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Research article

A novel bidirectional projection measures of circular intuitionistic fuzzy sets and its application to multiple attribute group decision-making problems

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Abstract: Atanassov recently proposed a new circular intuitionistic fuzzy set (CIFS) as an extension of intuitionistic fuzzy sets to express uncertain information by a circle with centered membership, non-membership, and radius r. Circular intuitionistic fuzzy sets can express uncertain information more flexibly than the intuitionistic fuzzy set. In this paper, we first propose a new method for calculating the radius r of CIFSs by ordinary least squares (OLS). We introduce some notions, such as modules of the circular intuitionistic fuzzy set and the cosine of the included angle between membership and non-membership vectors of the circular intuitionistic fuzzy set. Then, we define a new bidirectional projection measure of circular intuitionistic fuzzy sets, which takes into account the difference between different CIFSs in terms of membership degree and non-membership degree and radius r. The proposed bidirectional projection measures show superiority compared with some recent research works through numerical examples. Finally, the method is applied to a multi-attribute decision-making problem with group expert decision-making to prove the effectiveness and accuracy of the method.

Keywords: intuitionistic fuzzy sets; circular intuitionistic fuzzy sets; bidirectional projection measures; multi-attribute group decision making
Mathematics Subject Classification: 28E10, 90B50

1. Introduction

Multi-attribute group decision-making (MAGDM) [1–3] is an important area of decision-making research. For example, Xu et al. [1] used the interval-valued intuitionistic fuzzy set to deal with the group attribute decision-making problem. Akram et al. [2, 3] applied the Pythagorean fuzzy set and the VIKOR method to the group multi-attribute decision-making problem. Liu et al. [4] applied the fuzzy linguistic term set to the group multi-attribute decision-making problem. The multi-attribute group decision-making process is typically divided into three steps: collecting experts' views on the different attributes of the alternatives, aggregating expert opinions, and selecting the best alternative.

In the decision-making processes involving experts, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the inherent fuzziness in human cognition. In 1965, an American computer and cybernetics expert, Zadeh, proposed the concept of fuzzy sets [5] to solve fuzzy phenomena; ordinary fuzzy sets use a single value for membership. However, many researchers think that single-valued fuzzy sets make it challenging to represent fuzzy information accurately. Thus, many extended fuzzy sets have been proposed. For example, the interval fuzzy set [6] proposed by Guinness involves the replacement of membership from exact values to interval values. Atanassov introduced the concept of intuitionistic fuzzy sets (IFS) [7] in 1986, which considers both membership and non-membership for fuzzy sets. Non-membership represents fuzziness from the opposite side of things. Intuitionistic fuzzy sets can be more flexible than ordinary fuzzy sets to represent the fuzziness of information. Subsequently, intuitionistic fuzzy set theory has been used in various fields of research [8–10], such as in the fields of decision analysis, pattern recognition, clustering analysis, etc. For example, Demir [8] applied it in the multi-attribute decision-making (MADM), specifically for the security management of smart cities. Moslem [9] also applied it in the MADM model. Zeng et al. [10] applied the intuitionistic fuzzy sets to the pattern recognition problem. Kuo et al. [11] applied intuitionistic fuzzy sets for clustering analysis. It is Rahman [12] and Iqbal who applied intuitionistic fuzzy sets to the selection and optimization problem of railway systems.

However, to ensure the accuracy of the decision, it is often necessary to have several experts working together to provide a decision evaluation. Experts often struggle to give precise and consistent membership degree and non-membership degree in practical applications of intuitionistic fuzzy sets. In 1999, Atanassov extended the intuitionistic fuzzy set to an interval-valued intuitionistic fuzzy set (IVIFS) [13], which extension of membership degree and non-membership degree with exact values to interval values. The theoretical study of applying intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets to MADM problems has produced many results, and the main research focuses on score functions, distance measures, similarity measures, etc. Chen et al. [14] first proposed a score function on intuitionistic fuzzy numbers. Sahin et al. [15] proposed a new score function based on considering the hesitancy degree of interval-valued intuitionistic fuzzy sets for IVIFS. Kumar [16] converts IVIFVs into connection numbers (CNs), and develops a score function for ranking CNs. A large number of research results [17–19] have been presented on the distance between intuitionistic fuzzy sets. Szmidt and Kacprzyk [17] analogized vectors using the three parameters of the membership degree, non-membership degree, and hesitation degree of intuitionistic fuzzy sets, and proposed a distance formula for intuitionistic fuzzy sets. Zhang et al. [18] proposed a new distance measure for intuitionistic fuzzy sets, which significantly overcomes the problem of information loss. A similarity measure [19] is defined by using the weighted average distance between the intuitionistic fuzzy set and the two most fuzzy sets.

In addition to the aforementioned extensions of intuitionistic fuzzy sets, when the sum of membership degree and non-membership degree exceeds 1 in practical problems, the intuitionistic fuzzy sets will become inapplicable. To address this issue, Yager [20] proposed Pythagorean fuzzy sets, which makes the sum of the squares of the membership degree and the non-membership degree less than 1. It expands the value range of the membership degree and non-membership degree in the intuitionistic fuzzy set, and can provide decision-makers with a broader space for information expression. Subsequently, Yager extended the Pythagorean fuzzy set to the q-rung orthopair fuzzy set [21], this extension ensures that the sum of the q-th powers of the membership degree and the non-membership degree is less than 1. Considering that human opinions about a certain thing are not limited to simply “yes” or “no” , but also include “abstention” and “rejection”, Mahmood et al. [22] proposed the concept of spherical fuzzy sets. It allows the sum of the membership degree, non-membership degree, and hesitation degree to be greater than 1. Based on the above extended fuzzy sets, a great many relevant MADM and pattern recognition methods have also been developed. Sarfraz [23] proposed a new aggregation operator based on the interval-valued Pythagorean fuzzy set to address the priority ranking problem in MADM issues. Akram et al. [2] combined the Pythagorean fuzzy set and the rough set and proposed new AHP and TOPSIS decision-making methods. Asif et al. [24] proposed new operators, such as the Pythagorean fuzzy Hamacher interactive weighted averaging by using the Hamacher t-norm, and applied them to MADM problems. Liu et al. [25] proposed the q-rung orthopair fuzzy weighted averaging operator and the q-rung orthopair fuzzy weighted geometric operator to address the decision information. Pethaperumal et al. [26] proposed a new distance measure for q-rung orthopair MFSs, and they applied it to decision-making problems. Radovanovic et al. [27] introduced a hybrid multi-criteria decision-making (MCDM) model combining spherical fuzzy AHP and Grey MARCOS methods. The above extensions of fuzzy sets and intuitionistic fuzzy sets are all aimed at expressing the uncertainty and fuzziness in human information more accurately.

Then, since the projection measure can take into account not only the distance, but also the included angle between the evaluated objects, it is a very suitable method for dealing with decision problems. Xu et al. [28] introduce the concepts of intuitionistic fuzzy numbers in terms of modulus, angle of entrapment, projective measure, etc. Zeng et al. [29] used the projection measure of intuitionistic fuzzy numbers in the determination of expert weight information in decision problems. Zheng et al. [30] proposed the concept of bidirectional projection to compute the projection measure.

