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Research article

The coexistence of quasi-periodic and blow-up solutions for a class of impact oscillators without the twist condition

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Abstract: The purpose of this paper was the study of the dynamic behavior of impact oscillators:

$$\begin{cases} x'' + a(x)\,x^{2n+1} + \sum_{l=0}^{m} p_l(t)\,x^l = 0, \text{ for } x(t) > 0,\\ x(t) \geq 0,\\ x'(t_0^+) = -x'(t_0^-), \text{ if } x(t_0) = 0, \end{cases}$$

where the positive function $a(x)$ is a smooth T-periodic oscillator violating the monotone twist condition. We have proved that the above equation has an infinite number of bounded solutions as well as a solution that escapes to infinity in a finite amount of time.

Keywords: KAM theorem; impact oscillators; boundedness of solutions; quasi-periodic solutions; unbounded solutions
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1. Introduction and main results

In [1], Littlewood proposed the Duffing equation

$$x'' + G'(x) = p(t), p(t) = p(t+1)$$

concerning Lagrangian stability and boundedness of its solutions. We use Moser's twist theorem to prove boundedness when the Poincaré map has a monotone twist at infinity.

However, Zhiguo Wang and Yiqian Wang [2] have proved that the differential equation of the second order is

$$x'' + x^{2n+1}a(x) + p(t) = 0, \text{with } p(t) = p(t+1), a(x) = a(x+T) \text{ is smooth, and } a(x) > 0.$$

Despite violating the monotone twist condition, the upper equation remains stable in Lagrangian mechanics. For the Duffing equation $x'' + G'(x) = p(t)$, violating the monotone twist condition means that the ratio $\frac{G'(x)}{x}$ is not monotone, and one can see [3] for more details. The author [3] defines

$$A_+(k) = \text{the annulus bounded by } \Delta_{2k\pi - c_0} \text{ and } \Delta_{2k\pi}$$

and

$$A_-(k) = \text{the annulus bounded by } \Delta_{2k\pi+\pi-c_0} \text{ and } \Delta_{2k\pi+\pi},$$

where Δ_x is the closed orbit of the unperturbed system

$$x'' + x^{2n+1}(1 + ccosx) = 0,$$

and which passes through the point $(x, 0)$ in the $x - \dot{x}$ plane, where c_0 is a positive constant. The author pointed out that although the monotone twist condition is violated in the annulus $A_+(k) \bigcup A_-(k)$ (see [3] for details), the sign of $\frac{d^2 I}{dh^2}$ would not change in each annulus $A_+(k)$ ($A_-(k)$).

In [4], the authors showed that all solutions of the following equation can be found

$$x'' + x^{2n+1} + x^l p(t) = 0, \text{if } p(t) \in C^1, 0 \leq l \leq n,$$

and are bounded.

Zhiguo Wang and Yiqian Wang studied the following impact oscillators in [2]

$$\begin{cases} x'' + x^{2n+1} = p(t), \text{ for } x(t) > 0, \\ x(t) \geq 0, \ x'(t_0^+) = -x'(t_0^-), \text{ if } x(t_0) = 0. \end{cases}$$

They found that there are an infinite number of periodic and quasi-periodic solutions.

From a mechanical point of view, the previous equation shows how particles that are attached to nonlinear springs will bounce off a fixed barrier ($x = 0$). These types of systems are special cases of vibrating impact systems [5]. In addition, they are associated with the Fermi accelerator [6], the dual billiard [7], and certain models used in astronomy [8]. The application of many powerful mathematical tools is limited by the lack of smoothness caused by impact. However, the periodic and quasi-periodic motion of impact oscillators has been addressed in several recent papers, see refs. [9–11] and their references.

Let us start by studying the following impact system:

$$\begin{cases} x'' + a(x)\, x^{2n+1} + \sum_{l=0}^{m} p_l(t)\, x^l = 0, \text{ for } x(t) > 0, 0 \leq m \leq 2n, \\ x(t) \geq 0, \ x'(t_0^+) = -x'(t_0^-), \text{ if } x(t_0) = 0, \end{cases} \tag{1.1}$$

which has infinitely many bounded solutions. We suppose that the positive function $a(x)$ is a smooth T-periodic oscillating function that violates the monotone twist condition and the coefficient $p_l(t)$ is 1-periodic and C^5-smooth. Denote $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and then we have $p_l(t) \in C^5(\mathbb{T})$.

We can prove the first important result according to Moser's twist theorem.

Theorem 1.1. *Suppose $p_l(t) \in C^5(\mathbb{T}), 1 < n \in N, 0 \leq m \leq 2n-2$. Then every solution $x(t)$ of Eq* (1.1) *is bounded, i.e., it is in* $(-\infty, +\infty)$ *and satisfies*

$$\sup_{t\in R}(|x(t)| + |x'(t)|) < +\infty.$$

In addition, for any positive integer m, there are an infinite number of m-periodic solutions, as well as an infinite number of quasi-periodic solutions of large amplitude, i.e., there exists a very large $\omega^* > 0$ *such that for any irrational number* $\omega > \omega^*$ *which satisfies*

$$|\omega - \frac{p}{q}| \geq \varepsilon q^{-\frac{5}{2}}, \tag{1.2}$$

there exists a smooth function $F(\theta_1, \theta_2)$ *periodic with period 1 for all integers p and* $q \neq 0$ *with constant* $\varepsilon > 0$. *This function is such that* $x(t) = F(\theta_1 + \omega t, \theta_2 + t)$ *are solutions of Eq* (1.1).

Remark 1.1. *Here the existence of infinitely many periodic solutions can be obtained by the Aubry-Mather theory. We have chosen to omit it here because of the wealth of related results.*

Remark 1.2. *The* ω *set satisfying Eq* (1.2) *has a positive Lebesgue measure.*

Remark 1.3. *Consider the impact system:*

$$\begin{cases} x'' + x^{2n+1}(1 + ccosx) = p(t), \; for \; x(t) > 0, \; |c| < 1, \\ x(t) \geq 0, \\ x'(t_0^+) = -x'(t_0^-), \; if \; x(t_0) = 0, \end{cases} \tag{1.3}$$

where $1 < n \in N, p(t)$ *are smooth, and* $a(x + 2\pi) = a(x), p(t) = p(t + 1)$. *In this case, it is similar to statement 1.1, as Eq* (1.1) *is a generalization of Eq* (1.3).

Next, for the boundedness problem, during the past years, people have obtained many results via some Kolmogorov-Arnold-Moser (KAM) theorems, see refs [12–14]. In addition, there are many references on the study of nonsmooth oscillators using the KAM theory, such as [15–17]. However, to the best of our knowledge, there are few results [6, 18, 19] related to the unbounded problem. In [20], Wang obtained an unbounded solution of the following equation:

$$\ddot{x} + x^{2n+1} + p(t)\, x^l = 0,$$

with $p(t) \in C^0(S^1)$, $n \geq 2$, $2n + 1 > l \geq n + 2$.

Motivated by [2, 3, 14], we consider the blow-up solutions of Eq (1.1). For simplicity, we will only discuss the case where $p_l(t) = p(t)$, i.e.,

$$\begin{cases} \ddot{x} + a(x)\, x^{2n+1} + p(t)\, x^l = 0, \; \text{for} \; x(t) > 0, \\ x(t) \geq 0, \\ x'(t_0^+) = -x'(t_0^-), \; \text{if} \; x(t_0) = 0, \end{cases} \tag{1.4}$$

where $p(t) = p(t + 1)$, $n + 2 \leq l \leq 2n$, $a(x) = a(x + T) > 0$ are smooth.

From [3] we know that the monotone twist condition can be violated. Depending on the oscillation behavior of the potential $G(x) = \int_0^x a(s)s^{2n+1}ds$ of (1.4), the ratio $\frac{G'(x)}{x}$ may or may not be monotone. For example, it is monotone if $a(x) = 1$, which corresponds to a non-oscillating potential, and not monotone when $a(x) = 1 + c\cos x$, $0 < |c| < 1$, which corresponds to an oscillating potential [3]. The second main result is as follows.

