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Research article

Topological pressures of a factor map for iterated function systems

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Abstract: In this manuscript, we mainly investigate the topological pressure for iterated function systems on a compact metric space defined by Wang and Liao [Dynam. Syst., 2021, 36(3): 483–506]. Given a factor map, we establish a formula of topological pressure of a factor map, which generalizes Bowen's inequality in [Trans. Amer. Math. Soc., 1971, 153: 401–414.] to iterated function systems. Consequently, we further study the power rule of a topological pressure for iterated function systems.

Keywords: iterated function system; topological pressure; Biś's topological entropy; factor map
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1. Introduction

Let (X, T) be a topological dynamical system (TDS for short), where X is a compact metric space with metric d and T is a continuous map from X to itself. As we know, the topological entropy and its generalization called the topological pressure play important roles in the field of dynamical systems.

In 1971, Bowen [3] considered a factor map $\phi : (X, T) \to (Y, S)$ between two TDSs, and proved that

$$h_{top}(T) \leq h_{top}(S) + \sup_{y\in Y} h_{top}\left(T, \phi^{-1}(y)\right), \tag{1.1}$$

where $h_{top}(T, K)$ is the topological entropy of a compact subset $K \subset X$ [3]. Topological pressure was firstly introduced by Ruelle [21]. Later on, other definitions of topological pressure, via open covers and spanning sets, were proposed by Walters [23], and they were further explored by Pesin and Pitskel [20]. Pesin [19] utilized Carathéodory structures to give a dimensional definition for topological pressure. Recently, there have been generalizations of topological pressure for other systems, e.g., [11] for non-autonomous discrete dynamical systems and [16] for free semigroup actions.

Recently, Fang et al. [7] and Oprocha [18] further considered the topological entropy of subsets, and they extended the inequality (1.1) to the topological entropy for non-compact subsets of a factor map. Later on, a variety of versions of inequality (1.1) were established for different systems. Li et al. [13] generalized the results in [7, 18] to topological pressure, and established formulas for the topological pressure of non-compact subsets of a factor map. Some applications of the inequality can be seen in [8, 13]. Recently, Zhao et al. [28] proved an inequality of packing pressure of a factor map. Zhao et al. [29] gave an inequality of topological pressure under free semigroup action of a factor map. Liu et al. [15] established an inequality of Pesin-Pitskel topological pressure of a factor map for nonautonomous dynamical systems. For some more details about related concepts of entropy and pressure, see [4, 17, 19].

As it is well known, several versions of the topological entropy under free semigroup actions have been proposed by Biś [25], Bufetov [5], and Wang et al. [25]. Subsequently, analogous to the classical topological pressure, many scholars proposed different versions of topological pressure under free semigroup actions, which are natural generalizations of topological entropies under free semigroup actions. Motivated by Bufetov's entropy [5], Lin et al. [14] extended the notion of Bufetov's entropy to the concept of topological pressure of free semigroup actions (here we remark it as *pressure from [14]*), and then a partial variational principle for free semigroup actions was established. Furthermore, the opposite inequality was proved by Carvalho et al. [6], and the complete variational principle for free semigroup was obtained. Recently, Wang et al. [24] generalized the topological entropy in [25] to an another version of topological pressure of free semigroup actions (we remark it as *pressure from [24]*), and they also established a partial variational principle for this version of pressure of free semigroup actions. At the same time, analogous to the topological pressure given by the Carathéodory structure, which was given in Pesin's work [19]. Ma et al. [16] used the Carathéodory structure to propose the notions of topological pressure and topological entropy under free semigroup actions. Ju et al. [12] introduced the notions of the topological entropy and lower and upper capacity topological entropies of free semigroup actions on non-compact subsets, which generalized the concept of the topological entropy of free semigroup actions defined by Bufetov [5]. One can see some recent relevant results under free semigroup actions, such as for topological entropy [10, 16], for topologica pressure [26, 27, 30] and for inverse pressure [2].

In this paper, we mainly investigate the *pressure from [24]* of free semigroup actions with m generators. Let $\mathscr{F} = \{f_0, f_1, \ldots, f_{m-1}\}$ be a family of continuous self-maps on a compact metric space (X, d). The iterated function system $(X, \mathscr{F})$ (IFS for short) is the action of the semigroup generated by $\{f_0, f_1, \ldots, f_{m-1}\}$ on a compact metric space (X, d). Following the works of [3, 24, 29], notice that the authors [29] have investigated the *pressure from [14]* of a factor map for IFSs, while in the present paper, we shall continue this work with respect to *pressure from [24]*. More precisely, let $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ be two iterated function systems, where $\mathscr{F} = \{f_0, f_1, \ldots, f_{m-1}\}$, $\mathscr{G} = \{g_0, g_1, \ldots, g_{m-1}\}$. If there is a surjective and continuous map $\phi : X \to Y$ such that $\phi \circ f_i = g_i \circ \phi$ for any $0 \leq i \leq m-1$, then we say that $(Y, \mathscr{G})$ is a factor system of $(X, \mathscr{F})$ and ϕ is a factor map from $(X, \mathscr{F})$ to $(Y, \mathscr{G})$. Moreover, when ϕ is a homeomorphism, we say that $(X, \mathscr{F})$ is conjugate to $(Y, \mathscr{G})$. Given a factor map $\phi : X \to Y$, we establish an inequality for *pressure from [24]* of a factor map, which generalizes results in [3] to *pressure from [24]* for IFSs, and this inequality is different from the result in [29].

Besides, we further investigate the power rule of *pressure from [24]*. For an IFS $(X, \mathscr{F})$, denote $\mathscr{F}^k = \{h_1 \circ h_2 \circ \cdots \circ h_k : h_1, h_2, \ldots, h_k \in \mathscr{F}\}$. Then we explore the features associated with *pressure from [24]* of $(X, \mathscr{F}^k)$. We also derive a power rule formula for *pressure from [24]*, which extends the result in [22] to *pressure from [24]* for IFSs.

This manuscript is structured as follows. In Section 2, we recall several concepts of topological pressure of IFSs. In Section 3, we shall investigate the topological pressure of a factor map. In Section 4, we explore the power rule of a topological pressure.

2. Preliminaries

Let $C(X,\mathbb{R})$ denote the collection of all real-valued continuous functions of X equipped with the supremum norm. The sets of natural, nonnegative integers and real numbers are represented by $\mathbb{N}$, $\mathbb{Z}^+$, and $\mathbb{R}$, respectively. We adopt the notation $\#(\cdot)$ to represent the cardinality of a finite set.