Although extended intuitionistic fuzzy sets have more flexibility than intuitionistic fuzzy sets in expressing uncertain information, using a fixed value to represent the membership degree and non-membership degree lacks accuracy. Atanasov recently proposed a new extension of intuitionistic fuzzy sets called circular intuitionistic fuzzy sets (CIFS) [31]. CIFS fits the fuzziness and uncertainty of information to a circle in geometry, which has membership degree and non-membership degrees, and also uncertainty by adding a parameter radius. To some extent, CIFS retains the fuzziness and uncertainty of the evaluation information while maintaining its accuracy. Distance measures and similarity measures are very important to study in practical decision-making applications, existing distance measures, and similarity measures for intuitionistic fuzzy sets cannot be directly used for CIFS. Atanasov himself proposed four distances [32] on CIFS, based on the classical Euclidean and Hamming distances, among others, and the distance is calculated by employing parameters such as the radius r of membership and non-membership degrees. However, the specific meaning and origin of the distance formula has not been explained. Kahraman et al. [33, 34] transformed the CIFS into ordinary intuitionistic fuzzy set through the radius for calculation in decision-making problems. This will lead to the loss of some original decision-making information and cause deviations in the calculation of decision-making results. Xu et al. [35] defined a new circular intuitionistic fuzzy sets distance through the underlying relationships among the membership degree, non-membership degree, hesitation degree, and radius in CIFS. Subsequently, many extended fuzzy sets based on circular intuitionistic fuzzy sets have also been proposed. Khan et al. [36] proposed the circular

Pythagorean fuzzy set, and then generalized it to the new disc Pythagorean fuzzy set. They also gave the corresponding addition, subtraction, multiplication, and division operation rules, as well as the distance formula. Ashraf et al. [37] introduced the circular spherical fuzzy set and disc spherical fuzzy set, and established the ELECTRE method using the circular spherical fuzzy set. At the same time, circular intuitionistic fuzzy sets also have many applications in scoring methods [38] and pattern recognition [39]. It demonstrates the huge development potential of circular fuzzy sets.

Most of the existing methods only consider the distance and ignore the angle between different circular intuitionistic fuzzy sets when calculating the differences between them. In this paper, a new bidirectional projection measures of circular intuitionistic fuzzy sets will be given. Considering the differences in distance and angle between different CIFSs, the projection similarity is defined. When calculating the circular parameter r in CIFS, the method of taking the maximum value of different expert evaluations is used. When the value given by one of the experts is significantly different from that of the other experts, it will influence the value of r to a great extent. Therefore, in this paper, the calculation of the r parameter is improved by using the least squares method to fit the value of r.

The rest of the paper is organized as follows. The Section 2 briefly introduces some concepts of circular intuitionistic fuzzy sets and puts forward a new method for calculating the parameter r. Section 3 describes in detail the proposed bidirectional projection measurement method. In Section 4, an MADM method is established based on the proposed bidirectional projection measurement. Section 5 provides a computational example and makes a comparison with other methods. Finally, conclusions are given in Section 7.

2. Preliminaries of circular intuitionistic fuzzy sets

This part introduces the basic concepts of circular intuitionistic fuzzy sets and proposes a new method for calculating the parameter r.

Definition 1. *[31] Let X be a given fixed universe. A CIFS C in X is an object of the form:*

$$C = \{\langle x, \mu_C(x), \nu_C(x); r | x \in X\}, \tag{2.1}$$

where the functions

$$\mu_C : X \to [0,1], \quad \nu_C : X \to [0,1]$$

denote the membership degree and non-membership degree of C, respectively, and for any $x \in X$, there is

$$r : X \to |0, \sqrt{2}|$$

a circle around each element, and

$$0 \leq \mu_C(x) + \nu_C(x) \leq 1$$

holds. Further,

$$\pi_C(x) = 1 - \mu_C(x) - \nu_C(x)$$

is called the hesitancy degree of element x in circular intuitionistic fuzzy set C.

CIFS can be regarded as an extension of IFS. In geometry, IFS can be represented by a point, while CIFS can be represented by a circle with the center at $\mu_C(x), \nu_C(x)$ and a radius of r. The ways in which

CIFS and IFS express decision-maker information are different. Different experts may give different evaluations of the same decision problem. We need to synthesize the various opinions given by different experts. The following definition gives how to generate the corresponding circular intuitionistic fuzzy set when there is a set of intuitionistic fuzzy pairs.

Definition 2. *[31] A CIFS C in a fixed universe X, for $x_i \in X$, and a set of intuitionistic fuzzy pairs have the form $\{\langle m_{i,1}, n_{i,1}\rangle, \langle m_{i,2}, n_{i,2}\rangle, \ldots\}$. Then, we calculate the circular fuzzy set as follows:*

$$\langle \mu_C(x_i), \nu_C(x_i)\rangle = \left\langle \frac{\sum_{j=1}^{k_i} m_{i,j}}{k_i}, \frac{\sum_{j=1}^{k_i} n_{i,j}}{k_i} \right\rangle, \tag{2.2}$$

where k_i denotes the number of decision makers. Let

$$r_i = \max_{1\le j\le k_i} \left| \sqrt{\left(\mu_C(x_i) - m_{i,j}\right)^2 + \left(\nu_C(x_i) - n_{i,j}\right)^2} \right|, \tag{2.3}$$

where the radius of the C is obtained through the maximum Euclidean distance.

In Definition 2, the approach of calculating parameter r for Eq (2.3) by means of the maximum Euclidean distance is vulnerable to the influence of a few extremely incorrect decision makers. For example, four decision makers provide evaluation data $\{\langle 0.4, 0.5\rangle, \langle 0.5, 0.4\rangle, \langle 0.4, 0.4\rangle, \langle 1, 0\rangle\}$. Obviously, the fourth expert provided evaluation information that is significantly different from that of the other three experts,when calculating the parameter r, efforts should be made to minimize the influence of the fourth expert on the calculation. However, in practice, and thus, when calculating the parameter r using the maximum Euclidean distance, the information provided by the fourth expert is utilized to obtain r. This value of r exhibits an excessive disparity compared to the information from the evaluations of the other three experts.

Therefore, this paper proposes a method for obtaining the parameter of CIFS using the ordinary least squares

$$l_{ij} = \sqrt{\left(\mu_C(x_i) - m_{i,j}\right)^2 + \left(\nu_C(x_i) - n_{i,j}\right)^2}, \tag{2.4}$$

$$\min F(r_i) = \sum_{j=1}^{k_i} f_j^2(r_i) = \sum_{j=1}^{k_i} \left(l_{ij}^2 - r_i^2\right). \tag{2.5}$$

When the objective function reaches its minimum value, it means that, at this moment, the sum of the distances between the parameter r and all the original intuitionistic fuzzy pairs is the closest, and the parameter r preserves the information of the original intuitionistic fuzzy pairs to the greatest extent. However, when calculating the parameter r using the maximum value method, it may be significantly affected by extreme values, and the overall original values may be neglected.

The objective function is differentiated with respect to the variable r_i. When the derivative is equal to zero, the minimum value is obtained when

$$\frac{dF(r_i)}{dr_i} = 4k_i r_i^3 - 4 \times \left(\sum_{j=1}^{k_i} l_{ij}^2\right) \times r_i = 0,$$

$$k_i r_i^2 = \left(\sum_{j=1}^{k_i} l_{ij}^2\right), \tag{2.6}$$

$$r_i = \sqrt{\frac{\sum\limits_{j=1}^{k_i} l_{ij}^2}{k_i}}, \tag{2.7}$$

and we will redefine the calculation method of the parameter r of the CIFS below.

Definition 3. *A CIFS C in a fixed universe X, for $x_i \in X$, and a set of intuitionistic fuzzy pairs have the form $\{\langle m_{i,1}, n_{i,1}\rangle, \langle m_{i,2}, n_{i,2}\rangle, \ldots\}$. Then we calculate the circular fuzzy set as follows:*

$$\langle \mu_C(x_i), v_C(x_i)\rangle = \left\langle \frac{\sum_{j=1}^{k_i} m_{i,j}}{k_i}, \frac{\sum_{j=1}^{k_i} n_{i,j}}{k_i} \right\rangle, \tag{2.8}$$

where k_i denotes the number of decision makers. Let

$$r_i = \sqrt{\frac{\sum\limits_{j=1}^{k_i} l_{ij}^2}{k_i}}. \tag{2.9}$$

As illustrated in Figure 1, we can see that there are five possible shapes of the circle in the circular intuitionistic fuzzy set. Several circles have incomplete points because the values of membership degree and non-membership degree need to satisfy certain relationships in practical applications. We give the following Definition 4.

