 \cdot \chi_j(x) \\
&\leq C 2^{-2jn} \sum_{i=-\infty}^{j} \|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\
&\times \|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'} 2^{j\tau} \chi_j(x).
\end{aligned}$$

$$\begin{aligned}
&\|H_\tau(\zeta_1, \zeta_2) \cdot \chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} \\
&\leq C 2^{j\tau} \sum_{i=-\infty}^{j} \|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'} 2^{-2jn} \|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} \\
&\leq C \sum_{i=-\infty}^{j} \|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'} 2^{-j(2n-\tau)} \|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}. \qquad (3.1)
\end{aligned}$$

We postulate $\zeta = \chi_{B_j}$ and exploit the defining characteristics of the operator I_τ, yielding

$$I_\tau(\chi_{B_j})(x) \geq C 2^{j\tau} \chi_{B_j}(x),$$

$$\chi_{B_j}(x) \leq C 2^{-j\tau} I_\tau(\chi_{B_j})(x).$$

Applying the norm to both sides and leveraging the results encapsulated in Lemmas 2.11 and 2.16, we infer the following estimate:

$$\begin{aligned}
\|\chi_{B_j}\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\leq C 2^{-j\tau} \|I_\tau \chi_{B_j}\|_{L^{q(\cdot)}(w^{q(\cdot)})} \\
&\leq C 2^{-j\tau} \|\chi_{B_j}\|_{L^{p(\cdot)}(w^{p(\cdot)})} \\
&\leq C 2^{-j\tau} \|\chi_{B_j}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_{B_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\
&\leq C 2^{j(2n-\tau)} \|\chi_{B_j}\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\chi_{B_j}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}. \qquad (3.2)
\end{aligned}$$

To advance the argument, the substitution of Eq (3.2) into inequality (3.1), followed by the invocation of Lemma 2.15, furnishes the bound

$$\begin{aligned}\|H_\tau(\zeta_1,\zeta_2)\cdot\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\le C\sum_{i=-\infty}^{j}\|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\\ &\quad\times\|\chi_j\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_j\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\\ &\le C\sum_{i=-\infty}^{j}\|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\frac{\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}{\|\chi_j\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}\frac{\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}{\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}\\ &\le C\sum_{i=-\infty}^{j}2^{n\sigma_{33}(i-j)}2^{n\sigma_{44}(i-j)}\|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ &\le C\sum_{i=-\infty}^{j}2^{(n\sigma_{33}+n\sigma_{44})(i-j)}\|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.\end{aligned}$$

$$\begin{aligned}\|H_\tau(\zeta_1,\zeta_2)\cdot\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\le C\|\zeta_1\|_{\dot B^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot B^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=-\infty}^{j}2^{(n\sigma_{33}+n\sigma_{44})(i-j)}w(B_i)^{\lambda_1}w(B_i)^{\lambda_2}\\ &\quad\times\|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.\end{aligned}$$

Under the structural stipulations $\frac{1}{q(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}-\frac{\tau}{n}$ and $\lambda=\lambda_1+\lambda_2+\frac{\tau}{n}$, one may expound upon the norm relation of the characteristic function χ_i in a manner that accentuates the fundamental measure-theoretic decomposition:

$$\|\chi_i\|_{L^{q(\cdot)}(w^{q(\cdot)})}=w(B_i)^{\frac{1}{q(\cdot)}}=w(B_i)^{\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}-\frac{\tau}{n}}=\|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}w(B_i)^{-\frac{\tau}{n}}.$$

