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Research article

Flag-transitive 2-designs with block size 5 and alternating groups

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Abstract: This paper contributes to the classification of flag-transitive 2-designs with block size 5. In a recent paper, the flag-transitive automorphism groups of such designs are reduced to point-primitive groups of affine type and almost simple type, and a classification is given of such automorphism groups with sporadic socle. In the present paper, we classify such designs admitting a flag-transitive automorphism group whose socle is an alternating group. We prove that there are precisely six such designs and determine the corresponding automorphism groups.

Keywords: 2-design; alternating group; flag-transitive; automorphism group; primitive group; classification
Mathematics Subject Classification: 05B05, 05B25, 05E18, 20B25

1. Introduction

A 2-(v, k, λ) design (or a 2-design in short) is a structure $(\mathcal{P}, \mathcal{B})$ consisting of a set $\mathcal{P}$ of v elements (called points) and a set $\mathcal{B}$ of some k-subsets (called blocks) of $\mathcal{P}$, satisfying the condition that any two points of $\mathcal{P}$ are contained in exactly λ blocks in $\mathcal{B}$. The number of blocks containing a fixed point is proved to be a constant and denoted by the letter r. The total number $|\mathcal{B}|$ of blocks is denoted by the letter b. These v, k, λ, b and r are called the parameters of a design. If

$$|\mathcal{B}| = \binom{v}{k},$$

then $\mathcal{D}$ is called a complete design. We only consider incomplete designs in this paper. We shall also denote a 2-design by $\mathcal{D}$. A point-block pair (x, B) such that $x \in B$ is called a flag of a 2-design. We denote by $\mathcal{F}$ the set of all flags in a 2-design.

A permutation g of the point set $\mathcal{P}$ of a 2-design

$$\mathcal{D} = (\mathcal{P}, \mathcal{B})$$

is an automorphism if g preserves the block set $\mathcal{B}$. All automorphisms of $\mathcal{D}$ form a group with the natural product of permutations, which is defined as the full automorphism group of $\mathcal{D}$. We usually denoted this group by $\mathrm{Aut}(\mathcal{D})$. A group G is called an automorphism group of $\mathcal{D}$ if G is a subgroup of $\mathrm{Aut}(\mathcal{D})$. An automorphism group of a 2-design $\mathcal{D}$ is called flag-transitive on $\mathcal{D}$ if it acts transitively on $\mathcal{F}$. Similarly, a group is point-primitive if it is primitive on the point set $\mathcal{P}$, and one can define other types of transitive actions such as block-transitive and point-transitive.

In the present paper we shall adopt usual notations in finite permutation groups, which shall be consistent with those in [1] or [2], for example. Let G be an automorphism group of a 2-design

$$\mathcal{D} = (\mathcal{P}, \mathcal{B}).$$

Commonly, we write G_x as the stabilizer of the point $x \in \mathcal{P}$ in G and write G_B as the block-stabilizer of the block $B \in \mathcal{B}$ in G. We always denote by Ω_n a set of cardinality n. Moreover, we denote by $\mathrm{Alt}(\Omega_n)$ (respectively, $\mathrm{Sym}(\Omega_n)$) the alternating group (respectively, the symmetric group) on Ω_n. Sometimes, in abbreviation, we also adopt the notation $\mathrm{Alt}(n)$ (or even shorter, $\mathrm{A_n}$) and $\mathrm{Sym}(n)$ (or $\mathrm{S_n}$).

Classifying flag-transitive 2-designs is a long-term project. A very classic result on flag-transitive finite linear spaces, namely, 2-$(v, k, 1)$ designs, is given by Kantor [3] and a team of six [4]. Some classifications for symmetric designs with special automorphism groups are studied in [5–7], for 2-transitive groups and primitive groups of rank 3, respectively. Inspecting flag-transitive 2-designs with their block size a small number is also an important research direction. A 2-design with block size 2 has trivial structure, having the block set all the 2-subsets of points, and a block-transitive automorphism group of such a design is 2-homogeneous. Reductions for flag-transitive 2-designs with block sizes 3 and 4 are tackled in [8]. Then, flag-transitive 2-designs with block size 5 were studied by the first author and Zhou in [9]. They reduced such flag-transitive automorphism groups to point-primitive of affine type and almost simple type and gave a classification on the case that the groups have a sporadic simple socle.

The well-known Classification of Finite Simple Groups states that a finite non-abelian simple group is isomorphic to one of the groups in the following four types: (1) alternating groups; (2) classical simple groups; (3) exceptional simple groups of Lie type; and (4) sporadic simple groups. Since the authors in [9] have dealt with flag-transitive 2-designs with block size 5 in terms of sporadic simple groups. As a continuation of this classification project, in the current paper we will classify such 2-designs admitting a flag-transitive almost simple automorphism group whose socle is a simple alternating group. A complete classification is given in the following Theorem 1.1, as the main result of the paper.