Theorem 1.2. *There exists* $p(t) \in C^0(S^1)$, $n + 2 \leq l \leq 2n$, *and a sufficiently large* λ_0 *such that for the solution* $(x(t), x'(t))$ *with* $(x(0), x'(0)) = ((2\lambda_0)^{\frac{1}{n+2}}, 0)$ *of Eq* (1.4)*, there is a strictly increasing series* $\{T_i\}_{i=0}^{\infty}$ *such that* $\lim_{i \to \infty} T_i < 1$ *with* $T_0 = 0$ *and*

$$\min_{T_i \leq t \leq T_{i+1}} \lambda(x(t), x'(t)) \geq \lambda_0^{M^i},$$

where $M > 1$ and $\lambda(x, x')$ is the new action variable after coordinate transformations Ψ_1, Ψ_2, Ψ_3 in section 2.1 satisfying $(x(t), x'(t)) = \Psi_1 \circ \Psi_2 \circ \Psi_3(\phi(t), \lambda(t))$.

Corollary 1.1. *Equation (1.4) in Theorem 1.2 has a solution $(x(t), x'(t))$ which satisfies $(|x(t)| + |x'(t)|) \to +\infty$, as $t \to T_\infty < 1$.*

First, let us introduce the idea to prove the boundedness of the solutions of Eq (1.1). Using transformation theory, we can first characterize Eq (1.1) in terms of a Hamiltonian function $H^{(3)}(\phi, \lambda, t)$ (see Eq (2.8)) in action-angle variables, specified over the whole space $T \times R^+ \times R$. The action causes $H^{(3)}(\phi, \lambda, t)$ to be C^5 smooth in λ and t. It is only continuous in ϕ.

By changing the roles of the variables t and ϕ, we move away from a large disc in the space $D_r = \{(\phi, \lambda) \in T \times R^+, \lambda < r\}$ in the (ϕ, λ) plane, and $H^{(3)}(\phi, \lambda, t)$ is transformed into a perturbation of an integrable Hamiltonian system $\mathcal{H}(\theta, \rho, \tau)$ (see Eq (2.13)). This system is sufficiently smooth in θ and ρ. The Poincaré mapping concerning the new time parameter τ is closely related to a mapping called "twist mapping" in the region R^2/D_r and exhibits area-preserving properties.

The KAM theorem ensures the existence of arbitrarily large invariant curves that are diffeomorphic to circles and lie in the (x, y) plane, as presented in reference [21]. One can return to the equivalent system Eq (1.1), where it is obvious that any such curve is a basis for a time-periodic and flow-invariant cylinder in the extended phase space $(x, y, t) \in R^+ \times R \times R$. This restriction of the solutions leads internally to bounds on these solutions. This is true as long as the uniqueness of the initial value problem persists.

Then the idea for the proof of the infinite solutions of Eq (1.4) is as follows. Indeed, we construct simultaneously the function $p(t)$ and the solution $x(t)$ of Theorem 1.2. The first thing we notice is that when the curve is spiraling once around the origin, the action variable λ is increasing at some points in time and decreasing at other points in time. So we have no idea whether the increase in λ will be positive or negative. But we can construct a time $t_1 << 1$ and modify $p(t) \equiv 1$ at $[0, t_1]$ so that the increment is positive and $O(\frac{1}{\tau}\lambda_0^{\frac{l-n+1}{n+2}})$ if the starting point is $(\lambda(0), \phi(0)) = (\lambda_0, 0)$ is far enough from the origin, where the "jump" $\tau\,(0 < \tau < 1)$ is critical to modify $p(t)$ and our estimate. Inductively, we can construct a series of times $t_1, t_2, ..., t_i, t_{i+1}, \cdots$ and modify $p(t)$ on $[t_i, t_{i+1}]$, $i = 1, 2, \cdots,$ so that on each such interval $[t_i, t_{i+1}]$, the increment is positive and at least $O(\frac{1}{\tau}\lambda_0^{\frac{l-n+1}{n+2}})$. So we can build a time $T_1 \le \frac{1}{\tau'} < 1$, so that the curve spirals at least $O(\frac{1}{\tau'}\lambda_0^{\frac{n}{n+2}})$ times around the origin and $\lambda_1 = \lambda(T_1) > \lambda_0 + \frac{c}{\tau\tau'}\lambda_0^{\frac{l+1}{n+2}}$ where $\frac{l+1}{n+2} > 1$ and τ' are used to ensure that the time does not exceed 1. This completes an induction step: During the time interval $[0, T_1]$, λ increases from λ_0 to λ_1. Inductively, we can construct a series of times $T_1, T_2, ..., T_i, T_{i+1}, \cdots$, such that during the time interval $[T_k, T_{k+1}]$, λ increases from λ_k to λ_{k+1}, where $\lambda_{k+1} > \lambda_k + \frac{c}{\tau^k\tau'^k}\lambda_k^{\frac{l+1}{n+2}}$ with the jump $\frac{1}{\tau^k}$, where $T_{k+1} - T_k \le \frac{1}{\tau'^k}$. The reason why the jump is less and less is that we have to make sure that $p(t)$ is continuous. Since the exponent is $\frac{l+1}{n+2} > 1$, the smaller and smaller jump will not be able to stop the rapid increase of λ. If $\frac{1}{\tau'}$ is chosen small enough, we will find that $T_k \to T_\infty < 1$ as $k \to \infty$ and $\lambda_t \to +\infty$ as $t \to T_\infty$.

In the past, scholars studied either quasi-periodic solutions or blow-up solutions, but there were few research results on the coexistence of quasi-periodic solutions and blow-up solutions. In this paper, we obtained the result that quasi-periodic solutions and blow-up solutions coexist.

The article is structured as follows: In Section 2, the proof of Theorem 1.1 may be derived from Moser's twist theorem. To apply Moser's theorem, we provide the necessary estimates, which are proven in Section 3. These estimates are lengthy and complicated, but we think it is important to give the details because they are crucial for fitting the problem into the KAM theory framework. In Section 4, we construct the action-angle transformation and obtain some Lemmas. In Section 5, we construct $p(t) \in C^0(S^1)$ and a series of time T_k. We then obtain an unbounded solution and finish the proof of Theorem 1.2.

2. The proofs of Theorem 1.1

2.1. An only angularly continuous Hamiltonian

Equation (1.1) of the second-order model, excluding impacts, is equivalent to the following system of the one-order model:

$$\begin{cases} x' = y, \\ y' = -a(x)x^{2n+1} - \sum\limits_{l=0}^{m} p_l(t)x^l, \ 0 \le m \le 2n. \end{cases} \tag{2.1}$$

Equation (2.1) is a Hamiltonian system defined in the entirety of the phase plane XOY of the Hamiltonian function

$$H(x, y, t) = \frac{1}{2}y^2 + G(x) + \sum_{l=0}^{m} \frac{x^{l+1}}{l+1} p_l(t), \tag{2.2}$$

where

$$G(x) = \int_0^x a(s)s^{2n+1} ds. \tag{2.3}$$

Define $I = I(h) = \int_{\Gamma_h} \sqrt{2h - 2G(x)}dx$, where Γ_h represents the closed curve $\frac{1}{2}y^2 + G(x) = h$. We also define

$$S(x, I) = \begin{cases} \int_{-x_-}^{x} \sqrt{2h - 2G(x)}dx, \ y \ge 0, \\ I - \int_{-x_-}^{x} \sqrt{2h - 2G(x)}dx, \ y < 0, \end{cases}$$

where $G(-x_-) = h$.

Try changing the action angle like this:

$$y = \frac{\partial S}{\partial x}, \ \theta = \frac{\partial S}{\partial I},$$

and we get

$$\theta = \begin{cases} H_0'(I) \int_{-x_-}^{x} \frac{dx}{\sqrt{2h - 2G(x)}}, y \ge 0, \\ 1 - H_0'(I) \int_{-x_-}^{x} \frac{dx}{\sqrt{2h - 2G(x)}}, y < 0, \end{cases}$$

where H_0 is the inverse function of $I(h)$.