Subsequently, an invocation of Lemma 2.12 facilitates the reconfiguration of the preceding inequality into a more structurally refined form:

$$\begin{aligned}\|H_\tau(\zeta_1,\zeta_2)\cdot\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\le C\|\zeta_1\|_{\dot B^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot B^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=-\infty}^{j}2^{(n\sigma_{33}+n\sigma_{44})(i-j)}w(B_i)^{\lambda_1+\lambda_2+\frac{\tau}{n}}\|\chi_i\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\ &\le C\|\zeta_1\|_{\dot B^{q_1,\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot B^{q_2,\lambda_2}(w^{q_2(\cdot)})}\sum_{i=-\infty}^{j}2^{(n\sigma_{33}+n\sigma_{44})(i-j)}w(B_j)^{\lambda}\frac{w(B_i)^{\lambda}}{w(B_j)^{\lambda}}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\frac{\|\chi_i\|_{L^{q(\cdot)}(w^{q(\cdot)})}}{\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}}\\ &\le C\|\zeta_1\|_{\dot B^{q_1,\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot B^{q_2,\lambda_2}(w^{q_2(\cdot)})}\sum_{i=-\infty}^{j}2^{(n\sigma_{33}+n\sigma_{44})(i-j)}w(B_j)^{\lambda}\frac{w(B_i)^{\lambda}}{w(B_j)^{\lambda}}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\frac{\|\chi_i\|_{L^{q(\cdot)}(w^{q(\cdot)})}}{\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}}\\ &\le C\|\zeta_1\|_{\dot B^{q_1,\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot B^{q_2,\lambda_2}(w^{q_2(\cdot)})}\sum_{i=-\infty}^{j}2^{(\sigma_{33}+\sigma_{44}+\sigma+\lambda)n(i-j)}w(B_j)^{\lambda}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}.\end{aligned}$$

This sequence of transformations, meticulously orchestrated through a synthesis of functional analytic principles and weighted norm inequalities, underscores the interplay between the scaling properties of the weight functions and the underlying structure of the central Morrey space embeddings.

$$\|H_\tau(\zeta_1,\zeta_2)\|_{\dot B^{q(\cdot),\lambda}(w^{q(\cdot)})}\le C\|\zeta_1\|_{\dot B^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot B^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=-\infty}^{j}2^{(\sigma_{33}+\sigma_{44}+\sigma+\lambda)n(i-j)}.$$

By invoking the constraint $\lambda > -(\sigma_{33} + \sigma_{44} + \sigma)$, one may deduce the requisite boundedness condition, culminating in the refined estimate

$$\|H_\tau(\zeta_1, \zeta_2)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})} \leq C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}.$$

Theorem 3.2. *Assume that $q_1(\cdot)$ and $q_2(\cdot)$, both elements of the class $\mathfrak{P}(\mathbb{R}^n)$, adhere to the constraints prescribed by conditions (2.1) and (2.2) as delineated in Proposition 2.9. The variable exponent $q(\cdot)$ is then rigorously defined by the relation:*

$$\frac{1}{q(\cdot)} = \frac{1}{q_2(\cdot)} + \frac{1}{q_1(\cdot)} - \frac{\tau}{n}.$$

Let λ satisfy the expression $\lambda = \lambda_1 + \lambda_2 + \frac{\tau}{n}$ and impose the restriction $\tau < n(\sigma_{11} + \sigma_{22} - \sigma + \lambda)$, where the constants σ_{11}, σ_{22}, and σ correspond to those appearing in inequalities (2.4) and (2.5). Under these circumstances, the subsequent inequality holds:

$$\|H^*_\tau(\zeta_1, \zeta_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})} \leq C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}.$$

Proof. By invoking Hölder's inequality, we establish the following upper bound:

$$\begin{aligned}|H^*_\tau(\zeta_1, \zeta_2)(x) \cdot \chi_j(x)| &\leq \int_{R^n\setminus B_j}\int_{R^n\setminus B_j} \frac{1}{|y|^{2n-\tau}}|\zeta_1(y)||\zeta_2(y)|dy_1dy_2 \cdot \chi_j(x)\\ &\leq C\sum_{i=j+1}^{\infty} 2^{-i(2n-\tau)}\|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ &\times \|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\chi_j(x).\end{aligned}$$