Theorem 1.1. *Let $\mathcal{D}$ be a* 2-$(v, 5, \lambda)$ *design, and let G be an almost simple, flag-transitive, point-primitive automorphism group of $\mathcal{D}$ with alternating socle* $\mathrm{Alt}(n)$ *($n \geq 5$). Then one of the following holds:*

(1) *$\mathcal{D}$ is a unique* 2-$(10, 5, 8)$ *design with* $G = \mathrm{Alt}(6)$ *or* M_{10}.

(2) *$\mathcal{D}$ is a unique* 2-$(10, 5, 16)$ *design with* $G = \mathrm{Sym}(6)$, $\mathrm{PGL}_2(9)$, *or* $\mathrm{P\Gamma L}_2(9)$.

(3) *$\mathcal{D}$ is a unique* 2-$(21, 5, 12)$ *design with* $G = \mathrm{Alt}(7)$ *or* $\mathrm{Sym}(7)$.

(4) *$\mathcal{D}$ is a unique* 2-$(15, 5, 4)$ *design with* $G = \mathrm{Alt}(7)$.

(5) *$\mathcal{D}$ is a unique* 2-(15, 5, 12) *design with* $G = \mathrm{Alt}(7)$.

(6) *$\mathcal{D}$ is a unique* 2-(15, 5, 16) *design with* $G = \mathrm{Alt}(8)$.

Further, all the groups G are 2-transitive on $\mathcal{P}$, except the group G in (3)*, which acts on $\mathcal{P}$ with rank 3 and with the two suborbits of length 10.*

Remark 1.1. It is an important fact that for a fixed parameter set (v, k, λ), there could be many non-isomorphic designs with such parameters. In fact, for Theorem 1.1 (1), (2), and (4), check [10, Section II. 1.3] and we can see that there are more than 135922, 108, and 896 non-isomorphic designs with

$$(v, k, \lambda) = (10, 5, 8), (10, 5, 16) \text{ and } (15, 5, 4),$$

respectively. However, under the condition that the design admits a flag-transitive automorphism group with the socle a simple alternating group, there is only one design with each above parameter set.

Combining the above main theorem with the results in [9], we have the following immediate corollary.

Corollary 1. *If G is a flag-transitive automorphism group of a* 2-$(v, 5, \lambda)$ *design. Then G is point-primitive of affine type or almost simple type with socle a simple group of Lie type, or is one of the cases listed in Theorem 1.1 and [9, Theorem 2].*

According to the result, we see that the remaining work of classifying flag-transitive 2-$(v, 5, \lambda)$ designs is on the affine groups and almost simple groups with socle a simple group of Lie type. Hence, we have the following problem, which needs further study.

Problem. *Classify* 2-$(v, 5, \lambda)$ *designs admitting a flag-transitive, point-primitive automorphism group whose socle is an elementary abelian group or a simple group of Lie type.*

In the following, we will first collect some useful preliminary results in Section 2 and then prove Theorem 1.1 in Section 3.

2. Preliminary results

Before we start the proof of the main result, we first present some necessary preliminary results concerning 2-designs and their automorphisms.

The following facts about the parameters of a 2-design are well known and play a key role in the proof.

Lemma 2.1. [11, Section 2.1] *The parameters of a* 2-(v, k, λ) *design satisfy the following:*

(1) $r(k-1) = \lambda(v-1)$;

(2) $bk = vr$;

(3) $b \geq v$ *and* $r \geq k$;

(4) $r^2 > \lambda v$.

The following lemma concerns the non-trivial subdegrees of a flag-transitive group G on a 2-design $\mathcal{D}$. Note that a subdegree of G is the length of a non-trivial orbit of a point-stabilizer G_x, where $x \in \mathcal{P}$.

Lemma 2.2. *[12, Lemma 1] If G is a group acting flag-transitively on a 2-design, and d is a non-trivial subdegree of G, then $r \mid \lambda d$.*

In particular, for 2-designs with $k = 5$, by Lemma 2.1 (1) and Lemma 2.2, we have the following immediate corollary.

Lemma 2.3. *If G is a group acting flag-transitively on a 2-$(v, 5, \lambda)$ design, then for each non-trivial subdegree d of G, we have $v - 1 \mid 4d$.*

The following lemma is the simplified version of the classification of maximal subgroups of finite alternating groups and symmetric groups. More details shall be found in [13], where the authors particularly investigated the maximality of primitive groups, including the groups of affine type, almost simple type, diagonal type, and product action type.

