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Research article

The steepest descent method for fuzzy optimization problems under granular differentiability

Shexiang Hai* **and Liang He**

School of Science, Lanzhou University of Technology, Lanzhou 730050, China

* **Correspondence:** Email: haishexiang@lut.edu.cn; Tel: +8613893151451.

Abstract: This paper aimed to establish optimality conditions and develop an efficient solution method for unconstrained fuzzy optimization problems using the steepest descent method, based on the granular differentiability and granular convexity of fuzzy mappings. First, building upon the gr-difference for fuzzy numbers, the granular gradient, gr-differentiable and twice gr-differentiable of fuzzy mappings were researched. Furthermore, the efficient solution of fuzzy optimization was introduced under the granular convexity of fuzzy mappings. Finally, we proposed a steepest descent method for fuzzy optimization problems by employing the characterization function of fuzzy mappings. Our analysis demonstrated that the proposed method converged linearly under standard convexity conditions.

Keywords: fuzzy optimization; granular differentiability; efficient solution; steepest descent method
Mathematics Subject Classification: 03E72

1. Introduction

Optimization plays an important role in mathematics, and it also has wide application in practical problems. If the objective function is a real-valued function, the optimization is a classical optimization problem. However, a number of classical optimization problems often involve uncertain or imprecise data, so it is more suitable that the values are expressed with fuzzy numbers. In 1970, Bellman and Zadeh [1] established fuzzy optimization models. Since then, the fuzzy optimization problems have garnered increasing attention among researchers, leading to a growing body of studies in this field [2,3].

The differentiability and convexity of fuzzy-number-valued functions serve as a foundation for fuzzy optimization and have been extensively examined by numerous researchers. The differentiability and convexity of fuzzy-number-valued functions serve as foundational in fuzzy optimization and have been extensively examined by numerous researchers. In 1983, Puri and Ralescu [4] developed the Hukuhara differentiability (H-differentiability for short) for fuzzy mappings, which generalized and extended the concept of Hukuhara differentiability for set-valued mappings. Subsequently, Wang and Wu [5] proposed the directional derivative, differential and subdifferential of fuzzy mappings from R^n into E using the Hukuhara difference. In order to overcome the shortcoming of H-differentiability for fuzzy-number-valued functions, Bede and Gal [6] introduced and studied strongly generalized differentiability (g-differentiability) for fuzzy-number-valued functions in 2005. The Hukuhara difference of fuzzy numbers exists only under very restrictive conditions and may not always be defined [7]. Based on the g-difference [7, 8] for fuzzy sets which overcomes the shortcomings of the above-discussed concepts and always exists for any two fuzzy numbers, Bede and Stefanini [9] introduced and studied the generalized Hukuhara differentiability (gH-differentiability) for fuzzy-valued functions in 2013. With the development of theory and applications of fuzzy differentiation, increasing scholarly attention has been directed toward this research area. In 2016, based on the differentiability of endpoint functions for fuzzy functions Cano et al. [10] gave some characterizations of gH-differentiable fuzzy functions and introduced the gH^*-differentiability of fuzzy functions which was more general than the gH-differentiability. To study optimization problems with s-convex and s-differentiable fuzzy-number-valued functions, Hai et al. [11] introduced differentiability concepts for n-dimensional fuzzy-number-valued functions. They characterized the relationships among three types of differentiability: generalized differentiability (g-differentiability), support-function-wise differentiability (s-differentiability), and level-wise differentiability (l-differentiability). for handling the shortcoming of the computing complexity for fuzzy numbers. In 2018, Mazandarani et al. [12] introduced the concept of granular differentiability (gr-differentiability) for fuzzy-valued functions. This approach was developed by combining both the granular difference and horizontal membership function concepts. The authors further conducted a comprehensive analysis of gr-differentiability properties in comparison with other existing derivatives for fuzzy functions. Importantly, gr-differentiability is devoid of certain shortcomings and limitations inherent in fuzzy number arithmetic. Subsequently, Zhang et al. [13] produced the partial gr-derivative and gr-differentiability of fuzzy functions from R^n into E to present the Karush-Kuhn-Tucker type optimality conditions of the fuzzy relative optimal solution for fuzzy programming problems. Moreover, the convexity of fuzzy mappings is a considerably vital tool in fuzzy optimization, which has been investigated by numerous researchers [14, 15]. Nanda and Kar [16] first put forward the concept of convex fuzzy mapping, which laid a foundation for future research about fuzzy optimization. Subsequently, lots of researchers focused on investigating the convexity and generalized convexity of fuzzy mappings as a means to address challenges in fuzzy optimization. Furukawa [17], by using "fuzzy-max" order, introduced the concept of convexity and Lipschitz continuity for fuzzy mappings. Research shows that convex fuzzy mappings in R^n are Lipschitz continuous. Syau [18] studied the convexity of monotone fuzzy mappings in R^1 and R^2, and established two characterization theorems of convex fuzzy mapping. Zhang and Yuan [19] researched the operation properties of convex fuzzy mappings as proposed by Nanda and Kar, and put forward some important concepts as positive homogeneous fuzzy mapping and convex hull. Gong and Hai [11], utilizing the ordering of n-dimensional fuzzy number space, investigated the convexity of fuzzy mappings because most existing methods may not consider the convex fuzzy mappings on E^n. Shi and Pang [20] introduced some basic concepts in convex optimization theory as L-convex structure, L-neighborhood system, and fuzzy convex structure. Their research results revealed that every L- fuzzy set is an L-convex set.

With the development of fuzzy optimization theory, the solution of fuzzy optimization problems has gradually shifted to more complex environments, which contain multi-objective, multi-constraint, and non-convex fuzzy functions [21, 22]. Recently, numerical optimization methods are gradually introduced into the fuzzy environment. The existing numerical optimization algorithm for fuzzy optimization problems primarily rely on H-difference and gH-difference for fuzzy-numbers. For instance, Pirzada and Pathak [23] developed the Newton method to compute a non-dominated solution of an unconstrained multi-variable fuzzy optimization problem; Cano et al. [24] investigated the Newton method to solve the unconstrained fuzzy optimization problems using generalized Hukuhara differentiability of fuzzy mappings. However, these approaches are constrained by the limitations of generalized Hukuhara differentiability. Granular differentiability provides a more flexible framework that overcomes these restrictions. As a classical numerical optimization algorithm, the steepest descent method is widely used in unconstrained optimization problems because of its easy calculation and fast convergence speed. Its main idea is to find the optimal solution along the inverse gradient direction of the objective function. However, this method cannot be applied directly to solve unconstrained fuzzy optimization problems. Building upon prior research and recognizing the significance of the steepest descent method, we propose the characterization function

$$\Psi(t) = \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t, \mu, \alpha_F)\mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m}\mathrm{d}\mu$$

for fuzzy mappings. Our theoretical analysis establishes that any local minimizer of the real-valued function $\Psi(t)$ corresponds to a local efficient solution of the fuzzy optimization problem (FOP).

The purpose of this work is to develop the theory of the steepest descent method for unconstrained fuzzy optimization problems. From this perspective, the results presented will be useful for the development of the fuzzy optimization. First, we present the terminology used in the present paper. In Section 3, based on granular differentiability of fuzzy mappings, we propose the concepts of partial gr-derivatives, granular gradients, and twice gr-differentiability, and discuss their interrelationships. Additionally, granular convexity of fuzzy mappings is researched by utilizing gr-difference and ordering relation of fuzzy numbers [12, 25]. In Section 4, we introduce the steepest descent method to solve unconstrained fuzzy optimization problems under the characterization function

$$\Psi(t) = \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t, \mu, \alpha_F)\mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m}\mathrm{d}\mu$$

of fuzzy mappings. This method provides a robust tool for solving optimization problems in environments with imprecise data, offering potential applications in engineering and economics.

2. Preliminaries

In this paper, R denotes the set of all real numbers, R^n represents an n-dimensional Euclidean space, and $K \subseteq R^n$ is a convex set. E is the set of all fuzzy numbers on R. The μ-levels of $\widetilde{u}$ are given by

$$[\widetilde{u}]^\mu = [\underline{u}^\mu, \overline{u}^\mu],$$

where

$$\underline{u}^\mu, \overline{u}^\mu : [0, 1] \to R.$$

The definition domain of fuzzy number $\widetilde{u}$ is the interval $I \subseteq R$.