Denote

$$\Psi_1 : (\theta, I) \to (x, y). \tag{2.4}$$

Subsequently, the Hamiltonian H of Eq (2.2) is transformed into the following expression:

$$H^{(1)}(\theta, I, t) = H \circ \Psi_1 = H_0(I) + \sum_{l=0}^{m} \frac{x^{l+1}(I,\theta)}{l+1} p_l(t). \tag{2.5}$$

The impact case transforms the phase space into a half-plane $x \geq 0$ of the original phase plane XOY. When $x(t) = 0$, the smoothness and continuity of $H^{(1)}$ with variable θ is lost. Consequently, as outlined below, an alternative Hamiltonian, designated as $H^{(3)}$, will be pursued.

To normalize the angle variable at a later stage, the following definition is proposed:

$$\Psi_2 : (\phi_1, \lambda) \to (\theta, I) : \begin{cases} \theta = \frac{1}{2}\phi_1, \\ I = 2\lambda. \end{cases} \tag{2.6}$$

We get

$$H^{(2)}(\phi_1, \lambda, t) = H^{(1)} \circ \Psi_2 = H_0(2\lambda) + \sum_{l=0}^{m} \frac{x^{l+1}}{l+1}(2\lambda, \frac{1}{2}\phi_1) p_l(t). \tag{2.7}$$

The periodic extension of $H^{(2)}$ and ϕ_1 to the interval $[0, 1)$ defines a new Hamiltonian, which we denote by

$$H^{(3)}(\phi, \lambda, t) = H^{(2)} \circ \Psi_3 = H_0(2\lambda) + \sum_{l=0}^{m} \frac{x^{l+1}}{l+1}(2\lambda, \frac{1}{2}(\phi - [\phi])) p_l(t), \tag{2.8}$$

where

$$\Psi_3 : (\phi, \lambda) \to (\phi_1, \lambda) : \begin{cases} \phi_1 = \phi - [\phi], \\ \lambda = \lambda \end{cases} \tag{2.9}$$

where $[\phi]$ represents the greatest integer that is no greater than ϕ. We know that

$$\begin{cases} \phi' = \frac{\partial H^{(3)}}{\partial \lambda}, \text{ when } \phi \in (k, k+1), k \in Z, \\ \lambda' = -\frac{\partial H^{(3)}}{\partial \phi}, \text{ when } \phi \in (k, k+1), k \in Z, \\ \phi(t_0) = k, \lambda(t_0) = \lim_{t \to t_0} \lambda(t), \text{ when } \lim_{t \to t_0} \phi(t) = k, k \in Z, \end{cases} \tag{2.10}$$

is the corresponding Hamiltonian system.

It is evident that $H^{(3)}$ is periodic in ϕ with a period of 1, and C^∞ in ϕ when $\phi \notin Z$. However, it is only continuous at $\phi = \phi_0$, $\phi_0 \in Z$. In fact, $H^{(3)}|_{\phi=0} = H^{(3)}|_{\phi=1} = H_0(2\lambda)$ for $x(2\lambda, 0) = x(2\lambda, \frac{1}{2}) = 0$.

Then we provide the key lemma.

Lemma 2.1. *For every solution* $(\phi(t), \lambda(t))$ *of Eqs* (2.8) *and* (2.10) *with* $\lambda(t) \neq 0$, *the pair* $(x(t), x'(t)) = \Psi_1 \circ \Psi_2 \circ \Psi_3(\phi(t), \lambda(t))$ *is a continuous solution of Eq* (1.1) *with* $(x(t), x'(t)) \neq (0, 0)$, *and vice verse.*

Remark 2.1. *As stated in [22], a similar description is given for the equivalence of these systems. Therefore, the proof is omitted.*

2.2. *The new Hamiltonian is characterized by its sufficient differentiability in the action-angle variable*

In order to prove the boundedness of every solution of Eq (1.1), i.e., $|x(t)| + |x'(t)| < \infty$, by Lemma 2.1, we need to show the boundedness of $\lambda(t)$. We will consider the Poincaré mapping of the system Eq (2.15) below which is equivalent to the system Eq (2.10) by exchanging the positions of variables (ϕ, λ) and $(t, H^{(3)})$ to cope with the non-smoothness in ϕ. The next step is to use Moser's invariant curve theorem for similar considerations.

In this context, let a and A be suitable constants, without worrying about how big they are.

Lemma 2.2. *Denote* $\alpha = \frac{1}{2n+2}$. *For* λ *large enough,* $H^{(3)}$ *has the inverse function*

$$\lambda = [H^{(3)}(\phi, \cdot, t)]^{-1}(\rho) = \frac{1}{2}H_0^{-1}(\rho) + g(t, \rho, \phi),$$

and moreover,

$$|\partial_\rho^i \partial_t^j g| < A\rho^{m\alpha+(2\alpha-\frac{1}{2})+(3\alpha-1)i} \tag{2.11}$$

for $0 \le i + j \le 5, n \ge 1$, *provided* ρ *is large enough.*

We will approve the proof in Section 3. Then we exchange the roles of (ϕ, λ) and $(t, H^{(3)})$ by means of

$$\Psi_4 : (\phi, \lambda, t) \to (\theta, \rho, \tau) := (t, H^{(3)}(\lambda, \phi, t), \phi), \phi \notin Z. \tag{2.12}$$

This transformation again leads to a new Hamiltonian system, cf. [23], where the new Hamiltonian is

$$\mathcal{H}(\theta, \rho, \tau) = [H^{(3)}(\tau, \cdot, \theta)]^{-1}(\rho) = \frac{1}{2}H_0^{-1}(\rho) + g(\theta, \rho, \tau). \tag{2.13}$$

Therefore $\mathcal{H}(\theta, \rho, \tau)$ is C^5 in θ, C^∞ in ρ, and continuous in new time τ. This transformation is used by many authors, cf. [24, 25], etc.

2.3. Proof of Theorem 1.1

We know that the system (2.10) is the same as the Hamiltonian system

$$X_{\mathcal{H}} : \begin{cases} \frac{d\theta}{d\tau} = \frac{\partial \mathcal{H}}{\partial \rho} = \frac{1}{2}\frac{dH_0^{-1}(\rho)}{d\rho} + \frac{\partial g(\theta,\rho,\tau)}{\partial \rho}, \tau \notin Z, \\ \frac{d\rho}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \theta} = -\frac{\partial g(\theta,\rho,\tau)}{\partial \theta}, \tau \notin Z, \\ \theta(k) = \lim_{\tau \to k} \theta(\tau), \rho(k) = \lim_{\tau \to k} \rho(\tau), k \in Z. \end{cases} \tag{2.14}$$

Integrate the above system by τ from 0 to 1, and the Poincaré mapping is formed:

$$\Phi^1 : \begin{cases} \theta_1 = \theta + \gamma(\rho) + g_1(\theta, \rho), \\ \rho_1 = \rho + g_2(\theta, \rho) \end{cases} \tag{2.15}$$

where $\gamma(\rho) = \frac{1}{2}\frac{dH_0^{-1}(\rho)}{d\rho}$ and g_1, g_2 satisfy the following lemma.