Lemma 2.4. *[13, Theorem] Let G be an alternating group $\mathrm{Alt}(n)$ or a symmetric group $\mathrm{Sym}(n)$ on a set Ω_n. If M is a maximal subgroup of G with*

$$M \neq \mathrm{Alt}(n),$$

then one of the following holds:

(1)
$$M \cong (\mathrm{Sym}(s) \times \mathrm{Sym}(t)) \cap G$$
acts intransitively on Ω_n, where
$$n = s + t,$$
both s and t are positive integer, and $s \neq t$;

(2)
$$M \cong (\mathrm{Sym}(s) \wr \mathrm{Sym}(t)) \cap G$$
acts transitively and imprimitively on Ω_n, where
$$n = st,$$
both s and t are positive integer;

(3) *M acts primitively on Ω_n.*

Suppose that G is a flag-transitive group of automorphisms of a 2-design $\mathcal{D}$. From Lemma 2.1 (4) we easily obtain
$$|G| < |G_x|^3$$
by
$$r = \frac{|G_x|}{|G_{xB}|}$$
and
$$v = \frac{|G|}{|G_x|},$$
where (x, B) is a flag of $\mathcal{D}$. A subgroup H of a group G satisfying
$$|G| < |H|^3$$

is called large in G. Hence flag-transitive groups of a 2-design have large stabilizers. The following lemma for large maximal subgroups of alternating groups and symmetric groups is significant in the proof.

Lemma 2.5. [14, Theorem 2 and Proposition 6.1] *Let G be an alternating group* $\mathrm{Alt}(n)$ *or a symmetric group* $\mathrm{Sym}(n)$ $(n \geq 5)$. *If H is a large subgroup and is maximal in G acting primitively on Ω_n, then (G, H) is one of the following:*

(1) $(G, H) = (\mathrm{Sym}(n), \mathrm{Alt}(n))$;

(2) $G = \mathrm{Alt}(n)$ *and* (G, H) *is one of the following:* $(\mathrm{Alt}(5), D_{10})$, $(\mathrm{Alt}(6), \mathrm{PSL}_2(5))$, $(\mathrm{Alt}(7), \mathrm{PSL}_2(7))$, $(\mathrm{Alt}(8), \mathrm{AGL}_3(2))$, $(\mathrm{Alt}(9), 3^2.\mathrm{SL}_2(3))$, $(\mathrm{Alt}(9), \mathrm{P\Gamma L}_2(8))$, $(\mathrm{Alt}(10), \mathrm{M}_{10})$, $(\mathrm{Alt}(11), \mathrm{M}_{11})$, $(\mathrm{Alt}(12), \mathrm{M}_{12})$, $(\mathrm{Alt}(13), \mathrm{PSL}_3(3))$, $(\mathrm{Alt}(15), A_8)$, $(\mathrm{Alt}(16), \mathrm{AGL}_4(2))$, $(\mathrm{Alt}(24), \mathrm{M}_{24})$;

(3) $G = \mathrm{Sym}(n)$ *and* (G, H) *is one of the following:* $(\mathrm{Sym}(5), \mathrm{AGL}_1(5))$, $(\mathrm{Sym}(6), \mathrm{PGL}_2(5))$, $(\mathrm{Sym}(7), \mathrm{AGL}_1(7))$, $(\mathrm{Sym}(8), \mathrm{PGL}_2(7))$, $(\mathrm{Sym}(9), \mathrm{AGL}_2(3))$, $(\mathrm{Sym}(10), \mathrm{P\Gamma L}_2(9))$, $(\mathrm{Sym}(12), \mathrm{PGL}_2(11))$.

3. Proof of the main theorem

In this section we prove Theorem 1.1. Throughout the section, we always assume that G is an almost simple group with a socle an alternating group $\mathrm{Alt}(n)$ with $n \geq 5$. Such almost simple groups are known explicitly. It is well known that if $n \neq 6$, then G is $\mathrm{Alt}(n)$ or $\mathrm{Sym}(n)$. If $n = 6$, then $G = \mathrm{Alt}(6)$, $\mathrm{Sym}(6)$, M_{10}, $\mathrm{PGL}_2(9)$, or $\mathrm{P\Gamma L}_2(9)$. We first deal with the latter three exceptional cases, M_{10}, $\mathrm{PGL}_2(9)$, and $\mathrm{P\Gamma L}_2(9)$, in Section 3.2.

In the second step, we study the cases

$$G = \mathrm{Alt}(n)$$

and

$$G = \mathrm{Sym}(n)$$

in Sections 3.3–3.5. Since G acts primitively on $\mathcal{P}$ of the designs, the point-stabilizer G_x for $x \in \mathcal{P}$ is a maximal subgroup of G. We note that G has two transitive actions, respectively on $\mathcal{P}$ and Ω_n. According to Lemma 2.4, the groups G_x are divided into three different types with explicit structures according to their actions on Ω_n: intransitive action, imprimitive action, and primitive action. Our strategy of proving the non-existence of or constructing such designs is connecting the action of G on the point set $\mathcal{P}$ of the design and the action of G on the Ω_n. That is the action of G on the coset space of the corresponding maximal subgroups. We shall apply the known degrees and subdegrees in these actions and combine them with the parameter conditions to help us achieve the result. We handle these three types of maximal subgroups in Sections 3.3–3.5, respectively.

