Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(5): 10324–10339.
DOI: 10.3934/math.2025470
Received: 15 November 2024
Revised: 01 April 2025
Accepted: 07 April 2025
Published: 06 May 2025

Research article

Lower deviation probabilities for supercritical Markov branching processes with immigration

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Abstract: Let $\{Z(t); t \geq 0\}$ be a continuous-time supercritical branching process with immigration (MBPI) with the offspring mean $m(t)$. In this paper, we mainly research the lower deviation probabilities $\mathrm{P}(Z(t) = k_t)$ and $\mathrm{P}(0 \leq Z(t) \leq k_t)$ with $k_t/e^{m(t)} \to 0$ as $t \to \infty$. Moreover, we present the local limit theorem and some related estimates of the MBPIs. For our proofs, we use the well-known Cramér method to prove the large deviation of the sum of independent variables to satisfy our needs.

Keywords: lower deviation; supercritical; branching process; immigration; Cramér method
Mathematics Subject Classification: 60J27, 60J35

1. Introduction

Consider a continuous-time Markov branching process with immigration (MBPI), denoted $\{Z(t); t \geq 0\}$. This process comprises two components, including existing individuals and external immigration. Moreover, these two components are independently and identically distributed, following the evolutionary law determined by the branching rates $\{b_j; j \geq 0, j \neq 1\}$ and the immigration rates $\{a_j; j \geq 1\}$

$$\begin{cases} b_j \geq 0 (j \neq 1), & 0 < -b_1 = \sum_{j \neq 1} b_j < \infty, \\ a_j \geq 0 (j \neq 0), & 0 < -a_0 = \sum_{j \neq 0} a_j < \infty, \end{cases} \tag{1.1}$$

respectively. The corresponding Q-matrix $Q = \{q_{ij}; i, j \in \mathbb{Z}_+\}$ is defined as follows:

$$q_{ij} := \begin{cases} ib_{j-i+1} + a_{j-i} & if\ i \geq 0, j \geq i, \\ ib_0 & if\ i \geq 0, j = i-1, \\ 0 & otherwise. \end{cases} \tag{1.2}$$

Hence, the process above is completely determined by the infinitesimal generating functions $B(u) = \sum_{j=0}^{\infty} b_j u^j$ and $A(u) = \sum_{j=0}^{\infty} a_j u^j$ for $u \in [0, 1)$.

Throughout this paper, we assume that $b_0 = 0$ and $m := \sum_{j=0}^{\infty} jb_j < \infty$. By the definition of the branching rates $\{b_j; j \geq 0\}$ and the fact that $b_0 = 0$, we can deduce that $m > 0$, indicating that the Markov branching process is supercritical.

If $a_0 = 0$, then $Z(t)$ represents a pure branching process defined by $\{Z^0(t); t \geq 0\}$. From Athreya and Ney [1], we know that a normalized function $C(t)$ exists such that $W(t) := Z^0(t)/C(t) \to W$ as t tends to ∞, where W is a nondegenerate random variable and $C(t)$ satisfies $\lim_{t\to\infty} C(t+s)/C(t) = E[Z^0(s)] = e^{ms}$. Moreover, we find that $EW = 1$ holds if and only if the $L \log L$-moment condition holds. If this condition holds, W has a continuous density function $w(y)$ on $(0, \infty)$ such that the following global limit theorem holds:

$$\lim_{t\to\infty} P(W(t) \geq x) = \int_x^{\infty} w(y)dy, \ x > 0. \tag{1.3}$$

If $a_0 \neq 0$, according to Li et al. [2], we see that $Z(t+s)/Z(t)$ converges to e^{ms} in probability. Note that $Z(t)$ can be expressed as

$$Z(t+s) = \sum_{i=1}^{Z(t)} \xi_{t,i}(s) + Y(t) := Z^0(t) + Y(t), \tag{1.4}$$

where $\{\xi_{t,i}(s); t \geq 0, i \geq 1\}$ are independently and identically distributed (i.i.d.) random variables with the same law as $Z^0(s)$ and $Y(t)$ is the number of particles at moment $t+s$, being either immigrants or offspring of immigrants in $(t, t+s]$, which is independent of $\{\xi_{t,i}(s); i \geq 1\}$ and $Z(t)$. Obviously, the distribution of $Y(\cdot)$ is independent of t.

Here, we are interested in the lower deviation probabilities $P(Z(t) = k_t)$ and $P(0 \leq Z(t) \leq k_t)$ as $k_t/C(t) \to 0$ $(t \to \infty)$, since these probabilities characterize the evolution of the population when the population growth is below the average growth rate. Besides being of some interest in its own right, the asymptotic behavior of these probabilities is related to large deviations.

According to previous literature, large deviations are important and has drawn widespread attention from scholars. For the supercritical branching process, Athreya and Ney [3] considered the local limit theorem and some related aspects. Athreya [4] discussed the decay rates of $P(|Z_{n+1}/Z_n - \lambda| > \varepsilon)$(where λ denotes the offspring's mean) for a classical Galton Watson process $\{Z_n; n \geq 1\}$. Ney and Vidyashankar [5] considered the harmonic moments and large deviation rates in 2003. Moreover, Ney and Vidyashankar [6] considered the local limit theory and large deviations in 2004. Fleischmann and Wachtel [7] studied the lower deviation probabilities.

For the supercritical branching processes with immigration $\{X_n; n \geq 1\}$, Seneta [8] and Pakes [9] considered the supercritical Galton Watson process with immigration. Chu et al. [10] researched the small value probabilities. Liu and Zhang [11] studied the decay rates of $P(|X_{n+1}/X_n - \lambda| > \varepsilon)$. Sun and Zhang [12, 13] considered the convergence rates of harmonic moments and the lower deviations. Furthermore, Li and Zhang [14] focused on the harmonic moments and large deviations for a critical Galton Watson process with immigration.

