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Research article

First chen inequality for biwarped product submanifold of a Riemannian space forms

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Abstract: Within this paper, we have formulated the first Chen inequality for bi-warped product submanifolds within Riemannian space forms. This inequality intricately involved extrinsic invariants such as mean curvature and the lengths of the warping functions, while also incorporating intrinsic invariants like sectional curvature and δ-invariants. Furthermore, we extensively explored and analyzed the scenarios where equality conditions were met within the context of this inequality.

Keywords: bi-warped product; mean curvature; δ-invariant; scalar curvature; minimal sub-manifolds; Riemannian space forms; submanifolds
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1. Introduction

The theory of submanifolds has evolved naturally from the classical study of curves and surfaces in Euclidean space, employing the tools of differential calculus. The extensive applications of extrinsic and intrinsic Riemannian invariants of some sub-manifold, span across various scientific disciplines, particularly within the realm of general relativity. The main motivation for such a study lies in establishing the relationships between these invariants.

In the work [1], Chen demonstrated an inequality that involves these invariants for sub-manifolds within Riemannian space forms. Building upon Nash's theorem, researchers have derived geometric constraints, known as intrinsic and extrinsic invariants, for warped products in diverse space forms (see [2–4]). Of particular interest is Chen's first invariant, a significant intrinsic measure in our inequality, which was defined in [2].

$$\delta_{M^m}(x) = \tau(T_xM^m) - \inf\{K(\pi) : \pi \subset T_xM^m, x \in M^m, \dim \pi = 2\}, \tag{1.1}$$

where $K(\pi)$ is the sectional curvature of the plane section π, $\tau(T_xM^m)$ is the scalar curvature of T_xM^m.

Bishop and O'Neill [5] introduced the warped product of Riemannian manifolds to generate a broad class of complete manifolds characterized by negative curvature.

In [6], Chen and Dillen introduced the notion of multiply warped product manifolds and sub-manifolds as follows: Let $M_1, \ldots, M_k$ be Riemannian manifolds and let

$$M = M_1 \times \cdots \times M_k$$

be the Cartesian product of $M_1, \ldots, M_k$. For each i, denote by

$$\pi_i : M \to M_i,$$

the canonical projection of M onto M_i. When there is no confusion, we identify M_i with a horizontal lift of M_i in M via π_i. If

$$f_2, \ldots, f_k : M_1 \to \mathbb{R}^+$$

are positive real-valued functions, then

$$g(L_1, L_2) := g(\pi_{1*}L_1, \pi_{1*}L_2) + \sum_{i=2}^{k} (f_i \circ \pi_1)^2\, g(\pi_{i*}L_1, \pi_{i*}L_2)$$

defines a Riemannian metric g on M, called a multiply warped product metric, for any vector fields L_1, L_2 on M and π_{i*} denotes the push tangential map. The product manifold M endowed with this metric is denoted by

$$M_1 \times_{f_2} M_2 \times \cdots \times_{f_k} M_k.$$

In this case, the warping functions $f_2, \ldots, f_k$ are nonconstant functions on M_1. It is clear that if all $f_1, \ldots, f_k$ are constant except one function f_j such that $2 \leq j \leq n$, then M is a single warped product manifold. Also, if any two functions are not constant and all others are constant, then multiply warped product reduces to bi-warped product, which we discuss in this paper. Multiply warped products play crucial roles in both physics and differential geometry, particularly in the realm of relativity theory. Standard spacetime models such as Robertson-Walker and Schwarzschild are examples of warped products. Furthermore, elementary models describing the regions around stars and black holes often align with the warped product framework [7]. Additionally, many solutions to Einstein's field equations find expression in terms of warped products. [8].

In a study by [9], an enhanced form of the initial Chen inequality for Legendrian sub-manifolds within Sasakian space forms was introduced. Furthermore, the authors in [10] presented the first Chen inequality for general warped product sub-manifolds in a Riemannian space form. Similarly, in [11], Alghamdi et al. established a comparable inequality for warped product Legendrian sub-manifolds within Kenmotsu space forms. Additionally, in [12], Li et al. set forth a similar inequality for warped product sub-manifolds within $\mathbb{Q}^m_\epsilon \times \mathbb{R}$. Some applications of inequalities in other fields can be found on [13].

Building upon the research mentioned above, this paper introduces the first Chen inequality for bi-warped product sub-manifolds within Riemannian space forms. Additionally, we delve into the examination of the equality case and provide insights into the applications of these inequalities within the context of this study.

2. Preliminaries

In this section, we present the foundational mathematical framework required to establish the first Chen inequality for bi-warped product sub-manifolds in Riemannian space forms.

Consider the n-dimensional sub-manifold N^n of a Riemannian manifold (M^m, g) of dimension m. Let $\bar{\nabla}$ denote the Levi-Civita connection on M^m, and let ∇ be the induced connection on N^n. The Gauss and Weingarten formulas, which connect the geometry of the sub-manifold to that of the ambient manifold, are given as follows:

$$\bar{\nabla}_{L_1} L_2 = \nabla_{L_1} L_2 + h(L_1, L_2) \tag{2.1}$$

and

$$\bar{\nabla}_{L_1} \upsilon = -A_V L_1 + \nabla^{\perp}_{L_1} \upsilon, \tag{2.2}$$

where $L_1, L_2 \in TN^n, \upsilon \in T^{\perp}N^n$, and $h, \nabla^{\perp}, A_\upsilon$ denotes the second fundamental form, normal connection, and the shape operator, respectively.

The well-known equation of Gauss is given by

$$R(L_1, L_2, L_3, L_4) = \bar{R}(L_1, L_2, L_3, L_4) + g(h(L_1, L_4), h(L_2, L_3)) - g(h(L_1, L_3), h(L_2, L_4)), \tag{2.3}$$

for any $L_1, L_2, L_3, L_4 \in \Gamma(TM^m)$, where $\bar{R}$ and R are the curvature tensors of M^m and N^n, respectively. If we select two linearly independent tangent vectors $L_1, L_2 \in TM$, the sectional curvature of the 2-plane π spanned by L_1 and L_2 can be expressed in terms of the Riemannian curvature tensor $\bar{R}$ as follows:

$$\bar{K}(\pi) = \bar{K}(L_1 \wedge L_2) = \frac{g(\bar{R}(L_1, L_2)L_2, L_1)}{g(L_1, L_1)g(L_2, L_2) - (g(L_1, L_2))^2}. \tag{2.4}$$

If the 2-plane π is spanned by orthogonal unit vectors L_1 and L_2 from the tangent space T_xM^m, where $x \in M^m$, the previous definition can be expressed as:

$$\bar{K}(\pi) = \bar{K}_{M^m}(L_1 \wedge L_2) = g(\bar{R}(L_1, L_2)L_2, L_1). \tag{2.5}$$

It is important to note that the sectional curvature is independent of the choice of orthonormal basis for π and fully characterizes the Riemannian curvature tensor $\bar{R}$. Furthermore, if $\bar{K}(\pi)$ is constant for all planes π in T_xM^m and for all points $x \in M^m$, specifically

$$\bar{K}(\pi) = c,$$

we refer to $M^m(c)$ as a real space form.