3.1. A procedure of constructing designs

We mention a method (or a procedure) of using the algebraic computer system Magma [15] to rule out or construct designs with relatively small parameters (v, k, λ). The procedure contains the following steps:

Step 1. Output the group G acting on a set $\mathcal{P}$ by its coset action on the corresponding maximal subgroup.

Step 2. Check if G has a subgroup of order $|G_B|$. Denote by J such a subgroup as a candidate for the block stabilizer G_B.

Step 3. Find the orbits of J with length k. Denote such an orbit by O as a candidate for a block B.

Step 4. Apply the command `IsDesign`() to check if the incidence structure

$$\mathcal{D} = \{\mathcal{P}, O^G\}$$

forms a 2-design, where

$$O^G := \{O^g | g \in G\}.$$

Step 5. For the 2-designs with the same parameters (v, k, λ), apply the command `IsIsomorphic`() to check if they are isomorphic to each other.

This procedure will then be applied to rule out designs and construct designs with known parameters (v, k, λ) obtained in the proof of the main theorem.

3.2. The exceptional groups in the case $n = 6$

In this section we shall first assume that G has a socle Alt(6), but G is neither Alt(6) nor Sym(6). That is, $G = \mathrm{M}_{10}$, $\mathrm{PGL}_2(9)$, or $\mathrm{P\Gamma L}_2(9)$.

Lemma 3.1. *Let G be one of M_{10}, $\mathrm{PGL}_2(9)$, and $\mathrm{P\Gamma L}_2(9)$, and act flag-transitively on a 2-$(v, 5, \lambda)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$. Let $x \in \mathcal{P}$ and $B \in \mathcal{B}$. Then one of the following holds:*

(1) *$\mathcal{D}$ is a unique 2-$(10, 5, 8)$ design with*

$$G = \mathrm{M}_{10}, \ \ G_x \cong Z_3^2 : Q_8, \ \ \textit{and} \ \ G_B \cong Z_5 : Z_4;$$

(2) *$\mathcal{D}$ is a unique 2-$(10, 5, 16)$ design with*

$$G = \mathrm{PGL}_2(9), \ \ G_x = Z_3^2 : Z_8, \ \ \textit{and} \ \ G_B = D_{10},$$

or with

$$G = \mathrm{P\Gamma L}_2(9), \ \ G_x = Z_3^2 : (Z_8.Z_2), \ \ \textit{and} \ \ G_B = Z_5 : Z_4.$$

Further, in these cases, G acts 2-transitively on $\mathcal{P}$.

Proof. The maximal subgroups of M_{10}, $\mathrm{PGL}_2(9)$ and $\mathrm{P\Gamma L}_2(9)$ have explicit structures, which could be checked in [16], for example. We see that M_{10} has large maximal subgroups $\mathrm{AGL}_1(5)$, Z_3^2: Q_8 and Z_8: Z_2; $\mathrm{PGL}_2(9)$ has large maximal subgroups D_{20}, Z_3^2: Z_8 and D_{16}; and $\mathrm{P\Gamma L}_2(9)$ has large maximal subgroups Z_{10}: Z_4, Z_3^2: $(Z_8.Z_2)$ and $Z_8.(Z_2 \times Z_2)$. We will investigate all these cases by the basic properties of parameters in Lemma 2.1 and the elimination procedure presented in Section 3.1.

First we let

$$G = \mathrm{M}_{10}.$$

If the point-stabilizer G_x is $\mathrm{AGL}_1(5)$, then G is 2-transitive on $\mathcal{P}$, with

$$v = \frac{|G|}{|G_x|} = \frac{720}{20} = 36.$$

By the flag-transitivity, we have $r \mid |G_x|$, and so $r \in \{5, 10, 20\}$. None of these satisfies that

$$\lambda = \frac{r(k-1)}{v-1} = \frac{4r}{v-1}$$

(Lemma 2.1 (1)) is an integer. If

$$G_x = Z_3^2 : Q_8,$$

then

$$v = \frac{|G|}{|G_x|} = 10 \text{ and } 4r = 9\lambda.$$

As

$$gcd(4, 9) = 1,$$

we have $9 \mid r$. Note that $r \mid |G_x|$. So $r \in \{9, 18, 36, 72\}$. We indeed obtain 4 admissible parameter sets

$$(v, k, \lambda) : (10, 5, 4), (10, 5, 8), (10, 5, 16), \text{ and } (10, 5, 32).$$

For each of these parameters set, the possible order of the block stabilizer is 40, 20, 10, or 5, respectively. Implemented in Magma, we find that there is one subgroup of order 20 (up to conjugacy) such that it has two orbits of length 5, generating two 2-$(10, 5, 8)$ designs, admitting $G = \mathrm{M}_{10}$ as their automorphism group. We then check they are isomorphic to each other by Step 5 in Section 3.1.