Definition 2.1. *[12] Let $\widetilde{u} \in E$. The horizontal membership function*

$$u^{\mathrm{gr}} : [0,1] \times [0,1] \to I$$

is a representation of $\widetilde{u}$ as

$$u^{\mathrm{gr}}(\mu, \alpha_u) = x$$

in which "gr" stands for the granule of information included in $x \in I, \mu \in [0,1]$ is the membership degree of x in $\widetilde{u}, \alpha_u \in [0,1]$ is called relative-distance-measure variable, and

$$u^{\mathrm{gr}}(\mu, \alpha_u) = \underline{u}^{\mu} + \left(\bar{u}^{\mu} - \underline{u}^{\mu}\right)\alpha_u.$$

Remark 2.1. *We can also write*

$$u^{\mathrm{gr}}(\mu, \alpha_u) = (1 - \alpha_u)\underline{u}^{\mu} + \alpha_u \bar{u}^{\mu}$$

for any $\alpha_u \in [0,1]$, and α_u can determine points between the left and right endpoints of the μ-level sets of $\widetilde{u}$.

Remark 2.2. *[12] The horizontal membership function of $\widetilde{u} \in E$ is also denoted by*

$$\mathcal{H}(\widetilde{u}) \triangleq u^{\mathrm{gr}}(\mu, \alpha_u).$$

Moreover, the μ-level sets of the vertical membership function of $\widetilde{u}$ can be presented by

$$\mathcal{H}^{-1}\left(u^{\mathrm{gr}}(\mu, \alpha_u)\right) = [\widetilde{u}]^{\mu} = \left[\inf_{\beta \geq \mu} \min_{\alpha_u} u^{\mathrm{gr}}(\beta, \alpha_u), \sup_{\beta \geq \mu} \max_{\alpha_u} u^{\mathrm{gr}}(\beta, \alpha_u)\right],$$

which, in fact, represents the obtainable span of the information granule.

Definition 2.2. *[12] Let $\widetilde{u}, \widetilde{v} \in E$, whose horizontal membership functions are $u^{\mathrm{gr}}(\mu, \alpha_u)$ and $v^{\mathrm{gr}}(\mu, \alpha_v)$, respectively, as well as "$\odot$" denote one of the four basic operations. Then,*

$$\widetilde{u} \odot \widetilde{v} = \tilde{w} \in E,$$

such that

$$\mathcal{H}(\widetilde{w}) \triangleq u^{\mathrm{gr}}(\mu, \alpha_u) \odot v^{\mathrm{gr}}(\mu, \alpha_v).$$

It should be noted that $0 \notin v^{\mathrm{gr}}(\mu, \alpha_v)$ when "$\odot$" denotes the division operator. The difference between two fuzzy numbers is called gr-difference, which is denoted by $\ominus_{gr}$.

Remark 2.3. *[12] Let*

$$\widetilde{w} = \widetilde{u} \odot \widetilde{v}.$$

Then

$$[\widetilde{w}]^{\mu} = \mathcal{H}^{-1}\left(u^{\mathrm{gr}}(\mu, \alpha_u) \odot v^{\mathrm{gr}}(\mu, \alpha_v)\right)$$

always presents μ-level sets of the fuzzy number $\widetilde{w}$.

Remark 2.4. *[26] The horizontal membership function $\mathcal{H}$ is a linear mapping. Let $\widetilde{u}, \widetilde{v} \in E$ and $k \in R$. Then*

(1) $\mathcal{H}(\widetilde{u} \oplus \widetilde{v}) = u^{\mathrm{gr}}(\mu, \alpha_u) + v^{\mathrm{gr}}(\mu, \alpha_v) = \mathcal{H}(\widetilde{u}) + \mathcal{H}(\widetilde{v})$;

(2) $\mathcal{H}(k\widetilde{u}) = ku^{\mathrm{gr}}(\mu, \alpha_u) = k\mathcal{H}(\widetilde{u})$.

Definition 2.3. *[12] Let $\widetilde{u}, \widetilde{v} \in E$. The function*

$$D_{\mathrm{gr}} : E \times E \to R^{+} \cup \{0\}, D_{\mathrm{gr}}(\widetilde{u}, \widetilde{v}) = \sup_{\mu} \max_{\alpha_u, \alpha_v} \mid u^{\mathrm{gr}}(\mu, \alpha_u) - v^{gr}(\mu, \alpha_v) \mid$$

is a distance between two fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$.

Proposition 2.1. *Let $\widetilde{u}, \widetilde{v} \in E$. We have*

$$D_{\mathrm{gr}}(\widetilde{u}, \widetilde{v}) = D_{\mathrm{gr}}(\widetilde{u} \ominus_{gr} \widetilde{v}, 0).$$

Proof. The proof follows from Definition 2.3. □

Definition 2.4. *Let*

$$\widetilde{F} : K \to E.$$

The limit of $\widetilde{F}(t)$ as t approaches t_0 is $\widetilde{A} \in E$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever

$$0 < \|t - t_0\| < \delta$$

($\|\cdot\|$ is the Euclidean norm on R^n), we have

$$D_{\mathrm{gr}}(\widetilde{F}(t), \widetilde{A}) < \varepsilon$$

denoted by

$$\lim_{t \to t_0} \widetilde{F}(t) = \widetilde{A}.$$

Definition 2.5. *Let $\widetilde{F}$: $K \to E$, $t_0 \in K$, and $h \in R^n$ with $t_0 + h \in K$. $\widetilde{F}$ is said to be continuous at t_0, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $0 < \|h\| < \delta$, we have*

$$D_{\mathrm{gr}}(\widetilde{F}(t_0 + h), \widetilde{F}(t_0)) < \varepsilon.$$

The function $\widetilde{F}$ is continuous on K, if it is continuous at each $t \in K$.

If $K = [a, b]$, Definitions 2.4 and 2.5 coincide with [12, Definitions 12 and 13].

Theorem 2.1. *Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$, $t_0 \in K$. $\widetilde{F}$ is continuous at t_0 if and only if its horizontal membership function $F^{gr}(t, \mu, \alpha_F)$ is continuous at t_0 for any $\mu, \alpha_{u_i} \in [0, 1]$, where*

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$.

Proof. The proof follows from Definitions 2.3 and 2.5. □

Definition 2.6. *[27] The matrix*

$$\widetilde{A}=\left[\widetilde{a}_{ij}\right]_{n\times m}$$

is a fuzzy matrix if its entries are fuzzy numbers, i.e., $\widetilde{a}_{ij}\in E$ $(i=1,\cdots,n; j=1,\cdots,m)$. *Moreover, the horizontal membership function of the fuzzy matrix*

$$\widetilde{A}=\left[\widetilde{a}_{ij}\right]_{n\times m}$$

is defined as

$$\mathcal{H}(\widetilde{A})=\left[\mathcal{H}\left(\widetilde{a}_{ij}\right)\right]_{n\times m}.$$

3. The granular differentiability and granular convexity of fuzzy mappings

It is well-known that differentiability and convexity of fuzzy mappings play crucial roles in fuzzy optimization. In this section, based on the gr-difference for fuzzy numbers, we investigate the granular gradient, gr-differentiability, and twice gr-differentiability of fuzzy mappings. Furthermore, we examine the relationship between gr-differentiability of fuzzy mappings and differentiability of their corresponding characterization functions. Finally, building upon the partial order defined in [12, 25], we analyze the convexity properties of fuzzy mappings.

3.1. The granular differentiability of fuzzy mappings

Definition 3.1. *[12] The fuzzy mapping*

$$\widetilde{F}:[a,b]\to E$$

is said to be gr-differentiable at $t_0\in[a,b]$, *if there exists a fuzzy number* $\widetilde{u}\in E$, *such that*

$$\lim_{h\to 0}\frac{\widetilde{F}\left(t_0+h\right)\ominus_{gr}\widetilde{F}\left(t_0\right)}{h}=\widetilde{u}.$$

$\widetilde{u}$ *is called the gr-derivative of* $\widetilde{F}$ *at* t_0, *denoted by* $\frac{d\widetilde{F}(t_0)}{dt}$. *Moreover,* $\widetilde{F}$ *is said to be gr-differentiable on* $[a,b]$ *if it is gr-derivative at any* $t_0\in[a,b]$.