Here, then we have:

$$\bar{R}(L_1, L_2)L_3 = c(g(L_2, L_3)L_1 - g(L_1, L_3)L_2), \tag{2.6}$$

for any $L_1, L_2, L_3 \in \Gamma(TM^m(c))$.

Scalar curvature for M^m is defined in terms of the sectional curvature as:

$$\begin{aligned} \bar{\tau}(T_xM^m) &= \sum_{1\le i<j\le m} \bar{K}_{ij} \\ &\implies 2\bar{\tau}(T_xM^n) = \sum_{1\le i\ne j\le m} \bar{K}_{ij}. \end{aligned} \tag{2.7}$$

An interesting invariant for some manifold is the Chen's first invariant, which is defined as:

$$\bar{\delta}_{M^m}(x) = \bar{\tau}(T_xM^m) - \inf\{\bar{K}(\pi) : \pi \subset T_xM^m, x \in M^m, \dim \pi = 2\}. \tag{2.8}$$

Whenever we consider the above geometric objects such as the sectional curvature, scalar curvature, Chen's first invariant, etc., for the sub-manifold N^n, we simply denote them as K, τ and δ, respectively.

In particular, the scalar curvature $\tau(x)$ of N^n at x is identical with the scalar curvature of the tangent space T_xN^n of N^n at x, that is,

$$\tau(x) = \tau(T_xN^n).$$

Bi-warped product manifolds are special classes of manifolds. Let us consider,

$$N^n = N_1 \times_{f_1} N_2 \times_{f_2} N_3$$

as the bi-warped product sub-manifold of the Riemannian space form $M^m(c)$. We choose an orthonormal basis

$$\{\varkappa_1, \cdots, \varkappa_{n_1}, \varkappa_{n_1+1}, \cdots, \varkappa_{n_1+n_2}, \varkappa_{n_1+n_2+1}, \cdots, \varkappa_n\}$$

of T_xN^n, where

$$\{\varkappa_1, \cdots, \varkappa_{n_1}\}, \{\varkappa_{n_1+1}, \cdots, \varkappa_{n_1+n_2}\}, \{\varkappa_{n_1+n_2+1}, \cdots, \varkappa_n\}$$

are orthonormal bases of N_1, N_2, and N_3, respectively. Let $\{\varkappa_{n+2}, \cdots, \varkappa_m\}$ be an orthonormal basis of $T_x^{\perp}N^n$.

The coefficients of the second fundamental form h of N^n with respect to the above local frame are denoted as

$$h_{ij}^r = g(h(\varkappa_i, \varkappa_j), \varkappa_r), \tag{2.9}$$

where $i, j \in \{1, \cdots, n\}$ and $r \in \{n+1, \cdots, m\}$. The mean curvature vector H is defined with respect to the same local frame above, as

$$H = \frac{1}{n}\sum_{i=1}^{n} h(\varkappa_i, \varkappa_i). \tag{2.10}$$

We say that N^n is a minimal sub-manifold of M^m if H vanishes identically.

If f is a smooth function on M^m, then its gradient ∇f and Laplacian Δf are defined as

$$\begin{aligned} g(\nabla f, X) &= Xf, \\ \Delta f &= \sum_{i=1}^{m}((\nabla_{\varkappa_i}\varkappa_i)f - \varkappa_i\varkappa_i f). \end{aligned} \tag{2.11}$$

So from Eqs (2.3), (2.4), and (2.9), we get

$$\begin{aligned} K(\varkappa_i \wedge \varkappa_j) &= \bar{K}(\varkappa_i \wedge \varkappa_j) + \sum_{r=n+1}^{m} (g(h^r_{ii}\varkappa_r, h^r_{jj}\varkappa_r) - g(h^r_{ij}\varkappa_r, h^r_{ij}\varkappa_r)) \\ &\implies K(\varkappa_i \wedge \varkappa_j) = \bar{K}(\varkappa_i \wedge \varkappa_j) + \sum_{r=n+1}^{m} (h^r_{ii}h^r_{jj} - (h^r_{ij})^2), \end{aligned} \tag{2.12}$$

where $\bar{K}(\varkappa_i \wedge \varkappa_j)$ denotes the sectional curvature of the 2-plane spanned by $\varkappa_i$ and $\varkappa_j$ at x in the ambient manifold $M^m(c)$.

Taking the summation over the orthonormal frame of the tangent space of N^n in (2.12), we have

$$2\tau(T_xN^n) = 2\bar{\tau}(T_xN^n) + n^2\|H\|^2 - \|h\|^2. \tag{2.13}$$

The sectional curvature and warping functions are related by the following formulas [14]:

(i) $\sum_{a=1}^{n_1} \sum_{A=n_1+1}^{n_1+n_2} K_{aA} = \frac{n_2\Delta f_3}{f_3}$;

(ii) $\sum_{a=1}^{n_1} \sum_{b=n_1+n_2+1}^{n} K_{ab} = \frac{n_3\Delta f_4}{f_4}$;

(iii) $\sum_{A=n_1+1}^{n_1+n_2} \sum_{b=n_1+n_2+1}^{n} K_{Ab} = -\frac{n_2n_3}{f_3f_4} g_3(\nabla f_3, \nabla f_4)$.

So, we have

$$\begin{aligned} \tau(T_xN^n) &= \sum_{1\le i<j\le n} K_{ij} \\ &= \sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n_1+n_2} K_{aA} + \sum_{a=1}^{n_1}\sum_{b=n_1+n_2+1}^{n} K_{ab} + \sum_{A=n_1+1}^{n_1+n_2}\sum_{b=n_1+n_2+1}^{n} K_{Ab} \\ &\quad + \sum_{1\le a<a`\le n_1} K_{aa`} + \sum_{n_1+1\le A<A`\le n_1+n_2} K_{AA`} + \sum_{n_1+n_2+1\le b<b`\le n} K_{bb`} \\ &\implies \tau(T_xN^n) = \frac{n_2\Delta^1 f_1}{f_1} + \frac{n_3\Delta^1 f_2}{f_2} - n_2n_3 g(\nabla^1(\ln f_1), \nabla^1(\ln f_2)) \\ &\quad + \tau(T_xN_1) + \tau(T_xN_2) + \tau(T_xN_3). \end{aligned} \tag{2.14}$$

The following lemma will also be needed to prove our main result.