In recent years, the continuous-time Markov branching processes have drawn widespread attention. For example, Li et al. [15] researched the large deviation rates for Markov branching processes. For the Markov branching process with immigration, based on Li et al. [15], Li et al. [16] studied asymptotic properties. Inspired by [7] and [2], we deal with the asymptotic behavior of $P(Z(t) = k_t)$

and $P(0 \le Z(t) \le k_t)$, known as global and local lower deviation probabilities under the assumption of $E[Z(1)\log Z(1)] < \infty$. Moreover, we apply the Cramér method to analyze the large deviation of the sum of independent variables.

The rest of this paper is organized as follows. In Section 2, we state some necessary preliminaries, apply the Cramér method to the MBPIs, and present some related estimates. In Section 3, we list the main results. Section 4 is devoted to concrete and detailed proofs concerning the main results.

2. Preliminaries

Define $P^0(t) = (p^0_{ij}(t); i \ge 1, j \ge 1)$ as the transition function of the pure branching process without immigration $\{Z^0(t); t \ge 0\}$ and let $F^0(s,t) = \sum_{j=0}^{\infty} p^0_{1j}(t)s^j$ be the probability generating function of $\{Z^0(t); t \ge 0\}$ with the initial state $Z^0(0) = 1$.

Let $P(t) = (p_{ij}(t); i \ge 1, j \ge 1)$ be the transition function of the MBPI $\{Z(t); t \ge 0\}$ and $G_l(s,t) := E[s^{Z(t)}|Z(0) = l] = \sum_{j=0}^{\infty} p_{lj}(t)s^j$ with $G_l(s,0) = s^l$ for $0 \le s < 1$. Moreover, by Li et al. [16]

$$G_l(s,t) = H(s,t) \cdot [F^0(s,t)]^l, \quad l \in \mathbb{Z}_+, \tag{2.1}$$

where $H(s,t) = G_0(s,t)$. In particular, $G(s,t) := G_1(s,t) = H(s,t)F^0(s,t)$ with the initial state $Z(0) = 1$.

Proposition 2.1. *Define the function $Q(v) = \sum_{j=1}^{\infty} q_j v^j$ as the unique solution of*

$$B(v)Q'(v) + (A(v) - a_0 - b_1)Q(v) = 0, \qquad 0 \leqslant v < 1,$$

subject to

$$Q(0) = 0, \ \ Q'(0) = 1, \ \ Q(1) = \infty \ \ and \ \ Q(v) < \infty \ \ for \ \ 0 \leqslant v < 1,$$

where q_j satisfies

$$q_j := \begin{cases} p_{11}(t)\mathrm{e}^{-(b_1+a_0)t} & if \ j = 1, \\ \lim_{t\to\infty} p_{1j}(t)\mathrm{e}^{-(b_1+a_0)t} & if \ j \ge 2, \end{cases}$$

with $q_1 = 1$, $q_j \leqslant \Pi_{k=1}^{j-1}(1 + \frac{a_0}{kb_1})$ $(j \ge 2)$.

Proof. This follows from the Kolmogorov forward equation. □

Proposition 2.2. *For any $0 \le s < 1$ and $t > 0$, take*

$$R(s,t) := \frac{H(s,t)}{\mathrm{e}^{a_0 t}}, \ \ Q^0_l(s,t) := \frac{F^0(s,t)}{\mathrm{e}^{b_1 l t}}. \tag{2.2}$$

Then

$$Q_l(s,t) := \frac{G_l(s,t)}{\mathrm{e}^{(a_0+b_1 l)t}} = R(s,t)(Q^0(s,t))^l \nearrow R(s)(Q^0(s))^l =: Q^l(s), \ t \to \infty, \tag{2.3}$$

where $R(s) := \lim_{t\to\infty} R(s,t)$ and $Q^0(s) := \lim_{t\to\infty} Q^0(s,t)$.

Proof. By Liu and Zhang [11], together with the definition of $G(\cdot,\cdot)$, we can derive the results above. □

For convenience, we always assume that the following conditions $(A1)-(A3)$ hold throughout this paper:

(A1) $b_0 = 0$;

(A2) $0 < m < \infty$;

(A3) $E[Z(1)\log Z(1)] < +\infty$.

On this basis, we give the main results of this paper in Section 3. First, we will introduce the Cramér transformation, which is the most critical step in the famous Cramér method (see Petrov [17]). By this transformation, we can obtain some important related estimates for the subsequent proofs.

For the real random variable X, let $X(h)$ be the random variable resulting from the Cramér transform determined by the constant $h \in \mathbb{R}$. Then $X(h)$ satisfies the following:

$$E[\mathrm{e}^{iaX(h)}] = \frac{E[\mathrm{e}^{(h+ia)X}]}{E[\mathrm{e}^{hX}]}, \quad a \in \mathbb{R}, \tag{2.4}$$

where $E[\mathrm{e}^{hX}] < \infty$.

Set the random variable $X = Z(t)$. Following the equation (2.4) above, it can be inferred that for any $h \leq 0$, the Cramér transformation $Z(-h/\mathrm{e}^{mt})$ exists and satisfies

$$E[\mathrm{e}^{iaZ(-h/\mathrm{e}^{mt})}] = \frac{G_l(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}, t)}{G_l(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t)}, \quad a \in \mathbb{R}. \tag{2.5}$$

For the convenience of discussion, we first give the Cramér transformation of $Z^0(t)$ and $Y(t)$ separately, since the branching part $Z^0(t)$ and the pure immigration part $Y(t)$ are independent. For any $h \leq 0$ and $t \geq 0$, we define a sequence of random variables $\{X_i(h,t); i \geq 1\}$, which are i.i.d. according to the Cramér transform of $Z^0(t)$ determined by the constant $-h/\mathrm{e}^{mt}$, i.e.,

$$P(X_1(h,t) = k) = \frac{\mathrm{e}^{-kh/\mathrm{e}^{mt}}}{F^0(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t)} P(Z^0(t) = k), \ k \geq 1.$$