Theorem 3.1. *[12] Let*

$$\widetilde{F}:[a,b]\to E$$

include m distinct fuzzy numbers $\widetilde{u}_1,\widetilde{u}_2,\cdots,\widetilde{u}_m$. $\widetilde{F}$ *is gr-differentiable at* $t\in[a,b]$ *if and only if its horizontal membership function* $F^{gr}\left(t,\mu,\alpha_F\right)$ *is differentiable at t for any* μ, $\alpha_{u_i}\in[0,1]$,

$$\mathcal{H}\left(\frac{d\widetilde{F}(t)}{dt}\right)=\frac{\partial F^{gr}\left(t,\mu,\alpha_F\right)}{\partial t},$$

where

$$\alpha_F\triangleq\left(\alpha_{u_1},\alpha_{u_2},\cdots,\alpha_{u_m}\right)$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1,\widetilde{u}_2,\cdots,\widetilde{u}_m$.

Definition 3.2. *Let*

$$\widetilde{F} : K \to E$$

and

$$t_0 = \left(t_1^0, t_2^0, \cdots, t_n^0\right) \in K.$$

If fuzzy mapping

$$\widetilde{G}(t_i) = \widetilde{F}(t_1^0, \cdots, t_{i-1}^0, t_i, t_{i+1}^0, \cdots, t_n^0)$$

is gr-differentiable at t_i^0, then $\widetilde{F}$ is said to have the ith partial gr-derivative at t_0, denoted by $\frac{\partial \widetilde{F}(t_0)}{\partial t_i}$ and

$$\frac{\partial \widetilde{F}(t_0)}{\partial t_i} = \frac{d\widetilde{G}\left(t_i^0\right)}{dt}.$$

If $\frac{\partial \widetilde{F}(t)}{\partial t_i}$ has the jth partial gr-derivative at t_0 then $\widetilde{F}$ is said to have the second order (i, j) partial gr-derivative of $\widetilde{F}$ at t_0, denoted by $\frac{\partial^2 \widetilde{F}(t_0)}{\partial t_i \partial t_j}$.

Definition 3.3. *Let*

$$\widetilde{F} : K \to E$$

and

$$t_0 = \left(t_1^0, t_2^0, \cdots, t_n^0\right) \in K.$$

If all the partial gr-derivatives of $\widetilde{F}$ at t_0 exist, then the granular gradient of $\widetilde{F}$ at t_0 is defined by

$$\nabla \widetilde{F}(t_0) = (\frac{\partial \widetilde{F}(t_0)}{\partial t_1}, \frac{\partial \widetilde{F}(t_0)}{\partial t_2}, \cdots, \frac{\partial \widetilde{F}(t_0)}{\partial t_n})^T.$$

If all the second order partial gr-derivatives of $\widetilde{F}$ at t_0 exist, then the gr-Hessian matrix of $\widetilde{F}$ is denoted by $\nabla^2 \widetilde{F}(t_0)$ and

$$\nabla^2 \widetilde{F}(t_0) = \left[\frac{\partial^2 \widetilde{F}(t_0)}{\partial t_i \partial t_j}\right]_{n \times n} \quad (1 \leq i, j \leq n).$$

Definition 3.4. *Let*

$$\widetilde{F} : K \to E$$

include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$ and

$$t_0 = \left(t_1^0, t_2^0, \cdots, t_n^0\right) \in K.$$

The gradient of the horizontal membership function ∇F^{gr} for $\widetilde{F}$ at t_0 is defined by

$$\nabla F^{gr}(t_0, \mu, \alpha_F) = \left(\frac{\partial F^{gr}(t_0, \mu, \alpha_F)}{\partial t_1}, \frac{\partial F^{gr}(t_0, \mu, \alpha_F)}{\partial t_2}, \cdots, \frac{\partial F^{gr}(t_0, \mu, \alpha_F)}{\partial t_n}\right)^T,$$

for any $\mu, \alpha_{u_i} \in [0, 1]$, where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$.

Remark 3.1. *The definition of gradient for the horizontal membership function* ∇F^{gr} *coincides with the definition of gradient for the fuzzy mapping* $\widetilde{F}$ *in [13].*

Theorem 3.2. *Let* $\widetilde{F}$*:* $K \to E$ *and* $t_0 \in K$*. The granular gradient of* $\widetilde{F}$ *exists at* t_0 *if and only if the gradient of its horizontal membership function exists at* t_0 *and*

$$\mathcal{H}\left(\nabla\widetilde{F}(t_0)\right) = \nabla F^{gr}\left(t_0, \mu, \alpha_F\right),$$

for any $\mu, \alpha_{u_i} \in [0,1]$, *where*

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$.

Proof. According to Theorem 3.1 and Definition 3.2, for each $i(i = 1, 2, \cdots, n)$ and any μ, $\alpha_{u_i} \in [0,1]$, we can obtain

$$\mathcal{H}\left(\frac{\partial\widetilde{F}(t_0)}{\partial t_i}\right) = \frac{\partial F^{gr}(t_0, \mu, \alpha_F)}{\partial t_i},$$

where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2 \cdots, \widetilde{u}_m$. Therefore, we have from Definitions 3.3 and 3.4 that

$$\mathcal{H}\left(\nabla\widetilde{F}(t_0)\right) = \nabla F^{gr}\left(t_0, \mu, \alpha_F\right),$$

for any $\mu, \alpha_{u_i} \in [0,1]$, where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. □

Example 3.1. *Let*

$$\tilde{F}(t) = \widetilde{u}_1 t_1^2 + \widetilde{u}_2 t_2,$$

where

$$t = (t_1, t_2) \in R^2, \quad \widetilde{u}_1 = (-1, 1, 2, 5) \quad \textit{and} \quad \widetilde{u}_2 = (1, 3, 5, 6).$$

According to Definition 2.2 and Remark 2.4, the horizontal membership function of $\widetilde{F}$ *is as follows:*

$$\mathcal{H}(\widetilde{F}(t)) = (2\mu - 1 + (6 - 5\mu)\alpha_{u_1})t_1^2 + (2\mu + 1 + (5 - 3\mu)\alpha_{u_2})t_2$$

for any $\mu, \alpha_{u_1}, \alpha_{u_2} \in [0,1]$. *The gradient of the horizontal membership function for* $\widetilde{F}(t)$ *is*

$$\nabla F^{gr}\left(t_0, \mu, \alpha_F\right) = ((4\mu - 2 + (12 - 10\mu)\alpha_{u_1})t_1, 2\mu + 1 + (5 - 3\mu)\alpha_{u_2})^T.$$

It follows from Theorem 3.2 that the granular gradient of $\widetilde{F}(t)$ *is as follows:*

$$\nabla\widetilde{F}(t_0) = ((-2, 2, 4, 10)t_1, (1, 3, 5, 6))^T.$$

Definition 3.5. *Let $\widetilde{F}$: $K \to E$, $t_0 \in K$, $h \in R^n$ with $t_0 + h \in K$, and for a given $\delta > 0$, $\|h\| < \delta$. $\widetilde{F}$ is said to be gr-differentiable at t_0, if there exists a fuzzy vector $\nabla\widetilde{F}(t_0)$ such that*

$$\lim_{h\to 0} \frac{\widetilde{F}(t_0 + h) \ominus_{gr} \widetilde{F}(t_0) \ominus_{gr} \left\langle \nabla\widetilde{F}(t_0), h \right\rangle}{\|h\|} = 0,$$

where

$$\left\langle \nabla\widetilde{F}(t_0), h \right\rangle = \sum_{i=1}^{n} h_i \frac{\partial \widetilde{F}(t_0)}{\partial t_i}.$$

The fuzzy mapping $\widetilde{F}$ is gr-differentiable on K, if it is gr-differentiable at each $t \in K$.