In this section, we recall several versions of the notions of topological pressures for free semigroup actions given in [14, 24]. Denote by F_m^+ the set of all finite words of symbols $0, 1, \ldots, m-1$. For every $v \in F_m^+, |v|$ denotes the length of v. If $u, v \in F_m^+$, let vu be the word concatenation of v and u. It is obvious that F_m^+ associated with this concatenation is a free semigroup with m generators. We remark that $u \prec v$ if there is a word ρ with $v = u\rho$, furthermore, we remark that $u \preccurlyeq v$ if $u \prec v$ or $u = v$. Let $f_u = f_{u_{k-1}} \circ f_{u_{k-2}} \circ \cdots \circ f_{u_0}$ and $f_u^{-1} = f_{u_0}^{-1} \circ f_{u_1}^{-1} \circ \cdots \circ f_{u_{k-1}}^{-1}$ where $u = u_0u_1 \cdots u_{k-1} \in \{0, 1, \ldots, m-1\}^k$ and $f_{u_i}^{-1}$ is the preimage operator of f_{u_i} $(i = 0, \cdots, k-1)$.

Let $(X, \mathscr{F})$ be an IFS, $u \in F_m^+$. The max metric on X is given by $d_u(a, b) = \max_{v<u} d\left(f_v(a), f_v(b)\right)$. For any $\epsilon > 0$, we say that a subset $E \subset X$ is an $(\mathscr{F}, u, \epsilon)$-spanning set for X, if for any $a \in X$, there exists $b \in E$ with $d_u(a, b) < \epsilon$. For any $\epsilon > 0$, we say a subset $F \subset X$ is an $(\mathscr{F}, u, \epsilon)$-separated set of X, if for any $a, b \in F, a \neq b$, one has $d_u(a, b) \geq \epsilon$.

Given $n \in \mathbb{N}, u = u_0u_1 \cdots u_{n-1} \in F_m^+, |u| = n$, for $i \in \{0, \cdots, n-1\}$, let $f_{u|_i} = f_{u_{i-1}} \circ f_{u_{i-2}} \circ \cdots \circ f_{u_0}$, where $f_{u|_0} = id$. For any $\Psi \in C(X;\mathbb{R})$, any $u \in F_m^+$, $|u| = n, x \in X$, denote

$$S_{u,n}\Psi(x) := \sum_{i=0}^{n-1} \Psi\left(f_{u|_i}(x)\right).$$

Now, we recall the notions of the topological pressure given in [14, 24].

2.1. Topological pressure of IFSs

Inspired by Bufetov's entropy [5], Lin et al. [14] introduced the topological pressure of IFSs via spanning sets and separated sets. They obtained the equivalence of the definitions of topological pressure of IFSs, by using spanning sets or separated sets.

Definition 2.1. *[14] Given $\Psi \in C(X;\mathbb{R})$, define*

$$P_n(\Psi, \mathscr{F}, \epsilon) = \frac{1}{m^n} \sum_{|u|=n} \inf_{E_{u,n}} \left\{ \sum_{x \in E_{u,n}} e^{S_{u,n}\Psi(x)} : E_{u,n} \text{ is an } (\mathscr{F}, u, \epsilon)\text{-spanning set for } X \right\}.$$

The pressure from [14] *of IFSs is given by*

$$P(\Psi, \mathscr{F}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(\Psi, \mathscr{F}, \epsilon).$$

Definition 2.2. *[14] Given $\Psi \in C(X;\mathbb{R})$, define*

$$Q_n(\Psi, \mathscr{F}, \epsilon) = \frac{1}{m^n} \sum_{|u|=n} \sup_{F_{u,n}} \left\{ \sum_{x \in F_{u,n}} e^{S_{u,n}\Psi(x)} : F_{u,n} \text{ is an } (\mathscr{F}, u, \epsilon)\text{-separated set of } X \right\}.$$

Proposition 2.3. *[14] Let $(X, \mathscr{F})$ be an IFS, one has*

$$P(\Psi, \mathscr{F}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Q_n(\Psi, \mathscr{F}, \epsilon).$$

Remark 2.4. *When $\Psi = 0$, we remark that $h_B(\mathscr{F}) = P(0, \mathscr{F})$, where $h_B(\mathscr{F})$ is Bufetov's entropy [5]. Clearly, the notion of pressure from [14] of IFSs is still valid for any $\mathcal{F}$-subinvariant compact subset $K \subset X$.*

Later, Wang et al. [24] introduced another version of the topological pressure of IFSs, which is different from the notion of topological pressure given in [14].

Definition 2.5. *[24] For any $\Psi \in C(X; \mathbb{R})$, let*

$$P_n^W(\Psi, \mathscr{F}, \epsilon) = \frac{1}{m^n}\sum_{|u|=n}\log\inf_{E_{u,n}}\left\{\sum_{x\in E_{u,n}} e^{S_{u,n}\Psi(x)} : E_{u,n} \text{ is an } (\mathscr{F}, u, \epsilon)\text{-spanning set for } X\right\}.$$

The pressure from [24] of IFSs is given by

$$P^W(\Psi, \mathscr{F}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}P_n^W(\Psi, \mathscr{F}, \epsilon).$$

Similarly, Wang et al. [24] also obtained the equivalence for pressure from [24] of IFSs between the notions via spanning sets and separated sets.

Definition 2.6. *[24] Given $\Psi \in C(X; \mathbb{R})$, define*

$$Q_n^W(\Psi, \mathscr{F}, \epsilon) = \frac{1}{m^n}\sum_{|u|=n}\log\sup_{F_{u,n}}\left\{\sum_{x\in F_{u,n}} e^{S_{u,n}\Psi(x)} : F_{u,n} \text{ is an } (\mathscr{F}, u, \epsilon)\text{-separated set of } X\right\}.$$

Proposition 2.7. *[14] Let $(X, \mathscr{F})$ be an IFS, we have*

$$P^W(\Psi, \mathscr{F}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}Q_n^W(\Psi, \mathscr{F}, \epsilon).$$

Remark 2.8. *(1) It is clear that, the notion of pressure from [24] of IFSs still holds for any $\mathcal{F}$-subinvariant compact subset $K \subset X$ (i.e. $f(K) \subset K$ for all $f \in \mathcal{F}$).*

(2) It is not hard to check that $P^W(\Psi, \mathscr{F}) \le P(\Psi, \mathscr{F})$ [24].