Lemma 2.1. *[10] Let* $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta$ *be* $(n+1)(n \ge 2)$ *real numbers such that*

$$(\textstyle\sum_{i=1}^{n} \alpha_i)^2 = (n-1)(\sum_{i=1}^{n} \alpha_i^2 + \beta),$$

then $2\alpha_1\alpha_2 \ge \beta$, *with equality holds if, and only if,*

$$\alpha_1 + \alpha_2 = \alpha_3 = \cdots = \alpha_n.$$

3. First chen inequality for bi-warped product sub-manifolds of a Riemannian space forms

Theorem 3.1. *Let*

$$\phi : N^n = N_1 \times_{f_1} N_2 \times_{f_2} N_3 \to M^m(c)$$

be an isometric immersion of a bi-warped product sub-manifold N^n into a Riemannian space form $M^m(c)$. Then, for each point $x \in N^n$ and each plane section

$$\pi_i \subset T_x N_i^{n_i}, \quad n_i = dim\, N_i \geq 2$$

for $i = 1, 2, 3$, we have:
(1) *If $\pi_1 \subset T_x N_1$, then*

$$\begin{aligned}\delta_{N_1{}^{n_1}} \leq & \frac{n^2}{2}\|H\|^2 - \frac{n_2 \Delta f_1}{f_1} - \frac{n_3 \Delta f_2}{f_2} + n_2 n_3 g(\nabla(\ln f_1), \nabla(\ln f_2)) \\ & + \frac{n_1}{2}(n_1 + 2n_2 + 2n_3 - 1)c + n_2 n_3 c - c.\end{aligned}$$

(2) *If $\pi_2 \subset T_x N_2$, then*

$$\begin{aligned}\delta_{N_2{}^{n_2}} \leq & \frac{n^2}{2}\|H\|^2 - \frac{n_2 \Delta f_1}{f_1} - \frac{n_3 \Delta f_2}{f_2} + n_2 n_3 g(\nabla(\ln f_1), \nabla(\ln f_2)) \\ & + \frac{n_2}{2}(n_2 + 2n_1 + 2n_3 - 1)c + n_1 n_3 c - c.\end{aligned}$$

(3) *If $\pi_3 \subset T_x N_3$, then*

$$\begin{aligned}\delta_{N_3{}^{n_3}} \leq & \frac{n^2}{2}\|H\|^2 - \frac{n_2 \Delta f_1}{f_1} - \frac{n_3 \Delta f_2}{f_2} + n_2 n_3 g(\nabla(\ln f_1), \nabla(\ln f_2)) \\ & + \frac{n_3}{2}(n_3 + 2n_1 + 2n_2 - 1)c + n_1 n_2 c - c.\end{aligned}$$

the equality holds at $x \in N^n$ if, and only if, there exist an orthonormal basis $\{\varkappa_1, \cdots, \varkappa_n\}$ of $T_x N^n$ and an orthonormal basis $\{\varkappa_{n+1}, \cdots, \varkappa_n\}$ of $T_x^{\perp} N^n$ such that:

(a) $\pi_1 = span\{\varkappa_1.\varkappa_2\}, \pi_2 = span\{\varkappa_{n_1+1,\varkappa_{n_1+2}}\}, \pi_3 = span\{\varkappa_{n_1+n_2+1,n_1+n_2+2}\}$.

(b) The shape operator takes the following forms, where $O_{m\times n}$, denoting the zero matrix of order $m \times n$:
[i] If $\pi_1 \subset T_x N_1$, then for $r = n+1$, we have

$$A_{\varkappa_{n+1}} = \begin{pmatrix} BLOCK(I) & O_{n_1\times n_2} & O_{n_1\times n_3} \\ O_{n_2\times n_1} & BLOCK(II) & O_{n_2\times n_3} \\ O_{n_3\times n_1} & O_{n_3\times n_2} & BLOCK(III) \end{pmatrix},$$

where $BLOCK(I), BLOCK(II)$, and $BLOCK(III)$ are given, respectively, as follows:

$$\begin{pmatrix} \mu_1 & h_{12}^{n+1} & 0 & \cdots & 0_{1n_1} \\ h_{21}^{n+1} & \mu_2 & 0 & \cdots & 0_{2n_1} \\ 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{(n_1)(1)} & \cdots & \cdots & \cdots & \mu \end{pmatrix}, \begin{pmatrix} h_{(n_1+1)(n_1+1)}^{n+1} \cdots & \cdots & h_{(n_1+1)(n_1+n_2)}^{n+1} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h_{(n_1+n_2)(n_1+1)}^{n+1} & \cdots & h_{(n_1+n_2)(n_1+n_2)}^{n+1} \end{pmatrix}, \begin{pmatrix} h_{(n_1+n_2+1)(n_1+n_2+1)}^{n+1} \cdots & \cdots & h_{(n_1+n_2+1)(n)}^{n+1} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h_{(n)(n_1+n_2+1)}^{n+1} & \cdots & h_{(n)(n)}^{n+1} \end{pmatrix},$$

where

$$\mu = \mu_1 + \mu_2 = h_{11}^{n+1} + h_{22}^{n+1}.$$

Also, for $r \in \{n+2, \cdots, m\}$*,*

$$A_{\varkappa_r} = \begin{pmatrix} BLOCK(I) & O_{n_1\times n_2} & O_{n_1\times n_3} \\ O_{n_2\times n_1} & BLOCK(II) & O_{n_2\times n_3} \\ O_{n_3\times n_1} & O_{n_3\times n_2} & BLOCK(III) \end{pmatrix},$$

where $BLOCK(I), BLOCK(II)$ *and* $BLOCK(III)$ *are given, respectively, as*

$$\begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \cdots & 0_{1n_1} \\ h^r_{21} & -h^r_{11} & 0 & \cdots & 0_{2n_1} \\ 0 & 0 & 0_{33} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{(n_1)(1)} & \cdots & \cdots & \cdots & 0_{n_1n_1} \end{pmatrix}, \begin{pmatrix} h^r_{(n_1+1)(n_1+1)} \cdots & \cdots & h^r_{(n_1+1)(n_1+n_2)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h_{(n_1+n_2)(n_1+1)}cr & \cdots & h^r_{(n_1+n_2)(n_1+n_2)} \end{pmatrix}, \begin{pmatrix} h^r_{(n_1+n_2+1)(n_1+n_2+1)} \cdots & \cdots & h^r_{(n_1+n_2+1)(n)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h^r_{(n)(n_1+n_2+1)} & \cdots & h^r_{(n)(n)} \end{pmatrix}.$$