Similarly, we can apply the same argument for

$$G = \mathrm{PGL}_2(9)$$

(as it has the same orders of large maximal subgroups as M_{10}), and then we see that if

$$G_x \cong Z_3^2 : Z_8$$

and

$$v = \frac{|G|}{|G_x|} = 10,$$

then there is exactly one subgroup of order 10 (up to conjugacy) such that it has two orbits of length 5, generating two isomorphic 2-$(10, 5, 16)$ designs.

At last, for

$$G = \mathrm{P\Gamma L}_2(9),$$

we can easily rule out the two possible cases in terms of the point-stabilizer isomorphic to Z_{10}: Z_4 or $Z_8 \cdot (Z_2 \times Z_2)$, by the basic equation

$$\lambda = \frac{r(k-1)}{v-1} = \frac{4r}{v-1}$$

and

$$bk = vr = 5b.$$

If

$$G_x \cong Z_3^2 \cdot (Z_8 \cdot Z_2),$$

then we obtain 4 admissible parameter sets

$$(v, k, \lambda) : (10, 5, 4), (10, 5, 8), (10, 5, 16), \text{ and } (10, 5, 32).$$

We find that there is exactly one subgroup of order 10 (up to conjugacy) in G, such that it has two orbits of length 5 generating two isomorphic 2-$(10, 5, 16)$ designs. We also check that this 2-$(10, 5, 16)$ design is also isomorphic to the design arising from

$$G = \mathrm{PGL}_2(9).$$

Further, we notice that

$$\mathrm{PGL}_2(9) < \mathrm{P\Gamma L}_2(9) < \mathrm{Sym}(10)$$

is 2-transitive, see for example [10, Table 9.6.2]. □

In the following, we always suppose that

$$G = \mathrm{Alt}(n)$$

or $\mathrm{Sym}(n)$ (satisfying $n \geq 5$ by the simplicity of $\mathrm{Alt}(n)$) in Sections 3.3–3.5. For convenience, we make the following hypothesis:

Hypothesis 1. *Suppose that*

$$\mathcal{D} = (\mathcal{P}, \mathcal{B})$$

be a 2-$(v, 5, \lambda)$ *design, and suppose that*

$$G = \mathrm{Alt}(n)$$

or $\mathrm{Sym}(n)$*, acting as a flag-transitive automorphism group of* $\mathcal{D}$*, where* $n \geq 5$.

3.3. The action of G_x on Ω_n is intransitive

In this section, we study the case that the action of G_x on Ω_n is intransitive. One design is obtained in this case.

Lemma 3.2. *Suppose that Hypothesis 1 holds. If $x \in \mathcal{P}$ and G_x is intransitive on Ω_n, then $\mathcal{D}$ is a unique* 2-$(21, 5, 12)$ *design with*

$$G = \mathrm{Alt}(7), \quad G_x \cong (\mathrm{Sym}(2) \times \mathrm{Sym}(5)) \cap \mathrm{Alt}(7),$$

and

$$G_B \cong D_{10};$$

or with

$$G = \mathrm{Sym}(7), \quad G_x \cong \mathrm{Sym}(2) \times \mathrm{Sym}(5),$$

and

$$G_B \cong D_{20}.$$

Further, G acts on $\mathcal{P}$ with rank 3, and the two suborbits have length 10.

Proof. By Lemma 2.4 (1), the structure of G_x is

$$(\mathrm{Sym}(s) \times \mathrm{Sym}(t)) \cap G$$

where

$$n = s + t,$$

s and t are both positive integers, and $s \neq t$. This is the largest subgroup of G leaving a set of cardinality s (or t) invariant. We assume that $s < t$ (equivalently, $s < \frac{n}{2}$), without loss of generality. Since G is flag-transitive, it is easily shown that G_x cannot fix two points in $\mathcal{P}$. In fact, this can also be deduced from the maximality of G_x and that G is not a group of prime order. Further, since G_x is the largest group stabilizing a unique subset of cardinality s in Ω_n, and G acts transitively on all the s-subsets of Ω_n (denoted by $\Omega_n^{\{s\}}$) by its multiple transitivity (at least $(n-2)$-fold), this allows us to identify the point set $\mathcal{P}$ as $\Omega_n^{\{s\}}$. In addition, the action of G acting on $\mathcal{P}$ is equivalent to the action of G acting on $\Omega_n^{\{s\}}$. Now, $\mathcal{D}$ has

$$v = \binom{n}{s}$$

points. Further, one could see, for example, [17, Lemma 3.2] for subdegrees of G in this action: That is, the subdegrees are

$$d_i = \binom{s}{i}\binom{n-s}{s-i},$$

where $i = 0, 1, \cdots, s$. The rank of G in this action is equal to $s + 1$. If $s = 1$, then G acts on $\mathcal{P}$ in its natural action, and $\mathcal{D}$ is a complete design, which is not under our consideration. We then separate the subsequent proof into the following two cases.