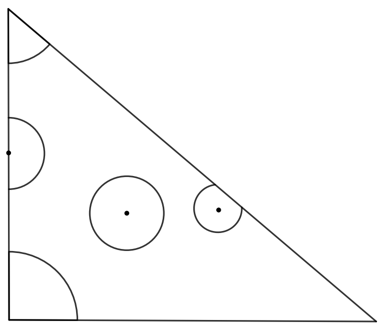

Figure 1. CIFS geometrical representation [31].

Definition 4. *[31] Let C be a circular intuitionistic fuzzy set, and*

$$L^* = \{\langle a, b\rangle \mid a, b \in [0, 1] \& a + b \leq 1\},$$
$$C = \{\langle x, \mu_C(x), v_C(x); r | \, x \in X\} = \{\langle x, O_r(\mu_C(x), v_C(x))\rangle \mid x \in X\},$$

where O_r represents a function with a radius of r and a center at $(\mu_C(x), v_C(x))$,

$$O_r(\mu_C(x), v_C(x)) = \left\{ \frac{\langle a, b\rangle \mid a, b \in [0.1]}{\sqrt{(\mu_C(x) - a)^2 + (v_C(x) - b)^2} \leq r} \right\} \cap L^*$$

$$= \left\{ \begin{array}{l} \langle a, b \rangle \mid a, b \in [0.1] \\ \sqrt{(\mu_C(x) - a)^2 + (v_C(x) - b)^2} \leq r \\ a + b \leq 1 \end{array} \right\}.$$

The following presents some basic operations of circular fuzzy sets that will be used in this paper.

Definition 5. *[35] Let*

$$C_1 = \left\{ \langle x, \mu_{C_1}(x), v_{C_1}(x); r \middle| x \in X \right\},$$
$$C_2 = \left\{ \langle x, \mu_{C_2}(x), v_{C_2}(x); r \middle| x \in X \right\}$$

be two circular intuitionistic fuzzy sets. Then,

$$C_1 \otimes C_2 = \left\{ x, \mu_{C_1}(x) * \mu_{C_2}(x), v_{C_1}(x) + v_{C_2}(x) - v_{C_1}(x) * v_{C_2}(x); \frac{r_1 + r_2}{2} \right\}. \tag{2.10}$$

Definition 6. *[40] Let*

$$C = \{ \langle x, \mu_C(x), v_C(x); r | \, x \in X \}$$

be a circular intuitionistic fuzzy set. We define $c = (u_c, v_c; r)$ *as a circular intuitionistic fuzzy value, and a score function* S_c *and an accuracy function* H_c *of the circular intuitionistic fuzzy value are defined as follows:*

$$S_c(c_r) = \left(u_c - v_c - \sqrt{2r} \right) / 3, \tag{2.11}$$
$$H_c(c_r) = u_c + v_c, \tag{2.12}$$

where $H_c(c_r) \in [0, 1]$.

Considering that the larger the value of r for an circular intuitionistic fuzzy value, the more negative influence it should have on the score of circular intuitionistic fuzzy value. Therefore, we subtract the parameter r when defining the scoring function.

Definition 7. *[40]*

Let

$$c_1 = (u_{c_1}, v_{c_1}; r)$$

and

$$c_2 = (u_{c_2}, v_{c_2}; r)$$

be two CIFVs, and the ranking rules are defined as follows:

1) If $S_c(c_1) \succ S_c(c_2)$*, then* $c_1 \succ c_2$*;*

2) If $S_c(c_1) = S_c(c_2)$ *and* $H_c(c_1) \succ H_c(c_2)$*, then* $c_1 \succ c_2$*;*

3) If $S_c(c_1) = S_c(c_2)$ *and* $H_c(c_1) = H_c(c_2)$*, then* $c_1 = c_2$.

3. Bidirectional projection measures of circular intuitionistic fuzzy set

For any variable x in the universe X, the circular intuitionistic fuzzy set can be regarded as a vector. When using distance to measure the similarity between two different circular intuitionistic fuzzy sets, the influence of the angle between vectors is not taken into account, and when considering the similarity between different circular intuitionistic fuzzy sets in this paper, both the included angle and the distance are taken into account. Currently, there is relatively little research on the projection model of two circular intuitionistic fuzzy sets. Therefore, this paper proposes a bidirectional projection measures of circular intuitionistic fuzzy sets.

3.1. Some existing projection models

Definition 8. *[28] Let*

$$\alpha = (\mu_\alpha, \nu_\alpha)$$

and

$$\beta = \left(\mu_\beta, \nu_\beta\right)$$

be two IFNs;

$$\pi_\alpha = 1 - \mu_\alpha - \nu_\alpha,$$

and then

$$|\alpha| = \sqrt{\mu_\alpha^2 + \nu_\alpha^2 + \pi_\alpha^2}, \tag{3.1}$$

$|\alpha|$ *is called the module of* α*. It is the same for* $|\beta|$.

$$\alpha \cdot \beta = \mu_\alpha\mu_\beta + \nu_\alpha\nu_\beta + \pi_\alpha\pi_\beta \tag{3.2}$$

is called the inner product of α *and* β*.*

$$\mathrm{Proj}_\beta(\alpha) = \frac{\mu_\alpha\mu_\beta + \nu_\alpha\nu_\beta + \pi_\alpha\pi_\beta}{\sqrt{\mu_\beta^2 + \nu_\beta^2 + \pi_\beta^2}} \tag{3.3}$$

is called the projection of α *on* β*.*

Definition 9. *[28] Let X be a finite universe,*

$$X = \{x_1, x_2, \ldots, x_n\},$$

and A and B be two IFSs in X. Then,

$$\mathrm{Proj}_B A = \frac{\sum_{i=1}^n \left(\mu_{\alpha_i}\mu_{\beta_i} + \nu_{\alpha_i}\nu_{\beta_i} + \pi_{\alpha_i}\pi_{\beta_i}\right)}{\sqrt{\sum_{i=1}^n \mu_{\beta_i}^2 + \nu_{\beta_i}^2 + \pi_{\beta_i}^2}} \tag{3.4}$$

is called the projection of A on B, and $(\mu_{\alpha_i}, \nu_{\alpha_i})$ *denotes the i-th intuitionistic fuzzy number in A, where*

$$\pi_{\alpha_i} = 1 - \mu_{\alpha_i} - \nu_{\alpha_i}.$$

As shown in Figure 2, the greater the projection value, the closer the intuitionistic fuzzy sets A and B are to each other. However, extensive research has revealed that, in certain cases, the above-mentioned method is unreasonable in determining the proximity between IFS A and B. For example, let A, B, and C be three intuitionistic fuzzy sets,

$$A = B = ([\mu_{\alpha_1}, \nu_{\alpha_1}], [\mu_{\alpha_2}, \nu_{\alpha_2}], ..., [\mu_{\alpha_n}, \nu_{\alpha_n}]),$$
$$C = ([2\mu_{c_1}, 2\nu_{c_1}], [2\mu_{c_2}, 2\nu_{c_2}], ..., [2\mu_{c_n}, 2\nu_{c_n}]),$$

we calculated and obtained

$$\mathrm{Proj}_B A = \sqrt{\sum_{i=1}^{n} \mu_{\alpha_i}^2 + v_{\alpha_i}^2 + \pi_{\alpha_i}^2}$$

and

$$\mathrm{Proj}_B C = 2\sqrt{\sum_{i=1}^{n} \mu_{\alpha_i}^2 + v_{\alpha_i}^2 + \pi_{\alpha_i}^2}.$$

In fact, A is closer to B than C, however, the value of $\mathrm{Proj}_B C$ is larger than $\mathrm{Proj}_B A$.