Theorem 3.3. *If $\widetilde{F}$: $K \to E$ is gr-differentiable at $t_0 \in K$, then $\widetilde{F}$ is continuous at t_0.*

Proof. Since $\widetilde{F}$ is gr-differentiable at $t_0 \in K$, there exists a fuzzy vector $\nabla\widetilde{F}(t_0)$, such that

$$\lim_{h\to 0} \frac{\widetilde{F}(t_0 + h) \ominus_{gr} \widetilde{F}(t_0) \ominus_{gr} \left\langle \nabla\widetilde{F}(t_0), h \right\rangle}{\|h\|} = 0,$$

which implies

$$\lim_{h\to 0} \left[\widetilde{F}(t_0 + h) \ominus_{gr} \widetilde{F}(t_0) \ominus_{gr} \left\langle \nabla\widetilde{F}(t_0), h \right\rangle \right] = 0.$$

Since

$$\lim_{h\to 0} \left\langle \nabla\widetilde{F}(t_0), h \right\rangle = 0,$$

we have

$$\lim_{h\to 0} [\widetilde{F}(t_0 + h) \ominus_{gr} \widetilde{F}(t_0)] = 0.$$

Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$; when $0 < \|h\| < \delta$, we have

$$D_{\mathrm{gr}}(\widetilde{F}(t_0 + h) \ominus_{gr} \widetilde{F}(t_0), 0) < \varepsilon.$$

According to Proposition 2.1, we can obtain

$$D_{\mathrm{gr}}(\widetilde{F}(t_0 + h), \widetilde{F}(t_0)) < \varepsilon.$$

This completes the proof. □

Definition 3.6. *[13] Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$, $t_0 \in K$, $h \in R^n$ with $t_0 + h \in K$, and for a given $\delta > 0$, $\|h\| < \delta$. The horizontal membership function F^{gr} of $\widetilde{F}$ is said to be differentiable at t_0, if there exist a vector $\nabla F^{gr}(t_0, \mu, \alpha_F)$ and a function*

$$\varepsilon(h, \mu, \alpha_F) : R^n \times [0, 1] \times [0, 1]^m \to R,$$

such that for any $\mu, \alpha_{u_i} \in [0, 1]$,

$$F^{gr}(t_0 + h, \mu, \alpha_F) = F^{gr}(t_0, \mu, \alpha_F) + \langle \nabla F^{gr}(t_0, \mu, \alpha_F), h \rangle + \|h\| \varepsilon(h, \mu, \alpha_F),$$

where

$$\lim_{h\to 0} \varepsilon(h, \mu, \alpha_F) = 0$$

and

$$\langle \nabla F^{gr}(t_0,\mu,\alpha_F), h\rangle = \sum_{i=1}^{n} h_i \frac{\partial F^{gr}(t_0,\mu,\alpha_F)}{\partial t_i},$$

where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$*. The horizontal membership function* F^{gr} *is differentiable on K, if it is differentiable at each* $t \in K$*.*

Proposition 3.1. *Let* $\widetilde{F}$*:* $K \to E$ *include m distinct fuzzy numbers* $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$*,* $t_0 \in K$*,* $h \in R^n$ *with* $t_0 + h \in K$*, and for a given* $\delta > 0$*,* $\|h\| < \delta$*. The horizontal membership function* F^{gr} *of* $\widetilde{F}$ *is differentiable at* t_0 *if, and only if, there exists a vector* $\nabla F^{gr}(t_0,\mu,\alpha_F)$ *such that*

$$\lim_{h\to 0} \frac{F^{gr}(t_0+h,\mu,\alpha_F) - F^{gr}(t_0,\mu,\alpha_F) - \langle \nabla F^{gr}(t_0,\mu,\alpha_F), h\rangle}{\|h\|} = 0,$$

for any $\mu, \alpha_{u_i} \in [0,1]$*, where*

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$*.*

Proof. The proof follows from Definition 3.6. □

Theorem 3.4. *Let* $\widetilde{F}$*:* $K \to E$ *include m distinct fuzzy numbers* $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$*,* $t_0 \in K$*.* $\widetilde{F}$ *is gr-differentiable at* t_0 *if, and only if, its horizontal membership function* F^{gr} *is differentiable at* t_0*.*

Proof. Only if: Let $h \in R^n$ with $t_0 + h \in K$, and for a given $\delta > 0$, $\|h\| < \delta$. We have from Definition 3.5 that

$$D_{\mathrm{gr}}\left(\frac{\widetilde{F}(t_0+h) \ominus_{gr} \widetilde{F}(t_0) \ominus_{gr} \left\langle \nabla\widetilde{F}(t_0), h\right\rangle}{\|h\|}, 0\right) < \varepsilon.$$

According to Remark 2.3 and Definition 2.3, for any $\varepsilon > 0$, we can obtain

$$\sup_{\mu} \max_{\alpha_F} \left|\frac{F^{gr}(t_0+h,\mu,\alpha_F) - F^{gr}(t_0,\mu,\alpha_F) - \langle \nabla F^{gr}(t_0,\mu,\alpha_F), h\rangle}{\|h\|}\right| < \varepsilon,$$

where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. Then we have

$$\left|\frac{F^{gr}(t_0+h,\mu,\alpha_F) - F^{gr}(t_0,\mu,\alpha_F) - \langle \nabla F^{gr}(t_0+h,\mu,\alpha_F), h\rangle}{\|h\|}\right| < \varepsilon,$$

for any $\mu, \alpha_{u_i} \in [0,1]$. This implies that

$$\lim_{h\to 0} \frac{F^{gr}(t_0+h,\mu,\alpha_F) - F^{gr}(t_0,\mu,\alpha_F) - \langle \nabla F^{gr}(t_0,\mu,\alpha_F), h\rangle}{\|h\|} = 0,$$

for any $\mu, \alpha_{u_i} \in [0,1]$. It follows from Proposition 3.1 that the horizontal membership function of $\widetilde{F}$ is differentiable at t_0.

If: According to Proposition 3.1, for any $\varepsilon > 0$, we have

$$\left|\frac{F^{gr}(t_0 + h, \mu, \alpha_F) - F^{gr}(t_0, \mu, \alpha_F) - \langle \nabla F^{gr}(t_0, \mu, \alpha_F), h\rangle}{\|h\|}\right| < \varepsilon,$$

for any $\mu, \alpha_{u_i} \in [0, 1]$, then

$$\sup_{\mu} \max_{\alpha_F} \left|\frac{F^{gr}(t_0 + h, \mu, \alpha_F) - F^{gr}(t_0, \mu, \alpha_F) - \langle \nabla F^{gr}(t_0, \mu, \alpha_F), h\rangle}{\|h\|}\right| < \varepsilon.$$

We have from Remark 2.3 and Definition 2.3 that

$$D_{\mathrm{gr}}\left(\frac{\widetilde{F}(t_0 + h) \ominus_{gr} \widetilde{F}(t_0) \ominus_{gr} \left\langle \nabla \widetilde{F}(t_0), h \right\rangle}{\|h\|}, 0\right) < \varepsilon.$$

Therefore $\widetilde{F}$ is gr-differentiable at t_0. □

Corollary 3.1. *Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$ and $t_0 \in K$. If the partial gr-derivative $\frac{\partial \widetilde{F}(t_0)}{\partial t_1}, \frac{\partial \widetilde{F}(t_0)}{\partial t_2}, \cdots, \frac{\partial \widetilde{F}(t_0)}{\partial t_n}$ exists on some neighborhood of t_0 and is continuous at t_0, then $\widetilde{F}$ is gr-differentiable at t_0.*

Proof. The proof follows from Theorems 2.1, 3.1, and 3.4. □

If the partial gr-derivative $\frac{\partial \widetilde{F}(t_0)}{\partial t_1}, \frac{\partial \widetilde{F}(t_0)}{\partial t_2}, \cdots, \frac{\partial \widetilde{F}(t_0)}{\partial t_n}$ of $\widetilde{F}$ is continuous at t_0, $\widetilde{F}$ is said to be a first-order continuous granular differentiable fuzzy mapping at t_0.

Definition 3.7. *Let $\widetilde{F}$: $K \to E$, $t_0 \in K$, $h \in R^n$ with $t_0 + h \in K$, and for a given $\delta > 0$, $\|h\| < \delta$. $\widetilde{F}$ is said to be twice gr-differentiable at t_0, if there exist a fuzzy vector $\nabla \widetilde{F}(t_0)$ and $n \times n$ symmetric fuzzy matrix $\nabla^2 \widetilde{F}(t_0)$ such that*

$$\lim_{h \to 0} \frac{\widetilde{F}(t_0 + h) \ominus_{gr} \widetilde{F}(t_0) \ominus_{gr} \left\langle \nabla \widetilde{F}(t_0), h \right\rangle \ominus_{gr} \frac{1}{2} h^T \nabla^2 \widetilde{F}(t_0) h}{\|h\|^2} = 0,$$

where

$$\nabla^2 \widetilde{F}(t_0) = \left[\frac{\partial^2 \widetilde{F}(t_0)}{\partial t_i \partial t_j}\right]_{n \times n}.$$

The fuzzy mapping $\widetilde{F}$ is twice gr-differentiable on K, if it is twice gr-differentiable at each $t \in K$.