Case 1. Assume that $s = 2$. Then

$$v = \frac{n(n-1)}{2}.$$

Pick two subdegrees

$$d_1 = 2(n-2)$$

and

$$d_0 = \frac{(n-2)(n-3)}{2}.$$

According to Lemma 2.3, we get that $\frac{n(n-1)}{2} - 1$ divides both $8(n-2)$ and $2(n-2)(n-3)$. It follows that $n = 7$, and so $v = 21$. By Lemma 2.1, one finds

$$r = 5\lambda \text{ and } b = 21\lambda.$$

Since r divides $|G|$, λ is a divisor of 48. Hence, we then get all the admissible parameters triples (λ, b, r). They are

$$(1, 21, 5), (2, 42, 10), (3, 63, 15), (4, 84, 20), (6, 126, 30), (8, 168, 40)$$
$$(12, 252, 60), (16, 336, 80), (24, 504, 120), (48, 1008, 240)$$

for

$$G = \mathrm{Sym}(7),$$

and all these are admissible for

$$G = \mathrm{Alt}(7)$$

except

$$(\lambda, b, r) = (16, 336, 80) \text{ and } (48, 1008, 240).$$

Now, by the elimination procedure presented in Section 3.1, we obtain a unique 2-$(21, 5, 12)$ design with flag-transitive groups Alt(7) and Sym(7), and

$$G_B \cong D_{10} \text{ or } D_{20},$$

respectively. Further, G has rank 3 on $\mathcal{P}$, with the two suborbits of length 10.

Case 2. Assume that $s \geq 3$. Then

$$n = s + t \geq 7.$$

Pick

$$d_{s-1} = s(n - s).$$

By Lemma 2.3, we obtain $v - 1 \mid 4s(n - s)$. We notice a fact that

$$v = \binom{n}{s} = \frac{n(n-1)\cdots(n-s+1)}{s!} = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-(s-1)}{s}.$$

Since $s \geq 3$, we have

$$v - 1 \geq \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} - 1 = \frac{n(n-1)(n-2)}{6} - 1.$$

Note that

$$\frac{n(n-1)(n-2)}{6} - 1 > n^2$$

for $n \geq 9$. Moreover, it is easily known that

$$4s(n - s) \leq 4 \cdot \frac{n}{2} \cdot \frac{n}{2} \leq n^2.$$

We have

$$v - 1 \geq \frac{n(n-1)(n-2)}{6} - 1 > n^2 \geq 4s(n - s)$$

for $n \geq 9$, which contradicts $v - 1 \mid 4s(n - s)$. Hence, n is either 7 or 8. If $n = 7$, then $s = 3$ and $t = 4$, and so

$$v - 1 = \binom{7}{3} - 1 = 34$$

and

$$4s(n - s) = 48,$$

which contradicts

$$v - 1 \mid 4s(n - s).$$

If $n = 8$, then $s = 3$ and $t = 5$, and so

$$v - 1 = \binom{8}{3} - 1 = 55$$

and

$$4s(n - s) = 60,$$

which is again impossible. □

3.4. The action of G_x on Ω_n is transitive and imprimitive

In the following, according to Lemma 2.4 (2), we tackle the case that the point stabilizer G_x is imprimitive on Ω_n. We find that there are two admissible designs and determine the corresponding automorphism groups.

Lemma 3.3. *Suppose that Hypothesis 1 holds. If $x \in \mathcal{P}$ and the point-stabilizer G_x is imprimitive on Ω_n, then one of the following holds:*

(1) *$\mathcal{D}$ is a unique* 2-(10, 5, 8) *design with* $G = \mathrm{Alt}(6)$, $G_x \cong (\mathrm{Sym}(3) \wr \mathrm{Sym}(2)) \cap \mathrm{Alt}(6)$, *and* $G_B \cong D_{10}$;

(2) *$\mathcal{D}$ is a unique* 2-(10, 5, 16) *design with* $G = \mathrm{Sym}(6)$, $G_x \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$, *and* $G_B \cong D_{10}$.

Further, in these two cases, G acts 2-transitively on $\mathcal{P}$.