(3) When $\Psi = 0$, we remark that $h^W(\mathscr{F}) = P^W(0, \mathscr{F})$, where $h^W(\mathscr{F})$ is the topological entropy given in [25], and $h^W(\mathscr{F})$ can also be presented by open covers [25]. More precisely, let C_X and C_X^o be the set of finite covers and finite open covers, respectively. For $n \in \mathbb{N}$ and $\mathscr{U}_1, \mathscr{U}_2, \dots, \mathscr{U}_n \in C_X$, we denote

$$\bigvee_{i=1}^{n}\mathscr{U}_i = \{U_1 \cap U_2 \cap \cdots \cap U_n : U_i \in \mathscr{U}_i, i \in \{0, \cdots, n-1\}\}.$$

For any non-empty subset $E \subset X$ and $\mathscr{U} \in C_X$, let $\mathscr{N}(\mathscr{U} \mid E)$ be the minimum among the cardinalities of the subsets of $\mathscr{U}$ that covers E, and we simply write $\mathscr{N}(\mathscr{U} \mid X)$ as $\mathscr{N}(\mathscr{U})$, define by

$$h_{top}(\mathscr{F}, \mathscr{U}) = \limsup_{n\to\infty}\frac{1}{n}\left[\frac{1}{m^n}\sum_{|u|=n}\log\mathscr{N}\left(\bigvee_{v\preccurlyeq u} f_v^{-1}\mathscr{U}\right)\right] \tag{2.1}$$

the topological entropy of $\mathscr{F}$ on the cover $\mathscr{U}$. The topological entropy of $\mathscr{F}$ is given by

$$h_{top}(\mathscr{F}) = \sup\left\{h_{top}\ (\mathscr{F}, \mathscr{U}) : \mathscr{U} \in C_X^o\right\}.$$

Following the idea of classical topological entropy [23], the authors [25] stated that $h_{top}(\mathscr{F}) = h^W(\mathscr{F})$, while, we point out that "$\limsup$*" can also be replaced by "*$\lim$*" in (2.1), and we put the statement in Appendix A.3.*

3. Topological pressure of a factor map

In this section, we study the topological pressure of a factor map, and several inequalities are established.

Theorem 3.1. *Let $\phi : (X, \mathscr{F}) \to (Y, \mathscr{G})$ be a factor map between two IFSs, and $\Psi \in C(Y; \mathbb{R})$. Then*

$$P^W(\Psi, \mathscr{G}) \leq P^W(\Psi \circ \phi, \mathscr{F}).$$

Proof. As $\phi : X \to Y$ is continuous, then for $\epsilon > 0$, there are $\tau > 0$ and $0 < \tau \leq \epsilon$ such that $\rho(\phi(x), \phi(y)) > \epsilon$ implies that $d(x, y) > \tau$. Given $u \in F_m^+$, we select an $(\mathscr{G}, u, \epsilon)$-separated set $Z_{u,n}(Y) \subset Y$. As ϕ is a surjective map, we can take $Z_{u,n}(X)$, and $Z_{u,n}(X)$ includes only one point from any $\phi^{-1}(z), z \in Z_{u,n}(Y)$, and does not contain any other points. As $Z_{u,n}(Y)$ is a $(\mathscr{G}, u, \epsilon)$-separated set, then we get that for $y_1 \neq y_2 \in Z_{u,n}(Y)$, $\rho_u(y_1, y_2) > \epsilon$. By definition of $Z_{u,n}(X)$, there exist $x_1, x_2 \in Z_{u,n}(X)$ such that $\phi(x_1) = y_1$, $\phi(x_2) = y_2$. This indicates that

$$d_u(x_1, x_2) > \tau.$$

Thus, $Z_{u,n}(X)$ is a $(u, \tau, \mathscr{F})$-separated set of X. Hence,

$$\begin{aligned}\sum_{y\in Z_{u,n}(Y)} e^{S_{u,n}\Psi(y)} &= \sum_{y\in\phi(Z_{u,n}(X))} e^{S_{u,n}\Psi(y)} = \sum_{x\in Z_{u,n}(X)} e^{S_{u,n}\Psi\circ\phi(x)} \\ &\leq \sup_{F_{u,n}(X)}\left\{\sum_{x\in Z_{u,n}(X)} e^{S_{u,n}\Psi\circ\phi(x)} : Z_{u,n}(X) \text{ is an } (\mathscr{F}, u, \tau)\text{-separated set of } X\right\}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\sup_{Z_{u,n}(Y)}\left\{\sum_{y\in Z_{u,n}(Y)} e^{S_{u,n}\Psi(y)} : Z_{u,n}(Y) \text{ is an } (\mathscr{G}, u, \epsilon)\text{-separated set of } Y\right\} \\ \leq\ &\sup_{Z_{u,n}(X)}\left\{\sum_{x\in Z_{u,n}(X)} e^{S_{u,n}\Psi\circ\phi(x)} : Z_{u,n}(X) \text{ is an } (\mathscr{F}, u, \tau)\text{-separated set of } X\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{m^n}\sum_{|u|=n}\log\sup_{Z_{u,n}(Y)}\left\{\sum_{y\in Z_{u,n}(Y)} e^{S_{u,n}\Psi(y)} : Z_{u,n}(Y) \text{ is an } (\mathscr{G}, u, \epsilon)\text{-separated set of } Y\right\} \\ \leq\ &\frac{1}{m^n}\sum_{|u|=n}\log\sup_{Z_{u,n}(X)}\left\{\sum_{x\in Z_{u,n}(X)} e^{S_{u,n}\Psi\circ\phi(x)} : Z_{u,n}(X) \text{ is an } (\mathscr{F}, u, \tau)\text{-separated set of } X\right\},\end{aligned}$$

and then $P^W(\Psi, \mathscr{G}, \epsilon) \leq P^W(\Psi \circ \phi, \mathscr{F}, \tau)$. This indicates that $P^W(\Psi, \mathscr{G}) \leq P^W(\Psi \circ \phi, \mathscr{F})$. □

As an immediate conclusion, we have.

Corollary 3.2. *Let $\phi : (X, \mathscr{F}) \to (Y, \mathscr{G})$ be a topological conjugacy between two IFSs, and $\Psi \in C(Y; \mathbb{R})$. Then*

$$P^W(\Psi, \mathscr{G}) = P^W(\Psi \circ \phi, \mathscr{F}).$$

Following the ideas given by Ghys, Langevin, and Walczak [9], in 2004 Biś [1] gave a concept of topological entropy of free semigroup actions. Let (X, d) be a compact metric space, $\mathscr{F} = \{f_0, \ldots, f_{m-1}\}$, $f_0, \ldots, f_{m-1}$ be continuous maps from X to itself, and $\mathscr{F}^n = \{g_1 \circ g_2 \circ \ldots \circ g_n : g_1, \ldots, g_n \in \mathscr{F}\}$. Let $K \subset X$, we say that $Z \subset X$ is an (n, ϵ)-spanning set of K, if for each $a \in K$, there is $b \in Z$ with

$$d_n(a, b) := \max\{d(f(a), f(b)) : f \in \mathscr{F}^n\} \le \epsilon.$$

Denote

$$s_n(\epsilon, K) = \max\{\#(Z) : Z \text{ is an } (n, \epsilon)\text{-spanning subset of } K\}.$$

Definition 3.3. *[1] The Biś's topological entropy is given by*

$$h_{Biś}(X, \mathscr{F}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, X).$$

Notice that, Biś's topological entropy is not lower than others [5, 25], i.e., $h^W(\mathscr{F}) \le h_B(\mathscr{F}) \le h_{\text{Biś}}(X, \mathscr{F} \cup \{id\})$.