The equation above can be rewritten as

$$E[\mathrm{e}^{iaZ^0(-h/\mathrm{e}^{mt})}] = E[\mathrm{e}^{iaX_1(h,t)}] = \frac{F^0(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}, t)}{F^0(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t)}.$$

For the pure immigrant part $Y(t)$, the Cramér transform is determined by the constant $-h/\mathrm{e}^{mt}$. We can define a random variable $T(h,t)$ that is independent of $X_i(h,t)$ and $Y(t)$. Thus, we have

$$P(T(h,t) = k) = \frac{\mathrm{e}^{-kh/\mathrm{e}^{mt}}}{H(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t)} P(Y(t) = k), \ k \geq 1,$$

which is equivalent to

$$E[\mathrm{e}^{iaT(h,t)}] = E[\mathrm{e}^{iaY(-h/\mathrm{e}^{mt})}] = \frac{H(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}, t)}{H(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t)}.$$

After the transformations above, we construct a sequence of independent random variables $\{S_l(h,t); t \geq 0, l \geq 1\}$ expressed as

$$S_l(h,t) := \sum_{i=1}^{l} X_i(h,t) + T(h,t), \ l \geq 1. \tag{2.6}$$

Thus

$$P(S_l(h,t)=k)=\frac{\mathrm{e}^{-kh/\mathrm{e}^{mt}}}{G_l(\mathrm{e}^{-h/\mathrm{e}^{mt}},t)}P(Z(t)=k|Z(0)=l),\ k\geq 1. \tag{2.7}$$

Proof. In order to prove that $S_l(h,t)$ and $Z(-h/\mathrm{e}^{mt},t)$ are identically distributed. It follows from (2.4) and the definition of $S_l(h,t)$ that

$$\begin{aligned} E\mathrm{e}^{iaS_l(h,t)} &= E\mathrm{e}^{ia[\sum_{i=1}^{l}X_i(h,t)+T(h,t)]} \\ &= \left(E\mathrm{e}^{iaZ^0(-h/\mathrm{e}^{mt})}\right)^l\cdot E\mathrm{e}^{iaY(-h/\mathrm{e}^{mt})} \\ &= \Big(\frac{E\mathrm{e}^{(-h/\mathrm{e}^{mt}+ia)Z^0(t)}}{E\mathrm{e}^{(-h/\mathrm{e}^{mt})Z^0(t)}}\Big)^l\cdot\frac{E\mathrm{e}^{(-h/\mathrm{e}^{mt}+ia)Y(t)}}{E\mathrm{e}^{(-h/\mathrm{e}^{mt})Y(t)}} \\ &= \left(\frac{F^0(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia},t)}{F^0(\mathrm{e}^{-h/\mathrm{e}^{mt}},t)}\right)^l\cdot\frac{H(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia},t)}{H(\mathrm{e}^{-h/\mathrm{e}^{mt}},t)} \\ &= \frac{G_l(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia},t)}{G_l(\mathrm{e}^{-h/\mathrm{e}^{mt}},t)}. \end{aligned}$$

On the other hand,

$$E\mathrm{e}^{iaS_l(h,t)}=\sum_{k=0}^{\infty}\mathrm{e}^{iak}P(S_l(h,t)=k).$$

If we compare the corresponding coefficients of the equations above, then (2.7) is established.

Denoting the characteristic function of $S_l(h,t)$ as $\Psi_S(a):=E[\mathrm{e}^{iaS_l(h,t)}]$, then we obtain

$$\Psi_S(a)=\frac{G_l(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia},t)}{G_l(\mathrm{e}^{-h/\mathrm{e}^{mt}},t)}. \tag{2.8}$$

By comparing (2.8) and (2.5), it can be found that $S_l(h,t)$ and $Z(-h/\mathrm{e}^{mt},t)$ are identically distributed. □

Definition 2.1. (Concentration function, [17], p. 38) For any $\lambda\geq 0$, the concentration function $Q(X;\lambda)$ of a random variable X is defined by the equality

$$Q(X;\lambda):=\sup_x P(x\leq X\leq x+\lambda). \tag{2.9}$$

Lemma 2.1. (Petrov [17], p. 38, Lemma 3) Suppose that X is a random variable with the characteristic function $\psi(t)$. Then for $\lambda\geq 0$ and $\theta>0$, we have

$$Q(X;\lambda)\leq\left(\frac{96}{95}\right)^2\max(\lambda,\frac{1}{\theta})\int_{-\theta}^{\theta}|\psi(t)|dt. \tag{2.10}$$

Lemma 2.2. *For any $h\geq 0$, $C(h)$ exists such that*

$$\sup_{t,k\geq 0}\mathrm{e}^{mt}P(S_l(h,t)=k)\leq C(h)l^{-1/2},\ \ l\geq l_0:=1+[1/\alpha], \tag{2.11}$$

where $\alpha=-\frac{\log\sigma}{m}$ and $\sigma:=\frac{\partial G(u,t)}{\partial u}\Big|_{(0,1)}=p_{11}(1)$.

Proof. First, if we recall Proposition 2.1, it is not difficult to find $p_{11}(t) = \mathrm{e}^{(b_1+a_0)t}$. By Lemma 2.2, we also have $\sigma := \frac{\partial G(u,t)}{\partial u}\Big|_{(0,1)} = p_{11}(1)$, and then we get $\sigma = e^{b_1+a_0}$ and thus it satisfies $p_{11}(t) = \sigma^t$.