[*ii*] *If* $\pi_2 \subset T_xN_2$*, then for* $r = n+1$*, we have*

$$A_{\varkappa_{n+1}} = \begin{pmatrix} BLOCK(I) & O_{n_1\times n_2} & O_{n_1\times n_3} \\ O_{n_2\times n_1} & BLOCK(II) & O_{n_2\times n_3} \\ O_{n_3\times n_1} & O_{n_3\times n_2} & BLOCK(III) \end{pmatrix},$$

where $BLOCK(I), BLOCK(II)$*, and* $BLOCK(III)$ *are given, respectively, as follows:*

$$\begin{pmatrix} h^{n+1}_{11} \cdots & \cdots & h^{n+1}_{(1)(n_1)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h^{n+1}_{(n_1)(1)} & \cdots & h^{n+1}_{(n_1)(n_1)} \end{pmatrix}, \begin{pmatrix} \mu_1 & h^{n+1}_{(n_1+1)(n_1+2)} & 0 & \cdots & 0_{(n_1+1)(n_1+n_2)} \\ h^{n+1}_{(n_1+2)(n_1+1)} & \mu_2 & 0 & \cdots & 0_{(n_1+2)(n_1+n_2)} \\ 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{(n_1+n_2)(n_1+1)} & \cdots & \cdots & \cdots & \mu \end{pmatrix}, \begin{pmatrix} h^{n+1}_{(n_1+n_2+1)(n_1+n_2+1)} \cdots & \cdots & h^{n+1}_{(n_1+n_2+1)(n)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h^{n+1}_{(n)(n_1+n_2+1)} & \cdots & h^{n+1}_{(n)(n)} \end{pmatrix},$$

where

$$\mu = \mu_1 + \mu_2 = h^{n+1}_{(n_1+1)(n_1+1)} + h^{n+1}_{(n_1+2)(n_1+2)}.$$

Also for $r \in \{n+2, \cdots, m\}$*,*

$$A_{\varkappa_r} = \begin{pmatrix} BLOCK(I) & O_{n_1\times n_2} & O_{n_1\times n_3} \\ O_{n_2\times n_1} & BLOCK(II) & O_{n_2\times n_3} \\ O_{n_3\times n_1} & O_{n_3\times n_2} & BLOCK(III) \end{pmatrix},$$

where $BLOCK(I), BLOCK(II)$*, and* $BLOCK(III)$ *are given, respectively, as*

$$\begin{pmatrix} h^r_{11} \cdots & \cdots & h^r_{(1)(n_1)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h^r_{(n_1)(1)} & \cdots & h^r_{(n_1)(n_1)} \end{pmatrix}, \begin{pmatrix} h^r_{(n_1+1)(n_1+1)} & h^r_{(n_1+1)(n_1+2)} & 0 & \cdots & 0_{(n_1+1)(n_1+n_2)} \\ h^r_{(n_1+2)(n_1+1)} & -h^r_{(n_1+1)(n_1+1)} & 0 & \cdots & 0_{(n_1+2)(n_1+n_2)} \\ 0 & 0 & 0_{(n_1+3)(n_1+3)} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{(n_1+n_2)(n_1+1)} & \cdots & \cdots & \cdots & 0_{(n_1+n_2)(n_1+n_2)} \end{pmatrix}, \begin{pmatrix} h^r_{(n_1+n_2+1)(n_1+n_2+1)} \cdots & \cdots & h^r_{(n_1+n_2+1)(n)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h^r_{(n)(n_1+n_2+1)} & \cdots & h^r_{(n)(n)} \end{pmatrix}.$$

[*iii*] *If* $\pi_3 \subset T_xN_3$*, then for* $r = n+1$*, we have*

$$A_{\varkappa_{n+1}} = \begin{pmatrix} BLOCK(I) & O_{n_1\times n_2} & O_{n_1\times n_3} \\ O_{n_2\times n_1} & BLOCK(II) & O_{n_2\times n_3} \\ O_{n_3\times n_1} & O_{n_3\times n_2} & BLOCK(III) \end{pmatrix},$$

where $BLOCK(I), BLOCK(II)$, and $BLOCK(III)$ are given, respectively, as follows:

$$\begin{pmatrix} h^{n+1}_{11}\cdots & \cdots & h^{n+1}_{(1)(n_1)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h^{n+1}_{(n_1)(1)} & \cdots & h^{n+1}_{(n_1)(n_1)} \end{pmatrix}, \begin{pmatrix} h^{n+1}_{(n_1+1)(n_1+1)}\cdots & \cdots & h^{n+1}_{(n_1+1)(n_1+n_2)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h^{n+1}_{(n_1+n_2)(n_1+1)} & \cdots & h^{n+1}_{(n_1+n_2)(n_1+n_2)} \end{pmatrix}, \begin{pmatrix} \mu_1 & h^{n+1}_{(n_1+n_2+1)(n_1+n_2+2)} & 0 & \cdots & 0_{(n_1+n_2+1)(n)} \\ h^{n+1}_{(n_1+n_2+2)(n_1+n_2+1)} & \mu_2 & 0 & \cdots & 0_{(n_1+n_2+2)(n)} \\ 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{(n)(n_1+n_2+1)} & \cdots & \cdots & \cdots & \mu \end{pmatrix},$$

where

$$\mu = \mu_1 + \mu_2 = h^{n+1}_{(n_1+n_2+1)(n_1+n_2+1)} + h^{n+1}_{(n_1+n_2+2)(n_1+n_2+2)}.$$