Consequently, taking the norm in the weighted variable exponent Lebesgue space, one arrives at

$$\begin{aligned}\|H^*_\tau(\zeta_1, \zeta_2)(x) \cdot \chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\leq C\sum_{i=j+1}^{\infty} 2^{-i(2n-\tau)}\|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ &\times \|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}.\end{aligned}$$

Employing the inequality (3.2), one deduces the refined estimate:

$$\begin{aligned}\|H^*_\tau(\zeta_1, \zeta_2)(x) \cdot \chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\leq C\sum_{i=j+1}^{\infty} \|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ &\times \|\chi_i\|^{-1}_{L^{q(\cdot)}(w^{q(\cdot)})}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\ &\leq C\sum_{i=j+1}^{\infty} 2^{n\sigma(j-i)}\|\zeta_1\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ &\leq C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=j+1}^{\infty} 2^{n\sigma(j-i)}\frac{w(B_i)^{\lambda_1}}{w(B_j)^{\lambda_1}}w(B_j)^{\lambda_1}\frac{w(B_i)^{\lambda_2}}{w(B_j)^{\lambda_2}}w(B_j)^{\lambda_2}\\ &\times \frac{\|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}}{\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}}\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\frac{\|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\end{aligned}$$

$$\leq C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=j+1}^{\infty}2^{(n\sigma-n\sigma_{11}-n\sigma_{22})(j-i)}\left(\frac{w(B_i)}{w(B_j)}\right)^{\lambda_1+\lambda_2}w(B_j)^{\lambda_1+\lambda_2}$$
$$\times\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.$$

In the present exposition, we invoke the fundamental identity $\frac{1}{q(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}-\frac{\tau}{n}$ in conjunction with the structural relation $\lambda=\lambda_1+\lambda_2+\frac{\tau}{n}$, which subsequently yields the expression

$$\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}=\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}w(B_j)^{-\frac{\tau}{n}}.$$

Consequently, leveraging the preceding formulation, we establish the upper bound

$$\begin{aligned}\|H_\tau^*(\zeta_1,\zeta_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}&\leq C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=j+1}^{\infty}2^{(n\sigma-n\sigma_{11}-n\sigma_{22})(j-i)}\\&\quad\times\left(\frac{w(B_i)}{w(B_j)}\right)^{\lambda_1+\lambda_2}w(B_j)^{\lambda_1+\lambda_2+\frac{\tau}{n}}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\&\leq C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=j+1}^{\infty}2^{(n\sigma-n\sigma_{11}-n\sigma_{22}-n\lambda+\tau)(j-i)}w(B_j)^{\lambda}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\end{aligned}$$

$$\|H_\tau^*(\zeta_1,\zeta_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})}\leq C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=j+1}^{\infty}2^{(n\sigma-n\sigma_{11}-n\sigma_{22}-n\lambda+\tau)(j-i)}.$$

In light of the imposed constraint $\tau<n(\sigma_{11}+\sigma_{22}-\sigma+\lambda)$, the summation converges, thereby furnishing the desired boundedness assertion

$$\|H_\tau^*(\zeta_1,\zeta_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})}\leq C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}.$$

4. Commutators associated with the n-dimensional bilinear fractional Hardy operator and its adjoint operator

Theorem 4.1. *Let $p(\cdot)\in\mathfrak{P}(\mathbb{R}^n)$ be a function that satisfies the hypotheses stipulated by conditions (2.1) and (2.2), as articulated in Proposition 2.9. The variable exponent $q(\cdot)$ is hereby defined via the relation*

$$\frac{1}{q(\cdot)}+\frac{\tau}{n}=\frac{1}{p(\cdot)},$$

where $\frac{1}{p(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}$.