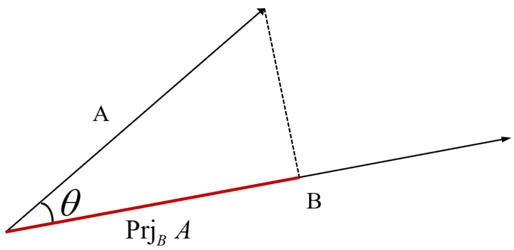

Figure 2. The projection of A on B.

In another case as shown in Figure 3, the projection value

$$\mathrm{Proj}_B A = \mathrm{Proj}_B C,$$

makes it impossible to distinguish which intuitionistic fuzzy set is closer to B.

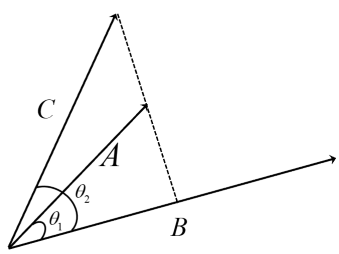

Figure 3. Illustration of the situation where the projection values are equal.

Hence, in order to overcome the shortcoming of the above projection model, a bidirectional projection measure has been proposed for interval numbers.

Definition 10. *[41] Let*

$$a = \left(\left[a_1^l, a_1^u\right], \left[a_2^l, a_2^u\right], \ldots, \left[a_n^l, a_n^u\right]\right)$$

and

$$b = \left(\left[b_1^l, b_1^u\right], \left[b_2^l, b_2^u\right], \ldots, \left[b_n^l, b_n^u\right]\right)$$

be two interval vectors. Then,

$$||a|| = \sqrt{\sum_{j=1}^{n} \left(a_j^l\right)^2 + \left(a_j^u\right)^2} \tag{3.5}$$

is called the module of a. It is the same for $||b||$.

$$a \cdot b = \sum_{j=1}^{n} \left(a_j^l b_j^l + a_j^u b_j^u\right) \tag{3.6}$$

is called the inner product of a and b.

$$\text{BProj}_b\, a = \frac{1}{1 + \left|\frac{a \cdot b}{||a||} - \frac{a \cdot b}{||b||}\right|} \tag{3.7}$$

is called the bidirectional projection between a and b.

The bidirectional projection measure includes both the distance and the angle between vectors a and b, as well as the bidirectional projection magnitude. Determine the proximity of a and b by projecting a and b onto each other. We find that when a = b, the projection value

$$\text{BProj}_b\, a = 1,$$

and for any two interval vectors a and b, the bidirectional projection measure

$$\text{BProj}_b\, a = \text{BProj}_a\, b$$

satisfies the symmetry of the similarity measure. However, it also has some drawbacks. For example, for the interval vectors

$$a = ([1, 0]), \quad b = ([0, 1]), \quad \text{and} \quad ||a|| = ||b||,$$

but the angle difference is large. Obviously,

$$\text{BProj}_b\, a = 1,$$

but it is unreasonable.

3.2. The bidirectional projection measure

In this paper, the bidirectional projection measure is extended to circular intuitionistic fuzzy sets. Since the membership and non-membership degrees of circular intuitionistic fuzzy sets are represented by a circle, we extend the modulus, projection values, etc., of circular intuitionistic fuzzy sets to interval values. Next, the bidirectional projection measure of circular intuitionistic fuzzy sets is presented as follows.

Definition 11. *Let*

$$X = \{x_1, x_2, \ldots, x_n\}$$

be a given universe,

$$C_1 = \left\{\langle x, \mu_{c_1}(x), \nu_{c_1}(x); r_1 \middle| x \in X\right\}$$

and

$$C_2 = \left\{\langle x, \mu_{c_2}(x), \nu_{c_2}(x); r_2 \middle| x \in X\right\}$$

are two CIFSs on X. Then,

$$\begin{aligned} \|C_1\| &= \left[\sqrt{\sum_{i=1}^{n} \mu_{c_1}(x_i)^2 + \nu_{c_1}(x_i)^2 - \sum_{i=1}^{n} r_1(x_i)},\ \sqrt{\sum_{i=1}^{n} \mu_{c_1}(x_i)^2 + \nu_{c_1}(x_i)^2 + \sum_{i=1}^{n} r_1(x_i)}\right] \\ &= \left[\left\|\mathrm{C}_1^l\right\|, \left\|\mathrm{C}_1^u\right\|\right] \end{aligned} \tag{3.8}$$

is called the module of C_1*, and* $\|C_1\|$ *is an interval number. The same is true with* $\|C_2\|$*.* $r_1(x_i)$ *denotes r in the CIFS* C_1 *corresponding to* x_i*, and*

$$\cos(C_1, C_2) = \frac{\sum_{i=1}^{n}\left[\mu_{c_1}(x_i)\,\mu_{c_2}(x_i) + \nu_{c_1}(x_i)\,\nu_{c_2}(x_i)\right]}{\sqrt{\sum_{i=1}^{n} \mu_{c_1}(x_i)^2 + \nu_{c_1}(x_i)^2}\sqrt{\sum_{i=1}^{n} \mu_{c_2}(x_i)^2 + \nu_{c_2}(x_i)^2}} \tag{3.9}$$

is called the cosine of the included angle between C_1 *and* C_2*.*

Definition 12. *Let*

$$X = \{x_1, x_2, \ldots, x_n\}$$

be a given universe, and

$$C_1 = \left\{\langle x, \mu_{c_1}(x), \nu_{c_1}(x); r_1 \middle| x \in X\right\}$$

and

$$C_2 = \left\{\langle x, \mu_{c_2}(x), \nu_{c_2}(x); r_2 \middle| x \in X\right\}$$

are two CIFSs on X. Then,

$$\begin{aligned} \mathrm{Proj}_{C_1} C_2 &= \|C_2\| \bullet \cos(C_1, C_2) \\ &= \left[\left\|C_2^l\right\| \cos(C1, C2), \left\|C_2^u\right\| \cos(C1, C2)\right] \\ &= \left[\mathrm{proj}_{C_1} C_2^l, \mathrm{proj}_{C_1} C_2^u\right] \end{aligned} \tag{3.10}$$

is called the projection of CIFS C_2 *on the CIFS* C_1*, and* $\mathrm{Proj}_{C_1} C_2$ *is an interval number.*

Definition 13. *Let*

$$X = \{x_1, x_2, \ldots, x_n\}$$

be a given universe, and

$$C_1 = \left\{\langle x, \mu_{c_1}(x), \nu_{c_1}(x); r_1 \middle| x \in X\right\}$$

and

$$C_2 = \left\{\langle x, \mu_{c_2}(x), \nu_{c_2}(x); r_2 \middle| x \in X\right\}$$

be two circular intuitionistic fuzzy sets on X. Then,

$$\mathrm{BProj}\left(C_1,C_2\right)=\frac{1}{1+\left|\mathrm{Proj}_{C_1}C_2-\|C_1\|\right|+\left|\mathrm{Proj}_{C_2}C_1-\|C_2\|\right|}\tag{3.11}$$

is called the bidirectional projection measure between C_1 and C_2, where

$$\left|\mathrm{Proj}_{C_1}C_2-\|C_1\|\right|=\frac{\left|\mathrm{proj}_{C_1}C_2^l-\left\|C_1^l\right\|\right|+\left|\mathrm{proj}_{C_1}C_2^u-\left\|C_1^u\right\|\right|}{2}$$

and

$$\left|\mathrm{Proj}_{C_2}C_1-\|C_2\|\right|=\frac{\left|\mathrm{proj}_{C_2}C_1^l-\left\|C_2^l\right\|\right|+\left|\mathrm{proj}_{C_2}C_1^u-\left\|C_2^u\right\|\right|}{2}.$$

Elements in the circular intuitionistic fuzzy set are represented by a circle. Their projection values, referring to the projection of ordinary vectors, are interval-valued, as shown in Figure 4.