Lemma 2.3. *If* ρ *is large enough, then*

$$\begin{aligned} &|\partial_\rho^i \partial_t^j g_1| < A\rho^{m\alpha+(2\alpha-\frac{1}{2})+(4\alpha-1)(i+1)}, \ for\ i + j \le 4, \\ &|\partial_\rho^i \partial_t^j g_2| < A\rho^{m\alpha+(2\alpha-\frac{1}{2})+(4\alpha-1)i}, \ for\ i + j \le 4. \end{aligned} \tag{2.16}$$

Proof. Set

$$\gamma(\rho, t) = \frac{t}{2}\frac{dH_0^{-1}(\rho)}{d\rho}$$

and set for the flow $(\theta(t), \rho(t)) = \Phi^t(\theta, \rho)$ with $\Phi^0 = id$:

$$\begin{cases} \theta_t = \theta + \gamma(\rho, t) + A(\theta, \rho, t), \\ \rho_t = \rho + B(\theta, \rho, t). \end{cases}$$

Then the integral equation

$$\Phi^t(\theta,\rho) = \Phi^0(\theta,\rho) + \int_0^t X_{\mathcal{H}} \circ \Phi^s ds$$

for the flow is equivalent to the following equation for A and B:

$$\begin{aligned} A(\theta,\rho,t) = & \ \tfrac{t}{2}\int_0^t\int_0^1 \frac{d^2H_0^{-1}(\rho+\tau B)}{d\rho^2}Bd\tau ds + \int_0^t \frac{\partial g(\theta+\gamma+A,\rho+B)}{\partial\rho}ds, \\ B(\theta,\rho,t) = & \ -\int_0^t \frac{\partial g(\theta+\gamma+A,\rho+B)}{\partial\theta}ds. \end{aligned} \tag{2.17}$$

It is easy to check that for any value of $\rho \geq \rho_0$, these equations have a unique solution in the space $|A|, |B| \leq 1$ using the construction principle. Also, A and B are smooth. The required estimates in Eq (2.16) can be verified from Eq (2.17) using induction.

Lemma 2.4. *The mapping $P = \Phi^1$(see Eq (2.15)) has the intersection property, i.e., if an embedded circle Γ in $R \times [0,1]$ is homotopic to a circle $\rho = const.$, then $P(\Gamma) \cap \Gamma \neq \emptyset$.*

Proof. One can see that the mapping Φ^1 in (2.15) is the time-one map of the Hamiltonian system (2.10); recall that the time-one map of a Hamiltonian system is symplectic, thus Φ^1 is symplectic, and hence, Φ^1 has the intersection property.

Proof of Theorem 1.1. Through Lemma 3.3 in Section 3, denote $G(x_+) = h$, and we can obtain an annulus $kT - c_0 \leq x_+ \leq kT$ when $a'(x) > 0$, or an annulus $kT + \frac{T}{2} - c_0 \leq x_+ \leq kT + \frac{T}{2}$ when $a'(x) < 0$, respectively, such that $\gamma(\rho)$ of Eq (2.15) satisfies $\gamma'(\rho) \neq 0, \forall\rho \in [a,b]$ and $\gamma(\rho) \in C^4[a,b]$. When $n > 1$, $0 \leq m \leq 2n-2$, by Lemma 2.3, we have $||g_1||_{C^4(A)} + ||g_2||_{C^4(A)} < \varepsilon$. Lemma 2.4 shows that Φ^1 has the intersection property. Therefore, Φ^1 meets the assumptions of Moser's twist theorem. Theorem 1.1 is complete.

3. Some estimates and the proof of Lemma 2.2

Lemmas 3.1, 3.2, 3.4–3.6 below are the direct results of [3]and [14].

Lemma 3.1. *It holds that*

$$ax^{2n+2} < G(x) < Ax^{2n+2},$$

and

$$|G^{(i)}(x)| < A|x|^{2n+1},$$

for any $i \geq 1$.

Lemma 3.2. *Denote $\alpha = \frac{1}{2n+2}$. There exist $a, A \in R$ and $0 < a < A$, for large $I > 0$, and we get*

$$ah^{\frac{1}{2}+\alpha} \leq I(h) \leq Ah^{\frac{1}{2}+\alpha},$$

$$ah^{\alpha-\frac{1}{2}} \leq I'(h) \leq Ah^{\alpha-\frac{1}{2}}.$$

If in the smooth T-periodic function $a(x) \not\equiv d$, d is a constant, then its maximum value points in $(0,T)$ must exist, and we use x^* to mark the largest point in $(0,T)$. Then we have the following lemma.

Lemma 3.3. *There exists $c_0 > 0$, such that*

1) $-Ah^{\frac{3}{2}\alpha-\frac{3}{2}} \leq I''(h) \leq -ah^{\frac{3}{2}\alpha-\frac{3}{2}}$, $kT - c_0 \leq x_+ \leq kT$, $a'(x) > 0$, $x \in (x^*, T)$;
2) $ah^{\frac{3}{2}\alpha-\frac{3}{2}} \leq I''(h) \leq Ah^{\frac{3}{2}\alpha-\frac{3}{2}}$, $kT + \frac{T}{2} - c_0 \leq x_+ \leq kT + \frac{T}{2}$, $a'(x) < 0$, $x \in (x^*, T)$;

where $G(x_+) = h$.

Proof. We denote

$$a(x) = \bar{a} + b(x),$$

where $\bar{a} = \frac{1}{T}\int_0^T a(s)ds$, $\frac{1}{T}\int_0^T b(s)ds = 0$. Then we get

$$\begin{aligned} G(x) &= \int_0^x a(s)s^{2n+1}ds \\ &= \int_0^x (\bar{a} + b(s))s^{2n+1}ds \\ &= \frac{\bar{a}}{2n+2}x^{2n+2} + \int_0^x b(s)s^{2n+1}ds \\ &= \frac{\bar{a}}{2n+2}x^{2n+2} + B(x)x^{2n+1} - (2n+1)\int_0^x B(s)s^{2n}ds, \end{aligned}$$

where $B'(x) = b(x)$.

From the above equality, we have

$$\begin{aligned} \frac{GG''}{G'^2} &= \frac{\frac{\bar{a}}{2n+2}x^{2n+2} + B(x)x^{2n+1} - (2n+1)\int_0^x B(s)s^{2n}ds}{a(x)x^{2n+1}} \\ &\times \frac{(2n+1)\bar{a}x^{2n} + (2n+1)b(x)x^{2n} + a'(x)x^{2n+1}}{a(x)x^{2n+1}} \\ &= \frac{2n+1}{2n+2}\frac{\bar{a}^2}{a^2(x)} + \frac{1}{2n+2}\frac{\bar{a}a'(x)x}{a^2(x)} + \frac{2n+1}{2n+2}\frac{\bar{a}b(x)}{a^2(x)} + \frac{B(x)a'(x)}{a^2(x)} + \frac{O(\frac{1}{x})}{a^2(x)}. \end{aligned}$$

Using the above equality, we get

$$\begin{aligned} I''(h) &= \frac{4}{h}\int_0^{x_+}(\frac{1}{2} - \frac{GG''}{G'^2})\frac{dx}{\sqrt{2h-2G(x)}} \\ &= \frac{4}{h}\int_0^{x_+}(\frac{1}{2} - \frac{2n+1}{2n+2}\frac{\bar{a}^2}{a^2(x)})\frac{dx}{\sqrt{2h-2G(x)}} \\ &- \frac{4}{h}\int_0^{x_+}\frac{1}{2n+2}\frac{\bar{a}a'(x)x}{a^2(x)}\frac{dx}{\sqrt{2h-2G(x)}} \\ &- \frac{4}{h}\int_0^{x_+}(\frac{2n+1}{2n+2}\frac{\bar{a}b(x)}{a^2(x)} + \frac{B(x)a'(x)}{a^2(x)})\frac{dx}{\sqrt{2h-2G(x)}} \\ &- \frac{4}{h}\int_0^{x_+}\frac{O(\frac{1}{x})}{a^2(x)}\frac{dx}{\sqrt{2h-2G(x)}} \\ &\doteq J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{3.1}$$

with

$$\begin{aligned} J_1 &= \frac{4}{h}\int_0^{x_+}(\frac{1}{2} - \frac{2n+1}{2n+2}\frac{\bar{a}^2}{a^2(x)})\frac{dx}{\sqrt{2h-2G(x)}}, \\ J_2 &= -\frac{4}{h}\int_0^{x_+}\frac{1}{2n+2}\frac{\bar{a}a'(x)x}{a^2(x)}\frac{dx}{\sqrt{2h-2G(x)}}, \\ J_3 &= -\frac{4}{h}\int_0^{x_+}(\frac{2n+1}{2n+2}\frac{\bar{a}b(x)}{a^2(x)} + \frac{B(x)a'(x)}{a^2(x)})\frac{dx}{\sqrt{2h-2G(x)}}, \\ J_4 &= -\frac{4}{h}\int_0^{x_+}\frac{O(\frac{1}{x})}{a^2(x)}\frac{dx}{\sqrt{2h-2G(x)}}. \end{aligned}$$