Assume further that $\lambda=\nu+\lambda_1+\lambda_2+\frac{\tau}{n}$, and that the inequality $\lambda>(\sigma_{33}+\sigma_{44}+\sigma)$ is satisfied, where σ_{33}, σ_{44}, and σ are constants consistent with those introduced in Lemma (2.15). Then, the following inequality holds:

$$\|[b,H_\tau](\zeta_1,\zeta_2)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})}\leq C\|b\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})},$$

where $b=(b_1,b_2)$ and $b\in CBMO^{q(\cdot),\nu}(w^{q(\cdot)})$.

Proof. By invoking the generalized Hölder inequality in conjunction with Lemma 2.17, we derive the ensuing estimate:

$$
\begin{aligned}
|[b_1,H_\tau](\zeta_1,\zeta_2)(x)\cdot\chi_j(x)| &\le \frac{1}{|x|^{2n-\tau}}\int_{B_j}\int_{B_j}|\zeta_1(y_1)\zeta_2(y_2)(b_1(x)-b_1(y_1))|dy_1dy_2\cdot\chi_j(x)\\
&\le \frac{1}{|x|^{2n-\tau}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|\zeta_1(y_1)\zeta_2(y_2)(b_1(x)-(b_1)_{B_i}+(b_1)_{B_i}-b_1(y_1))|dy_1dy_2\cdot\chi_j(x)\\
&\le \frac{1}{|x|^{2n-\tau}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|\zeta_1(y_1)\zeta_2(y_2)(b_1(x)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
&+\frac{1}{|x|^{2n-\tau}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|\zeta_1(y_1)\zeta_2(y_2)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
&= I+II.
\end{aligned}
$$

The term I can then be estimated by employing the generalized Hölder inequality over variable exponent weighted Lebesgue spaces. Explicitly, one obtains:

$$
\begin{aligned}
I =&\frac{1}{|x|^{2n-\tau}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|\zeta_1(y_1)\zeta_2(y_2)(b_1(x)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
\le& C2^{-j(2n-\tau)}\sum_{i=-\infty}^{j}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}|(b_1(x)-(b_1)_{B_i})\chi_j(x)|.
\end{aligned}
$$

$$
\begin{aligned}
\|I\|_{L^{q(\cdot)}(w^{q(\cdot)})} \le& C2^{-j(2n-\tau)}\sum_{i=-\infty}^{j}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\|(b_1(x)-(b_1)_{B_i})\chi_j(x)\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
\le& C2^{-j(2n-\tau)}\sum_{i=-\infty}^{j}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}(j-i)\|b_1\|_{BMO}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}.
\end{aligned}
\tag{4.1}
$$

In a similar fashion, the term II may be estimated as follows:

$$
\begin{aligned}
II =& \frac{1}{|x|^{2n-\tau}}\sum_{i=-\infty}^{j}\int_{B_j}\int_{B_j}|\zeta_1(y_1)\zeta_2(y_2)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
\le& C2^{-j(2n-\tau)}\sum_{i=-\infty}^{j}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|b_1\|_{BMO}\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}|\chi_j(x)|
\end{aligned}
$$

$$
\begin{aligned}
\|II\|_{L^{q(\cdot)}(w^{q(\cdot)})} \le& C2^{-j(2n-\tau)}\sum_{i=-\infty}^{j}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|b_1\|_{BMO}\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}.
\end{aligned}
\tag{4.2}
$$

By virtue of inequalities (4.1) and (4.2), we deduce the following bound:

$$\|[b_1, H_\tau](\zeta_1, \zeta_2) \cdot \chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} \leq C 2^{-j(2n-\tau)} \sum_{i=-\infty}^{j} (j-i) \|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \times \|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'} \|b_1\|_{BMO} \|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}.$$

$$\|[b_1, H_\tau](\zeta_1, \zeta_2) \cdot \chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} \leq C \sum_{i=-\infty}^{j} (j-i) \|b_1\|_{BMO} \|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'} 2^{-j(2n-\tau)} \|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}. \tag{4.3}$$