Definition 3.8. *[13] Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$, $t_0 \in K$, $h \in R^n$ with $t_0 + h \in K$, and for a given $\delta > 0$, $\|h\| < \delta$. The horizontal membership function F^{gr} of $\widetilde{F}$ is said to be twice differentiable at t_0, if there exist a vector $\nabla F^{gr}(t_0, \mu, \alpha_F)$ and $n \times n$ symmetric matrix $\nabla^2 F^{gr}(t_0, \mu, \alpha_F)$ such that*

$$F^{gr}(t_0 + h, \mu, \alpha_F) = F^{gr}(t_0, \mu, \alpha_F) + \langle \nabla F^{gr}(t_0, \mu, \alpha_F), h\rangle + \frac{1}{2} h^T \nabla^2 F^{gr}(t_0, \mu, \alpha_F) h + O\left(\|h\|^2\right),$$

where the $n \times n$ matrix $\nabla^2 F^{gr}(t_0, \mu, \alpha_F)$ is the Hessian matrix for the horizontal membership function F^{gr} at t_0 and

$$\nabla^2 F^{gr}(t_0, \mu, \alpha_F) = \begin{bmatrix} \frac{\partial^2 F^{gr}(t_0,\mu,\alpha_F)}{\partial t_1^2} & \cdots & \frac{\partial^2 F^{gr}(t_0,\mu,\alpha_F)}{\partial t_1 \partial t_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 F^{gr}(t_0,\mu,\alpha_F)}{\partial t_n \partial t_1} & \cdots & \frac{\partial^2 F^{gr}(t_0,\mu,\alpha_F)}{\partial t_n^2} \end{bmatrix},$$

for any $\mu, \alpha_{u_i} \in [0, 1]$, where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. The horizontal membership function F^{gr} is twice differentiable on K, if it is twice differentiable at each $t \in K$.

Theorem 3.5. *Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$ and $t_0 \in K$. $\widetilde{F}$ is twice gr-differentiable at t_0 if, and only if, its horizontal membership function F^{gr} is twice differentiable at t_0, and*

$$\mathcal{H}\left(\nabla^2 \widetilde{F}(t_0)\right) = \nabla^2 F^{gr}(t_0, \mu, \alpha_F),$$

for any $\mu, \alpha_{u_i} \in [0, 1]$, where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$.

Proof. The proof follows from Theorem 3.4 and Definitions 3.7 and 3.8, and the proof progress is similar to Theorem 3.4. □

Definition 3.9. *Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. The real-valued function*

$$\phi(t, \mu) = \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t, \mu, \alpha_F) \mathrm{d}\alpha_{u_1} \mathrm{d}\alpha_{u_2} \ldots \mathrm{d}\alpha_{u_m}$$

is said to be the characterization function of $\widetilde{F}$, where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$ and $\mu \in [0, 1]$.

Remark 3.2. *Let*

$$\widetilde{F}(t) = \widetilde{u}.$$

According to Remark 2.1, we have

$$\phi(t, \mu) = \frac{\underline{u}^\mu + \overline{u}^\mu}{2}.$$

Definition 3.10. *Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. Denote*

$$\Psi(t) = \int_0^1 \phi(t, \mu) \mathrm{d}\mu = \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t, \mu, \alpha_F) \mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m} \mathrm{d}\mu,$$

where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$.

Theorem 3.6. *Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$ and $t_0 \in K$. If $\widetilde{F}$ is first-order continuous granular differentiable at t_0, then the real-valued function*

$$\Psi(t) = \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t, \mu, \alpha_F) \mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m} \mathrm{d}\mu$$

is differentiable at t_0, where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$.

Proof. Let $\widetilde{F}$ be first-order continuous granular differentiable at t_0. We have from Corollary 3.1 that $\widetilde{F}$ is gr-differentiable at t_0, thus $\widetilde{F}$ is continuous at t_0. It follows from Theorem 2.1 that its horizontal membership function $F^{gr}(t, \mu, \alpha_F)$ is continuous at t_0 for any $\mu, \alpha_{u_i} \in [0, 1]$, where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. Since $\frac{\partial \widetilde{F}(t)}{\partial t_i}$ $(i = 1, 2, \cdots, n)$ is continuous at t_0, we can obtain $\frac{\partial F^{gr}(t_0, \mu, \alpha_F)}{\partial t_i}$ $(i = 1, 2, \cdots, n)$ is continuous at t_0 for any $\mu, \alpha_{u_i} \in [0, 1]$, where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. Therefore, $\Psi(t)$ is differentiable at t_0. □

3.2. The granular convexity of fuzzy mappings

Definition 3.11. *[12, 25] Let $\widetilde{u}, \widetilde{v} \in E$, and we say that*

i) $\widetilde{u} \geq \widetilde{v}$, if $\mathcal{H}(\widetilde{u}) \geq \mathcal{H}(\widetilde{v})$ for any $\alpha_u = \alpha_v \in [0, 1]$ and $\mu \in [0, 1]$;

ii) $\widetilde{u} = \widetilde{v}$, if $\mathcal{H}(\widetilde{u}) = \mathcal{H}(\widetilde{v})$ for any $\alpha_u = \alpha_v \in [0, 1]$ and $\mu \in [0, 1]$;

iii) $\widetilde{u} > \widetilde{v}$, if $\mathcal{H}(\widetilde{u}) > \mathcal{H}(\widetilde{v})$ for any $\alpha_u = \alpha_v \in [0, 1]$ and $\mu \in [0, 1]$.

$\widetilde{u}$ and $\widetilde{v}$ are said to be comparable if

$$\mathcal{H}(\widetilde{u}) \geq \mathcal{H}(\widetilde{v})$$

or

$$\mathcal{H}(\widetilde{u}) \leq \mathcal{H}(\widetilde{v})$$

for any

$$\alpha_u = \alpha_v \in [0, 1]$$

and $\mu \in [0, 1]$.

Definition 3.12. *Let $\widetilde{F}$: $K \to E$. $\widetilde{F}$ is said to be granular convex on K if*

$$\widetilde{F}(\lambda t_1 + (1 - \lambda) t_2) \leq \lambda \widetilde{F}(t_1) + (1 - \lambda) \widetilde{F}(t_2),$$

for any $\lambda \in [0, 1]$ and $t_1, t_2 \in K$ with $t_1 \neq t_2$.

Proposition 3.2. *Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$ and $\widetilde{F}$ as granular convex on K. Then, the real-valued function*

$$\Psi(t) = \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t, \mu, \alpha) \mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m} \mathrm{d}\mu$$

is convex on K, where

$$\alpha \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$ and $\mu, \alpha_{u_i} \in [0, 1]$.

Proof. Let $\widetilde{F}$ be granular convex on K. We have from Definitions 3.11 and 3.12 and Remark 2.4 that

$$\mathcal{H}(\widetilde{F}(\lambda t_1 + (1 - \lambda) t_2)) \le \lambda \mathcal{H}(\widetilde{F}(t_1)) + (1 - \lambda) \mathcal{H}(\widetilde{F}(t_2)),$$

for any $\lambda \in [0, 1]$ and $t_1, t_2 \in K$ with $t_1 \neq t_2$. That is

$$F^{gr}(\lambda t_1 + (1 - \lambda) t_2, \mu, \alpha) \le \lambda F^{gr}(t_1, \mu, \alpha) + (1 - \lambda) F^{gr}(t_2, \mu, \alpha),$$

for any $\lambda, \mu, \alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m} \in [0, 1]$ and $t_1, t_2 \in K$ with $t_1 \neq t_2$, where

$$\alpha \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m}).$$

According to Definition 3.10, we have

$$\Psi(\lambda t_1 + (1 - \lambda) t_2) \le \lambda \Psi(t_1) + (1 - \lambda) \Psi(t_2),$$

for any $\lambda \in [0, 1]$ and $t_1, t_2 \in K$ with $t_1 \neq t_2$. Therefore, Ψ is convex on K. □

Remark 3.3. *The convexity of $\Psi(t)$ does not necessarily imply that the corresponding fuzzy function $\widetilde{F}$ is granular convex.*

Example 3.2. *Let*

$$\widetilde{F}(t) = \widetilde{u}_1 e^{t_1} + \widetilde{u}_2 t_2, \quad \widetilde{u}_1 = (-1, 5, 6), \quad \widetilde{u}_2 = (1, 3, 7) \quad \textit{and} \quad t = (t_1, t_2) \in R^2.$$