Similarly to the proof of Lemma 2.1 in [3], it is easy to see that

$$J_1 = O(h^{\alpha-\frac{3}{2}}),$$

$$J_3 = O(h^{\alpha-\frac{3}{2}}),$$

and

$$J_4 = O(h^{-\frac{3}{2}}).$$

If the following inequality holds:

$$\begin{array}{lll} -Ah^{\frac{3}{2}\alpha-\frac{3}{2}} & \leq & J_2 \leq -ah^{\frac{3}{2}\alpha-\frac{3}{2}}, \\ kT & - & c_0 \leq x_+ \leq kT,\ a'(x) > 0,\ x \in (x^*, T), \end{array} \tag{3.2}$$

the proof of Lemma 3.3 can be reduced. Next, we will only prove (3.2). Due to 2) of the lemma, the proof is similar. Let $x_+ = kT - \delta \in [kT - c_0, kT]$ with $c_0 > 0$ to be determined later. Choose $c_1 = c_1(x_+) \in (0, 1)$ such that $c_1 x_+ = kT - T + x^*$.

We denote

$$J_2 = -\frac{4\bar{a}}{\sqrt{2}(2n+2)h}\tilde{J}_2 \tag{3.3}$$

and

$$\begin{aligned} \tilde{J}_2 &= \int_0^{x_+} \frac{a'(x)x}{a^2(x)} \frac{dx}{\sqrt{h - G(x)}} \\ &= \int_0^{c_1 x_+} \frac{a'(x)x}{a^2(x)} \frac{dx}{\sqrt{h - G(x)}} + \int_{c_1 x_+}^{x_+} \frac{a'(x)x}{a^2(x)} \frac{dx}{\sqrt{h - G(x)}} \\ &\doteq J_{21} + J_{22} \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} J_{21} &= \int_0^{c_1 x_+} \frac{a'(x)x}{a^2(x)} \frac{dx}{\sqrt{h - G(x)}} \\ &= -\int_0^{c_1 x_+} \frac{x}{\sqrt{h - G(x)}} d\frac{1}{a(x)} \\ &= \frac{x}{a(x)\sqrt{h - G(x)}}\Big|_{c_1 x_+}^0 + \int_0^{c_1 x_+} \frac{1}{a(x)}\Big(\frac{1}{\sqrt{h - G(x)}} + \frac{xG'(x)}{\sqrt{(h - G(x))^3}}\Big)dx \\ &= -\frac{c_1 x_+}{a(c_1 x_+)\sqrt{h - G(c_1 x_+)}} + \int_0^{c_1 x_+} \frac{1}{a(x)}\Big(\frac{1}{\sqrt{h - G(x)}} + \frac{xG'(x)}{\sqrt{(h - G(x))^3}}\Big)dx \\ &= O(h^{\alpha-\frac{1}{2}}), \end{aligned}$$

and

$$\begin{aligned} J_{22} &= \int_{c_1 x_+}^{x_+} \frac{a'(x)x}{a^2(x)} \frac{dx}{\sqrt{h - G(x)}} \\ &= \int_{kT-T+x^*}^{kT-\delta} \frac{a'(x)x}{a^2(x)} \frac{dx}{\sqrt{h - G(x)}} \\ &= \int_\delta^{T-x^*} \frac{a'(-x)(kT - x)}{a^2(-x)\sqrt{h - G(kT - x)}} dx \\ &= kT \int_\delta^{T-x^*} \frac{a'(-x)}{a^2(-x)\sqrt{h - G(kT - x)}} dx - \int_\delta^{T-x^*} \frac{a'(-x)x}{a^2(-x)\sqrt{h - G(kT - x)}} dx \\ &\doteq I_1 + I_2. \end{aligned}$$

After a simple calculation, we have

$$I_2 = O(h^{\frac{\alpha}{2}-\frac{1}{2}})$$

and by $\delta < x < T - x^*$, we get $x^* < T - x < T - \delta$, and then $a'(-x) = a'(T - x) > 0$. Therefore

$$ah^{\frac{3\alpha}{2}-\frac{1}{2}} \le I_1 \le Ah^{\frac{3\alpha}{2}-\frac{1}{2}}.$$

From the fact that $kT \in [x_+, x_+ + T] \subset [ah^\alpha, Ah^\alpha]$, it holds for small δ that

$$ah^{\frac{3\alpha}{2}-\frac{1}{2}} \le J_{22} \le Ah^{\frac{3\alpha}{2}-\frac{1}{2}}.$$

Combining (3.3), (3.4), and the last inequality, there exists $c_0 > 0$, such that (3.2) holds, and thus the proof is complete.

For higher derivatives of $I(h)$, from the definition of $I(h)$, [26], and Lemma 3.1, we have:

Lemma 3.4. *For large* $I > 0$, *we have*

$$|I^{(i)}(h)| \le Ah^{\alpha+\frac{1}{2}-i(1-\alpha)}, \ i \ge 3.$$

Lemma 3.5. *For large* $I > 0$, *we have*

$$aI^{\frac{2}{2\alpha+1}} \le H_0(I) \le AI^{\frac{2}{2\alpha+1}},$$

$$aI^{\frac{1-2\alpha}{2\alpha+1}} \le H_0'(I) \le AI^{\frac{1-2\alpha}{2\alpha+1}},$$

$$|H_0''(I)| \le AI^{\frac{-3\alpha}{2\alpha+1}},$$

and

$$|H_0^{(i)}(I)| \le AI^{\frac{2+3(i-1)\alpha}{2\alpha+1}-i}, \text{for each } i \ge 3.$$

Lemma 3.6. *For* $i, j \ge 0$, *we have*

$$|\partial_I^i \partial_\theta^j x(I,\theta)| \le AI^{a(i,j)-i}, \tag{3.5}$$

where

$$a(i,j) = \begin{cases} \dfrac{2\alpha}{2\alpha+1}, \ i = 0, j = 0, 1, \\ \dfrac{(4i + 2\max\{j-1, 0\})\alpha}{2\alpha+1}, \ \textit{otherwise}. \end{cases} \tag{3.6}$$

Proof. The proof method is like the one in Lemma 3.4 in [3], so we will not go into detail about it here.

Lemma 3.7. *For* $i, j \ge 0$, *we have*

$$|\partial_I^i \partial_\theta^j x^{l+1}(I,\theta)| \ \le \ AI^{a(i,j) - i + \dfrac{2l\alpha}{2\alpha+1}}. \tag{3.7}$$

Proof. We know $\partial_I^i \partial_\theta^j x^{l+1}(I,\theta)$ is the sum of terms

$$(x^{l+1})^{(\nu)} \partial_I^{s_1} \partial_\theta^{t_1} x(I,\theta) \cdots \partial_I^{s_\nu} \partial_\theta^{t_\nu} x(I,\theta)$$

with $\nu \ge 1, s_1, t_1, \cdots, s_\nu, t_\nu \ge 0, s_1 + \cdots + s_\nu = i, t_1 + \cdots + t_\nu = j$.

If you add the above inequality to Lemma 3.6, you can easily show that (3.7) is true.