Analogous to the topological entropy of a subset given by Bowen [4], the notion of Biś's topological entropy can be extended to a subset as follows. Let $K \subset X$; the Biś's topological entropy of K is given by

$$h_{\text{Biś}}(K, \mathscr{F}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, K).$$

By using Biś's topological entropy, we establish an inequality about *pressure from [24]* of a factor map for IFSs, which is similar to Bowen's inequality [3].

Theorem 3.4. *Let $\phi : (X, \mathscr{F}) \to (Y, \mathscr{G})$ be a factor map between two IFSs, and $\Psi \in C(Y; \mathbb{R})$. Then*

$$P^W(\Psi \circ \phi, \mathscr{F}) \le P^W(\Psi, \mathscr{G}) + \sup_{y \in Y} h_{Biś}\left(\phi^{-1}(y), \mathscr{F} \cup \{id\}\right).$$

Especially, if $P^W(\Psi, \mathscr{G}) < +\infty$, then

$$P^W(\Psi \circ \phi, \mathscr{F}) - P^W(\Psi, \mathscr{G}) \le \sup_{y \in Y} h_{Biś}\left(\phi^{-1}(y), \mathscr{F} \cup \{id\}\right).$$

Proof. For any $\tau > 0$, denote

$$\text{Var}(\Psi, \tau) = \sup\{|\Psi(x) - \Psi(y)| : d(x, y) < \tau, x, y \in Y\}.$$

Assume that $\beta = \sup_{y \in Y} h_{\text{Biś}}\left(\phi^{-1}(y), \mathscr{F} \cup \{id\}\right) < \infty$, otherwise, there is nothing to prove. For $\epsilon > 0, n \in \mathbb{N}$, and each $y \in Y$, let $F_y \subset \phi^{-1}(y)$ be an (n, ϵ)-spanning set of $\phi^{-1}(y)$. By the definition of $h_{\text{Biś}}\left(\phi^{-1}(y), \mathscr{F} \cup \{id\}\right)$, one has $h_{\text{Biś}}\left(\phi^{-1}(y), \mathscr{F} \cup \{id\}\right) \geq h_{\text{Biś}}\left(\phi^{-1}(y), \mathscr{F} \cup \{id\}, \epsilon\right)$ where $h_{\text{Biś}}\left(\phi^{-1}(y), \mathscr{F} \cup \{id\}, \epsilon\right) = \limsup_{n\to\infty} \frac{1}{n} \log s_n\left(\epsilon, \phi^{-1}(y)\right)$.

For above $\epsilon > 0$. Take any $\zeta > 0$. Then for every $y \in Y$, there is $m(y)$ such that

$$\frac{\log M\left(m(y), \epsilon, \mathscr{F}, \phi^{-1}(y)\right)}{m(y)} \leq h_{\text{Biś}}\left(\phi^{-1}(y), \mathscr{F} \cup \{id\}, \epsilon\right) + \zeta \leq \beta + \zeta, \tag{3.1}$$

where $M\left(m(y), \epsilon, \mathscr{F}, \phi^{-1}(y)\right) = \#(F_y)$, and F_y is the minimal cardinality $(m(y), \epsilon)$-spanning set of $\phi^{-1}(y)$.

Next, for $u \in F_m^+$, we define

$$D_n(\mathscr{F}, u, z, 2\epsilon) := \{c \in X : d_u(c, z) < 2\epsilon\},$$

where $|u| = n$ and $d_u(c, z) = \max_{v<u} d\left(f_v(c), f_v(z)\right)$. Since F_y is an $(m(y), \epsilon)$-spanning set of $\phi^{-1}(y)$, this indicates that F_y is also an $(\mathscr{F}, u, \epsilon)$-spanning set for $\phi^{-1}(y)$ for every $u \in F_m^+$ with $|u| = m(y)$. Here we remark that $F_y^u := F_y$. Hence, $U_y = \bigcup_{z\in F_y^u} D_{m(y)}(\mathscr{F}, u, z, 2\epsilon) \supset \phi^{-1}(y)$.

For every $y \in Y$, we have $\left(X \backslash U_y\right) \cap \bigcap_{r>0} \phi^{-1}\left(\overline{B_\rho(y, r)}\right) = \emptyset$, where $B_\rho(y, r) = \{z \in Y : \rho(z, y) < r\}$. By the finite intersection property, there is $W_y = B_\rho(y, r)$ such that $\phi^{-1}\left(W_y\right) \subset U_y$. Let $\left\{W_{y_1}, W_{y_2}, \ldots, W_{y_p}\right\}$ covers Y and δ be its Lebesgue number under metric ρ.

Based on the notion of $P^W(\Psi, \mathscr{G})$, then there is an $(\mathscr{G}, u, \delta)$-spanning set $E_{u,n}$ of Y such that

$$\frac{1}{n}\left[\frac{1}{m^n} \sum_{|u|=n} \log \sum_{y\in E_{u,n}} e^{S_{u,n}\Psi(y)}\right] < P^W(\mathscr{G}, \Psi). \tag{3.2}$$

Suppose above δ is small enough such that for every $i \in [1, p]$, $u = |m(y_i)|$, we have

$$\operatorname{Var}\left(S_{u,m(y_i)}\Psi, \delta\right) < \epsilon. \tag{3.3}$$

For every $y \in E_{u,n}, 0 \leq j \leq n-1$, take $c_j(y) \in \left\{y_1, \ldots, y_p\right\}$ such that

$$\overline{B_\delta\left(g_{u|_j}(y)\right)} = \left\{z \in Y : \rho\left(g_{u|_j}(y), z\right) \leq \delta\right\} \subset W_{c_j(y)},$$

where $g_{u|_j}(y) = g_{u_j} \circ \cdots \circ g_{u_0}, u = u_0 \cdots u_j \cdots u_{n-1}$.

Recursively, define $t_0(y) = 0$ and $t_{s+1}(y) = t_s(y) + m\left(c_{t_s(y)}(y)\right)$ until one gets $t_{k(y)+1}(y) \geq n$, and take $k(y) = k$. For $y \in E_{u,n}, x_0 \in F^u_{c_{t_0(y)}(y)}, \ldots, x_k \in F^u_{c_{t_k(y)}(y)}$, denote

$$\begin{aligned}&V(y; x_0, \ldots, x_k)\\ :=& \left\{x \in X : d\left(f_{u|_{t+t_s}}(x), f_{u|_t}(x_s)\right) < 2\epsilon, \forall t \in [0, m(c_{t_s(y)}(y) - 1] \text{ and } s \in [0, k(y)]\right\}.\end{aligned}$$

Claim. (I) Denote $\mathscr{V} = \left\{V(y; x_0, \ldots, x_s) : y \in E_{u,n}, x_s \in F^u_{c_{t_s(y)}(y)}, s \in [0, k(y)]\right\}$, then $\mathscr{V}$ covers X. (II) For every $u \in F_m^+$, with $|u| = n$, then any $(\mathscr{F}, u, 4\epsilon)$-separated set intersects each element of $\mathscr{V}$ in at most one point.