First, we prove that (2.11) holds when $l = l_0 = [1/\alpha] + 1$. Taking $X = S_{l_0}(h,t)$, $\lambda = 1/2$, and $\theta = \pi$ in (2.10) together with (2.8), we have

$$Q(S_{l_0}(h,t); \frac{1}{2}) \le C \int_{-\pi}^{\pi} \frac{|G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}, t)|}{G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t)} da, \tag{2.12}$$

where C is a constant independent of h and t. Note that $S_{l_0}(h,t)$ is a non-negative integer random variable, then from (2.9), the inequality above is equivalent to

$$\sup_{k\ge1} P(S_{l_0}(h,t) = k) \le C \int_{-\pi}^{\pi} \frac{|G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}, t)|}{G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t)} da. \tag{2.13}$$

Consider the denominator part of the inequality above, owing to (1.4) and the branching property, we have

$$G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t) = \left(E[\mathrm{e}^{-h(Z^0(t)/\mathrm{e}^{mt})}]\right)^{l_0} (E[\mathrm{e}^{-h(Y(t)/\mathrm{e}^{mt})}]) \xrightarrow{t\to\infty} E[\mathrm{e}^{-h(W^{*l_0}+I)}] > 0, \tag{2.14}$$

where W^{*l_0} represents the l-fold convolution of W.

The convergence above is uniform for h in a compact subset of $\mathbb{R}_+$. Thus for any fixed h, a positive number t_0 exists such that $G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t) > \delta > 0$ for all $t \ge t_0$, which implies $\inf_{t>0} G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}}, t) > 0$. According to (2.12), a constant $C(h)$ exists such that

$$\sup_{k\ge1} P(S_{l_0}(h,t) = k) \le C(h) \int_{-\pi}^{\pi} |G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}, t)| da. \tag{2.15}$$

Now we only need to prove that the right side of the inequality above is bounded.

$$\begin{aligned}\int_{-\pi}^{\pi} |G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}, t)| da =& (\int_{-\pi}^{-\pi\mathrm{e}^{-mt}} + \int_{-\pi\mathrm{e}^{-mt}}^{\pi\mathrm{e}^{-mt}} + \int_{\pi\mathrm{e}^{-mt}}^{\pi}) |G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}, t)| da \\ :=& I_1 + I_2 + I_3.\end{aligned} \tag{2.16}$$

Clearly,

$$I_2 = \int_{-\pi\mathrm{e}^{-mt}}^{\pi\mathrm{e}^{-mt}} \left|G_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}, t)\right| da \le 2\pi\mathrm{e}^{-mt}. \tag{2.17}$$

It follows, by (2.3), that $\frac{G_l(s,t)}{\mathrm{e}^{(a_0+b_1 l)t}} \uparrow Q_l(s)$ as $t \uparrow \infty$, and then

$$I_3 \le \int_{\pi\mathrm{e}^{-mt}}^{\pi} Q_{l_0}(\mathrm{e}^{-h/\mathrm{e}^{mt}+ia}) \mathrm{e}^{(a_0+b_1 l_0)t} da. \tag{2.18}$$

It is easy to find that when t is large enough, a constant $C_1(h)$ exists such that the integrand is bounded. We can get $I_3 \le \pi\mathrm{e}^{-mt}\mathrm{e}^{(a_0+b_1 l_0)t} C_1(h)$. Taking $u = -a$ for I_1, after a similar analysis, $I_1 \le \pi\mathrm{e}^{-mt}\mathrm{e}^{(a_0+b_1 l_0)t} C_2(h)$ is obatined. Then $C_3(h)$ exists such that $C_1(h) + C_2(h) \le 2C_3(h)$. Hence

$$I_1 + I_2 + I_3 \le 2\pi\mathrm{e}^{-mt}[\mathrm{e}^{(a_0+b_1 l_0)t} C_3(h) + 1]. \tag{2.19}$$

By the definition of a_0, b_1, and m, the right side of (2.19) is bounded in $t \in (0, \infty)$. Due to (2.13), one has

$$\sup_{t\geq 0, k\geq 1} e^{mt} P(S_l(h,t) = k) \leq C(h). \tag{2.20}$$

When $l > l_0$, the result also holds as shown below. The sequence $\{X_i(h,t)\}_{i\geq 1}$ given by the Cramér transformation is a sequence of independently and identically distributed random variables. Set $S^0_{l_0}(h,t) := \sum_{i=1}^{l_0} X_i(h,t)$, then $S^0_{l_0}(h,t)$ is also nondegenerate, since $X_i(h,t)$ is nondegenerate. According to (98) in [7], for $j \geq 1$, $D_1(h) > 0$, and $D_2(h) > 0$, we have

$$\sup_{t\geq 0, k\geq 1} e^{mt} P(S^0_{jl_0}(h,t) = k) \leq \frac{D_1(h)}{\sqrt{j}} = \frac{D_2(h)}{\sqrt{jl_0}}. \tag{2.21}$$

According to Lemma 1 from Chapter III of Petrov [17], for any two independent random variables X and Y, the inequality $Q(X+Y;\lambda) \leq Q(\min(X,Y);\lambda)$ holds for $\forall\ \lambda > 0$. Now, letting $X = S^0_l(h,t)$ and $Y = T(h,t)$, by (2.6), we have

$$\begin{aligned} \sup_{t\geq 0, k\geq 1} e^{mt} P(S_l(h,t) = k) &\leq \sup_{t\geq 0, k\geq 1} e^{mt} P(S^0_l(h,t) = k) \\ &\leq \sup_{t\geq 0, k\geq 1} e^{mt} P(S^0_{[l/l_0]l_0}(h,t) = k). \end{aligned} \tag{2.22}$$

Taking $j = [l/l_0] \geq 1$ in (2.19), the inequality (2.11) above also holds for every $l > l_0$. □

Lemma 2.3. *The constants $\delta > 0$ and $D > 0$ exists such that*

$$e^{mt} P(Z(t) = k | Z(0) = l) \leq D e^{-\delta l} l^{-\frac{1}{2}} e^{k e^{-mt}}, \quad t > 0, k \geq 1, l \geq l_0 := [\frac{1}{\alpha}] + 1,$$

where α is defined as in Lemma 2.2.