It follows from Definition 2.2 and Remark 2.4 that the horizontal membership function of $\widetilde{F}(t)$ is as follows:

$$\mathcal{H}(\widetilde{F}(t)) = [6\mu - 1 + 7(1 - \mu)\alpha_{u_1}] e^{t_1} + [2\mu + 1 + 6(1 - \mu)\alpha_{u_2}] t_2,$$

for any $\mu, \alpha_{u_1}, \alpha_{u_2} \in [0, 1]$. Therefore,

$$\begin{aligned}
\Psi(t) &= \int_0^1 \int_0^1 \int_0^1 F^{gr}(t, \mu, \alpha_F) \mathrm{d}\alpha_{u_1} \mathrm{d}\alpha_{u_2} \mathrm{d}\mu \\
&= \int_0^1 \int_0^1 \int_0^1 \{[6\mu - 1 + 7(1 - \mu)\alpha_{u_1}] e^{t_1} + [2\mu + 1 + 6(1 - \mu)\alpha_{u_2}] t_2\} \mathrm{d}\alpha_{u_1} \mathrm{d}\alpha_{u_2} \mathrm{d}\mu \\
&= \frac{15}{4} e^{t_1} + \frac{7}{2} t_2,
\end{aligned}$$

where

$$\alpha_F = (\alpha_{u_1}, \alpha_{u_2})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2$. *Let*

$$p = (t_1, t_2), \quad q = (s_1, s_2) \in R^2.$$

We can obtain

$$\begin{aligned}
\Psi(\lambda p + (1-\lambda)q) &= \Psi(\lambda t_1 + (1-\lambda)s_1, \lambda t_2 + (1-\lambda)s_2) \\
&= \frac{15}{4}e^{\lambda t_1 + (1-\lambda)s_1} + \frac{7}{2}[\lambda t_2 + (1-\lambda)s_2] \\
&\leq \frac{15}{4}\lambda e^{t_1} + \frac{15}{4}(1-\lambda)e^{s_1} + \frac{7}{2}\lambda t_2 + \frac{7}{2}(1-\lambda)s_2 \\
&= \lambda(\frac{15}{4}e^{t_1} + \frac{7}{2}t_2) + (1-\lambda)(\frac{15}{4}e^{s_1} + \frac{7}{2}s_2) \\
&= \lambda\Psi(p) + (1-\lambda)\Psi(q),
\end{aligned}$$

for any $\lambda \in [0, 1]$, *which implies that* $\Psi(t)$ *is convex on* R^2. *If*

$$\mu = \alpha_{u_1} = \alpha_{u_2} = 0,$$

we can obtain

$$\mathcal{H}(\widetilde{F}(t)) = -e^{t_1} + t_2.$$

Then, for any $p = (t_1, t_2), q = (s_1, s_2) \in R^2$,

$$\begin{aligned}
\mathcal{H}(\widetilde{F}(\lambda p + (1-\lambda)q)) &= -e^{\lambda t_1 + (1-\lambda)s_1} + (\lambda t_2 + (1-\lambda)s_2) \\
&\geq -\lambda e^{t_1} - (1-\lambda)e^{s_1} + \lambda t_2 + (1-\lambda)s_2 \\
&= \lambda(-e^{t_1} + t_2) + (1-\lambda)(-e^{s_1} + s_2) \\
&= \lambda\mathcal{H}(\widetilde{F}(p)) + (1-\lambda)\mathcal{H}(\widetilde{F}(q)),
\end{aligned}$$

for any $\lambda \in [0, 1]$. *Thus,* $\widetilde{F}(t)$ *is not granular convex on R.*

Theorem 3.7. *[13] Let* $\widetilde{F}$*:* $K \to E$ *include m distinct fuzzy numbers* $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$*, and the horizontal membership function of* $\widetilde{F}(t)$ *be a twice differentiable. Then,* $\widetilde{F}$ *is a gr-convex fuzzy mapping on K if, and only if, for all* $t \in K$*, the gr-Hessian matrix* $\nabla^2 F^{gr}(t, \mu, \alpha)$ *is a positive semidefinite matrix for any* $\mu, \alpha_{u_i} \in [0, 1]$*, where*

$$\alpha \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$.

4. Unconstrained fuzzy optimization and steepest descent method

The steepest descent method, as a cornerstone of classical optimization theory, has been widely applied in scientific and engineering due to its simplicity and efficiency in locating local minima. In this section, the steepest method is used to solve the unconstrained fuzzy optimization problem. We study the efficient solution of such problems and present a detailed algorithmic framework for the steepest descent method in unconstrained fuzzy optimization, accompanied by a rigorous convergence analysis.

Let $\widetilde{F}$ be a first-order continuous granular differentiable fuzzy mapping including m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. We consider the following fuzzy optimization problem

$$\text{(FOP)} \quad \min \widetilde{F}(t), \quad t \in K.$$

Definition 4.1. *Let $K \subseteq R^n$ be an open set and $t^* \in K$. If there does not exist any*

$$t \in N_\delta(t^*) \cap K,$$

such that

$$\widetilde{F}(t) \prec \widetilde{F}(t^*),$$

t^* is said to be a local efficient solution of the FOP, where $N_\delta(t^*)$ is a δ-neighborhood of t^*.

Definition 4.2. *Let $K \subseteq R^n$ be an open set and $t^* \in K$. If there does not exist any $t \in K/t^*$ such that*

$$\widetilde{F}(t) \prec \widetilde{F}(t^*),$$

t^* is said to be a global efficient solution of the FOP, where K/t^* is a feasible set excluding point t^*.

Theorem 4.1. *Let $\widetilde{F}$: $K \to E$ include m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$ and $t^* \in K$. If t^* is a local minimizer of the real-valued function $\Psi(t)$, then t^* is a local efficient solution of the FOP.*

Proof. t^* is not a local efficient solution of the FOP. Then for any $\delta > 0$, there exists a point

$$\bar{t} \in N_\delta(t^*) \cap K,$$

such that

$$\widetilde{F}(\bar{t}) \prec \widetilde{F}(t^*).$$

By Definition 3.9, we have

$$\mathcal{H}(\widetilde{F}(\bar{t})) < \mathcal{H}(\widetilde{F}(t^*)),$$

for any $\mu, \alpha_{u_i} \in [0, 1]$, where

$$\alpha \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. This implies that

$$F^{gr}\left(\bar{t}, \mu, \alpha\right) < F^{gr}\left(t^*, \mu, \alpha\right),$$

for any $\mu, \alpha_{u_i} \in [0, 1]$, where

$$\alpha \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_m})$$

with regard to the distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$. Therefore,

$$\int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(\bar{t}, \mu, \alpha) \mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m} \mathrm{d}\mu < \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t^*, \mu, \alpha) \mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m} \mathrm{d}\mu.$$

Thus

$$\Psi(\bar{t}) < \Psi(t^*),$$

i.e., t^* is not a local efficient solution of the FOP, which is contradictory. □

Theorem 4.2. *Let $\widetilde{F}$: $K \to E$ be a granular convex fuzzy mapping. The local efficient solution t^* of $\widetilde{F}$ is also the global efficient solution of $\widetilde{F}$.*

Proof. t^* is not the global efficient solution. Then, there exists $\bar{t} \in K/t^*$ such that

$$\widetilde{F}(\bar{t}) \prec \widetilde{F}(t^*).$$

Moreover, $\widetilde{F}$ is a granular convex fuzzy mapping, then we have from Remark 2.3 and Definition 3.11 that

$$\begin{aligned}\mathcal{H}\left(\widetilde{F}\left(\lambda\bar{t} + (1-\lambda)t^*\right)\right) &\leq \lambda\mathcal{H}\left(\widetilde{F}\left(\bar{t}\right)\right) + (1-\lambda)\mathcal{H}\left(\widetilde{F}(t^*)\right) \\ &< \lambda\mathcal{H}\left(\widetilde{F}(t^*)\right) + (1-\lambda)\mathcal{H}\left(\widetilde{F}(t^*)\right) \\ &= \mathcal{H}(\widetilde{F}(t^*).\end{aligned}$$

For $\lambda \in [0, 1]$ with

$$\lambda\bar{t} + (1-\lambda)t^* \in N_\delta(t^*) \cap K,$$

we have

$$\widetilde{F}\left(\lambda\bar{t} + (1-\lambda)t^*\right) \prec \widetilde{F}(t^*),$$

which is contradictory. □

4.1. The steepest descent method

In this section, we employ the steepest descent method to obtain an efficient solution for the FOP and assume that $\widetilde{F}$: $K \to E$ satisfies the following two conditions:

i) $\widetilde{F}$ includes m distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m$ and is well-defined at each iteration point of $\{t^{(k)}\}$, where $i = 0, 1, 2, \cdots$.

ii) The fuzzy mapping $\widetilde{F}$ is first-order continuous granular differentiable.