Proof of Claim. (I). Given $x \in X$, as $E_{u,n}$ is an $(\mathscr{G}, u, \delta)$-spanning set of Y, then there is a $y \in E$ with

$\rho_u(y,\phi(x)) = \max_{v<u}\rho\left(g_{u|_j}(y), g_{u|_j}(\phi(x))\right) < \delta$ for each $j \in [0, n-1]$. For every $s \in [0, k(y)]$, we have $\phi \circ f_{u|_{t_s(y)}}(x) = g_{u|_{t_s(y)}}(\phi(x)) \in W_{c_s(y)}$. Hence, there exists $x_s \in F^u_{c_{t_s(y)}(y)}$ such that $d\left(f_{u|_{t+t_s(y)}}(x), f_{u|_t}(x_s)\right) < 2\epsilon$ for every $t \in [0, m(c_{t_s(y)}(y)-1]$ and $s \in [0, k(y)]$. This indicates that $x \in V(y; x_0, \ldots, x_k)$. We finish the proof of *Claim (I)*.

(II). For every $z, c \in V(y; x_0, \ldots, x_k)$, every $t \in [0, m(c_{t_s(y)}(y)-1]$ and $s \in [0, k(y)]$, one has

$$\begin{aligned} & d\left(f_{u|_{t+t_s(y)}}(z),\ f_{u|_{t+t_s(y)}}(c)\right) \\ \le\ & d\left(f_{u|_{t+t_s(y)}}(z), f_{u|_t}(x_s)\right) + d\left(f_{u|_{t+t_s(y)}}(x_s), f_{u|_{t+t_s(y)}}(z), f_{u|_t}(c)\right) \\ <\ & 4\epsilon. \end{aligned}$$

We finish the proof of *Claim (II)*.

For any $(\mathscr{F}, u, 4\epsilon)$-separated set $H_{u,n}$ of X, we are to estimate the upper bound of $\sum_{x\in H_{u,n}} e^{S_{u,n}\Psi\circ\phi(x)}$. For every $y \in E_{u,n}$, denote $\mathscr{V}_y := \left\{V(y; x_0, \ldots, x_k) : x_i \in F^u_{c_{t_i(y)}(y)}, \forall i \in [0, k]\right\}$. Set $\mathscr{V} = \cup_{y\in E_{u,n}}\mathscr{V}_y$, where $\#\left(\mathscr{V}_y\right) = \prod_{i=0}^{k=k(y)} M\left(m\left(c_{t_i(y)}(y)\right), \epsilon, \mathscr{F}, \phi^{-1}(y)\right)$. By (3.1), we have

$$\#\left(\mathscr{V}_y\right) \le \exp((n+A)(\beta+\zeta)), \tag{3.4}$$

where $A = \max\{m(y_1), \cdots, m(y_p)\}$.

For every $x \in H_{u,n}$, there exist $y \in E_{u,n}$, $x_0 \in F^u_{c_{t_1(y)}(y)}, \ldots, x_k \in F^u_{c_{t_k(y)}(y)}$ with $x \in V(y; x_0, \ldots, x_k)$. Thus

$$\begin{aligned} S_{u,n}\Psi\circ\phi(x) &= \sum_{r=0}^{n-1}\Psi\left(\phi\circ f_{u|_r}(x)\right) = \sum_{i=0}^{k}\sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1}\Psi\left(\phi\circ f_{u|_r}(x)\right) \\ &= \sum_{i=0}^{k}\sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1}\left(\Psi\left(\phi\circ f_{u|_r}(x_i)\right) + \left(\Psi\left(\phi\circ f_{u|_r}(x)\right) - \Psi\left(\phi\circ f_{u|_r}(x_i)\right)\right)\right) \\ &= \sum_{i=0}^{k}\sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1}\Psi\left(\phi\circ f_{u|_r}(x_i)\right) \\ &\quad + \sum_{i=0}^{k}\sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1}\left(\Psi\left(\phi\circ f_{u|_r}(x)\right) - \Psi\left(\phi\circ f_{u|_r}(x_i)\right)\right). \end{aligned}$$

Notice that for any $r \in [0, m\left(c_{t_i(y)}(y)\right) - 1]$ and $i \in [0, k]$, one has

$$\Psi\left(\phi\circ f_{u|_r}(x)\right) - \Psi\left(\phi\circ f_{u|_r}(x_i)\right) \le \mathrm{Var}(\Psi\circ\phi, 2\epsilon).$$

This indicates that

$$\begin{aligned} S_{u,n}\Psi\circ\phi(x) &\le \sum_{i=0}^{k}\sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1}\Psi\left(\phi\circ f_{u|_r}(x_i)\right) + n\,\mathrm{Var}(\Psi\circ\phi, 2\epsilon) \\ &= \sum_{i=0}^{k} S_{u,m\left(c_{t_i(y)}(y)\right)}\Psi\circ\phi(x_i) + n\,\mathrm{Var}(\Psi\circ\phi, 2\epsilon). \end{aligned}$$

Combining the above arguments, for $i \in [0, k(y)]$, we have

$$
\begin{aligned}
S_{u,m\left(c_{t_i(y)}(y)\right)}\Psi \circ (\phi(x_i)) &= \sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1} \Psi\left(\phi \circ f_{u|_r}(x_i)\right) \\
&= \sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1} \Psi\left(g_{u|_r}(y)\right) + \sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1} \left(\Psi\left(\phi \circ f_{u|_r}(x_i)\right) - \Psi\left(\phi \circ f_{u|_r}(x)\right)\right) \\
&\quad + \sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1} \left(\Psi\left(\phi \circ f_{u|_r}(x)\right) - \Psi\left(g_{u|_r}(y)\right)\right) \\
&= \sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1} \Psi\left(g_{u|_r}(y)\right) + \sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1} \left(\Psi\left(\phi \circ f_{u|_r}(x_i)\right) - \Psi\left(\phi \circ f_{u|_r}(x)\right)\right) \\
&\quad + \sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1} \left(\Psi\left(g_{u|_r} \circ \phi(x)\right) - \Psi\left(g_{u|_r}(y)\right)\right) \\
&\le \sum_{r=0}^{m\left(c_{t_i(y)}(y)\right)-1} \Psi\left(g_{u|_r}(y)\right) + m\left(c_{t_i(y)}(y)\right) \operatorname{Var}(\Psi \circ \phi, 2\epsilon) + m\left(c_{t_i(y)}(y)\right)\epsilon \\
&= S_{u,m\left(c_{t_i(y)}(y)\right)}\Psi\left(g_{u|_{t_i(y)}}(y)\right) + m\left(c_{t_i(y)}(y)\right) \operatorname{Var}(\Psi \circ \phi, 2\epsilon) + m\left(c_{t_i(y)}(y)\right)\epsilon.
\end{aligned}
$$