For $r \in \{n+2, \cdots, m\}$,

$$A_{\varkappa_r} = \begin{pmatrix} BLOCK(I) & O_{n_1\times n_2} & O_{n_1\times n_3} \\ O_{n_2\times n_1} & BLOCK(II) & O_{n_2\times n_3} \\ O_{n_3\times n_1} & O_{n_3\times n_2} & BLOCK(III) \end{pmatrix},$$

where $BLOCK(I), BLOCK(II)$, and $BLOCK(III)$ are given, respectively, as

$$\begin{pmatrix} h^{r}_{11}\cdots & \cdots & h^{r}_{(1)(n_1)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h^{r}_{(n_1)(1)} & \cdots & h^{r}_{(n_1)(n_1)} \end{pmatrix}, \begin{pmatrix} h^{r}_{(n_1+1)(n_1+1)}\cdots & \cdots & h^{r}_{(n_1+1)(n_1+n_2)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h_{(n_1+n_2)(n_1+1)}cr & \cdots & h^{r}_{(n_1+n_2)(n_1+n_2)} \end{pmatrix}, \begin{pmatrix} h^{r}_{(n_1+n_2+1)(n_1+n_2+1)} & h^{r}_{(n_1+n_2+1)(n_1+n_2+2)} & 0 & \cdots & 0_{(n_1+n_2+1)(n)} \\ h^{r}_{(n_1+n_2+2)(n_1+n_2+1)} & -h^{r}_{(n_1+n_2+1)(n_1+n_2+1)} & 0 & \cdots & 0_{(n_1+n_2+2)(n)} \\ 0 & 0 & 0_{(n_1+n_2+3)(n_1+n_2+3)} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{(n)(n_1+n_2+1)} & \cdots & \cdots & \cdots & 0_{nn} \end{pmatrix}.$$

(4) *If equality of* (i) *or* (ii) *or* (iii) *holds, then*

$$N_1 \times_{f_1} N_2 \times_{f_2} N_3$$

is mixed totally geodesic in $M^m(c)$. Moreover,

$$N_1 \times_{f_1} N_2 \times_{f_2} N_3$$

is both D_1, D_2, and D_3-minimal. Thus,

$$N_1 \times_{f_1} N_2 \times_{f_2} N_3$$

is a minimal bi-warped product sub-manifold of $M^m(c)$.

Proof. For simplicity we are providing the complete proof of (1). The other two parts viz., (2) and (3) can be proved in a similar way together with the equality cases.

We start our proof by considering a point $x \in N^n$ and let $\pi_1 \subset T_xN_1$ be a 2-plane. We choose an orthonormal basis

$$\{\varkappa_1, \cdots, \varkappa_{n_1}, \varkappa_{n_1+1}, \cdots, \varkappa_{n_1+n_2}, \varkappa_{n_1+n_2+1}, \cdots, \varkappa_n\}$$

of T_xN^n, where

$$\{\varkappa_1, \cdots, \varkappa_{n_1}\}, \{\varkappa_{n_1+1}, \cdots, \varkappa_{n_1+n_2}\}, \{\varkappa_{n_1+n_2+1}, \cdots, \varkappa_n\}$$

are orthonormal bases of N_1, N_2, and N_3, respectively, and $\{\varkappa_{n+1}, \cdots, \varkappa_m\}$ is an orthonormal basis of $T^{\perp}_xN^n$. First, put

$$\pi_1 = \text{span}\{\varkappa_1, \varkappa_2\}$$

such that the normal vector $\varkappa_{n+1}$ is in the direction of the mean curvature vector H. We have from (2.3) and (2.6)

$$n^2\|H\|^2 = 2\tau(T_xM^n) + \|h\|^2 - n(n-1)c,$$

which gives

$$\begin{aligned}(\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 =&2\tau(T_xM^n) + \|h\|^2 - n(n-1)c - (\sum_{A=n_1+1}^{n_1+n_2} h_{AA}^{n+1})^2 - (\sum_{b=n_1+n_2+1}^{n} h_{bb}^{n+1})^2\\ &- 2\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n_1+n_2} h_{aa}^{n+1}h_{AA}^{n+1} - 2\sum_{a=1}^{n_1}\sum_{b=n_1+n_2+1}^{n} h_{aa}^{n+1}h_{bb}^{n+1}2\sum_{A=n_1+1}^{n_1+n_2}\sum_{b=n_1+n_2+1}^{n} h_{AA}^{n+1}h_{bb}^{n+1}.\end{aligned}$$

Assume

$$\begin{aligned}\tau_1 =&2\tau(T_xM^n) - \frac{n_1-2}{n_1-1}(\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 - (\sum_{A=n_1+1}^{n_1+n_2} h_{AA}^{n+1})^2 - (\sum_{b=n_1+n_2+1}^{n} h_{bb}^{n+1})^2\\ &- 2\sum_a\sum_A h_{aa}^{n+1}h_{AA}^{n+1} - 2\sum_a\sum_b h_{aa}^{n+1}h_{bb}^{n+1} - 2\sum_A\sum_b h_{AA}^{n+1}h_{bb}^{n+1} - n(n-1)c.\end{aligned}$$

Thus,

$$\begin{aligned}(n_1-1)\tau_1 &= (n_1-1)\Big[2\tau(T_xM^n) - \frac{(n_1-1)-1}{n_1-1}(\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 - (\sum_{A=n_1+1}^{n_1+n_2} h_{AA}^{n+1})^2\\ &- (\sum_{b=n_1+n_2+1}^{n} h_{bb}^{n+1})^2 - 2\sum_a\sum_A h_{aa}^{n+1}h_{AA}^{n+1} - 2\sum_a\sum_b h_{aa}^{n+1}h_{bb}^{n+1}\\ &- 2\sum_A\sum_b h_{AA}^{n+1}h_{bb}^{n+1} - n(n-1)c\Big]\\ &\implies (n_1-1)\tau_1 = (n_1-1)[2\tau(T_xM^n) - (\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 - (\sum_{A=n_1+1}^{n_1+n_2} h_{AA}^{n+1})^2\\ &- (\sum_{b=n_1+n_2+1}^{n} h_{bb}^{n+1})^2 - 2\sum_a\sum_A h_{aa}^{n+1}h_{AA}^{n+1} - 2\sum_a\sum_b h_{aa}^{n+1}h_{bb}^{n+1}\\ &- 2\sum_A\sum_b h_{AA}^{n+1}h_{bb}^{n+1} - n(n-1)c] - (\sum_{a=1}^{n_1} h_{aa}^{n+1})^2\\ &\implies (n_1-1)\tau_1 = (n_1-1)[-\|h\|^2] - (\sum_{a=1}^{n_1} h_{aa}^{n+1})^2\\ &\implies (\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 = (n_1-1)[\tau_1 + \|h\|^2]\\ &\implies (\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 = (n_1-1)[\tau_1 + \sum_{i,j=1}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2]\end{aligned}$$

$$\implies (\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 = (n_1 - 1)\Big[\tau_1 + \sum_{a=1}^{n_1}(h_{aa}^{n+1})^2 + \sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2 + \sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2$$
$$+ \sum_{\substack{i,j=1\\i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\Big].$$