In order to advance further, the substitution of (3.2) into (4.3) furnishes the ensuing inequality:

$$\begin{aligned}
\|[b_1, H_\tau](\zeta_1, \zeta_2) \cdot \chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\leq C \sum_{i=-\infty}^{j} (j-i) \|b_1\|_{BMO} \|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'} \\
&\times \|\chi_j\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\chi_j\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))} \\
&\leq C \sum_{i=-\infty}^{j} (j-i) \|b_1\|_{BMO} \|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \frac{\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_j\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}} \frac{\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}{\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}} \\
&\leq C \sum_{i=-\infty}^{j} 2^{n\sigma_{33}(i-j)} 2^{n\sigma_{44}(i-j)} (j-i) \|b_1\|_{BMO} \|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\
&\leq C \sum_{i=-\infty}^{j} 2^{(n\sigma_{33}+n\sigma_{44})(i-j)} (j-i) \|b_1\|_{BMO} \|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.
\end{aligned}$$

$$\begin{aligned}
\|[b_1, H_\tau](\zeta_1, \zeta_2) \cdot \chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\leq C \|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})} \|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})} \sum_{i=-\infty}^{j} 2^{(n\sigma_{33}+n\sigma_{44})(i-j)} (j-i) \|b_1\|_{BMO} w(B_i)^{\lambda_1} w(B_i)^{\lambda_2} \\
&\times \|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\
&\leq C \|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})} \|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})} \sum_{i=-\infty}^{j} 2^{(n\sigma_{33}+n\sigma_{44})(i-j)} (j-i) \frac{\|(b_1 - (b_1)_{b_i})\chi_{B_i}\|_{L^{q(\cdot)}(w^{q(\cdot)})}}{\|\chi_{B_i}\|_{L^{q(\cdot)}(w^{q(\cdot)})}} \\
&\times w(B_i)^{\lambda_1} w(B_i)^{\lambda_2} \|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\
&\leq C \|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})} \|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})} \sum_{i=-\infty}^{j} 2^{(n\sigma_{33}+n\sigma_{44})(i-j)} (j-i) \frac{\|(b_1 - (b_1)_{b_i})\chi_{B_i}\|_{L^{q(\cdot)}(w^{q(\cdot)})}}{w(B_i)^{\nu} \|\chi_{B_i}\|_{L^{q(\cdot)}(w^{q(\cdot)})}} \\
&\times w(B_i)^{\nu} w(B_i)^{\lambda_1} w(B_i)^{\lambda_2} \|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\
&\leq C \|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})} \|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})} \sum_{i=-\infty}^{j} 2^{(n\sigma_{33}+n\sigma_{44})(i-j)} (j-i) \|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})} \\
&\times w(B_i)^{\nu+\lambda_1+\lambda_2} \|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.
\end{aligned}$$

Under the stipulated conditions $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)} - \frac{\tau}{n}$ and $\lambda = \nu + \lambda_1 + \lambda_2 + \frac{\tau}{n}$, the norm expression can be reformulated as follows:

$$\|\chi_i\|_{L^{q(\cdot)}(w^{q(\cdot)})} = w(B_i)^{\frac{1}{q(\cdot)}} = w(B_i)^{\frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)} - \frac{\tau}{n}} = \|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} w(B_i)^{-\frac{\tau}{n}}.$$

The preceding inequality may be reformulated into an equivalent yet structurally refined expression, as follows:

$$\begin{aligned}\|[b_1,H_\tau](\zeta_1,\zeta_2)\cdot\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\le C\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\\ &\quad\times\sum_{i=-\infty}^{j}2^{(n\sigma_{33}+n\sigma_{44})(i-j)}w(B_i)^{\nu+\lambda_1+\lambda_2+\frac{\tau}{n}}\|\chi_i\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\ &\le C\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\|\zeta_1\|_{\dot{B}^{q_1,\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2,\lambda_2}(w^{q_2(\cdot)})}\\ &\quad\times\sum_{i=-\infty}^{j}2^{(n\sigma_{33}+n\sigma_{44})(i-j)}w(B_j)^{\lambda}\frac{w(B_i)^{\lambda}}{w(B_j)^{\lambda}}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\frac{\|\chi_i\|_{L^{q(\cdot)}(w^{q(\cdot)})}}{\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}}\\ &\le C\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_1(\cdot)})}\\ &\quad\times\sum_{i=-\infty}^{j}2^{(\sigma_{33}+\sigma_{44}+\sigma+\lambda)n(i-j)}w(B_j)^{\lambda}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}.\end{aligned}$$