Hence, we deduce that

$$
\begin{aligned}
&S_{u,n}\Psi \circ \phi(x) - n \operatorname{Var}(\Psi \circ \phi, 2\epsilon) \\
&\le \sum_{i=0}^{k} \left(S_{u,m\left(c_{t_i(y)}(y)\right)}\Psi\left(g_{u|_{t_i(y)}}(y)\right) + m\left(c_{t_i(y)}(y)\right) \operatorname{Var}(\Psi \circ \phi, 2\epsilon) + m\left(c_{t_i(y)}(y)\right)\epsilon\right),
\end{aligned}
$$

and then

$$
S_{u,n}\Psi \circ \phi(x) \le S_{u,n}\Psi(y) + 2n \operatorname{Var}(\Psi \circ \phi, 2\epsilon) + n\epsilon. \tag{3.5}
$$

It follows that

$$
\sum_{x \in H_{u,n}} e^{S_{u,n}\Psi\circ\phi(x)} \le \sum_{y \in E_{u,n}} \#\left(\mathscr{V}_y\right) \exp\left(S_{u,n}\Psi(y) + 2n \operatorname{Var}(\Psi \circ \phi, 2\epsilon) + n\epsilon\right). \tag{3.6}
$$

Combining the above arguments, we obtain

$$
\begin{aligned}
&\frac{1}{m^n} \sum_{|u|=n} \log \sum_{x \in H_{u,n}} e^{S_{u,n}\Psi\circ\phi(x)} \\
&\le \frac{1}{m^n} \sum_{|u|=n} \log \sum_{y \in E_{u,n}} \#\left(\mathscr{V}_y\right) \exp\left(S_{u,n}\Psi(y) + 2n \operatorname{Var}(\Psi \circ \phi, 2\epsilon) + n\epsilon\right) \\
&\le \frac{1}{m^n} \sum_{|u|=n} \log \sum_{y \in E_{u,n}} \exp((n + A)(\beta + \zeta)) \cdot \exp\left(S_{u,n}\Psi(y) + 2n \operatorname{Var}(\Psi \circ \phi, 2\epsilon) + n\epsilon\right).
\end{aligned}
$$

Hence, we deduce that

$$\frac{1}{m^n}\sum_{|u|=n}\log\sup_{H_{u,n}}\left\{\sum_{x\in H_{u,n}} e^{S_{u,n}\Psi\circ\phi(x)} : H_{u,n} \text{ is an } (\mathscr{F}, u, 4\epsilon)\text{-separated subset}\right\}$$
$$\le \frac{1}{m^n}\sum_{|u|=n}\log\sum_{y\in E_{u,n}} e^{S_{u,n}\Psi(y)} + \frac{1}{m^n}\sum_{|u|=n}((n+A)(\beta+\zeta) + 2n\operatorname{Var}(\Psi\circ\phi, 2\epsilon) + n\epsilon).$$

Thus,

$$Q_n^W(\Psi\circ\phi, \mathscr{F}, 4\epsilon) \le ((n+A)(\beta+\zeta) + 2n\operatorname{Var}(\Psi\circ\phi, 2\epsilon) + n\epsilon) + \frac{1}{m^n}\sum_{|u|=n}\log\sum_{y\in E_{u,n}} e^{S_{u,n}\Psi(y)}.$$

This indicates that

$$\frac{1}{n}\log Q_n^W(\Psi\circ\phi, \mathscr{F}, 4\epsilon)$$
$$\le (1+\frac{A}{n})(\beta+\zeta) + 2\operatorname{Var}(\Psi\circ\phi, 2\epsilon) + \epsilon + \frac{1}{n}\cdot\frac{1}{m^n}\sum_{|u|=n}\log\sum_{y\in E_{u,n}} e^{S_{u,n}\Psi(y)}$$
$$< (1+\frac{A}{n})(\beta+\zeta) + 2\operatorname{Var}(\Psi\circ\phi, 2\epsilon) + \epsilon + P^W(\mathscr{G}, \Psi).$$

Taking $n\to\infty$, $\epsilon\to 0$,

$$P^W(\Psi\circ\phi, \mathscr{F}) \le P^W(\Psi, \mathscr{G}) + \beta + \zeta.$$

By the arbitrariness of ζ, we derive that

$$P^W(\Psi\circ\phi, \mathscr{F}) \le P^W(\Psi, \mathscr{G}) + \beta = P^W(\Psi, \mathscr{G}) + \sup_{y\in Y} h_{\text{Biś}}\left(\phi^{-1}(y), \mathscr{F}\cup\{id\}\right).$$

□

Particularly, taking $\Psi = 0$, we get:

Corollary 3.5. *Let $\phi : (X, \mathscr{F}) \to (Y, \mathscr{G})$ be a factor map between two IFSs. Then*

$$h^W(\mathscr{F}) \le h^W(\mathscr{G}) + \sup_{y\in Y} h_{Biś}\left(\phi^{-1}(y), \mathscr{F}\cup\{id\}\right).$$

Especially, if $h^W(\mathscr{G}) < +\infty$, then

$$h^W(\mathscr{F}) - h^W(\mathscr{G}) \le \sup_{y\in Y} h_{Biś}\left(\phi^{-1}(y), \mathscr{F}\cup\{id\}\right).$$

Moreover, we consider the question of whether the following inequalities hold?

Question 3.6.

$$P^W(\Psi\circ\phi, \mathscr{F}) \le P^W(\Psi, \mathscr{G}) + \sup_{y\in Y} h_B\left(\phi^{-1}(y), \mathscr{F}\right)$$

or

$$P^W(\Psi\circ\phi, \mathscr{F}) \le P^W(\Psi, \mathscr{G}) + \sup_{y\in Y} h^W\left(\phi^{-1}(y), \mathscr{F}\right).$$

4. Power rule of topological pressure

In this section, we explore power rules of pressure from [24]. Let $(X,\mathscr{F})$ be an IFS, where $\mathscr{F}=\{f_0,f_1,\ldots,f_{m-1}\}$. We continue to study the *pressure from [24]* of $(X,\mathscr{F}^k)$, where

$$\mathscr{F}^k=\{h_1\circ h_2\circ\cdots\circ h_k : h_1,h_2,\ldots,h_k\in\mathscr{F}\}.$$