Proof. It can be obtained from (2.7) that

$$P(Z(t) = k | Z(0) = l) = e^{kh/e^{mt}} G_l(e^{-h/e^{mt}}, t) P(S_l(h,t) = k),\ k \geq 1. \tag{2.23}$$

Multiplying both sides of (2.23) by e^{mt}, specifying $h = 1$, and using the definition of the generating function (2.1) together with Lemma 2.2, for $l \geq l_0$, we have

$$\begin{aligned} e^{mt} P(Z(t) = k | Z(0) = l) &\leq e^{kh/e^{mt}} G_l(e^{-1/e^{mt}}, t) \sup_{t,k} e^{mt} P(S_l(1,t) = k) \\ &\leq e^{k/e^{mt}} [F^0_l(e^{-1/e^{mt}}, t) H(e^{-1/e^{mt}}, t)] C(1) l^{-\frac{1}{2}} \\ &\leq C(1) (F^0(e^{-1/e^{mt}}, t))^l l^{-\frac{1}{2}} e^{k/e^{mt}}, \end{aligned} \tag{2.24}$$

where $C(1)$ is defined in Lemma 2.2. Note that

$$F^0(e^{-1/e^{mt}}, t) = E[e^{-Z^0(t)/e^{mt}}] \xrightarrow{t\to\infty} E[e^{-W}] \in (0,1).$$

There is a constant $\delta > 0$ such that $\sup_{t>0} F^0(e^{-1/e^{mt}}, t) \leq e^{-\delta} < 1$, which is proved by substituting the inequality above into (2.24). □

For sake of the subsequent discussions, we define the Laplace transform of the normalized random variables V, W, and I as follows:

$$\phi_V(u) := E[\mathrm{e}^{-uV}], \quad \phi_W(u) := E[\mathrm{e}^{-uW}], \quad \phi_I(u) := E[\mathrm{e}^{-uI}], \quad u \geq 0. \tag{2.25}$$

Moreover, we have $\phi_V(u) = \phi_W(u)\phi_I(u)$, since $Z^0(t)$ and $Y(t)$ are independent.

Proposition 2.3. (*Iterative functional equations for the Laplace transform*) *If* $b_0 = 0$*, then the following functional equations hold:*

$$\begin{aligned} \phi_V(\mathrm{e}^{ms}u) &= G(\phi_W(u), s), \quad u \geq 0, s \geq 0; \\ \phi_W(\mathrm{e}^{ms}u) &= F^0(\phi_W(u), s), \quad u \geq 0, s \geq 0; \\ \phi_I(\mathrm{e}^{ms}u) &= H(\phi_W(u), s), \quad u \geq 0, s \geq 0. \end{aligned} \tag{2.26}$$

Proof. The conclusion can be obtained from (2.1) together with (2.25). □

Record the characteristic function as $\phi^l_W(a)\phi_I(a) := (E[\mathrm{e}^{iaW}])^l E[\mathrm{e}^{iaI}]$, where $\phi^l_W(a)\phi_I(a)$ is the characteristic function of $w^{*l} * i(a)$.

Lemma 2.4. *For any* $x \in [a, b]$*, there are constants* $C > 0$ *and* $\lambda > 0$ *such that*

$$w^{*l} * i(x) \leq C\lambda^l, \quad l \geq 1. \tag{2.27}$$

Proof. This follows from Lemma 5 in Athreya [4]. □

Lemma 2.5. *Suppose that* $E[Z(1)\log Z(1)] < +\infty$ *and* $b_1 l + a_0 + m < 0$*. Then for* $0 < a < b$ *and* $l \geq 1$*,*

$$\lim_{t\to\infty} \frac{1}{2\pi} \int_{-\pi\mathrm{e}^{mt}}^{\pi\mathrm{e}^{mt}} G_l(\mathrm{e}^{ix/\mathrm{e}^{mt}}, t)\mathrm{e}^{-ixh}dx = w^{*l} * i(h) \tag{2.28}$$

and this converges uniformly on $[a, b]$.

Proof. This is achieved by the Fourier transform and decomposing the integral,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{2\pi} \int_{-\pi\mathrm{e}^{mt}}^{\pi\mathrm{e}^{mt}} G_l(\mathrm{e}^{ix/\mathrm{e}^{mt}}, t)\mathrm{e}^{-ixh}dx \\ &= \lim_{t\to\infty} \frac{1}{2\pi} \Big(\int_{-\pi\mathrm{e}^{mt}}^{-\pi} + \int_{-\pi}^{\pi} + \int_{\pi}^{\pi\mathrm{e}^{mt}}\Big) G_l(\mathrm{e}^{ix/\mathrm{e}^{mt}}, t)\mathrm{e}^{-ixh}dx \\ &:= H_1 + H_2 + H_3. \end{aligned}$$

Applying the dominated convergence theorem (see [18], Theorem 16, p. 89), we have

$$\begin{aligned} H_2 &= \lim_{t\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_{I(t)}(x)(\phi_{W(t)}(x))^l \mathrm{e}^{-ixh}dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_I(x)(\phi_W(x))^l \mathrm{e}^{-ixh}dx, \end{aligned}$$

where $\phi_{I(t)}(x) := E[\mathrm{e}^{ixI(t)}]$ and $\phi_{W(t)}(x) := E[\mathrm{e}^{ixW(t)}]$.

For

$$H_3 = \lim_{t\to\infty} \frac{1}{2\pi} \int_{\pi}^{\pi e^{mt}} G_l(e^{ix/e^{mt}}, t)e^{-ixh}dx,$$

the family of functions $\{e^{ix\xi}; \xi \geq 1\}$ is equicontinuous with respect to x and gives

$$F^0(e^{ixe^{-mt}}, t) \overset{t\to\infty}{\longrightarrow} \phi_W(x) \ \ and \ \ H(e^{ixe^{-mt}}, t) \overset{t\to\infty}{\longrightarrow} \phi_I(x)$$

uniformly with respect to $x \geq \pi$. Hence, we can examine $\int_{\pi}^{\pi e^{mt}} G_l(e^{ix/e^{mt}})e^{-ixh}dx$ by replacing $\int_{\pi}^{\pi e^{mt}} \phi_I(x)(\phi_W(x))^l e^{-ixh}dx$.