The preceding inequality admits a further refinement into a more succinct and structurally elucidative formulation, expressed as follows:

$$\|[b_1,H_\tau](\zeta_1,\zeta_2)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})}\le C\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=-\infty}^{j}2^{(\sigma_{33}+\sigma_{44}+\sigma+\lambda)n(i-j)}.$$

By invoking the constraint $\lambda>-(\sigma_{33}+\sigma_{44}+\sigma)$, one secures the subsequent fundamental bound:

$$\|[b_1,H_\tau](\zeta_1,\zeta_2)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})}\le C\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}.$$

In an analogous manner, a straightforward adaptation of the preceding argument yields the corresponding estimate:

$$\|[b_2,H_\tau](\zeta_1,\zeta_2)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})}\le C\|b_2\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}.$$

Theorem 4.2. *Consider $q_1(\cdot)$ and $q_2(\cdot)$, elements of the class $\mathfrak{P}(\mathbb{R}^n)$, satisfying the constraints enunciated in conditions (2.1) and (2.2) as expounded in Proposition 2.9. The variable exponent $q(\cdot)$ is prescribed through the relation:*

$$\frac{1}{q(\cdot)}=\frac{1}{q_2(\cdot)}+\frac{1}{q_1(\cdot)}-\frac{\tau}{n}.$$

Let the parameter λ be given as $\lambda=\nu+\lambda_1+\lambda_2+\frac{\tau}{n}$, and suppose that the inequality $\lambda<n\sigma_{11}+n\sigma_{22}-n\sigma-\frac{\tau}{n}$ is satisfied, where the constants σ_{11}, σ_{22}, and σ are those appearing in inequalities (2.4) and (2.5). Under these assumptions, the following inequality holds:

$$\|[b,H_\tau^*](\zeta_1,\zeta_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})}\le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})},$$

where $b=(b_1,b_2)$ and $b\in CBMO^{q(\cdot),\nu}$.

Proof. By invoking Hölder's inequality, one deduces the following bound:

$$
\begin{aligned}
|[b_1, H_\tau^*](\zeta_1,\zeta_2)(x)\cdot\chi_j(x)| &\le \int_{R^n\setminus B_j}\int_{R^n\setminus B_j}\frac{1}{|y|^{2n-\tau}}|\zeta_1(y)\zeta_2(y)(b_1(x)-b_1(y_1))|dy_1dy_2\cdot\chi_j(x)\\
&= \int_{R^n\setminus B_j}\int_{R^n\setminus B_j}\frac{1}{|y|^{2n-\tau}}|\zeta_1(y)\zeta_2(y)(b_1(x)-(b_1)_{B_i}+(b_1)_{B_i}-b_1(y_1))|dy_1dy_2\cdot\chi_j(x)\\
&\le \int_{R^n\setminus B_j}\int_{R^n\setminus B_j}\frac{1}{|y|^{2n-\tau}}|\zeta_1(y)\zeta_2(y)(b_1(x)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
&+ \int_{R^n\setminus B_j}\int_{R^n\setminus B_j}\frac{1}{|y|^{2n-\tau}}|\zeta_1(y)\zeta_2(y)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
&= I + II.
\end{aligned}
$$