Theorem 4.1. *Let $(X,\mathscr{F})$ be an IFS with $\mathscr{F}=\{f_0,f_1,\ldots,f_{m-1}\}$. Let $\Psi\in C(Y;\mathbb{R})$ and $\Psi\geq 0$. Then*

$$P^W(\Psi,\mathscr{F}^k)\leq k\cdot P^W(\Psi,\mathscr{F}).$$

Proof. Take $u=u_0u_1\cdots u_{k-1}u_k\cdots u_{nk-1}\in F_m^+$, where $|u|=nk$, and $f_u=f_{u_{nk-1}}\circ\cdots\circ f_{u_0}$. Let

$$h_{\tau_i}=f_{u_{(i+1)k-1}}\circ\cdots\circ f_{u_{ik+1}}\circ f_{u_{ik}}$$

for $i=0,1,\ldots,n-1$ and $\tau=\tau_0\tau_1\cdots\tau_{n-1}$. It is clear that $h_{\tau_i}\in\mathscr{F}^k$. Next, we are to show

$$P^W(\Psi,\mathscr{F}^k)\leq k\cdot P^W(\Psi,\mathscr{F}).$$

For any $\epsilon>0$, assume E is an $(\mathscr{F},u,\epsilon)$-spanning set of X. Then for every $a\in X$, there is $b\in E$ such that $d_u(a,b)=\max_{v<u} d\left(f_v(a),f_v(b)\right)<\epsilon$. This implies that

$$d_\tau(a,b)=\max_{\tau'<\tau} d\left(h_{\tau'}(a),h_{\tau'}(b)\right)<\epsilon.$$

Hence, we deduce that E is also an $(\mathscr{F}^k,\tau,\epsilon)$-spanning set of X. Moreover, one has

$$S_{\tau,n}\Psi(x)\Big|_h\triangleq\sum_{i=0}^{n-1}\Psi\left(h_{\tau|_i}(x)\right)\leq\sum_{i=0}^{nk-1}\Psi\left(f_{u|_i}(x)\right)\triangleq S_{u,nk}\Psi(x)\Big|_f, \tag{4.1}$$

and notice that $\Psi\geq 0$, then we have

$$\begin{aligned}
&\inf_{Z_{\tau,n}}\left\{\sum_{x\in Z_{\tau,n}} e^{S_{\tau,n}\Psi(x)|_h} : Z_{\tau,n}\text{ is an }(\mathscr{F}^k,\tau,\epsilon)\text{-spanning set for }X\right\}\\
\leq&\inf_{Z_{\tau,n}}\left\{\sum_{x\in Z_{\tau,n}} e^{S_{u,nk}\Psi(x)|_f} : Z_{\tau,n}\text{ is an }(\mathscr{F}^k,\tau,\epsilon)\text{-spanning set for }X\right\}\\
\leq&\inf_{Z_{u,nk}}\left\{\sum_{x\in Z_{u,nk}} e^{S_{u,nk}\Psi(x)|_f} : Z_{u,nk}\text{ is an }(\mathscr{F},u,\epsilon)\text{-spanning set for }X\right\}.
\end{aligned}$$

This indicates that

$$\begin{aligned}
&\sum_{|\tau|=n}\log\inf_{Z_{\tau,n}}\left\{\sum_{x\in Z_{\tau,n}} e^{S_{\tau,n}\Psi(x)|_h} : Z_{\tau,n}\text{ is an }(\mathscr{F}^k,\tau,\epsilon)\text{-spanning set for }X\right\}\\
\leq&\sum_{|u|=nk}\log\inf_{Z_{u,nk}}\left\{\sum_{x\in Z_{u,nk}} e^{S_{u,nk}\Psi(x)|_f} : Z_{u,nk}\text{ is an }(\mathscr{F},u,\epsilon)\text{-spanning set for }X\right\}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
P_n^W(\Psi, \mathscr{F}^k, \epsilon) &= \frac{1}{(m^k)^n} \sum_{|\tau|=n} \log \inf_{Z_{\tau,n}} \left\{ \sum_{x\in E_{\tau,n}} e^{S_{\tau,n}\Psi(x)|_h} : Z_{\tau,n} \text{ is an } (\mathscr{F}^k, \tau, \epsilon)\text{-spanning set for } X \right\} \\
&\leq \frac{1}{m^{nk}} \sum_{|u|=nk} \log \inf_{Z_{u,nk}} \left\{ \sum_{x\in Z_{u,nk}} e^{S_{u,nk}\Psi(x)|_f} : Z_{u,nk} \text{ is an } (\mathscr{F}, u, \epsilon)\text{-spanning set for } X \right\} \\
&\leq P_{nk}^W(\Psi, \mathscr{F}, \epsilon).
\end{aligned}
$$

This yields that

$$
\begin{aligned}
P^W(\Psi, \mathscr{F}^k) &= \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} P_n^W(\Psi, \mathscr{F}^k, \epsilon) \\
&\leq \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} P_{nk}^W(\Psi, \mathscr{F}, \epsilon) \\
&= k \cdot \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{nk} P_{nk}^W(\Psi, \mathscr{F}, \epsilon) \\
&= k \cdot P^W(\Psi, \mathscr{F}).
\end{aligned}
$$

□

Particularly, taking $\Psi = 0$, we have.

Corollary 4.2. *Let $(X, \mathscr{F})$ be an IFS with $\mathscr{F} = \{f_0, f_1, \dots, f_{m-1}\}$. Then*

$$h^W(\mathscr{F}^k) \leq k \cdot h^W(\mathscr{F}).$$

A.

Lemma A.1. *[23] Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_{n+p} \leq a_n + a_p$ for all n, p. Then $\lim_{n\to\infty} a_n/n$ exists and equals $\inf_n a_n/n$. (The limit could be $-\infty$, but if the (a_n) is bounded from below, then the limit will be non-negative.)*

Lemma A.2. *[23] For $\mathscr{U}, \mathscr{V} \in C_X$, we have $\mathscr{N}(\mathscr{U} \vee \mathscr{V}) \leq \mathscr{N}(\mathscr{U}) \cdot \mathscr{N}(\mathscr{V})$. Moreover, for any continuous map $f : X \to X$, we have $\mathscr{N}(f^{-1}\mathscr{U}) \leq \mathscr{N}(\mathscr{U})$.*

Theorem A.3. *Let $(X, \mathscr{F})$ be an IFS with $\mathscr{F} = \{f_0, f_1, \dots, f_{m-1}\}$. If $\mathscr{U} \in C_X$, then*

$$\lim_{n\to\infty} \frac{1}{n} \left[\frac{1}{m^n} \sum_{|w|=n} \log \mathscr{N} \left(\bigvee_{\nu \preccurlyeq w} f_\nu^{-1} \mathscr{U} \right) \right]$$

exists and is equal to $\inf_n \frac{1}{n} \left[\frac{1}{m^n} \sum_{|w|=n} \log \mathscr{N} \left(\bigvee_{\nu\preccurlyeq w} f_\nu^{-1} \mathscr{U} \right) \right]$.