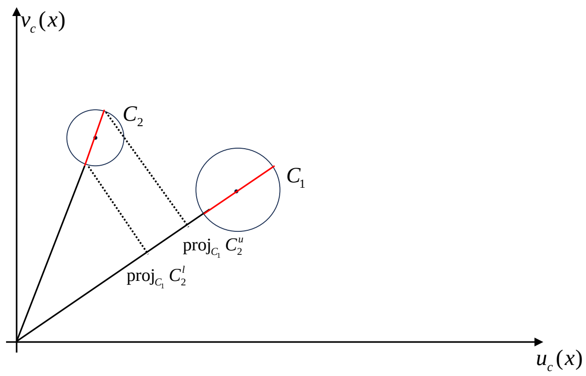

Figure 4. The Projection value of circular intuitionistic fuzzy set.

The bidirectional projection measure takes into account not only the distance between different circular intuitionistic fuzzy sets, but also the angle between them. The bidirectional projection measure satisfies the following theorems. It can be seen that when the modulus length of the circular intuitionistic fuzzy set is larger, it indicates that the relationship between membership and non-membership is more explicit, and we obtain more information. While the magnitude of the angle reflects the proportional relationship between the membership degree and the non-membership degree, these factors will undoubtedly affect the similarity between different circular fuzzy sets.

Theorem 1. *Let C_1 and C_2 be two circular intuitionistic fuzzy sets on the universe*

$$X=\{x_1,x_2,\ldots,x_n\}$$

and $\mathrm{BProj}(C_1,C_2)$ *be the bidirectional projection measure on the CIFS.* $\mathrm{BProj}(C_1,C_2)$ *satisfies the following three properties:*

a) $0\le \mathrm{BProj}\left(C_1,C_2\right)\le 1.$

b) $\text{BProj}(C_1, C_2) = \text{BProj}(C_2, C_1)$.

c) $\text{BProj}(C_1, C_2) = 1$ *if and only if* $C_1 = C_2$.

Proof. a) The conclusion clearly holds as obtained through Eq (3.11).

b) Through Eq (3.11), Theorem 1 b) clearly holds as well.

c) When

$$C_1 = C_2, \quad \left|\text{Proj}_{C_1} C_2 - \|C_1\|\right| = 0, \quad \left|\text{Proj}_{C_2} C_1 - \|C_2\|\right| = 0,$$

substituting into Eq (3.11), we can obtain that

$$\text{BProj}(C_1, C_2) = 1.$$

The converse also holds.

□

3.3. Example and comparison analysis

In order to demonstrate the rationality of calculating the parameter r of the circular fuzzy set using the least squares method and the rationality of the bidirectional projection measure based on the circular fuzzy set. First, an example is given to compare the integrity of different fuzzy sets in representing human fuzzy information. Another example is provided to compare the superiority of the bidirectional projection similarity method based on circular intuitionistic fuzzy sets over other measurement methods.

Example 1. *Suppose a company wants to evaluate the work capabilities of two employees. Corresponding fuzzy membership functions are established. The greater the membership degree and the smaller the non-membership degree, the stronger the work capability is indicated. The universe*

$$X = \{x_1, x_2\}$$

represents two employees, and four experts conduct evaluations on them and provide the corresponding membership degrees and non-membership degrees,

$$\{x_1 :< 0.2, 0.2 >, < 0.6, 0.1 >, < 0.6, 0.2 >, < 0.8, 0.1 >\}.$$

The evaluation information of the second employee is

$$\{x_2 :< 0.55, 0.2 >, < 0.5, 0.1 >, < 0.6, 0.2 >, < 0.55, 0.1 >\},$$

respectively using the intuitionistic fuzzy set, the interval-valued intuitionistic fuzzy set, the circular fuzzy set with parameter r calculated in [31], and the circular fuzzy set with parameter r calculated by the least squares method to represent the evaluation information.

By observing Table 1, we can find that when using intuitionistic fuzzy sets to represent the decision-making information of the two employees, it is completely consistent.

Table 1. Representation of evaluation information in fuzzy sets.

	x_1	x_2
IFS [7]	$\langle x_1, 0.55, 0.15\rangle$	$\langle x_2, 0.55, 0.15\rangle$
IVIFS [13]	$\langle x_1, [0.2, 0.8], [0.1, 0.2]\rangle$	$\langle x_2, [0.5, 0.6], [0.1, 0.2]\rangle$
CIFS [31]	$\langle x_1, 0.55, 0.15; 0.3536\rangle$	$\langle x_2, 0.55, 0.15; 0.0707\rangle$
CIFS(OLS)	$\langle x_1, 0.55, 0.15; 0.2236\rangle$	$\langle x_2, 0.55, 0.15; 0.0612\rangle$

However, it is obvious that the evaluation information provided by the four experts is completely inconsistent. The evaluation information interval for the first employee is relatively large, while the evaluation interval for the second employee is more concentrated. Although the IVIFS makes a distinction in terms of numerical values, we find that the evaluation intervals of the two employees differ significantly, and in fact the actual gap is not that large. The CIFS retains as much information as possible from all the original evaluations. The method of calculating the parameter r in [31] is vulnerable to the influence of extreme evaluation information. However, choosing the least squares method can effectively avoid this problem.

Example 2. *Utilize the existing distances and measures of circular fuzzy sets, as well as the bidirectional projection similarity proposed in this paper, to compare the distances between different circular fuzzy sets. The results are shown in Table 2.*

Table 2. Comparison of distance measures between CIFSs.

	1	2	3	4	5
A	$\langle x, 0.1, 0.1; 0\rangle$	$\langle x, 0.1, 0.1; 0\rangle$	$\langle x, 0.3, 0.4; 0.1\rangle$	$\langle x, 0.1, 0.1; 0.2\rangle$	$\langle x, 0.1, 0.1; 0.5\rangle$
B	$\left\langle x, \frac{3}{10}\sqrt{2} + 0.1, 0.1; 0\right\rangle$	$\langle x, 0.4, 0.4; 0.2\rangle$	$\langle x, 0.4, 0.3; 0.1\rangle$	$\langle x, 0.3, 0.4; 0.1\rangle$	$\langle x, 0.3, 0.4; 0.1\rangle$
$E_2(A, B)$ [32]	0.2207	0.2207	0.05	0.1628	0.2689
$\mathrm{BProj}_B A$ [41]	0.7550	0.7021	1	0.7380	0.7380
$\mathrm{BProj}(A, B)$	0.6328	0.5608	0.9615	0.5836	0.5568

(1) By comparing the circular fuzzy sets of the first group and the second group, we find that the membership degree in $\left\langle x, \frac{3}{10}\sqrt{2} + 0.1, 0.1; 0\right\rangle$ is larger than that in $\langle x, 0.3, 0.4; 0.1\rangle$, and the non-membership degree in $\left\langle x, \frac{3}{10}\sqrt{2} + 0.1, 0.1; 0\right\rangle$ is smaller than that in $\langle x, 0.3, 0.4; 0.1\rangle$,so $\left\langle x, \frac{3}{10}\sqrt{2} + 0.1, 0.1; 0\right\rangle$ should be greater than $\langle x, 0.3, 0.4; 0.1\rangle$. However, the distances calculated by using the Euclidean distance in [32] are equal. It is not reasonable. It can be seen that the bidirectional projection measure proposed in this paper can effectively distinguish them.

(2) By observing the circular fuzzy sets in the third group, it can be found that when A and B are completely different, the similarity calculated using the method in [41] is 1, indicating complete similarity, which does not conform to the actual situation. However, the similarity calculated by using the bidirectional projection similarity proposed in this paper is 0.9615, which is in line with the actual situation.

