Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(4): 9882–9899.
DOI: 10.3934/math.2025452
Received: 02 December 2024
Revised: 22 March 2025
Accepted: 15 April 2025
Published: 27 April 2025

Research article

On a dependent risk model perturbed by mixed-exponential jump-diffusion processes

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Abstract: In the present paper, we investigate a dependent risk model perturbed by a mixed-exponential jump-diffusion process, in which the claim inter-arrival times and claim sizes are dependent through Farlie-Gumbel-Morgenstern (FGM) copula. The expected discounted penalty (EDP) functions are studied when ruin is caused by a claim or the jump-diffusion process. The Laplace transforms satisfied by the EDP functions are obtained, then we give the corresponding defective renewal equations. The analytical expressions for the EDP functions are derived when the claim sizes follow exponential distributions, and a numerical example for the ruin probabilities are also provided.

Keywords: expected discounted penalty function; dependence; mixed-exponential jump-diffusion process; defective renewal equation; Laplace transform
Mathematics Subject Classification: 91B30, 91B70

1. Introduction

Consider the jump-diffusion process $\{L(t), t \geq 0\}$ as follows:

$$-L(t) = ct + \sigma B(t) + \sum_{i=1}^{M(t)} Y_i, \tag{1.1}$$

where constant $c > 0$ is the drift and $\sigma > 0$ is volatility for the diffusion term, $\{M(t), t \geq 0\}$ is a Poisson process with rate $\kappa > 0$, and $\{B(t), t \geq 0\}$ is a standard Wiener process with $B(0) = 0$. The random variables $\{Y_i, i \geq 1\}$ are independent and identically distributed (i.i.d.) with probability density function (p.d.f.) $f_Y(y)$. The processes $\{B(t)\}$, $\{M(t)\}$, and $\{Y_i\}$ are assumed to be independent.

The process $\{L(t), t \geq 0\}$ is a special case of Lévy processes. The corresponding exit problems for this kind of process are widely used in finance and actuarial science. In the theory of mathematical finance, see for example, Kou [1], Kou and Wang [2], Gao and Yin [3], Alili and Kyprianou [4], Cai and Kou [5]. In the setting of risk theory, Gerber [6] first proposed the perturbed risk process to model the investment of the surplus. Since then, such risk models have received a lot of attentions, see for example, (Dufresne and Gerber [7], Tsai [8, 9], Zhang and Yang [10], Adékambi and Takouda [11]). However, Aït-Sahalia and Jacod [12] proved that the normality behavior was insufficient to describe the leptokurtic property through empirical study. Chi [13] extended the classical risk process perturbed by diffusion to include the jumps. By using the Wiener-Hopf factorization method, they gave the analytical expression for the EDP function. Chi and Lin [14] further investigated the jump-diffusion risk model and the EDP function under the threshold dividend strategy. Yin et al. [15] studied the hyper-exponential jump-diffusion processes, and the explicit expressions of the dividend formulae are investigated under different strategies. Yin et al. [16] further considered the mixed-exponential jump-diffusion processes and gave some applications in finance and insurance. Zhang et al. [17] proposed a dependent risk model perturbed by a jump-diffusion process, in which dependence between the interclaim time and claim amounts is determined by some bivariate distribution. We remark that the risk models with different dependent structures have attracted more and more attentions in recent years, see for example, (Boudreault, et al. [18], Chadjiconstantinidis and Vrontos [19], Xie and Zou [20]).

The aim of the present paper is two-fold. First, motivated by Zhang et al. [17], we study the EDP functions for a dependent risk model perturbed by mixed-exponential jump-diffusion process, in which the claim amounts and claim inter-arrival times are dependent through FGM copula. Second, we obtain analytical expressions for the Laplace transforms and defective renewal equations satisfied by the EDP functions.

Consider the surplus process $\{U(t), t \geq 0\}$ for an insurance company given by

$$U(t) = u - L(t) - \sum_{i=1}^{N(t)} X_i,$$

where $u \geq 0$ is initial surplus and $L(t)$ is defined by (1.1). $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, and interclaim times $\{W_i, i \geq 1\}$ are identically distributed as the canonical random variable W with p.d.f. f_W and cumulative distribution function (c.d.f.) F_W. The claim amounts $\{X_i, i \geq 1\}$ are i.i.d. positive random variables with p.d.f. f_X, c.d.f. F_X. Suppose that $\{L(t), t \geq 0\}$ is independent of $\{N(t), t \geq 0\}$ and $\{X_i, i \geq 1\}$.

The sequence $\{(W_i, X_i), i \geq 1\}$ is assumed to be i.i.d. and distributed like a canonical random vector (W, X). However, the dependence structure between $\{W_i, i \geq 1\}$ and $\{X_i, i \geq 1\}$ through the FGM copula. More precisely, the joint p.d.f. $f_{W,X}$ of (W, X) has the following form

$$f_{W,X}(t, x) = f_X(x)f_W(t) + \theta f_X(x)f_W(t)(1 - 2F_X(x))(1 - 2F_W(t)), (t, x) \in R^+ \times R^+,$$

where $-1 \leq \theta \leq 1$. The readers are referred to Chadjiconstantinidis and Vrontos [19] for more details on the FGM copula. It is easy to see that $f_{X,W}(x, t)$ can be further rewritten as

$$f_{W,X}(t, x) = \lambda e^{-\lambda t} f_X(x) + \theta(2\lambda e^{-2\lambda t} - \lambda e^{-\lambda t})h_X(x), \tag{1.2}$$

where $h_X(x) = (1 - 2F_X(x))f_X(x)$.