Notice that $|e^{-ixh}| = 1$ and thus

$$\begin{aligned} \frac{1}{2\pi} \int_{\pi}^{\pi e^{mt}} \phi_I(x)(\phi_W(x))^l e^{-ixh}dx \leq & \ \frac{1}{2\pi} \int_{\pi}^{\pi e^{mt}} \phi_I(x)(\phi_W(x))^l dx \\ = & \ \frac{1}{2\pi} \int_{\pi}^{\pi e^{mt}} G_l(\phi(xe^{-mt}), t)dx \\ = & \ \frac{1}{2\pi} \int_{\pi e^{-mt}}^{\pi} e^{mt} G_l(\phi(u), t)du. \end{aligned} \tag{2.29}$$

The last equation is obtained by substituting $x = ue^{mt}$.

Assume that

$$G_l(\phi(u), t) = \sum_{j=0}^{\infty} p_{lj}(t)\phi^j(u) = Q_l(\phi(u), t)e^{(b_1 l + a_0)t},$$

where $Q_l(\cdot, \cdot)$ is as defined in (2.3). Then

$$H_3 \leq \lim_{t\to\infty} \frac{1}{2\pi} \int_{\pi e^{-mt}}^{\pi} Q_l(\phi(u), t)e^{(b_1 l + a_0 + m)t}du.$$

Due to the assumption $a_0 + b_1 l + m < 0$, it is obvious that $e^{(a_0+b_1 l+m)t} \to 0$ as $t \to \infty$. Moreover, the interval of integration is also finite. Combining this with the convergence of $Q_l(\cdot, \cdot)$, we have $H_3 < \infty$. In the same way, we obtain $H_1 < \infty$. Therefore

$$\frac{1}{2\pi} \int_{-\pi e^{mt}}^{\pi e^{mt}} G_l(e^{ix/e^{mt}}, t)e^{-ixh}dx$$

converges to

$$\lim_{t\to\infty} \frac{1}{2\pi} \int_{-\pi e^{mt}}^{\pi e^{mt}} \phi_I(x)(\phi_W(x))^l e^{-ixh}dx$$

uniformly with respect to h. Finally, since $\phi_W(x)$ and $\phi_I(x)$ are the Fourier transformation of the probability density functions $w(x)$ and $i(x)$, combining these with the conclusion of Lemma 8 in Dubuc and Seneta [19] completes the conclusion. □

3. Main results

Throughout this paper, we suppose that the generating function $B(u)$ is aperiodic, i.e., the greatest common divisor of the set $\{i-j; i \neq j, b_i b_j > 0, i \geqslant 1, i \geqslant 1\}$ is 1. Assume that the sequence $\{k_t\}$ satisfies $k_t \to \infty$ and $k_t e^{-mt} \to 0$ as $t \to \infty$. Define $s_2 := \frac{\log k_t}{m}$. Then we certainly have $s_2 < t$ for a large enough t.

In the subsequent discussions, we always assume that the conditions $(A1)-(A3)$ hold. On this basis, we give the main results of this paper.

Theorem 3.1. *If we suppose that the generating function $B(u)$ is aperiodic and the assumptions $(A1)-(A3)$ hold, then*

$$P(Z(t) = k_t) = e^{-mt} v(k_t/e^{mt})(1 + o(1)),$$

where the sequence $\{k_t\}$ satisfies $k_t \to \infty$ and $k_t = o(e^{mt})$ as $t \to \infty$.

Theorem 3.2. *Suppose that the assumptions $(A1) - (A3)$ hold and the generating function $B(u)$ is aperiodic, then*

$$P(0 \leq Z(t) \leq k_t) = F_V(k_t/e^{mt})(1 + o(1)),$$

where the sequence $\{k_t\}$ are defined as in Theorem 3.1 and $F_V(x) = P(V \leq x)$.

Depending on Lemma 2.5, it is easy to conclude the following local limit theorem.

Theorem 3.3. (Local limit theorem) Suppose that the assumptions $(A1)-(A3)$ hold and the generating function $B(u)$ is aperiodic if the integer sequence $\{k_t\}$ satisfies $k_t \to \infty$, $k_t/e^{mt} \to h$, and $h > 0$ is a constant, then

$$\lim_{t\to\infty} e^{mt} P(Z(t) = k_t | Z(0) = l) = w^{*l} * i(h).$$

4. Proofs of the main results

In this section, we present detailed proofs related to the main results.

4.1. Proof of Theorem 3.1

Proof. According to the Markov property

$$\begin{aligned} P(Z(t) = k_t) &= \sum_{l=1}^{\infty} P(Z(s_1) = l) P(Z(t) = k_t | Z(s_1) = l) \\ &= \sum_{l=1}^{\infty} P(Z(s_1) = l) P(Z(s_2) = k_t | Z(0) = l), \end{aligned} \tag{4.1}$$

where $s_1 + s_2 = t$. There is an integer $N > 1$ such that

$$\begin{aligned} & P(Z(t) = k_t) \\ =& \sum_{l=1}^{N-1} P(Z(s_1) = l) P(Z(s_2) = k_t | Z(0) = l) + \sum_{l=N}^{\infty} P(Z(s_1) = l) P(Z(s_2) = k_t | Z(0) = l) \\ :=& I_1(N, t) + I_2(N, t). \end{aligned}$$