Proof. Suppose that

$$G_x \cong (\mathrm{Sym}(s) \wr \mathrm{Sym}(t)) \cap G,$$

acting as the largest imprimitive group on Ω_n (in the natural action of G). Then Ω_n has a point-partition

$$\Sigma = \{\Gamma_1, \Gamma_2, \cdots, \Gamma_t\}$$

with $|\Gamma_i| = s$ and $n = st$, which is invariant under the action of G_x. It is obvious that

$$G_x = G_\Sigma.$$

According to the analysis in [18, Section 3], the group G_x has only one invariant non-trivial point-partition of Ω_n. Denote by $P^s_{\Omega_n}$ the set of all the partitions of Ω_n into t parts, each of length s. Then it will be straightforward to verify that the action of G on $\mathcal{P}$ is equivalent to the action of G on $P^s_{\Omega_n}$ (in the natural induced action of G on Ω_n). Hence, we can identify $\mathcal{P}$ with $P^s_{\Omega_n}$ and identify the action of G on $\mathcal{P}$ with the action of G on $P^s_{\Omega_n}$. Thus,

$$v = |P^s_{\Omega_n}|,$$

which could be calculated directly, or equals $\frac{|G|}{|G_x|}$ as G acts transitively on $P^s_{\Omega_n}$. Then, we have

$$v = \frac{n!}{(s!)^t t!} = \frac{\binom{ts}{s}\binom{(t-1)s}{s}\cdots\binom{3s}{s}\binom{2s}{s}}{t(t-1)\cdots 2\cdot 1} = \binom{ts-1}{s-1}\binom{(t-1)s-1}{s-1}\cdots\binom{3s-1}{s-1}\binom{2s-1}{s-1}.$$

If $s = 2$, see, for example, [19, Section 3.2], and we know that there exist suborbits of G (acting on the coset space of G_x in G) with degree

$$d_j = 2^{j-1}\binom{t}{j},$$

where $j = 2, \cdots, t$. For $s \geq 3$, there exist subdegrees

$$d_j = s^j\binom{t}{j},$$

where $j = 2, \cdots, t$.

Case 1. Assume that $s = 2$. Then it would be straightforward to see that

$$v = (2t-1)(2t-3)\cdot\cdots\cdot 3\cdot 1.$$

Since

$$n = st \geq 5,$$

we have $t \geq 3$. Pick subdegree

$$d_2 := t(t-1).$$

By Lemma 2.3, we have

$$v - 1 \mid 4t(t-1).$$

If $t \geq 4$, then

$$v - 1 > 4t(t-1),$$

which is a contradiction. If $t = 3$, then

$$v - 1 = 14$$

does not divide

$$4t(t-1) = 24,$$

a contradiction.

Case 2. Assume that $s \geq 3$. Pick subdegree

$$d_2 := s^2\frac{t(t-1)}{2}.$$

By Lemma 2.3, we have $v - 1 \mid 2s^2t(t-1)$. It is not hard to see that

$$\binom{ts-1}{s-1} = \frac{(ts-1)(ts-2)\cdots(ts-(s-1))}{(s-1)(s-2)\cdots 3\cdot 2\cdot 1} > t^{s-1}.$$

This implies

$$v > t^{s-1}(t-1)^{s-1}\cdots 3^{s-1}\cdot 2^{s-1} = (t!)^{s-1}.$$

So

$$(t!)^{s-1} \leq v - 1 \leq 2s^2t(t-1).$$

If $t \geq 3$, then

$$(t(t-1))^{s-2}\cdot(t-2)!^{s-1} \leq 2s^2.$$

Only

$$(s, t) = (3, 3)$$

is possible. Now

$$v = \frac{(st)!}{(s!)^t t!} = \frac{9!}{(3!)^3 3!} = 280$$

and

$$2s^2 t(t-1) = 108,$$

which contradicts $v - 1 \mid 2s^2 t(t-1)$. Thus, $t = 2$. Now

$$2^{s-1} \leq v - 1 \mid 2s^2 t(t-1) = 4s^2.$$

We obtain

$$3 \leq s \leq 9.$$

We tackle these cases one by one:

Subcase 1. If $s = 3$, then

$$v = \frac{(2s)!}{2 \cdot (s!)^2} = \frac{6!}{2 \cdot (3!)^2} = 10$$

and

$$4s^2 = 36.$$

This case satisfies $v - 1 \mid 4s^2$. Implemented by Magma, we indeed find a unique 2-$(10, 5, 8)$ design admitting a flag-transitive group Alt(6) and a unique 2-$(10, 5, 16)$ design admitting a flag-transitive group Sym(6). Both of the block stabilizers are

$$G_B \cong D_{10}.$$

We also notice that

$$\mathrm{Alt}(6) < \mathrm{Sym}(6) < \mathrm{Sym}(10)$$

is 2-transitive (see [10, Table 9.62]).