The proof of Lemma 2.2. $H^{(3)}(\phi,\lambda,t)$ is defined in $R\times R^{+}\times R$, and is 1-periodic in ϕ and t. First, consider the existence of the inverse function of $H^{(3)}(\phi,\lambda,t)$ with the second variable. Denote

$$\rho = H^{(3)}(\phi,\lambda,t) = H_0(2\lambda) + \sum_{l=0}^{m}\frac{x^{l+1}}{l+1}(2\lambda,\frac{1}{2}(\phi-[\phi]))p_l(t).$$

For λ large enough, by Lemma 3.5, we have $\frac{\partial\rho}{\partial\lambda} > \lambda^{\frac{0.5-\alpha}{0.5+\alpha}} > 0$, where $\frac{0.5-\alpha}{0.5+\alpha} > 0$, thus $\lim\limits_{\lambda\to\infty}\rho = \infty$ and ρ has its inverse function

$$\lambda = [H^{(3)}(\phi,\cdot,t)]^{-1}(\rho) = \frac{1}{2}H_0^{-1}(\rho) + g(t,\rho,\phi).$$

Next, we will provide the estimates in summary, for ρ large enough,

$$|\partial_\rho^i\partial_t^j g| < A\rho^{m\alpha+(2\alpha-\frac{1}{2})+(3\alpha-1)i},\ \text{for } 0\le i+j\le 5.$$

Rewrite

$$\rho = H_0\Big(2(\frac{1}{2}H_0^{-1}(\rho)+g)\Big) + \sum_{l=0}^{m}\frac{x^{l+1}}{l+1}\Big(2(\frac{1}{2}H_0^{-1}(\rho)+g),\frac{1}{2}(\phi-[\phi])\Big)p_l(t) \tag{3.8}$$

into the following form:

$$2g\int_0^1 H_0'\Big(H_0^{-1}(\rho)+2\tau g\Big)d\tau + \sum_{l=0}^{m}\frac{x^{l+1}}{l+1}\Big(H_0^{-1}(\rho)+2g,\frac{1}{2}(\phi-[\phi])\Big)p_l(t) = 0. \tag{3.9}$$

If ρ is large, g is well determined by the contraction principle. Moreover, by the implicit function theorem, g is smooth in ρ, for large ρ. We can easily obtain $|g(t,\rho,\phi)| < A\rho^{m\alpha+2\alpha-\frac{1}{2}}$.

By Eq (3.9), we have

$$g = -\frac{1}{2}\sum_{l=0}^{m}\frac{x^{l+1}}{l+1}\Big(H_0^{-1}(\rho)+2g,\frac{1}{2}(\phi-[\phi])\Big)p_l(t)\frac{1}{\int_0^1 H_0'\Big(H_0^{-1}(\rho)+2\tau g\Big)d\tau}. \tag{3.10}$$

Applying ∂_ρ^i to Eq (3.10), the left-hand side is $\partial_\rho^i g$ and the right-hand side is the algebraic sum of terms

$$-\frac{1}{2}\partial_\rho^{i_1}\frac{x^{l+1}}{l+1}\Big(H_0^{-1}(\rho)+2g,\frac{1}{2}(\phi-[\phi])\Big)p_l(t)\partial_\rho^{i_2}\frac{1}{\int_0^1 H_0'\Big(H_0^{-1}(\rho)+2\tau g\Big)d\tau},$$

with $i_1+i_2=i$.

We know $-\frac{1}{2}\partial_\rho^{i_1}\frac{x^{l+1}}{l+1}\Big(H_0^{-1}(\rho)+2g,\frac{1}{2}(\phi-[\phi])\Big)p_l(t)$ is the algebraic sum of terms

$$-\frac{1}{2}\frac{(x^{l+1})^{(\nu)}}{l+1}\prod_{k=1}^{\nu}\partial_\rho^{\nu_k}(H_0^{-1}(\rho)+2g)p_l(t),\ \nu\ge 1,\sum_{k=1}^{\nu}\nu_k = i_1$$

and $\partial_\rho^{i_2}\frac{1}{H_0'\big(H_0^{-1}(\rho)+2\tau g\big)}$ is the algebraic sum of terms

$$\partial_\lambda^{\mu}\frac{1}{H_0'(\lambda)}\prod_{k=1}^{\mu}\partial_\rho^{\mu_k}(H_0^{-1}(\rho)+2\tau g),\ \mu\ge 1,\ \sum_{k=1}^{\mu}\mu_k = i_2,\ \nu+\mu\le i.$$

Thus, by Lemma 3.7 and

$$|\partial_\rho^{i_2}\frac{1}{H_0'(\rho)}| < A\rho^{\frac{(i_2+1)(2\alpha-1)}{2\alpha+1}},$$

we can get

$$|\partial_\rho^i g| < A\rho^{m\alpha+(2\alpha-\frac{1}{2})+(3\alpha-1)i}.$$

Next differentiating $\partial_\rho^i g$ with respect to t, we have

$$|\partial_\rho^i\partial_t^j g| < A\rho^{m\alpha+(2\alpha-\frac{1}{2})+(3\alpha-1)i}, \text{ for } 0\le i+j\le 5.$$

Thus when $n \ge 1$, for $0 \le i + j \le 5$, Eq (2.11) has been proved.

4. Action-angle transformation and some lemmas

In this section, we are concerned with the blow-up solutions of the system (1.4). Recall that by similar variable transformations as in Section 2.1, the system (1.4) can be determined by

$$\begin{cases} \frac{d\phi}{dt} = \frac{\partial H_3}{\partial\lambda} = 2H_0'(2\lambda) + x^l(2\lambda, \frac{1}{2}(\phi-[\phi]))\frac{\partial x(2\lambda,\frac{1}{2}(\phi-[\phi]))}{\partial\lambda}(p(t)-1), \text{ when } \phi\in(k,k+1), \\ \frac{d\lambda}{dt} = -\frac{\partial H_3}{\partial\phi} = -x^l(2\lambda,\frac{1}{2}(\phi-[\phi]))\frac{\partial x(2\lambda,\frac{1}{2}(\phi-[\phi]))}{\partial\phi}(p(t)-1), \text{ when } \phi\in(k,k+1), \\ \phi(t_0) = k, \lambda(t_0) = \lim_{t\to t_0}\lambda(t), \text{ when } \lim_{t\to t_0}\phi(t) = k, k\in Z, \end{cases} \tag{4.1}$$

where

$$H_3(\lambda,\phi,t) = H_2\circ\Psi_3 = H_0(2\lambda) + \frac{x^{l+1}}{l+1}(2\lambda, \frac{1}{2}(\phi-[\phi]))(p(t)-1). \tag{4.2}$$

4.1. Some lemmas

Similarly to the Lemmas 3.1 and 3.2 in [14], it is easy to imply that there exist some constants $a_i > 0$, $i = 1, 2, 3, 4, 5$, such that

$$a_1 I^{\frac{n}{n+2}} < H_0'(I) < a_2 I^{\frac{n}{n+2}}, \tag{4.3}$$

$$|x^l(I,\theta)| < a_3 I^{\frac{l}{n+2}}, \tag{4.4}$$

$$|\frac{\partial x}{\partial I}(I,\theta)| < a_4 I^{\frac{-n}{n+2}}, \tag{4.5}$$

$$|\frac{\partial x}{\partial\theta}(I,\theta)| < a_5 I^{\frac{1}{n+2}}, \tag{4.6}$$

$$\frac{\partial x}{\partial\theta} \ge 0, \text{when } y\ge 0;\ \frac{\partial x}{\partial\theta} < 0, \text{when } \ y<0. \tag{4.7}$$

We assume that l is an even number, and then define $p_0(t)$ to be a piecewise continuous function, where t_k is the unique time which satisfies $\theta(t_k) = k - \frac{1}{2}$, $I_k = I_{t_k}$ for the solution of the new system corresponding with $p_0(t)$.

$$p_0(t)=\begin{cases}1, & [0,t_{\frac{1}{2}}],\\ 1-\sigma, & (t_{\frac{1}{2}},t_1],\\ 1, & (t_1,1].\end{cases} \tag{4.8}$$

Remark 4.1. *By* (4.1) *and* (4.8)*, it is easy to imply that* $\lambda_{\frac{1}{2}}=\lambda_0$.

In the following, all c_i are independent of the steps in the induction process.