From Lemma 2.1 with

$$\alpha_a = h_{aa}^{n+1},$$
$$\beta = [\tau_1 + \sum_{a=1}^{n_1}(h_{aa}^{n+1})^2 + \sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2 + \sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2$$
$$+ \sum_{\substack{i,j=1\\i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2],$$

we have

$$h_{11}^{n+1}h_{22}^{n+1} \geq \frac{1}{2}[\tau_1 + \sum_{a=1}^{n_1}(h_{aa}^{n+1})^2 + \sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2 + \sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2 + \sum_{\substack{i,j=1\\i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2].$$

Assume

$$\pi_1 =< \varkappa_1, \varkappa_2 > .$$

Hence,

$$K(\pi_1) = c + \sum_{r=n+1}^{m}(h_{11}^r h_{22}^r - (h_{12}^r)^2)$$
$$\implies K(\pi_1) = c + h_{11}^{n+1}h_{22}^{n+1} + \sum_{r=n+2}^{m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{m}(h_{12}^r)^2$$
$$\implies K(\pi_1) \geq c + \frac{1}{2}[\tau_1 + \sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2 + \sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2 + \sum_{\substack{i,j=1\\i\neq j}}^{n}(h_{ij}^{n+1})^2$$
$$+ \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2] - \sum_{r=n+1}^{m}(h_{12}^r)^2 + \sum_{r=n+2}^{m} h_{11}^r h_{22}^r.$$

Arranging the terms of the right hand side in the last inequality, we derive

$$K(\pi_1) \geq c + \frac{1}{2}\tau_1 + \frac{1}{2}\sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2 + \frac{1}{2}\sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2$$
$$+ \sum_{r=n+2}^{m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{m}(h_{12}^r)^2 + \sum_{\substack{i,j=1\\i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2.$$

Applying [10, Lemma 2] with the last four terms of the above inequality, we find that

$$K(\pi_1) \geq c + \frac{1}{2}\tau_1 + \frac{1}{2}\sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2 + \frac{1}{2}\sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2 + \frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2$$
$$+ \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2 + \frac{1}{2}(h_{11}^{r} + h_{22}^{r})^2 + \sum_{r=n+1}^{m}\sum_{j+3}^{n}((h_{1j}^{r})^2 + (h_{2j}^{r})^2)$$
$$\Longrightarrow \; K(\pi_1) \geq c + \frac{1}{2}\tau_1 + \frac{1}{2}\sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2 + \frac{1}{2}\sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2 + \frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2,$$

and substituting

$$\frac{1}{2}\tau_1 = \tau(T_xM^n) - (\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 + \frac{1}{2(n_1-1)}(\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 - \frac{1}{2}(\sum_{A=n_1+1}^{n_1+n_2} h_{AA}^{n+1})^2 - \frac{1}{2}(\sum_{b=n_1+n_2+1}^{n} h_{bb}^{n+1})^2$$
$$- \sum_{a}\sum_{A} h_{aa}^{n+1}h_{AA}^{n+1} - \sum_{a}\sum_{b} h_{aa}^{n+1}h_{bb}^{n+1} - \sum_{A}\sum_{b} h_{AA}^{n+1}h_{bb}^{n+1} - n(n-1)c,$$

we get

$$K(\pi_1) \geq c + \tau(T_xM^n) - (\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 + \frac{1}{2(n_1-1)}(\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 - \frac{1}{2}(\sum_{A=n_1+1}^{n_1+n_2} h_{AA}^{n+1})^2$$
$$- \frac{1}{2}(\sum_{b=n_1+n_2+1}^{n} h_{bb}^{n+1})^2 - \sum_{a}\sum_{A} h_{aa}^{n+1}h_{AA}^{n+1} - \sum_{a}\sum_{b} h_{aa}^{n+1}h_{bb}^{n+1}$$
$$- \sum_{A}\sum_{b} h_{AA}^{n+1}h_{bb}^{n+1} - n(n-1)c + \frac{1}{2}\sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2$$
$$+ \frac{1}{2}\sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2 + \frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2$$
$$\Longrightarrow \; K(\pi_1) \geq c + \tau(T_xM^n) + \frac{1}{2(n_1-1)}(\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 - n(n-1)c + \frac{1}{2}\sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2$$
$$+ \frac{1}{2}\sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2 + \frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2 - \frac{n^2}{2}\|H\|^2.$$

Using (2.14), we have

$$K(\pi_1) \geq c + \frac{n_2\Delta^1 f_1}{f_1} + \frac{n_3\Delta^1 f_2}{f_2} - n_2n_3g(\nabla^1(\ln f_1), \nabla^1(\ln f_2)) + \tau(T_xN_1) + \tau(T_xN_2) + \tau(T_xN_3)$$
$$+ \frac{1}{2(n_1-1)}(\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 - n(n-1)c + \frac{1}{2}\sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2 + \frac{1}{2}\sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2$$

$$+\frac{1}{2}\sum_{\substack{i,j=3\\i\neq j}}^{n}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2-\frac{n^2}{2}\|H\|^2$$

$$\implies \tau(T_xN_1)-K(\pi_1)\leq\frac{n^2}{2}\|H\|^2-[c+\frac{n_2\Delta^1 f_1}{f_1}+\frac{n_3\Delta^1 f_2}{f_2}-n_2n_3g(\nabla^1(\ln f_1),\nabla^1(\ln f_2))$$

$$+\tau(T_xN_2)+\tau(T_xN_3)+\frac{1}{2(n_1-1)}(\sum_{a=1}^{n_1}h_{aa}^{n+1})^2-n(n-1)c+\frac{1}{2}\sum_{A=n_1+1}^{n_1+n_2}(h_{AA}^{n+1})^2$$

$$+\frac{1}{2}\sum_{b=n_1+n_2+1}^{n}(h_{bb}^{n+1})^2+\frac{1}{2}\sum_{\substack{i,j=3\\i\neq j}}^{n}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2].$$

From (2.3), we have

$$-2\tau_2(T_xN_2)=-2\bar{\tau}_2(T_xN_2)+\sum_{r=n+1}^{m}\sum_{A,A'=n_1+1}^{n_1+n_2}(h_{AA'}^{r})^2-\sum_{r=n+1}^{m}(h_{(n_1+1)(n_1+1)}^{r}+\cdots+h_{(n_1+n_2)(n_1+n_2)}^{r})^2$$

and

$$-2\tau_3(T_xN_3)=-2\bar{\tau}_3(T_xN_3)+\sum_{r=n+1}^{m}\sum_{b,b'=n_1+n_2+1}^{n}(h_{bb'}^{r})^2-\sum_{r=n+1}^{m}(h_{(n_1+n_2+1)(n_1+n_2+1)}^{r}+\cdots+h_{nn}^{r})^2.$$