Next, we analyze the rate of convergence of $I_1(N,t)$ and $I_2(N,t)$. We begin with the second part $I_2(N,t)$. For a sufficiently large N such that $N \geq l_0$, by Lemma 2.3, there are $D > 0$ and $\delta > 0$ such that

$$\begin{aligned} &\mathrm{e}^{ms_2}\sum_{l=N}^{\infty} P(Z(s_1)=l)P(Z(s_2)=k_t|Z(0)=l) \\ \leq & D\sum_{l=N}^{\infty} \mathrm{e}^{-\delta l} l^{-1/2} \mathrm{e}^{k_t/\mathrm{e}^{ms_2}} P(Z(s_1)=l) \\ \leq & DN^{-1/2}\sum_{l=N}^{\infty} \mathrm{e}^{-\delta l} \mathrm{e}^{k_t/\mathrm{e}^{ms_2}} P(Z(s_1)=l) \\ \leq & DN^{-1/2} G(\mathrm{e}^{-\delta}, s_1)\mathrm{e}^{k_t/\mathrm{e}^{ms_2}} \\ \leq & DN^{-1/2}\sigma^{s_1}. \end{aligned} \tag{4.2}$$

The last inequality holds by $G(e^{-\delta}; s_1) = Ee^{-Z(s_1)\delta} < 1 \leq C\sigma^{s_1}$ together with the definition of σ, where s_1 does not grow when t and C can be chosen appropriately.

The treatment of $I_1(N,t)$ is given below. On the one hand, according to Lemma 2.5, for $l \geq 1$,

$$\lim_{t\to\infty} \frac{1}{2\pi}\int_{-\pi \mathrm{e}^{mt}}^{\pi \mathrm{e}^{mt}} G_l(\mathrm{e}^{iy/\mathrm{e}^{mt}}, t)\mathrm{e}^{-iyx} dy = w^{*l} * i(x) \tag{4.3}$$

holds uniformly on $[\mathrm{e}^{-m}, 1]$.

On the other hand, if $\frac{y_{s_2}}{\mathrm{e}^{ms_2}} \to a$ as $s_2 \to \infty$, where $a > 0$ is a constant, using the inversion formula

$$P(Z(s_2) = y_{s_2}|Z(0) = l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_l(\mathrm{e}^{iu}, s_2)\mathrm{e}^{-iuy_{s_2}} du.$$

Then $k_t = 1$ for all t. Setting $u = v/\mathrm{e}^{ms_2}$, we have

$$\mathrm{e}^{ms_2} P(Z(s_2) = k_t|Z(0) = l) = \frac{1}{2\pi}\int_{-\pi \mathrm{e}^{ms_2}}^{\pi \mathrm{e}^{ms_2}} G_l(\mathrm{e}^{iv/\mathrm{e}^{ms_2}}, s_2)\mathrm{e}^{-iv} dv. \tag{4.4}$$

According to (4.3)–(4.4) and Lemma 2.5

$$\lim_{t\to\infty}[\mathrm{e}^{ms_2} P(Z(s_2) = k_t|Z(0) = l) - w^{*l} * i(1)] = 0. \tag{4.5}$$

Hence

$$\begin{aligned} &\mathrm{e}^{ms_2}\sum_{l=1}^{N-1} P(Z(s_1)=l)P(Z(s_2)=k_t|Z(0)=l) \\ = & \left[\sum_{l=1}^{N-1} P(Z(s_1)=l) w^{*l} * i(1)\right](1+o(1)). \end{aligned} \tag{4.6}$$

Thus, for any fixed N

$$\begin{aligned} I_1(N,t) &= \mathrm{e}^{-ms_2}\sum_{l=1}^{N-1} P(Z(s_1)=l)w^{*l} * i(1)[1+o(1)] \\ &= \mathrm{e}^{-ms_2}\Big(\sum_{l=1}^{\infty} - \sum_{N}^{\infty}\Big)P(Z(s_1)=l)w^{*l} * i(1)[1+o(1)]. \end{aligned} \tag{4.7}$$

With Lemma 2.4, there are $C > 0$ and $\lambda \in (0,1)$ such that $w^{*l} * i(1) \le C\lambda^l$ for all $l > 1$, and hence

$$\mathrm{e}^{-ms_2} \sum_{l=N}^{\infty} P(Z(s_1) = l) w^{*l} * i(1) \le C\mathrm{e}^{-ms_2} \sum_{l=N}^{\infty} P(Z(s_1) = l)\lambda^l. \tag{4.8}$$

By (4.8) and the definition of $G(\cdot,\cdot)$, and taking the constant $\lambda_1 \in (\lambda, 1)$, similarly to the proof of inequality (4.2), we have

$$\begin{aligned} \sum_{l=N}^{\infty} P(Z(s_1) = l) w^{*l} * i(1) &\le C(\lambda/\lambda_1)^N \sum_{l=N}^{\infty} P(Z(s_1) = l)\lambda_1^l \\ &\le C(\lambda/\lambda_1)^N G(\lambda_1, s_1) \\ &\le C\mathrm{e}^{-\delta N}\sigma^{s_1}, \end{aligned} \tag{4.9}$$

where δ is a positive constant.

According to (4.1)–(4.2), (4.6), and (4.9)

$$P(Z(t) = k_t) = \mathrm{e}^{-ms_2}[\sum_{l=1}^{\infty} P(Z(s_1) = l) w^{*l} * i(1)][1 + o(1)] + O(\mathrm{e}^{-ms_2} N^{-1/2}\sigma^{s_1}). \tag{4.10}$$

If we take the equality $\phi(u\mathrm{e}^{ms_1}) = \phi_W(u\mathrm{e}^{ms_1})^l \phi_I(u\mathrm{e}^{ms_1}) = G(\phi(u), s_1)$ in the form of a density function, then for any $x \ge 0$

$$\sum_{l=1}^{\infty} P(Z(s_1) = l) w^{*l} * i(x) = v(x/\mathrm{e}^{ms_1})/\mathrm{e}^{ms_1} = v(xk_t/\mathrm{e}^{mt})/\mathrm{e}^{m(t-s_2)}. \tag{4.11}$$