Proof. By Lemma A.1, we just need to show that

$$
\begin{aligned}
&\frac{1}{m^{n_1+n_2}} \sum_{|\mu|=n_1+n_2} \log \mathscr{N} \left(\bigvee_{\nu\preccurlyeq\mu} f_\nu^{-1} \mathscr{U} \right) \\
&\leq \frac{1}{m^{n_1}} \sum_{|\mu^{(1)}|=n_1} \log \mathscr{N} \left(\bigvee_{\nu\preccurlyeq\mu^{(1)}} f_\nu^{-1} \mathscr{U} \right) + \frac{1}{m^{n_2}} \sum_{|w^{(2)}|=n_2} \log \mathscr{N} \left(\bigvee_{\nu\preccurlyeq\mu^{(2)}} f_\nu^{-1} \mathscr{U} \right).
\end{aligned}
$$

Take any $\mu = i_0 \cdots i_{n_1-1} i_{n_1} \cdots i_{n_1+n_2-1}, \mu^{(1)} = i_0 \cdots i_{n_1-1}, \mu^{(2)} = i_{n_1} \cdots i_{n_1+n_2-1}$, i.e.,

$$|\mu| = n_1 + n_2, \quad \left|\mu^{(1)}\right| = n_1, \quad \left|\mu^{(2)}\right| = n_2.$$

Since $f_\mu = f_{i_{n_1+n_2-1}} \circ \cdots f_{i_{n_1}} \circ f_{i_{n_1}-1} \cdots \circ f_{i_{n_0}}, f_\mu^{-1} = f_{i_0}^{-1} \cdots \circ f_{i_{n_1}-1}^{-1} \circ f_{i_{n_1}}^{-1} \cdots \circ f_{i_{n_1+n_2-1}}^{-1}$, and then we have

$$\begin{aligned}
\log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu} f_\nu^{-1}(\mathscr{U})\right) &= \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(1)}} f_\nu^{-1}\mathscr{U} \vee f_{\mu^{(1)}}^{-1}\left(\bigvee_{\nu \preccurlyeq \mu^{(2)}} f_\nu^{-1}\mathscr{U}\right)\right) \\
&\leq \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(1)}} f_\nu^{-1}\mathscr{U}\right) + \log \mathscr{N}\left(f_{\mu^{(1)}}^{-1}\left(\bigvee_{\nu \preccurlyeq \mu^{(2)}} f_\nu^{-1}\mathscr{U}\right)\right) \\
&\leq \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(1)}} f_\nu^{-1}\mathscr{U}\right) + \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(2)}} f_\nu^{-1}\mathscr{U}\right).
\end{aligned}$$

Hence, combining the above arguments, we have

$$\begin{aligned}
&\frac{1}{m^{n_1+n_2}} \sum_{|\mu|=n_1+n_2} \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu} f_\nu^{-1}\mathscr{U}\right) \\
\leq &\frac{1}{m^{n_1+n_2}} \sum_{|\mu|=n_1+n_2} \left(\log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(1)}} f_\nu^{-1}\mathscr{U}\right) + \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(2)}} f_\nu^{-1}\mathscr{U}\right)\right) \\
= &\frac{1}{m^{n_1+n_2}} \left(\sum_{\left|w^{(1)}\right|=n_1, \left|\mu^{(2)}\right|=n_2} \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(1)}} f_\nu^{-1}\mathscr{U}\right) + \sum_{\left|\mu^{(1)}\right|=n_1, \left|\mu^{(2)}\right|=n_2} \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(2)}} f_\nu^{-1}\mathscr{U}\right)\right) \\
= &\frac{1}{m^{n_1+n_2}} \left(m^{n_2} \sum_{\left|\mu^{(1)}\right|=n_1} \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(1)}} f_\nu^{-1}\mathscr{U}\right) + m^{n_1} \sum_{\left|\mu^{(2)}\right|=n_2} \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(2)}} f_\nu^{-1}\mathscr{U}\right)\right) \\
= &\frac{1}{m^{n_1}} \sum_{\left|\mu^{(1)}\right|=n_1} \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(1)}} f_\nu^{-1}\mathscr{U}\right) + \frac{1}{m^{n_2}} \sum_{\left|\mu^{(2)}\right|=n_2} \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu^{(2)}} f_\nu^{-1}\mathscr{U}\right).
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{m^n} \sum_{|w|=n} \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu} f_\nu^{-1}\mathscr{U}\right)\right] = \inf_n \frac{1}{n}\left[\frac{1}{m^n} \sum_{|\mu|=n} \log \mathscr{N}\left(\bigvee_{\nu \preccurlyeq \mu} f_\nu^{-1}\mathscr{U}\right)\right].$$

We finished the proof. □

By Theorem A.3, we have the following results, and their proofs are standard; one can refer to [23, Chapter 7] for some details.

Theorem A.4. *Let $(X, \mathscr{F})$ be an IFS with $\mathscr{F} = \{f_0, f_1, \ldots, f_{m-1}\}$, and $\{\mathscr{U}_n\}_{n=0}^\infty \subset C_X^o$ with $\lim_{n\to\infty} diam\,(\mathscr{U}_n) = 0$, where $diam\,(\mathscr{U}_n) = \sup\{diam(U), U \in \mathscr{U}_n\}$. Then $\lim_{n\to\infty} h^W(\mathscr{F}, \mathscr{U}_n) = h^W(\mathscr{F})$.*

Corollary A.5. *Let $(X, \mathscr{F})$ be an IFS with $\mathscr{F} = \{f_0, f_1, \ldots, f_{m-1}\}$. Then*

$$h^W(\mathscr{F}) = \lim_{\delta\to 0}\{h^W(\mathscr{F}, \mathscr{U}) : diam\,(\mathscr{U}) < \delta\}.$$

Theorem A.6. *Let $(X, \mathscr{F})$ be an IFS with $\mathscr{F} = \{f_0, f_1, \dots, f_{m-1}\}$. Then*

$$h^W(\mathscr{F}) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}P_n^W(0,\mathscr{F},\epsilon) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}Q_n^W(0,\mathscr{F},\epsilon).$$

Remark A.7. *Notice that, in [25], the definition of $h^W(\mathscr{F})$ is given as follows.*

$$h^W(\mathscr{F}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}P_n^W(0,\mathscr{F},\epsilon) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}Q_n^W(0,\mathscr{F},\epsilon).$$

B. Conclusions

In this research, we discuss the topological pressure for iterated function systems on a compact metric space. Firstly, a formula of topological pressure of a factor map is established, which generalizes the result in [Trans. Amer. Math. Soc., 1971, 153: 401–414]. Finally, we also study the power rule of a topological pressure for iterated function systems. These results enrich the theory of topological pressure for iterated function systems.

Author contributions

Zhongxuan Yang: Conceptualization, Writing-original draft, Methodology, Writing-review & editing; Xiaojun Huang: Conceptualization, Methodology, Funding acquisition; Jiajun Zhang: Conceptualization, Writing-original draft, Methodology, Writing-review & editing. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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