So,

$$\begin{aligned}\tau(T_xN_1)-K(\pi_1)\;&\leq\frac{n^2}{2}\|H\|^2-\frac{n_2\Delta^1 f_1}{f_1}-\frac{n_3\Delta^1 f_2}{f_2}+n_2n_3g(\nabla^1(\ln f_1),\nabla^1(\ln f_2))+(\frac{n^2}{2}-\frac{n}{2}-1)c\\&-\bar{\tau}_2(T_xN_2)-\bar{\tau}_3(T_xN_3)-\frac{1}{2}\sum_{i,j=3}^{n}(h_{ij}^{n+2})^2-\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2-\frac{1}{2}\sum_{A}(h_{AA}^{n+1})^2\\&-\frac{1}{2}\sum_{b}(h_{bb}^{n+1})^2+\frac{1}{2}\sum_{r=n+1}^{m}\sum_{A,A'=n_1+1}^{n_1+n_2}(h_{AA'}^{r})^2+\frac{1}{2}\sum_{r=n+1}^{m}\sum_{b,b'=n_1+n_2+1}^{n}(h_{bb'}^{r})^2\\&\implies \tau(T_xN_1)-K(\pi_1)\leq\frac{n^2}{2}\|H\|^2-\frac{n_2\Delta^1 f_1}{f_1}-\frac{n_3\Delta^1 f_2}{f_2}+n_2n_3g(\nabla^1(\ln f_2),\nabla^1(\ln f_2))\\&+(\frac{n^2}{2}-\frac{n}{2}-1)c-\bar{\tau}_2(T_xN_2)-\bar{\tau}_3(T_xN_3)-\frac{1}{2}\sum_{\substack{a,a'=3\\a\neq a'}}^{n_1}(h_{aa'}^{n+1})^2-\frac{1}{2}\times 2\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n_1+n_2}(h_{aA}^{n+1})^2\\&-\frac{1}{2}\times 2\sum_{a=3}^{n_1}\sum_{b=n_1+n_2+1}^{n}(h_{ab}^{n+1})^2-\frac{1}{2}\times 2\sum_{A=n_1+1}^{n_1+n_2}\sum_{b=n_1+n_2+1}^{n}(h_{Ab}^{n+1})^2-\frac{1}{2}\sum_{r=n+2}^{m}\sum_{a,a'=3}^{n_1}(h_{aa'}^{r})^2\\&-\frac{1}{2}\times 2\sum_{r=n+2}^{m}\sum_{a}\sum_{A}(h_{aA}^{r})^2-\frac{1}{2}\times 2\sum_{r=n+2}^{m}\sum_{a}\sum_{b}(h_{ab}^{r})^2-\frac{1}{2}\times 2\sum_{r=n+2}^{m}\sum_{A}\sum_{b}(h_{Ab}^{r})^2.\end{aligned}$$

Putting

$$n=(n_1+n_2+n_3),\quad \bar{\tau}_2(T_xN_2)=\frac{n_2(n_2-1)}{2}$$

and

$$\bar{\tau}_3(T_xN_3) = \frac{n_3(n_3-1)}{2},$$

we have our inequality. Clearly, equalities hold if, and only if:

(i)

$$\alpha_1 + \alpha_2 = \alpha_3 = \cdots = \alpha_n,$$

which implies

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{n_1n_1}^{n+1},$$
$$\mu = \mu_1 + \mu_2 = h_{11}^{n+1} + h_{22}^{n+1}.$$

(ii)

$$\sum_{r=n+2}^{m}(h_{11}^r + h_{22}^r)^2 + \sum_{r=n+1}^{m}\sum_{j=3}^{n}((h_{1j}^r)^2 + (h_{2j}^r)^2) = 0.$$

(iii)

$$(\sum_{a=1}^{n_1} h_{aa}^{n+1})^2 = \sum_{r=n+1}^{m}(h_{(n_1+1)(n_1+1)}^r + \cdots + h_{(n_1+n_2)(n_1+n_2)}^r)^2 = \sum_{r=n+1}^{m}(h_{(n_1+n_2+1)(n_1+n_2+1)}^r + \cdots + h_{nn}^r)^2.$$

(iv)

$$\begin{aligned}&\sum_{\substack{a,a'=3\\a\neq a'}}^{n_1}(h_{aa'}^{n+1})^2 + \sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n_1+n_2}(h_{aA}^{n+1})^2 + \sum_{a=3}^{n_1}\sum_{b=n_1+n_2+1}^{n}(h_{ab}^{n+1})^2\\&+\sum_{A=n_1+1}^{n_1+n_2}\sum_{b=n_1+n_2+1}^{n}(h_{Ab}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{a,a'=3}^{n_1}(h_{aa'}^r)^2 + \sum_{r=n+2}^{m}\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n_1+n_2}(h_{aA}^r)^2\\&+\sum_{r=n+2}^{m}\sum_{a=3}^{n_1}\sum_{b=n_1+n_2+1}^{n}(h_{ab}^r)^2 + \sum_{r=n+2}^{m}\sum_{A=n_1+1}^{n_1+n_2}\sum_{b=n_1+n_2+1}^{n}(h_{Ab}^r)^2 = 0.\end{aligned}$$

That is,

$$A_{\varkappa_{n+1}} = \begin{pmatrix} BLOCK(I) & O_{n_1\times n_2} & O_{n_1\times n_3}\\ O_{n_2\times n_1} & BLOCK(II) & O_{n_2\times n_3}\\ O_{n_3\times n_1} & O_{n_3\times n_2} & BLOCK(III)\end{pmatrix},$$

where $BLOCK(I), BLOCK(II)$, and $BLOCK(III)$ are given, respectively, as follows:

$$\begin{pmatrix}\mu_1 & h_{12}^{n+1} & 0 & \cdots & 0_{1n_1}\\ h_{21}^{n+1} & \mu_2 & 0 & \cdots & 0_{2n_1}\\ 0 & 0 & \mu & \cdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0_{(n_1)(1)} & \cdots & \cdots & \cdots & \mu\end{pmatrix}, \begin{pmatrix} h_{(n_1+1)(n_1+1)}^{n+1} & \cdots & \cdots & h_{(n_1+1)(n_1+n_2)}^{n+1}\\ \vdots & & \cdots & \vdots\\ \vdots & & \cdots & \vdots\\ \vdots & & \cdots & \vdots\\ h_{(n_1+n_2)(n_1+1)}^{n+1} & & \cdots & h_{(n_1+n_2)(n_1+n_2)}^{n+1}\end{pmatrix}, \begin{pmatrix} h_{(n_1+n_2+1)(n_1+n_2+1)}^{n+1} & \cdots & \cdots & h_{(n_1+n_2+1)(n)}^{n+1}\\ \vdots & & \cdots & \vdots\\ \vdots & & \cdots & \vdots\\ \vdots & & \cdots & \vdots\\ h_{(n)(n_1+n_2+1)}^{n+1} & & \cdots & h_{(n)(n)}^{n+1}\end{pmatrix}.$$