(3) By observing the circular intuitionistic fuzzy sets in the fourth and fifth groups, it can be found that only the parameter r of A is different. Through the results, it is found that the parameter r will affect the distance and similarity between circular fuzzy sets, which demonstrates the effectiveness of CIFS compared with intuitionistic fuzzy sets and extended intuitionistic fuzzy sets.

While the aforementioned example does not encompass all possible cases of circular intuitionistic fuzzy sets, it is representative of a subset of such sets. It can be observed that some existing measures for distinguishing different circular intuitionistic fuzzy sets exhibit certain limitations. The bidirectional projection measure proposed in this paper fully considers the effects of angle and distance on the similarity of circular intuitionistic fuzzy sets, and it is found to be more reasonable than some of the existing methods by example.

4. Application of bidirectional projection measure in group decision making

In this section, for a group multi-attribute decision-making (GMADM) problem, this paper proposes a GMADM method based on the bidirectional projection measure of circular intuitionistic fuzzy sets. The basic procedure of the proposed method is illustrated in the following Figure 5. The detailed basic steps of the proposed method are as follows:

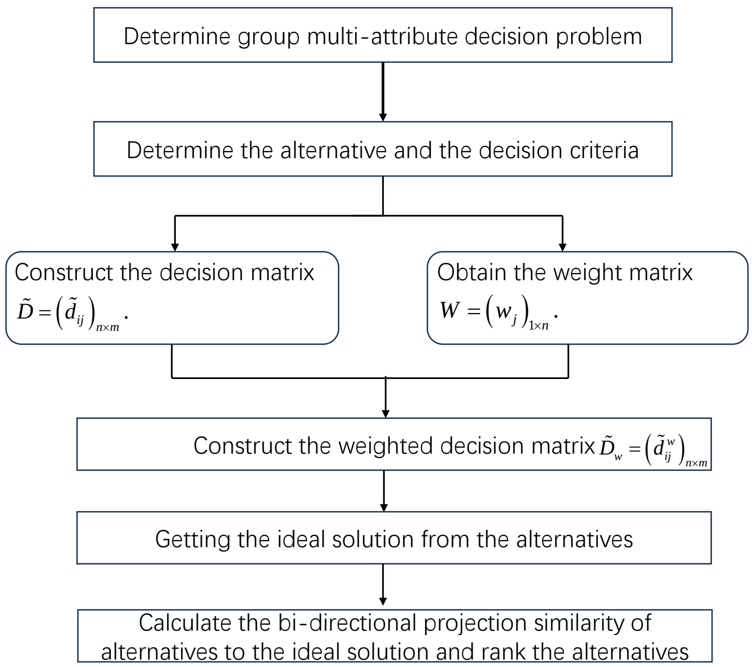

Figure 5. Proposed methodology for MAGDM using CIFS and bidirectional projection.

Step 1. For MADM problem, first, the set of alternatives

$$A = \{A_1, A_2, \dots, A_m\}$$

is determined for the problem. The set of decision criteria are

$$C = \{C_1, C_2, \dots\dots, C_n\},$$

and the weight vector of the criterion is

$$W = \{w_1, w_2, \dots\dots, w_n\}.$$

DMs represent a group of experts in various fields, denoted by $DM1, DM2, \ldots, DMk$.

Step 2. First, the evaluation linguistic of each expert for each criterion of all alternatives was collected, and then the evaluation linguistic was transformed into corresponding intuitionistic fuzzy pairs according to Table 3.

Table 3. Linguistic value intuitionistic fuzzy number conversion.

Linguistic terms	IFNs for alternatives
Certainly High Value-(CHV)	$\langle 0.9, 0.1\rangle$
Very High Value-(VHV)	$\langle 0.8, 0.15\rangle$
High Value-(HV)	$\langle 0.7, 0.25\rangle$
Above Average Value-(AAV)	$\langle 0.6, 0.35\rangle$
Average Value-(AV)	$\langle 0.5, 0.45\rangle$
Under Average Value-(UAV)	$\langle 0.4, 0.55\rangle$
Low Value-(LV)	$\langle 0.3, 0.65\rangle$
Very Low Value-(VLV)	$\langle 0.2, 0.75\rangle$
Certainly Low Value-(CLV)	$\langle 0.1, 0.9\rangle$

Table 3 presents the transformation relationship between human language evaluation values and intuitionistic fuzzy numbers, and transforms the evaluation languages of experts into intuitionistic fuzzy numbers. Here, we obtain k matrices

$$\tilde{D}_k = \left(\tilde{d}_{ijk}\right)_{n\times m},$$

where

$$\tilde{d}_{ijk} = \left(m_{ijk}, n_{ijk}\right),$$

and $\tilde{d}_{ijk}$ represents the performance evaluation of the i-th alternative on the j-th criterion by the k-th expert.

Step 3. Use Eq (2.3) to aggregate intuitionistic fuzzy pairs of the same alternative with the same criterion in k matrices

$$\tilde{D}_k = \left(\tilde{d}_{ijk}\right)_{n\times m}$$

to get the corresponding intuitionistic fuzzy numbers $\left\langle \mu_{ij}, v_{ij}\right\rangle$. Then, the corresponding r length is calculated using Eq (2.7), and, subsequently, a matrix

$$\tilde{D} = \left(\tilde{d}_{ij}\right)_{n\times m}$$

is obtained, where

$$\tilde{d}_{ij} = \left(\mu_{ij}, v_{ij}; r_{ij}\right)$$

is a circular intuitionistic fuzzy number representing the evaluation value of the j-th criterion for the i-th alternative.

Step 4. The weight information of each attribute value is also evaluated through the linguistic values provided by experts, such as "very important" and "unimportant", and so on. The linguistic weight information of different criteria is quantified into the corresponding intuitionistic fuzzy numbers based on Table 4. Then, the r-length is calculated using Eq (2.7). We obtain the corresponding weight information matrix

$$W = \left(w_j\right)_{1\times n},$$

where

$$w_j = \left(u_j, v_j; r_j\right)$$

represents the weight of the j-th criterion.

Table 4. Weighted information intuitionistic fuzzy number transformation.

Linguistic terms	IFNs for criteria
Certainly high importance-(CHI)	$\langle 0.9, 0.1\rangle$
Very high importance-(VHI)	$\langle 0.8, 0.15\rangle$
High importance-(HI)	$\langle 0.7, 0.25\rangle$
Above average importance-(AAI)	$\langle 0.6, 0.35\rangle$
Average importance-(AI)	$\langle 0.5, 0.45\rangle$
Under average importance-(UAI)	$\langle 0.4, 0.55\rangle$
Low importance-(LI)	$\langle 0.3, 0.65\rangle$
Very low importance-(VLI)	$\langle 0.2, 0.75\rangle$
Certainly low importance-(CLI)	$\langle 0.1, 0.9\rangle$

Step 5. Construct a weighted decision matrix

$$\tilde{D}_w = \left(\tilde{d}_{ij}^w\right)_{n\times m},$$

where

$$\left(\tilde{d}_{ij}^w\right)_{n\times m} = \tilde{d}_{ij} \otimes w_j$$

is calculated by Eq (2.8).

Step 6. Identifying an ideal solution

$$Y = \{\mathrm{y}_1, \mathrm{y}_2, \ldots.., \mathrm{y}_n\},$$

we categorize the n criteria into two types: one type is the cost criteria N_1, and the other type is the benefit criteria N_2.

$$\begin{aligned} Y &= \left\{\left\langle\left(\max_{1\le\le\le n} g_{ij} \mid j \in N_1\right), \left(\min_{1\le i\le m} g_{ij} \mid j \in N_2\right)\right\rangle \mid j = 1, 2, \ldots n\right\} \\ &= \{\mathrm{y}_1, \mathrm{y}_2, \ldots.., \mathrm{y}_n\}. \end{aligned}$$

Use the method in Definition 7 to compare the ordinal relations of circular intuitionistic fuzzy sets.