It follows from Theorem 3.6 that real-valued function

$$\Psi(t) = \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t, \mu, \lambda)\mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m}\mathrm{d}\mu$$

is differentiable and $\Psi(t^{(k)}), \nabla\Psi(t^{(k)})$ are well-defined for all $k = 0, 1, 2, \cdots$. According to Taylor's formula, we can approximate the real-valued function

$$\Psi(t) = \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t, \mu, \alpha)\mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m}\mathrm{d}\mu$$

at $t^{(k)}$ by a linear function

$$\mathbf{h}(t) = \Psi(t^{(k)}) + \nabla\Psi(t^{(k)})^T(t - t^{(k)}).$$

Let

$$t = t^{(k)} + v,$$

and we have

$$\mathbf{h}(t^{(k)} + v) = \Psi(t^{(k)}) + \nabla\Psi(t^{(k)})^T v, \tag{4.1}$$

where $v \in R^n$ with $||v|| = 1$. We assume that $d^{(k)}$ is the steepest descent direction of $\Psi(t)$ at the point $t^{(k)}$ and satisfies

$$d^{(k)} := \arg\min_v \{\Psi(t^{(k)}) + \nabla\Psi(t^{(k)})^T v\}. \tag{4.2}$$

Since $\Psi(t^{(k)})$ is a constant, the above equation is equivalent to

$$\min \quad \nabla\Psi(t^{(k)})^T v.$$

Theorem 4.3. $\frac{-\nabla\Psi(t^{(k)})}{||\nabla\Psi(t^{(k)})||}$ *is the steepest descent direction at point* $t^{(k)}$.

Proof. The proof follows as the classical proof of the steepest descent method. □

Let $\alpha^{(k)}$ be the step length at the point $t^{(k)}$ with regard to the steepest descent direction $d^{(k)}$ and satisfy

$$\alpha^{(k)} := \arg\min_{\alpha \geq 0} \Psi(t^{(k)} + \alpha d^{(k)}).$$

We can obtain the next iteration point

$$t^{(k+1)} = t^{(k)} + \alpha^{(k)} d^{(k)}. \tag{4.3}$$

Thus, we can generate a sequence to approximate the minimizer of function $\Psi(t)$ through (4.3) with an initial point. We can employ the termination condition of the above process of generating the sequence $\{t^{(k)}\}$ as

$$||t^{(k+1)} - t^{(k)}|| < \varepsilon$$

or

$$||\nabla\Psi(t^{(k)})|| < \varepsilon,$$

where ε is a prespecified tolerance level of acceptable solution. The algorithmic implementation of the above steepest descent method is presented in Algorithm 1.

Algorithm 1 The steepest descent method.

Input: The objective function $\Psi(t) = \int_0^1 \int_0^1 \cdots \int_0^1 F^{gr}(t, \mu, \alpha) \mathrm{d}\alpha_{u_1} \ldots \mathrm{d}\alpha_{u_m} \mathrm{d}\mu$, the tolerance value ε, the initial point $x^{(0)}$.

Output: t^* that meets the accuracy requirements.

1: Initialization $k \leftarrow 0, t^{(0)} \leftarrow x^{(0)}$.
2: **while** $||g^{(k)}|| > \varepsilon$ **do**
3: Compute gradient $g^{(k)} \leftarrow \nabla\Psi(t^{(k)})$.
4: Determine search direction $d^{(k)} \leftarrow -\frac{g^{(k)}}{||g^{(k)}||}$.
5: Compute the step length $\alpha^{(k)} \leftarrow \arg\min_{\alpha \geq 0} \Psi(t^{(k)} + \alpha d^{(k)})$.
6: Update iterate $t^{(k+1)} \leftarrow t^{(k)} + \alpha^{(k)} d^{(k)}$.
7: Increment counter $k \leftarrow k + 1$.
8: **end while**
9: **return** $t^* \leftarrow t^k$.

The convergence of the steepest method with the positive definite quadratic fuzzy number-valued objective function is as follows.

Theorem 4.4. *Let*

$$\widetilde{F}(t) = \frac{1}{2}t^T\widetilde{A}t - t^T\widetilde{B},$$

where $\widetilde{A}$ is a symmetric $n\times n$ fuzzy matrix

$$\widetilde{A} = \left[\widetilde{a}_{ij}\right]_{n\times n},$$

and

$$\left[\mathcal{H}\left(\widetilde{a}_{ij}\right)\right]_{n\times n} = [a_{ij}^{gr}(\mu, \alpha_{a_{ij}})]_{n\times n}$$

is a positive definite matrix for any $\mu, \alpha_{a_{ij}} \in [0,1]$, and $\widetilde{B}$ is an $n\times 1$ fuzzy matrix $[\widetilde{b_i}]_{n\times 1}$. Then, the steepest descent method is linearly convergent with respect to the iterated sequence $\{t^{(k)}\}$ and the efficient solution is t^*.

Proof. We have from Definition 2.2 that

$$\mathcal{H}(F(t)) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{gr}(\mu, \alpha_{a_{ij}})t_it_j - \sum_{i=1}^{n} b_i^{gr}(\mu, \alpha_{b_i})t_i,$$

where $\mu, \alpha_{a_{ij}}, \alpha_{b_i} \in [0,1]$ and $i, j = 1, 2, \cdots, n$. The real-valued function is

$$\begin{aligned}\Psi(t) =& \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^1\int_0^1\cdots\int_0^1 a_{ij}^{gr}(\mu, \alpha_{a_{ij}})t_it_j d\alpha_{a_{11}}\cdots d\alpha_{a_{nn}}d\mu \\ &- \sum_{i=1}^{n}\int_0^1\int_0^1\cdots\int_0^1 b_i^{gr}(\mu, \alpha_{b_i})t_i d\alpha_{b_1}\cdots d\alpha_{b_n}d\mu \\ =& \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^1(\overline{a_{ij}}(\mu) + \underline{a_{ij}}(\mu))d\mu\cdot t_it_j - \frac{1}{2}\sum_{i=1}^{n}\int_0^1(\overline{b_i}(\mu) + \underline{b_i}(\mu))d\mu\cdot t_i \\ =& \frac{1}{2}t^TAx - t^TB,\end{aligned}$$

where A is an $n\times n$ real symmetric matrix $[\frac{1}{2}\int_0^1(\overline{a_{ij}}(\mu) + \underline{a_{ij}}(\mu))d\mu]_{n\times n}$, B is an $n\times 1$ real matrix $[\frac{1}{2}\int_0^1(\overline{b_i}(\mu) + \underline{b_i}(\mu))d\mu]_{n\times 1}$. Since $[a_{ij}^{gr}(\mu, \alpha_{a_{ij}})]_{n\times n}$ is a positive definite matrix for any $\mu, \alpha_{a_{ij}} \in [0,1]$, we have

$$t^T[a_{ij}^{gr}(\mu, \alpha_{a_{ij}})]_{n\times n}t = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{gr}(\mu, \alpha_{a_{ij}})t_it_j > 0$$

for any $t \in R^n$ and $x \neq 0$. We can obtain

$$\int_0^1\int_0^1\cdots\int_0^1\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{gr}(\mu, \alpha_{a_{ij}})t_it_j d\alpha_{a_{11}}\cdots d\alpha_{a_{nn}}d\mu = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^1(\overline{a_{ij}}(\mu) + \underline{a_{ij}}(\mu))d\mu t_it_j > 0.$$

Thus, A is a positive real matrix. It follows from the classical conclusion that

$$(t^{(k+1)} - t^*)^TA(t^{(k+1)} - t^*) \leq (\frac{t_1 - t_n}{t_1 + t_n})^2(t^{(k)})^TA(t^{(k)}),$$

where t_1 and t_n are, respectively, the smallest and largest eigenvalues of A. Therefore, the steepest descent method is linearly convergent. □