Lemma 4.1. *If* λ_0 *is sufficiently large, then*

$$t_1-t_{\frac{1}{2}}\le c_1\lambda_0^{-\frac{n}{n+2}}.$$

Proof. By (4.7), we have $\frac{\partial x}{\partial\phi}\ge 0$, and when $t\in[t_{\frac{1}{2}},t_1]$, λ is an increasing function on this interval. We know that $\phi(t_{\frac{1}{2}})=0,\phi(t_1)=1$. By (4.1), (4.3), (4.4), and (4.5), if λ_0 is sufficiently large, we get

$$\begin{aligned} t_1-t_{\frac{1}{2}} &= \int_0^1\frac{d\phi}{2H_0'(2\lambda)-\sigma x^l\frac{\partial x}{\partial\lambda}}\\ &\le \int_0^1\frac{d\phi}{2a_1(2\lambda)^{\frac{n}{n+2}}-\sigma a_3a_4(2\lambda)^{\frac{l-n}{n+2}}}\\ &\le \int_0^1\frac{d\phi}{(2\lambda_0)^{\frac{n}{n+2}}(2a_1-\sigma a_3a_4(2\lambda_0)^{\frac{l-2n}{n+2}})}\\ &\le \frac{(2\lambda_0)^{-\frac{n}{n+2}}}{a_1}\le c_1\lambda_0^{-\frac{n}{n+2}}.\end{aligned}$$

Lemma 4.2. *If* λ_0 *is sufficiently large, then*

$$\lambda_1\le\lambda_0(1+c_2\sigma\lambda_0^{\frac{l-1-2n}{n+2}}).$$

Proof. By (4.1), (4.4), and (4.6), one has

$$\begin{aligned} t_1-t_{\frac{1}{2}} &= \int_{t_{\frac{1}{2}}}^{t_1}dt\\ &= \int_{\lambda_{\frac{1}{2}}}^{\lambda_1}\frac{d\lambda}{\sigma x^l\frac{\partial x}{\partial\phi}}\\ &\ge \int_{\lambda_{\frac{1}{2}}}^{\lambda_1}\frac{d\lambda}{\sigma a_3a_5(2\lambda)^{\frac{l+1}{n+2}}}\\ &= \frac{(2\lambda_1)^{\frac{n+1-l}{n+2}}-(2\lambda_{\frac{1}{2}})^{\frac{n+1-l}{n+2}}}{2\sigma a_3a_5\frac{n+1-l}{n+2}}.\end{aligned}$$

Because $n+1-l<0$, by the above inequality, Remark 4.1, and Lemma 4.1, we have

$$\frac{\lambda_1^{\frac{n+1-l}{n+2}}-\lambda_{\frac{1}{2}}^{\frac{n+1-l}{n+2}}}{\frac{n+1-l}{n+2}}\le\sigma\frac{a_3a_5}{a_1}\lambda_0^{-\frac{n}{n+2}},$$

i.e.,

$$\begin{aligned}
\lambda_1^{\frac{n+1-l}{n+2}} &\geq \lambda_0^{\frac{n+1-l}{n+2}} + \sigma \frac{a_3 a_5}{a_1} \frac{(n+1-l)}{n+2} \lambda_0^{-\frac{n}{n+2}} \\
&= \lambda_0^{\frac{n+1-l}{n+2}} (1 + \sigma \frac{a_3 a_5}{a_1} \frac{(n+1-l)}{n+2} \lambda_0^{\frac{l-1-2n}{n+2}}).
\end{aligned}$$

Then

$$\begin{aligned}
\lambda_1 &\leq \lambda_0 (1 + \sigma \frac{a_3 a_5}{a_1} \frac{(n+1-l)}{n+2} \lambda_0^{\frac{l-1-2n}{n+2}})^{\frac{n+2}{n+1-l}} \\
&\leq \lambda_0 (1 + 2\sigma \frac{a_3 a_5}{a_1} \lambda_0^{\frac{l-1-2n}{n+2}}) \\
&\leq \lambda_0 (1 + c_2 \sigma \lambda_0^{\frac{l-1-2n}{n+2}}).
\end{aligned}$$

Theorem 4.1.

$$\lambda_1 \geq \lambda_0 (1 + c_3 \sigma \lambda_0^{\frac{l-1-2n}{n+2}}).$$

Proof. If $t \in [t_{\frac{1}{2}}, t_1]$, by (4.1)–(4.5), there exists $a_6 > 0$, such that

$$\phi'(t) < a_6 \lambda_0^{\frac{n}{n+2}}. \tag{4.9}$$

Denote

$$x^l = \lambda^{\frac{l}{n+2}} q_1(\lambda, \phi),\ \frac{\partial x}{\partial \phi} = \lambda^{\frac{1}{n+2}} q_2(\lambda, \phi),$$

and then if $\phi \in (\frac{3}{4}, \frac{5}{4})$, we have $q_2(\lambda, \phi) > 0$ and if $q_1(\lambda, \phi) = 0$, then $\phi = 1$. Thus

$$\int_{\frac{3}{4}}^{\frac{5}{4}} q_1(\lambda, \phi) q_2(\lambda, \phi) d\phi > 0. \tag{4.10}$$

By (4.1), (4.9), and (4.10), we get

$$\begin{aligned}
\lambda_1 - \lambda_{\frac{1}{2}} &= \int_{\lambda_{\frac{1}{2}}}^{\lambda_1} d\lambda \\
&= \int_{t_{\frac{1}{2}}}^{t_1} \sigma x^l \frac{\partial x}{\partial \phi} dt \\
&\geq \sigma (2\lambda_0)^{\frac{l+1}{n+2}} \int_{t_{\frac{1}{2}}}^{t_1} q_1(\lambda, \phi) q_2(\lambda, \phi) dt \\
&= \sigma (2\lambda_0)^{\frac{l+1}{n+2}} \int_{\frac{3}{4}}^{\frac{5}{4}} \frac{q_1(\lambda, \tau) q_2(\lambda, \tau)}{\phi'(t)} d\tau \\
&\geq \frac{\sigma (2\lambda_0)^{\frac{l+1-n}{n+2}}}{a_6} \int_{\frac{3}{4}}^{\frac{5}{4}} q_1(\lambda, \tau) q_2(\lambda, \tau) d\tau \\
&\geq c_3 \sigma \lambda_0^{\frac{l-n+1}{n+2}}.
\end{aligned}$$

Lemma 4.3. *If I_0 is sufficiently large, then*

$$\exists\, \eta > \frac{n}{n+2} \text{ such that } t_1 - t_{\frac{1}{2}} > 2\lambda_0^{-\eta}.$$

Proof. By (4.1)–(4.5), and Lemma 4.2, we get

$$\begin{aligned} t_1 - t_{\frac{1}{2}} &= \int_{\frac{3}{4}}^{\frac{5}{4}} \frac{d\phi}{H_0'(2\lambda) - \sigma x^l \frac{\partial x}{\partial \lambda}} \\ &> \frac{1}{2} \frac{1}{a_2(2\lambda_1)^{\frac{n}{n+2}} + \sigma a_3 a_4 (2\lambda_1)^{\frac{l-n}{n+2}}} \\ &\geq \frac{1}{2} \frac{1}{(2\lambda_1)^{\frac{n}{n+2}} (a_2 + \sigma a_3 a_4 \lambda_1^{\frac{l-2n}{n+2}})} \\ &> \frac{1}{16a_2} \lambda_0^{-\frac{n}{n+2}} \\ &> 2\lambda_0^{-\eta}, \end{aligned}$$

where $\eta > \frac{n}{n+2}$.

Now we change the piecewise continuous function $p_0(t)$ of (4.8) into a continuous function:

$$p^0(t) = \begin{cases} 1, & [0, t_{\frac{1}{2}}], \\ \sigma(t_{\frac{1}{2}} - t)\lambda_0^{\eta} + 1, & (t_{\frac{1}{2}}, t_{\frac{1}{2}} + \lambda_0^{-\eta}], \\ 1 - \sigma, & (t_{\frac{1}{2}} + \lambda_0^{-\eta}, t_1 - \lambda_0^{-\eta}], \\ 1 + (t - t_1)\lambda_0^{\eta}\sigma, & (t_1 - \lambda_0^{-\eta}, t_1] \\ 1, & (t_1, 1]. \end{cases} \tag{4.11}$$

It is easy to check that Lemmas 4.1–4.3 and Theorem 4.1 still hold with $\widetilde{c_i}$ after this modification in view of $\lambda_0^{-\eta} \ll \lambda_0^{-\frac{n}{n+2}}$.