Subcase 2. If $s = 4$, then

$$v = \frac{(2s)!}{2 \cdot (s!)^2} = 35$$

and

$$4s^2 = 64.$$

This does not satisfy $v - 1 \mid 4s^2$.

Subcase 3. If $s = 5$, then $v = 126$, and $4s^2 = 100$. We have

$$v - 1 > 4s^2,$$

a contradiction. If $s = 6$, then $v = 462$, and $4s^2 = 144$. We have

$$v - 1 > 4s^2,$$

a contradiction. If $s = 7$, then $v = 1716$, and $4s^2 = 196$. If $s = 8$, then $v = 6435$, and $4s^2 = 256$. If $s = 9$, then $v = 24310$, and $4s^2 = 324$. For all cases with $s \geq 5$, we have

$$v - 1 > 4s^2,$$

a contradiction to $v - 1 \mid 4s^2$. □

3.5. *The action of G_x on Ω_n is primitive*

In this final section, we deal with the case of Lemma 2.4 (3), that is, G_x is a primitive group on Ω_n in the natural action of G. We obtain three different designs, as listed in the lemma below.

Lemma 3.4. *Suppose that Hypothesis 1 holds. If $x \in \mathcal{P}$ and the point-stabilizer G_x of G acts primitively on Ω_n, then we obtain the following:*

(1) *$\mathcal{D}$ is a unique* 2-(15, 5, 4) *design with $G = \mathrm{Alt}(7)$, $G_x \cong \mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, and $G_B \cong \mathrm{Alt}(5)$;*

(2) *$\mathcal{D}$ is a unique* 2-(15, 5, 12) *design with $G = \mathrm{Alt}(7)$, $G_x \cong \mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, and $G_B \cong Z_5{:}\ Z_4$;*

(3) *$\mathcal{D}$ is a unique* 2-(15, 5, 16) *design with $G = \mathrm{Alt}(8)$, $G_x \cong \mathrm{AGL}_3(2)$, and $G_B \cong \mathrm{Sym}(5)$.*

Further, in these three cases, G acts 2-transitively on $\mathcal{P}$.

Proof. Since a flag-transitive group of automorphisms has large point-stabilizer, the group G and the possible point-stabilizer G_x are listed in Lemma 2.5. In the meantime, the possible number of points

$$v = |\mathcal{P}|$$

equals $\frac{|G|}{|G_x|}$. By the five steps of the procedure of elimination and construction given in Section 3.1, and with the implementation in Magma, we see that only the two cases

$$(G, G_x) = (\mathrm{Alt}(7), \mathrm{PSL}(2, 7))$$

and (Alt(8), AGL(3,2)) yield flag-transitive 2-designs with $k = 5$. For

$$(G, G_x) = (\mathrm{Alt}(7), \mathrm{PSL}(2, 7)),$$

we obtain two designs with parameters

$$(v, k, \lambda) = (15, 5, 4) \text{ and } (15, 5, 12),$$

and the block stabilizers are

$$G_B \cong \mathrm{Alt}(5)$$

and

$$G_B \cong Z_5 : Z_4,$$

respectively. For

$$(G, G_x) = (\mathrm{Alt}(8), \mathrm{AGL}(3, 2)),$$

we obtain a unique design with parameters

$$(v, k, \lambda) = (15, 5, 16),$$

with

$$G_B \cong \mathrm{Sym}(5).$$

Further, we note that

$$G = \mathrm{Alt}(7)$$

and Alt(8) are 2-transitive on $\mathcal{P}$ (see [10, Table 9.62]). □

Proof of Theorem 1.1. In Lemmas 3.1–3.4, we inspect the three different types of maximal subgroups of G, as listed in Lemma 2.4. Further, by Step 5 in Section 3.1, we check that the 2-$(10, 5, 8)$ designs in Lemmas 3.1 and 3.3 are isomorphic. Besides, the 2-$(10, 5, 16)$ designs in Lemmas 3.1 and 3.3 are also isomorphic. Therefore, there are a total of six different 2-designs with block size 5, admitting flag-transitive groups whose socle is a simple alternating group. □

4. Conclusions

In Section 3, we prove the main result of this paper: give a complete classification of 2-designs with block size 5, admitting a flag-transitive automorphism group Alt(n), Sym(n), M_{10}, $PGL_2(9)$, or $P\Gamma L_2(9)$. We think that the techniques and methods used in the proof of the current paper might also be useful in the study of 2-designs satisfying other conditions, even in other objects such as transitive graphs.

Author contributions

Jiaxin Shen: conceptualization, writing—original draft preparation, writing—review and editing; Yuqing Xia: writing—original draft preparation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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