Example 4.1. *We consider the following fuzzy optimization problem:*

$$\min \widetilde{F}(t), \quad t = (t_1, t_2) \in R^2.$$

Let

$$\widetilde{F}(t) = \widetilde{u}_1 t_1^2 + \widetilde{u}_2(t_2^2 - t_1 t_2) - \widetilde{u}_3 t_1,$$

where

$$\widetilde{u}_1 = (3, 8, 9), \ \widetilde{u}_2 = (3, 4, 5) \quad \text{and} \quad \widetilde{u}_3 = (-2, 8, 10)$$

are triangular fuzzy numbers. The horizontal membership function of $\widetilde{F}(t)$ is

$$\begin{aligned}\mathcal{H}(\widetilde{F}(t)) &= F^{gr}(t, \mu, \alpha_F) \\ &= [5\mu + 3 + 6(1-\mu)\alpha_{u_1}]t_1^2 + [3 + \mu + 2(1-\mu)\alpha_{u_2}](t_2^2 - t_1 t_2) - [10\mu - 2 + 12(1-\mu)\alpha_{u_3}]t_1,\end{aligned}$$

for any $\mu, \alpha_{u_1}, \alpha_{u_2}, \alpha_{u_3} \in [0, 1]$, where

$$\alpha_F \triangleq (\alpha_{u_1}, \alpha_{u_2}, \alpha_{u_3})$$

with regard to distinct fuzzy numbers $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$. Therefore,

$$\begin{aligned}\Psi(t) = &\int_0^1 \int_0^1 \int_0^1 F^{gr}(t, \mu, \alpha_F) \mathrm{d}\alpha_{u_1} \mathrm{d}\alpha_{u_2} \mathrm{d}\alpha_{u_3} \mathrm{d}\mu \\ = &\int_0^1 \int_0^1 \int_0^1 \int_0^1 \{[5\mu + 3 + 6(1-\mu)\alpha_{u_1}]t_1^2 + [3 + \mu + 2(1-\mu)\alpha_{u_2}](t_2^2 - t_1 t_2) \\ &- [10\mu - 2 + 12(1-\mu)\alpha_{u_3}]t_1\} \mathrm{d}\alpha_{u_1} \mathrm{d}\alpha_{u_2} \mathrm{d}\alpha_{u_3} \mathrm{d}\mu \\ = &7t_1^2 + 4(t_2^2 - t_1 t_2) - 6t_1.\end{aligned}$$

According to Algorithm 1, with the initial point

$$t^0 = (0, 1)$$

and the tolerance value 10^{-5}, the efficient solution of $\Psi(t)$ is obtained after 9 iterations as

$$\bar{t} = (0.50000, 0.25000).$$

Detailed iterations are provided in Table 1.

Table 1. Iteration data table and dates from MATELAB R2021b.

Iteration: k	iteration points: $t^{(k)}$	direction: $d^{(k)}$	step length: $\lambda^{(k)}$	functional value: $\Psi(t^{(k)})$
1	(0.642633,0.485893)	(-1.053292,-1.316614)	0.822972	-1.269592
2	(0.479054,0.281419)	(0.418923,-0.335138)	0.261855	-1.490348
3	(0.505975,0.259882)	(-0.044125,-0.055156)	0.034476	-1.499596
4	(0.499123,0.251316)	(0.017550,-0.014040)	0.010970	-1.499983
5	(0.500250,0.250414)	(-0.001848,-0.002311)	0.001444	-1.499999
6	(0.499963,0.250055)	(0.000735,-0.000588)	0.000460	-1.500000
7	(0.500010,0.250017)	(-0.000077,-0.000097)	0.000061	-1.500000
8	(0.499998,0.250002)	(0.000031,-0.000025)	0.000019	-1.500000
9	(0.500000,0.250001)	(-0.000003,-0.000004)	0.000003	-1.500000

We have from Theorem 4.1 that $\bar{t}$ is a local efficient solution of $\widetilde{F}(t)$. It follows from Definition 3.8 that the Hessian matrix for the horizontal membership function F^{gr} is

$$\nabla^2 F^{gr}(t,\mu,\alpha_F) = \begin{bmatrix} 6+10\mu+12(1-\mu)\alpha_{u_1} & -3-\mu-2(1-\mu)\alpha_{u_2} \\ -3-\mu-2(1-\mu)\alpha_{u_2} & 6+2\mu+4(1-\mu)\alpha_{u_2} \end{bmatrix}.$$

Thus,

$$\min_{\mu,\alpha_{u_1}\in[0,1]} \{6+10\mu+12(1-\mu)\alpha_{u_1}\} = 6 > 0$$

and

$$\min_{\mu,\alpha_{u_1},\alpha_{u_2}\in[0,1]} |\nabla^2 F^{gr}(t,\mu,\alpha_F)| = 27 > 0.$$

Then, $\nabla^2 F^{gr}(t,\mu,\lambda)$ is a positive definite matrix. According to Theorem 3.7, $\widetilde{F}(t)$ is a gr-convex fuzzy function. We have from Theorem 4.2 that $\bar{t}$ is a global efficient solution of $\widetilde{F}(t)$ (Figure 1 shows the variation of the functional value $\Psi(t^{(k)})$ with respect to the number of iterations K under the steepest descent method with the objective function

$$\widetilde{F}(t_1,t_2) = (3,8,9)t_1^2 + (3,4,5)(t_2^2 - t_1t_2) - (-2,8,10)t_1.$$

In the first few steps, the objective function decreases rapidly; but when approaching the efficient solution of $\widetilde{F}(t)$, the convergence is slow).

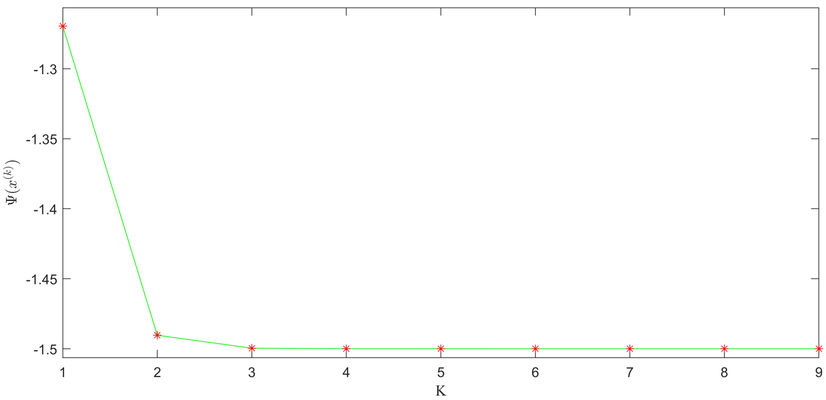

Figure 1. The iteration process of the steepest descent method with the objective function $\widetilde{F}(t_1,t_2) = (3,8,9)t_1^2 + (3,4,5)\,(t_2^2 - t_1t_2) - (-2,8,10)t_1$.

Furthermore, the horizontal membership function of $\widetilde{F}(t)$ at

$$\bar{t} = (0.50000, 0.25000)$$

is

$$\mathcal{H}(\widetilde{F}(\bar{t})) = \frac{1}{4}[5\mu+3+6(1-\mu)\alpha_{u_1}] - \frac{1}{16}[3+\mu+2(1-\mu)\alpha_{u_2}] - \frac{1}{2}[10\mu-2+12(1-\mu)\alpha_{u_3}].$$

According to Remark 2.2, we can obtain

$$\widetilde{F}(\bar{t}) = (-\frac{73}{16}, -\frac{9}{4}, \frac{49}{16}).$$

5. Conclusions

We define granular differentiability of fuzzy mappings from R^n to E, provide the characterization of differentiability for the horizontal membership function, and discuss the granular convexity of fuzzy mappings. Granular differentiability and convexity overcome limitations of the Hukuhara difference, resulting in improved convergence properties and efficiency. Based on $\Psi(t)$, we have proved that the proposed method achieves linear convergence for quadratic fuzzy optimization problems, thereby ensuring both reliability and efficiency in practical implementations. The numerical experiments validate the theoretical findings and demonstrate rapid convergence of the algorithm. The research in this paper not only enriches the fuzzy optimization theory, but also provides new ideas and technical means for fuzzy numerical computation. Furthermore, the paper provides a robust tool for solving optimization problems in environments with imprecise data, offering potential applications in engineering and economics. In the future, we will apply this method in multi-objective, multi-constraint, and non-convex fuzzy optimization problems and explore its potential for wide application in practical problems. Additionally, based on both our current work and the foundation established in [28], we will explore more advanced optimization algorithms for fuzzy optimization problems.

Author contributions

Shexiang Hai: conceptualization, funding acquisition, project administration, resources, supervision, validation, formal analysis, writing-review and editing; Liang He: conceptualization, data curation, investigation, methodology, software, validation, writing-original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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