$$
\begin{aligned}
I &= \int_{R^n\setminus B_j}\int_{R^n\setminus B_j}\frac{1}{|y|^{2n-\tau}}|\zeta_1(y)\zeta_2(y)(b_1(x)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
II &= \int_{R^n\setminus B_j}\int_{R^n\setminus B_j}\frac{1}{|y|^{2n-\tau}}|\zeta_1(y)\zeta_2(y)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
|I| &\le C\sum_{i=j+1}^{\infty}2^{-i(2n-\tau)}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)})'}\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}(b_1(x_1)-(b_1)_{B_i})\cdot\chi_j(x).\\
\|I\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\le C\sum_{i=j+1}^{\infty}2^{-i(2n-\tau)}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\|(b_1(x_1)-(b_1)_{B_i})\cdot\chi_j(x)\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\le C\sum_{i=j+1}^{\infty}2^{-i(2n-\tau)}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\|b_1\|_{BMO}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}. \qquad (4.4)
\end{aligned}
$$

Now,

$$
\begin{aligned}
II &= \int_{R^n\setminus B_j}\int_{R^n\setminus B_j}\frac{1}{|y|^{2n-\tau}}|\zeta_1(y)\zeta_2(y)(b_1(y_1)-(b_1)_{B_i})|dy_1dy_2\cdot\chi_j(x)\\
&\le C\sum_{i=j+1}^{\infty}2^{-i(2n-\tau)}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|(b_1(y_1)-(b_1)_{B_i})\cdot\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\cdot\chi_j(x).
\end{aligned}
$$

By invoking Lemma 2.17, one establishes the ensuing bound:

$$
\begin{aligned}
\|II\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\le C\sum_{i=j+1}^{\infty}2^{-i(2n-\tau)}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|(b_1(y_1)-(b_1)_{B_i})\cdot\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\le C\sum_{i=j+1}^{\infty}2^{-i(2n-\tau)}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|b_1\|_{BMO}\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}.
\end{aligned}
\qquad (4.5)
$$

By synthesizing the implications of Eqs (4.4) and (4.5), one arrives at the following estimate:

$$\begin{aligned}\|[b_1, H_\tau^*](\zeta_1,\zeta_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} \le C\sum_{i=j+1}^{\infty} 2^{-i(2n-\tau)}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|b_1\|_{BMO}\|\chi_i\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\\ \times\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_i\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}.\end{aligned}$$

Subsequently, by invoking inequality (3.2) in conjunction with Lemma 2.15, one derives:

$$\begin{aligned}[b]\|[b_1, H_\tau^*](\zeta_1,\zeta_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\le C\sum_{i=j+1}^{\infty}\|b_1\|_{BMO}\|\zeta_{1i}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\zeta_{2i}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\times\|\chi_i\|^{-1}_{L^{q(\cdot)}(w^{q(\cdot)})}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\
&\le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=j+1}^{\infty}2^{n\sigma(j-i)}\frac{\|(b_1-(b_1)_{b_i})\|_{L^{q(\cdot)}(w^{q(\cdot)})}}{\|\chi_{B_i}\|_{L^{q(\cdot)}(w^{q(\cdot)})}}\\
&\times\frac{w(B_i)^{\lambda_1}}{w(B_j)^{\lambda_1}}w(B_j)^{\lambda_1}\frac{w(B_i)^{\lambda_2}}{w(B_j)^{\lambda_2}}w(B_j)^{\lambda_2}\\
&\times\frac{\|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}}{\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}}\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\frac{\|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=j+1}^{\infty}2^{n\sigma(j-i)}\frac{\|(b_1-(b_1)_{b_i})\|_{L^{q(\cdot)}(w^{q(\cdot)})}}{w(B_i)^{\nu}\|\chi_{B_i}\|_{L^{q(\cdot)}(w^{q(\cdot)})}}\\
&\times w(B_i)^{\nu}\frac{w(B_i)^{\lambda_1}}{w(B_j)^{\lambda_1}}w(B_j)^{\lambda_1}\frac{w(B_i)^{\lambda_2}}{w(B_j)^{\lambda_2}}w(B_j)^{\lambda_2}\\
&\times\frac{\|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}}{\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}}\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\frac{\|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\sum_{i=j+1}^{\infty}2^{n\sigma(j-i)}\frac{\|(b_1-(b_1)_{b_i})\|_{L^{q(\cdot)}(w^{q(\cdot)})}}{w(B_i)^{\nu}\|\chi_{B_i}\|_{L^{q(\cdot)}(w^{q(\cdot)})}}\\
&\times\frac{w(B_i)^{\nu}}{w(B_j)^{\nu}}w(B_j)^{\nu}\frac{w(B_i)^{\lambda_1}}{w(B_j)^{\lambda_1}}w(B_j)^{\lambda_1}\frac{w(B_i)^{\lambda_2}}{w(B_j)^{\lambda_2}}w(B_j)^{\lambda_2}\\
&\times\frac{\|\chi_i\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}}{\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}}\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\frac{\|\chi_i\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\\
&\times\sum_{i=j+1}^{\infty}2^{(n\sigma-n\sigma_{11}-n\sigma_{22})(j-i)}\left(\frac{w(B_i)}{w(B_j)}\right)^{\lambda_1+\lambda_2+\nu}w(B_j)^{\lambda_1+\lambda_2+\nu}\\
&\times\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.\end{aligned}$$