5. The proof of Theorem 1.2

We will modify $p^0(t)$ inductively and denote the function obtained and the corresponding solution with (λ_0, ϕ_0) as the initial point by p^i and $\phi^i(t), \lambda^i(t)$ with $\phi^i(t_i) = i, \lambda^i(t_i) = \lambda_i$.

Suppose we have obtained $p^0, p^1, \ldots, p^i$. p^{i+1} is constructed by modifying p^i on the interval $[t_i, t_{i+1}]$, where t_{i+1} satisfies $\phi^{i+1}(t_{i+1}) = i+1$ in the same way as above if we regard λ_i, t_i as λ_0, t_0. All the lemmas are true after the modification.

In the process of constructing p^i, we keep the jump $\sigma = 1/\tau(\tau > 2)$ unchanged until $i = j_1$. Then we let $\sigma = 1/\tau^2$ and keep it unchanged until $i = j_2$. Inductively, we choose $\sigma = 1/\tau^k$ when $\phi \in [j_{k-1}, j_k]$, where $j_0 = 0, j_1, j_2, \ldots$ satisfies

$$
\begin{aligned}
&T_0 = 0;\\
&t \in [0, t_{j_1}],\ \sigma = \tfrac{1}{\tau},\ j_1 = [O(\tfrac{\lambda_0^{\frac{n}{n+2}}}{\tau'})],\ T_1 = t_{j_1};\\
&t \in (t_{j_1}, t_{j_2}],\ \sigma = \tfrac{1}{\tau^2},\ j_2 - j_1 = [O(\tfrac{\lambda_{j_1}^{\frac{n}{n+2}}}{\tau'})],\ T_2 = t_{j_2};\\
&t \in (t_{j_2}, t_{j_3}],\ \sigma = \tfrac{1}{\tau^3},\ j_3 - j_2 = [O(\tfrac{\lambda_{j_2}^{\frac{n}{n+2}}}{\tau'^2})],\ T_3 = t_{j_3};\\
&\cdots\cdots\\
&t \in (t_{j_{k-1}}, t_{j_k}],\ \sigma = \tfrac{1}{\tau^k},\ j_k - j_{k-1} = [O(\tfrac{\lambda_{j_{k-1}}^{\frac{n}{n+2}}}{\tau'^k})],\ T_k = t_{j_k};\\
&\cdots\cdots
\end{aligned}
\tag{5.1}
$$

Then we can imply that

$$
\begin{aligned}
&\lambda_{j_1} > \lambda_0 + \frac{c'_{j_1}}{\tau\tau'}\lambda_0^{\frac{l+1}{n+2}};\\
&\lambda_{j_2} > \lambda_{j_1} + \frac{c'_{j_2}}{\tau\tau'}\lambda_{j_1}^{\frac{l+1}{n+2}};\\
&\lambda_{j_3} > \lambda_{j_2} + \frac{c'_{j_3}}{\tau^2\tau'^2}\lambda_{j_2}^{\frac{l+1}{n+2}};\\
&\cdots\cdots\\
&\lambda_{j_k} > \lambda_{j_{k-1}} + \frac{c'_{j_k}}{\tau^{k-1}\tau'^{k-1}}\lambda_{j_{k-1}}^{\frac{l+1}{n+2}};\\
&\cdots\cdots
\end{aligned}
\tag{5.2}
$$

Lemma 5.1.

$$\lim_{k\to\infty} T_k < 1,\ \textit{if } \tau' \textit{ is large enough.}$$

Proof. First, we will show

$$T_{k+1} - T_k < \frac{c^1_{j_{k+1}}}{\tau'^k},\ k = 0, 1, \cdots \tag{5.3}$$

In fact, by (4.1), we have

$$t_{\frac{1}{2}} < \frac{\lambda_0^{\frac{-n}{n+2}}}{2a_1}$$

and

$$T_1 = t_{j_1} < j_1 \cdot 2t_{\frac{1}{2}} < [O(\frac{\lambda_0^{\frac{n}{n+2}}}{\tau'})] \cdot \frac{\lambda_0^{\frac{-n}{n+2}}}{a_1} < \frac{c^1_1}{\tau'}.$$

If $T_k - T_{k-1} < \frac{c^1_{j_k}}{\tau'^{k-1}}$, then

$$\begin{aligned} T_{k+1}-T_k &= t_{j_{k+1}}-t_{j_k} < (j_{k+1}-j_k)\cdot 2(t_{j_{k+\frac{1}{2}}}-t_{j_k}) \\ &< [O(\frac{\lambda_{j_k}^{\frac{n}{n+2}}}{\tau'^k})]\cdot 2c_{j_{k+1}}\lambda_{j_k}^{\frac{-n}{n+2}} \\ &< \frac{c^1_{j_{k+1}}}{\tau'^k}, \end{aligned}$$

and therefore, if τ' is large enough, we have

$$\begin{aligned} \lim_{k\to\infty} T_k &= \lim_{k\to\infty}[T_1+(T_2-T_1)+(T_3-T_2)+\cdots+(T_k-T_{k-1})+\cdots] \\ &\leq \frac{c^1_1}{\tau'}+\frac{c^1_{j_2}}{\tau'}+\frac{c^1_{j_3}}{\tau'^2}+\cdots+\frac{c^1_{j_k}}{\tau'^{k-1}}+\cdots \\ &< 1. \end{aligned}$$

Lemma 5.2.

$$\lambda_{j_k} > 2\lambda_0^{M^k},\ M>1.$$

Proof. By (5.2), one has

$$\lambda_{j_1} > \lambda_0 + \frac{c'_{j_1}}{\tau\tau'}\lambda_0^{\frac{l+1}{n+2}} > 2\lambda_0^M, M = 1+\frac{l-n-1}{2(n+2)}.$$

If

$$\lambda_{j_{k-1}} > 2\lambda_0^{M^{k-1}},$$

then

$$\begin{aligned} \lambda_{j_k} &> \lambda_{j_{k-1}} + \frac{c'_{j_k}}{\tau^{k-1}\tau'^{k-1}}\lambda_{j_{k-1}}^{\frac{l+1}{n+2}} \\ &> \frac{c'_{j_k}}{\tau^{k-1}\tau'^{k-1}}(2\lambda_0^{M^{k-1}})^{\frac{l+1}{n+2}} \\ &= 2\lambda_0^{M^k}\frac{c'_{j_k}}{\tau^{k-1}\tau'^{k-1}}2^{\frac{l+1}{n+2}-1}\lambda_0^{M^{k-1}\cdot\frac{l+1}{n+2}-M^k} \\ &= 2\lambda_0^{M^k}\frac{c'_{j_k}}{\tau^{k-1}\tau'^{k-1}}2^{\frac{l-n-1}{n+2}}\lambda_0^{\frac{l-n-1}{2(n+2)}\cdot M^k} \\ &> 2\lambda_0^{M^k}. \end{aligned}$$

Proof of Theorem 1.2. According to Lemma 5.2, one can see that

$$\min_{t\in[T_i,T_{i+1}]}\lambda(t) \geq \frac{1}{2}\lambda_{j_i} \geq \lambda_0^{M^i}.$$

Thus Theorem 1.2 has been proved.

Since $M>1$, we imply that $\lambda(t)\to+\infty$ as $i\to+\infty$. Therefore, Eq (1.4) in Theorem 1.2 possesses an unbounded solution $(x(t),x'(t))$ satisfying $(|x(t)|+|x'(t)|)\to+\infty$, as $t\to T_\infty<1$.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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