By setting $x = k_t/\mathrm{e}^{ms_2} = 1$ in the equality above, then (4.10) becomes

$$P(Z(t) = k_t) = \mathrm{e}^{-mt} v(k_t/\mathrm{e}^{mt})[1 + o(1)] + O(\mathrm{e}^{-ms_2} N^{-1/2}\sigma^{s_1}). \tag{4.12}$$

Let t go to infinity first and then let N go to infinity in the equality above. The proof is completed. □

4.2. Proof of Theorem 3.2

Proof. By the Markov property, we have

$$P(Z(t) \le k_t) = \sum_{l=1}^{\infty} P(Z(s_1) = l) P(Z(s_2) \le k_t | Z(0) = l). \tag{4.13}$$

According to the branching property combined with the independence between immigration and branching, it follows that

$$\begin{aligned} P(Z(s_2) \le k_t | Z(0) = l) &= P(Z^0(s_2) + Y(s_2) \le k_t | Z(0) = l) \\ &\le P(Z^0(s_2) \le k_t | Z(0) = l) P(Y(s_2) \le k_t) \\ &\le [P(Z^0(s_2) \le k_t)]^l. \end{aligned} \tag{4.14}$$

By (1.4)

$$P(Z^0(s_2) \le k_t) = P(\frac{Z^0(s_2)}{e^{ms_2}} \le 1) \to \int_0^1 w(x)dx, \quad t \to \infty,$$

where $w(x)$ is continuous in $(0, \infty)$. Thus, a constant $\eta \in (0, 1)$ and a sufficiently large t exist such that $P(Z^0(s_2) \le k_t) \le \eta$. By formula (4.14) and a sufficiently large N, we have C and $\delta > 0$

$$\begin{aligned} \sum_{l=N}^{\infty} P(Z(s_1) = l)P(Z(s_2) \le k_t | Z(0) = l) &\le \sum_{l=N}^{\infty} P(Z(s_1) = l)\eta^l \\ &\le C\sigma^{s_1} e^{-\delta N}. \end{aligned} \tag{4.15}$$

If we assume $F_W(x) = P(W \le x)$ and $F_I(x) = P(I \le x)$, then by (1.3), together with the continuity of the distribution function

$$P(Z(s_2) \le x e^{ms_2} | Z(0) = l) \to F_W^{*l} * F_I(x)$$

uniformly in $x > 0$. Hence, we have

$$\lim_{t \to \infty} \sup_{k \ge 1} |P(Z(s_2) \le k_t | Z(0) = l) - F_W^{*l} * F_I(1)| = 0. \tag{4.16}$$

According to (4.13), (4.15), and (4.16)

$$P(Z(t) \le k_t) = [\sum_{l=1}^{\infty} P(Z(s_1) = l)F_W^{*l} * F_I(1)](1 + o(1)) + O(\sigma^{s_1} e^{-\delta N}). \tag{4.17}$$

Moreover, by the definition of s_2 and the inequality $F_W^{*l} * F_I(1) \ge F_W^{*l} * F_I(1/e^m)$, there is a constant $C \in [0, 1]$ such that

$$\sum_{l=1}^{\infty} P(Z(s_1) = l)F_W^{*l} * F_I(1) \ge P(Z(s_1) = 1)F_W^{*l} * F_I(1/e^m) \ge C\sigma^{s_1}.$$

Hence, (4.17) can be simplified to

$$P(Z(t) \le k_t) = [\sum_{l=1}^{\infty} P(Z(s_1) = l)F_W^{*l} * F_I(1)][1 + o(1) + O(e^{-\delta N})]. \tag{4.18}$$

Integrating both sides of the density function (4.11), we have

$$F_V(a/e^{mk}) = \sum_{l=1}^{\infty} P(Z(k) = l)F_W^{*l} * F_I(a). \tag{4.19}$$

Taking $k = s_1$ and $a = 1$ in equation above and substituting this into (4.18)

$$P(Z(t) \le k_t) = F_V(k_t/e^{mt})(1 + o(1) + O(e^{-\delta N})). \tag{4.20}$$

Letting $t \to \infty$ and $N \to \infty$ completes this proof. □

4.3. Proof of Theorem 3.3

Proof. This is similar to the proofs in [20] (Theorem 7.1, p. 105)

$$k_t e^{-mt} \to h(t \to \infty),$$

then by the inversion formula

$$P(Z(t) = k_t | Z(0) = l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_l(\mathrm{e}^{iu}, t)\mathrm{e}^{-iuk_t} du.$$

If we set $u = v\mathrm{e}^{-mt}$, then

$$\mathrm{e}^{mt} P(Z(t) = k_t | Z(0) = l) = \frac{1}{2\pi} \int_{-\pi \mathrm{e}^{mt}}^{\pi \mathrm{e}^{mt}} G_l(\mathrm{e}^{iv/\mathrm{e}^{mt}}, t)\mathrm{e}^{-ivk_t/\mathrm{e}^{mt}} dv.$$

Therefore

$$\begin{aligned} \lim_{t\to\infty} [\mathrm{e}^{mt} P(Z(t) = k_t | Z(0) = l)] &= \lim_{t\to\infty} \frac{1}{2\pi} \int_{-\pi \mathrm{e}^{mt}}^{\pi \mathrm{e}^{mt}} G_l(\mathrm{e}^{iv/\mathrm{e}^{mt}}, t)\mathrm{e}^{-ivk_t/\mathrm{e}^{mt}} dv \\ &= w^{*l} * i(h). \end{aligned}$$

The proof is completed. □

5. Conclusions

In this paper, we discuss a continuous-time supercritical branching process with immigration (MBPI). We mainly research the local lower deviation probabilities and the global lower deviation probabilities, obtain some related results such as local limit theorem and some related estimates of the MBPIs, which generalized the results of discrete-time branching processes to continuous-time cases.

Author contributions

Juan Wang: Conceptualization, writing—review and editing, funding acquisition; Chao Peng: Conceptualization, writing—original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is substantially supported by the National Natural Sciences Foundation of China (No.11901392).

Conflict of interest

The authors declare no conflicts of interest.

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