Here, we utilize the identity $\frac{1}{q(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{q_2(\cdot)}-\frac{\tau}{n}$, along with the relationship $\lambda=\nu+\lambda_1+\lambda_2+\frac{\tau}{n}$, to express the following equivalence:

$$\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}=\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}w(B_j)^{-\frac{\tau}{n}}.$$

Consequently, we deduce the inequality:

$$\begin{aligned}\|[b_1, H_\tau^*](\zeta_1,\zeta_2)(x)\cdot\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})} &\le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\sum_{i=j+1}^{\infty}2^{(n\sigma-n\sigma_{11}-n\sigma_{22})(j-i)}\\ &\quad\times\left(\frac{w(B_i)}{w(B_j)}\right)^{\nu+\lambda_1+\lambda_2} w(B_j)^{\nu+\lambda_1+\lambda_2+\frac{\tau}{n}}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\\ &\le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\\ &\quad\times\sum_{i=j+1}^{\infty}2^{(n\sigma-n\sigma_{11}-n\sigma_{22}-\lambda+\frac{\tau}{n})(j-i)}w(B_j)^{\lambda}\|\chi_j\|_{L^{q(\cdot)}(w^{q(\cdot)})}\end{aligned}$$

$$\|[b_1, H_\tau^*](\zeta_1,\zeta_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})} \le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}\sum_{i=j+1}^{\infty}2^{(n\sigma-n\sigma_{11}-n\sigma_{22}-\lambda+\frac{\tau}{n})(j-i)}.$$

By invoking the condition $\lambda < n\sigma_{11} + n\sigma_{22} - n\sigma - \frac{\tau}{n}$, we obtain the requisite conclusion:

$$\|[b_1, H_\tau^*](\zeta_1,\zeta_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})} \le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\|b_1\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}.$$

In a similar vein, we may swiftly approximate the ensuing result:

$$\|[b_2, H_\tau^*](\zeta_1,\zeta_2)(x)\|_{\dot{B}^{q(\cdot),\lambda}(w^{q(\cdot)})} \le C\|\zeta_1\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\|\zeta_2\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\|b_2\|_{CBMO^{q(\cdot),\nu}(w^{q(\cdot)})}.$$

5. Conclusions

This treatise furnishes substantial advancements in the analytical exploration of bilinear fractional Hardy operators within the framework of weighted central Morrey spaces endowed with variable exponents. The findings herein elucidate profound insights into the boundedness characteristics of commutators linked to H_τ (or H_τ^*) in conjunction with the weighted λ-central BMO function, thereby enriching the theoretical landscape of such operators.

Author contributions

The authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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