Step 7. Calculate Bproj (A_i, Y), and rank the alternatives based on the value.

5. Example analysis

5.1. Illustrative example

The supplier selection example from [33] was chosen for calculations to test the effectiveness of the new approach. The example presented is a fast-moving consumer goods supplier selection problem. The detailed steps of the proposed algorithm are illustrated as follows:

Step 1. First, the set of alternative suppliers for the fast-moving consumer goods (FMCG) is determined, comprising four candidates. There are four alternatives (A_1, A_2, A_3, and A_4). They respectively represent four suppliers. Based on the literature review and expert opinions, seven decision criteria ($\{C_1, C_2, \ldots, C_7\}$) were determined for the fast-moving consumer goods (FMCG) supplier selection problem. These seven criteria, are respectively, price, quality, performance, delivery, flexibility, relationship closeness, and reputation. A panel of experts was established to evaluate the problem, consisting of five experts, namely DM_1–DM_5.

Step 2. Based on the linguistic evaluation values provided in Table 3, the 5 experts evaluated the 7 criteria for the four suppliers, resulting in Table 5. Then, according to Table 3, the linguistic values are transformed into the corresponding intuitionistic fuzzy pairs, resulting in Table 6.

Table 5. Evaluations of alternative suppliers by experts.

Criteria	DMs	A1	A2	A3	A4
C1	DM1	AAV	UAV	HV	LV
	DM2	HV	LV	VHV	VLV
	DM3	HV	UAV	AAV	LV
	DM4	AAV	AV	AAV	LV
	DM5	AV	UAV	HV	VLV
C2	DM1	AV	AAV	VHV	LV
	DM2	AAV	AV	HV	UAV
	DM3	UAV	HV	AAV	UAV
	DM4	AV	AAV	AV	AV
	DM5	UAV	AV	AAV	AV
C3	DM1	AV	HV	AV	LV
	DM2	AAV	AV	AV	UAV
	DM3	AAV	AV	AAV	UAV
	DM4	UAV	AAV	AAV	AV
	DM5	AV	AAV	AV	AV
C4	DM1	HV	LV	VHV	AV
	DM2	HV	VLV	VHV	AV
	DM3	AAV	VLV	HV	AAV
	DM4	AV	LV	HV	UAV
	DM5	AV	LV	AAV	AV
C5	DM1	LV	AV	AV	UAV
	DM2	UAV	AV	AV	UAV
	DM3	UAV	UAV	AAV	AV
	DM4	AV	AAV	AAV	AAV
	DM5	LV	AAV	AV	AV
C6	DM1	HV	AAV	AAV	AAV
	DM2	HV	AAV	HV	AAV
	DM3	VHV	AV	HV	AV
	DM4	HV	AV	AAV	AAV
	DM5	AAV	AAV	HV	AAV
C7	DM1	AV	AV	HV	VHV
	DM2	UAV	AV	HV	VHV
	DM3	UAV	AAV	VHV	HV
	DM4	AV	HV	AAV	HV
	DM5	AV	HV	HV	HV

Table 6. Transformed decision matrix with IFNs.

Criteria	DMs	A1	A2	A3	A4
C1	DM1	(0.6, 0.35)	(0.4, 0.55)	(0.7, 0.25)	(0.3, 0.65)
	DM2	(0.7, 0.25)	(0.3, 0.65)	(0.8, 0.15)	(0.2, 0.75)
	DM3	(0.7, 0.25)	(0.4, 0.55)	(0.6, 0.35)	(0.3, 0.65)
	DM4	(0.6, 0.35)	(0.5, 0.45)	(0.6, 0.35)	(0.3, 0.65)
	DM5	(0.5, 0.45)	(0.4, 0.55)	(0.7, 0.25)	(0.2, 0.75)
C2	DM1	(0.5, 0.45)	(0.6, 0.35)	(0.8, 0.15)	(0.3, 0.65)
	DM2	(0.6, 0.35)	(0.5, 0.45)	(0.7, 0.25)	(0.4, 0.55)
	DM3	(0.4, 0.55)	(0.7, 0.25)	(0.6, 0.35)	(0.4, 0.55)
	DM4	(0.5, 0.45)	(0.6, 0.35)	(0.5, 0.45)	(0.5, 0.45)
	DM5	(0.4, 0.55)	(0.5, 0.45)	(0.6, 0.35)	(0.5, 0.45)
C3	DM1	(0.5, 0.45)	(0.7, 0.25)	(0.5, 0.45)	(0.3, 0.65)
	DM2	(0.6, 0.35)	(0.5, 0.45)	(0.5, 0.45)	(0.4, 0.55)
	DM3	(0.6, 0.35)	(0.5, 0.45)	(0.6, 0.35)	(0.4, 0.55)
	DM4	(0.4, 0.55)	(0.6, 0.35)	(0.6, 0.35)	(0.5, 0.45)
	DM5	(0.5, 0.45)	(0.6, 0.35)	(0.5, 0.45)	(0.5, 0.45)
C4	DM1	(0.7, 0.25)	(0.3, 0.65)	(0.8, 0.15)	(0.5, 0.45)
	DM2	(0.7, 0.25)	(0.4, 0.55)	(0.8, 0.15)	(0.5, 0.45)
	DM3	(0.6, 0.35)	(0.3, 0.65)	(0.7, 0.25)	(0.6, 0.35)
	DM4	(0.5, 0.45)	(0.3, 0.65)	(0.7, 0.25)	(0.4, 0.55)
	DM5	(0.5, 0.45)	(0.3, 0.65)	(0.6, 0.35)	(0.5, 0.45)
C5	DM1	(0.3, 0.65)	(0.5, 0.45)	(0.5, 0.45)	(0.4, 0.55)
	DM2	(0.4, 0.55)	(0.5, 0.45)	(0.5, 0.45)	(0.4, 0.55)
	DM3	(0.4, 0.55)	(0.4, 0.55)	(0.6, 0.35)	(0.5, 0.45)
	DM4	(0.5, 0.45)	(0.6, 0.35)	(0.6, 0.35)	(0.6, 0.35)
	DM5	(0.3, 0.65)	(0.6, 0.35)	(0.5, 0.45)	(0.5, 0.45)
C6	DM1	(0.7, 0.25)	(0.6, 0.35)	(0.6, 0.35)	(0.6, 0.35)
	DM2	(0.7, 0.25)	(0.6, 0.35)	(0.7, 0.25)	(0.6, 0.35)
	DM3	(0.8, 0.15)	(0.5, 0.45)	(0.7, 0.25)	(0.5, 0.45)
	DM4	(0.7, 0.25)	(0.5, 0.45)	(0.6, 0.35)	(0.6, 0.35)
	DM5	(0.6, 0.35)	(0.6, 0.35)	(0.7, 0.25)	(0.6, 0.35)
C7	DM1	(0.5, 0.45)	(0.5, 0.45)	(0.7, 0.25)	(0.8, 0.15)
	DM2	(0.4, 0.55)	(0.5, 0.45)	(0.7, 0.25)	(0.8, 0.15)
	DM3	(0.4, 0.55)	(0.6, 0.35)	(0.8, 0.15)	(0.7, 0.25)
	DM4	(0.5, 0.45)	(0.7, 0.25)	(0.6, 0.35)	(0.7, 0.25)
	DM5	(0.5, 0.45)	(0.7, 0.25)	(0.7, 0.25)	(0.7, 0.25)

Step 3. According to Eqs (2.3) and (2.7), the 5 matrices $\tilde{D}_k$ are aggregated to obtain the circular intuitionistic fuzzy set decision matrix

$$\tilde{D} = \left(\tilde{d}_{ij}\right)_{7\times 3},$$

as shown in Table 7.