For $r \in \{n+2, \cdots, m\}$, since

$$h^r_{11} + h^r_{22} = \sum_{j=3}^{n} h^r_{1j} = \sum_{j=3}^{n} h^r_{2j} = \sum_{a,a`=3}^{n_1} h^r_{aa`} = \sum_{a}\sum_{A} h^r_{aA} = \sum_{a}\sum_{b} h^r_{ab} = \sum_{A}\sum_{b} h^r_{Ab} = 0,$$

$$A_{\varkappa_r} = \begin{pmatrix} BLOCK(I) & O_{n_1\times n_2} & O_{n_1\times n_3} \\ O_{n_2\times n_1} & BLOCK(II) & O_{n_2\times n_3} \\ O_{n_3\times n_1} & O_{n_3\times n_2} & BLOCK(III) \end{pmatrix},$$

where $BLOCK(I), BLOCK(II)$, and $BLOCK(III)$ are given, respectively, as

$$\begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \cdots & 0_{1n_1} \\ h^r_{21} & -h^r_{11} & 0 & \cdots & 0_{2n_1} \\ 0 & 0 & 0_{33} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{(n_1)(1)} & \cdots & \cdots & \cdots & 0_{n_1n_1} \end{pmatrix}, \begin{pmatrix} h^r_{(n_1+1)(n_1+1)} \cdots & \cdots & h^r_{(n_1+1)(n_1+n_2)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h_{(n_1+n_2)(n_1+1)}cr & \cdots & h^r_{(n_1+n_2)(n_1+n_2)} \end{pmatrix}, \begin{pmatrix} h^r_{(n_1+n_2+1)(n_1+n_2+1)} \cdots & \cdots & h^r_{(n_1+n_2+1)(n)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ h^r_{(n)(n_1+n_2+1)} & \cdots & h^r_{(n)(n)} \end{pmatrix}.$$

This completes the inequality part of (1).

If equality of (i) or (ii) or (iii) holds, N^n is mixed totally geodesic in $M^m(c)$. Moreover, N^n is N_1-minimal, N_2-minimal, and N_3-minimal. Thus, N^n is a minimal warped product sub-manifold in the Riemannian space form $M^m(c)$. □

The above theorem provides partial answer to the Chen problem of finding a necessary condition for the bi-warped product sub-manifold of a Riemannian space form to be minimal. The conditions are stated in the following corollaries.

Corollary 3.1. *Let*

$$\phi : N^n = N_1 \times_{f_1} N_2 \times_{f_2} N_3$$

be an isometric immersion of a bi-warped product sub-manifold N^n into a Riemannian space form $M^m(c)$. Then, for each point $x \in N^n$,

$$\delta_{N_1{}^{n_1}} + \frac{n_2\Delta f_1}{f_1} + \frac{n_3\Delta f_2}{f_2} \leq n_2n_3 g(\nabla(\ln f_1), \nabla(\ln f_2)) + \frac{n_1}{2}(n_1 + 2n_2 + 2n_3 - 1)c + n_2n_3c - c,$$

and if the equality holds, then the immersion ϕ is minimal.

Corollary 3.2. *Let*

$$\phi : N^n = N_1 \times_{f_1} N_2 \times_{f_2} N_3$$

be an isometric immersion of a bi-warped product sub-manifold N^n into a Riemannian space form $M^m(c)$. Then, for each point $x \in N^n$,

$$\delta_{N_2{}^{n_2}} + \frac{n_2\Delta f_1}{f_1} + \frac{n_3\Delta f_2}{f_2} \leq n_2n_3 g(\nabla(\ln f_1), \nabla(\ln f_2)) + \frac{n_2}{2}(n_2 + 2n_1 + 2n_3 - 1)c + n_1n_3c - c,$$

and if the equality holds, then the immersion ϕ is minimal.

Corollary 3.3. *Let*

$$\phi : N^n = N_1 \times_{f_1} N_2 \times_{f_2} N_3$$

be an isometric immersion of a bi-warped product sub-manifold N^n into a Riemannian space form $M^m(c)$. Then, for each point $x \in N^n$,

$$\delta_{N_3{}^{n_3}} + \frac{n_2 \Delta f_1}{f_1} + \frac{n_3 \Delta f_2}{f_2} \leq n_2 n_3 g(\nabla(\ln f_1), \nabla(\ln f_2)) + \frac{n_3}{2}(n_3 + 2n_1 + 2n_2 - 1)c + n_1 n_2 c - c,$$

and if the equality holds, then the immersion ϕ is minimal.

4. Conclusions

If the warping function is $f_1 = 1$, then the first warped product becomes an ordinary product and a whole N^n becomes a simply warped product sub-manifold of Riemannian space form $M^m(c)$. So, Theorem 3.1 gives the results of [10]. The main limitation for our article is that we only investigated on Riemannian space forms without any additional structures on it. It should be noted that imposing additional structures like Sasakian, Kenmotsu, etc. may extend our results.

The Chen delta invariant has applications in physics, particularly in the study of topological field theories. In algebraic topology it measures the extent to which a loop in space fails to be the boundary of a surface. If a loop is the boundary of a surface, then the Chen delta invariant is zero [11]. We investigated the effect of warping functions on the Chen's delta invariant for bi-warped product sub-manifolds on Riemannian space forms. So finding the Chen inequality for various space forms with additional structures will be an interesting area to discover.

Author contributions

B. Bag: conceptualization, methodology, formal analysis, investigation, writing—original draft preparation, writing—review and editing; M. Ali Khan: validation, writing—original draft preparation, writing—review and editing, project administration, funding acquisition; T. Pal: methodology, validation, formal analysis; S. K. Hui: conceptualization, validation, investigation, supervision, project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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