Table 7. Circular intuitionistic fuzzy decision matrix.

	A1	A2	A3	A4
C1	(0.62;0.33;0.1058)	(0.4;0.55;0.0894)	(0.68;0.27;0.1058)	(0.26;0.69;0.0693)
C2	(0.48;0.47;0.1058)	(0.58;0.37;0.1058)	(0.64;0.31;0.1442)	(0.42;0.53;0.1058)
C3	(0.52;0.43;0.1058)	(0.58;0.37;0.1058)	(0.54;0.41;0.0693)	(0.42;0.53;0.1058)
C4	(0.6;0.35;0.1265)	(0.26;0.69;0.0566)	(0.72;0.23;0.1058)	(0.5;0.45;0.0894)
C5	(0.38;0.57;0.1058)	(0.52;0.43;0.1058)	(0.54;0.41;0.0693)	(0.48;0.47;0.1058)
C6	(0.7;0.25;0.0894)	(0.56;0.39;0.0693)	(0.666;0.29;0.0693)	(0.58;0.37;0.0566)
C7	(0.46;0.49;0.0693)	(0.6;0.35;0.1265)	(0.7;0.25;0.0894)	(0.74;0.21;0.0693)

Step 4. Based on the weight information of the criteria provided by the decision-makers, Table 8 is obtained. Based on the transformation relationship of linguistic values into intuitionistic fuzzy numbers provided in Table 4, an intuitionistic fuzzy set weight information matrix is obtained. Then, by applying Eq (2.7), the radius r of the circular intuitionistic fuzzy set is calculated to derive the weight information matrix of the circular intuitionistic fuzzy set (Table 9).

Table 8. Weight information matrix.

Criterion	DM1	DM2	DM3	DM4	DM5	Type
C1	AAI	AI	AI	AAI	AI	Cost
C2	VHI	CHI	CHI	VHI	CHI	Benefit
C3	HI	VHI	HI	HI	VHI	Benefit
C4	VHI	VHI	HI	HI	VHI	Benefit
C5	HI	HI	VHI	VHI	HI	Benefit
C6	AAI	HI	AAI	AI	HI	Benefit
C7	AAI	AAI	AI	AAI	AI	Benefit

Table 9. Circular intuitionistic fuzzy sets criteria weight information matrix.

Criteria	Criteria weight
C1	(0.537, 0.412; 0.0693)
C2	(0.859, 0.120; 0.0548)
C3	(0.738, 0.211; 0.0693)
C4	(0.758, 0.192; 0.0693)
C5	(0.738, 0.211; 0.0693)
C6	(0.615, 0.334; 0.1058)
C7	(0.558, 0.392; 0.0693)

Step 5. Obtain the weighted decision matrix (Table 10) using Eq (2.8).

Table 10. Weighted decision matrix.

Criteria	A1	A2	A3	A4
C1	(0.33, 0.61; 0.09)	(0.21, 0.74; 0.08)	(0.37, 0.57; 0.09)	(0.14, 0.82; 0.07)
C2	(0.41, 0.53; 0.08)	(0.50, 0.45; 0.08)	(0.55, 0.39; 0.1)	(0.36, 0.59; 0.08)
C3	(0.38, 0.55; 0.09)	(0.43, 0.50; 0.09)	(0.40, 0.53; 0.07)	(0.31, 0.63; 0.09)
C4	(0.45, 0.47; 0.10)	(0.20, 0.75; 0.06)	(0.55, 0.38; 0.09)	(0.38, 0.56; 0.08)
C5	(0.28, 0.66; 0.09)	(0.38, 0.55; 0.09)	(0.40, 0.53; 0.07)	(0.35, 0.58; 0.09)
C6	(0.43, 0.50; 0.10)	(0.34, 0.59; 0.09)	(0.41, 0.53; 0.09)	(0.36, 0.58; 0.08)
C7	(0.26, 0.69; 0.07)	(0.33, 0.60; 0.10)	(0.39, 0.54; 0.08)	(0.41, 0.52; 0.07)

Step 6. Determining the ideal solution Y (Table 11).

Table 11. Ideal solution.

Criteria	Value
C1	(0.14, 0.82; 0.07)
C2	(0.55, 0.39; 0.1)
C3	(0.43, 0.50; 0.09)
C4	(0.55, 0.38; 0.09)
C5	(0.40, 0.53; 0.07)
C6	(0.43, 0.50; 0.10)
C7	(0.41, 0.52; 0.07)

Step 7. Calculate the bidirectional projection similarity of each alternative to the ideal solution (Table 12).

Table 12. Bidirectional projection similarity of each alternative to the ideal solution.

	A1	A2	A3	A4
Bproj (A_i, Y)	0.8884	0.8572	0.8993	0.8814
Rank	2	4	1	3

The ranking of the alternative options is as follows:

$$A_3 \succ A_1 \succ A_4 \succ A_2.$$

5.2. Comparative analysis

In [33], the calculated results are:

$$A_3 \succ A_4 \succ A_2 \succ A_1,$$

and we found that the optimal selection results are consistent with those of this paper, but the comparison results between the first and fourth options differ. Observing the weighted decision matrix in Table 10, we can see that A_1 has 4 criteria superior to A_4, while the other criteria where A_4 is better have only a small margin. It can be inferred that A_1 should be superior to A_4. At the same time, we use the intuitionistic fuzzy set projection method in [28] to compute the results for comparison (Table 13).

Table 13. Comparison of methods and rankings.

Reference	Method	Ranking
Xu and Hu [28]	IF PROJECTION	$A_3 \succ A_2 \succ A_1 \succ A_4$,
Kahraman and Alkan [33]	C-IF TOPSIS	$A_3 \succ A_4 \succ A_2 \succ A_1$,
Proposed method	Bidirectional projection measures for CIFS	$A_3 \succ A_1 \succ A_4 \succ A_2$,

The optimal choice for all three methods is A_3, indicating the effectiveness of the method proposed in this paper. Furthermore, through the previous analysis, it can be concluded that A_1 should be superior to A_2 and A_4, which also demonstrates the rationality and accuracy of the method proposed in this paper.

6. Managerial implications

This paper pioneers a method of using the geometric projection approach in CIFS to calculate the similarity, providing research ideas for the subsequent calculation of the similarity of related extended circular fuzzy sets. Additionally, relevant decision-making methods are presented, offering a new model for GMADM problems.

7. Conclusions

Circular intuitionistic fuzzy sets have a stronger capability to express the uncertainty in expert evaluations compared to intuitionistic fuzzy sets. This paper proposes a novel method for calculating the parameter r of CIFS, and applies it to the definition of bidirectional projection measure for CIFS. We introduces a new method for calculating the similarity between circular intuitionistic fuzzy sets and demonstrates its rationality and correctness through examples. The new bidirectional projection measure method not only considers the distance between CIFS, but also takes into account the angles between them. The new bidirectional projection measure method fills the gap in the research of projection measures within the field of circular intuitionistic fuzzy sets. There are still some shortcomings in the research presented in this paper. There has been no further detailed discussion on various relationships between different circles, such as tangency, intersection, and inclusion, and the relationship between these relationships and the similarity among them. This can serve as a future research direction. Moreover, when using the operation operators between circular intuitionistic fuzzy sets, there has been no further comparison of their rationality. This can also be considered as a future research direction.

In further research, this method can be further extended to other types of decision-making data, such as Pythagorean fuzzy sets, spherical fuzzy sets, and so on, consider the process of incorporating parameter r into other extended fuzzy sets. This method can be applied to more complex decision-making problems. And, we may consider whether it can be applied to more problems, such as pattern recognition, risk assessment, and other issues.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest regarding the publication for the paper.

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