Furthermore, the jumps $\{Y_i\}$ are assumed to follow mixed-exponentially distribution, that is,

$$f_Y(y) = p_u \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} \mathbb{I}_{(y>0)} + q_d \sum_{j=1}^{n} \vartheta_j \theta_j e^{\theta_j y} \mathbb{I}_{(y<0)},$$

where $p_u \geq 0$, $q_d = 1-p_u$, $\sum_{i=1}^{m} p_i = 1$, $\sum_{j=1}^{n} \vartheta_j = 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m$, and $0 < \theta_1 < \theta_2 < \cdots < \theta_n$. $\mathbb{I}_{(A)}$ is the indicator function of the event A.

Let $\tau = \inf_{t\geq 0}\{t, U(t) \leq 0\}$ be the time of ruin, where $\tau = \infty$ if ruin does not occur in finite time. To ensure that ruin will not occur almost surely, the following safety loading condition is needed, that is,

$$c + \kappa\Big(p_u \sum_{i=1}^{m} \frac{p_i}{\eta_i} + q_d \sum_{i=1}^{n} \frac{-\vartheta_j}{\theta_j}\Big) > \lambda EX. \tag{1.3}$$

In this article, we aim to evaluate the EDP function proposed by Gerber and Shiu [21]. For $\delta \geq 0$, the EDP function is defined by

$$\phi(u) = E[e^{-\delta\tau}\omega(U(\tau-), \mid U(\tau) \mid)\mathbb{I}_{(\tau<\infty)} \mid U(0) = u],$$

where $U(\tau-)$ is the surplus prior to ruin, $\mid U(\tau) \mid$ is the deficit at ruin, and $\omega(x, y)$ is the penalty function for $x, y \geq 0$. From practical perspectives, He et al. [22] presented comprehensive comments on the EDP function in the existing actuarial literature.

Note that ruin can be caused by either a claim or the jump-diffusion process (1.1). Thus, the EDP function can be decomposed as

$$\phi(u) = \phi_\zeta(u) + \phi_\varepsilon(u),$$

where

$$\phi_\zeta(u) = E[e^{-\delta\tau}\omega(U(\tau-), \mid U(\tau) \mid)\mathbb{I}_{(\tau<\infty, U(\tau)=0)} \mid U(0) = u], \tag{1.4}$$

$$\phi_\varepsilon(u) = E[e^{-\delta\tau}\omega(U(\tau-), \mid U(\tau) \mid)\mathbb{I}_{(\tau<\infty, U(\tau)<0)} \mid U(0) = u]. \tag{1.5}$$

Without loss of generality, it is supposed that $\omega(0, 0) = 1$.

The rest of the paper is organized as follows. Section 2 gives analytical expression for a q-potential measure related to the jump-diffusion process. In Section 3, we obtain some integral expressions and the Laplace transforms of the EDP functions. The defective renewal equations are obtained in Section 4. Section 5 gives explicit expressions when the individual claim amounts follow exponential distributions, and a numerical example is also provided. Finally, some discussions are made in Section 6.

2. Some preliminaries

Unless otherwise stated, we add a hat above a letter to indicate its Laplace transform.

Define $\bar{L}(t) = \sup_{0\leq s\leq t} L(s)$. Let e_q follow an exponential distribution with parameter q and $e_0 = \infty$. For $q \geq 0$, the Wiener-Hopf factorization implies that $\bar{L}(e_q)$ and $L(e_q) - \bar{L}(e_q), q \geq 0$ are independent and infinitely divisible. Therefore, we have

$$E[\exp(-s\bar{L}(e_q))]E[\exp(-s[L(e_q) - \bar{L}(e_q)])] = E[\exp(-sL(e_q))] = \frac{q}{q - G(s)}, \tag{2.1}$$

where

$$G(s) = \frac{\sigma^2}{2}s^2 + cs + \kappa\Big(p_u \sum_{i=1}^{m} \frac{p_i\eta_i}{\eta_i - s} + q_d \sum_{j=1}^{n} \frac{\vartheta_j\theta_j}{\theta_j + s} - 1\Big).$$

For $q > 0$, the equation $G(s) = q$ has exactly $m + 1$ positive roots $-\beta_{1,q}, -\beta_{2,q}, \cdots, -\beta_{m+1,q}$ and $n + 1$ negative roots $-\gamma_{1,q}, -\gamma_{2,q}, \cdots, -\gamma_{n+1,q}$ satisfying

$$0 < -\beta_{1,q} < \eta_1 < -\beta_{2,q} < \cdots < \eta_m < -\beta_{m+1,q} < \infty,$$

$$0 < \gamma_{1,q} < \theta_1 < \gamma_{2,q} < \cdots < \theta_n < \gamma_{n+1,q} < \infty,$$

see Yin et al. [16] for more details. Then, (2.1) can be rewritten as

$$\hat{f}_{-,q}(s)\hat{f}_{+,q}(s) = \frac{q}{q - G(s)} = \frac{\frac{2q}{\sigma^2}\prod_{i=1}^{m}(\eta_i - s)\prod_{j=1}^{n}(\theta_j + s)}{\prod_{i=1}^{m+1}(s + \beta_{i,q})\prod_{j=1}^{n+1}(s + \gamma_{j,q})}, \tag{2.2}$$

where $f_{+,q}$ and $f_{-,q}$ are the p.d.f. of $\bar{L}(e_q)$ and $L(e_q) - \bar{L}(e_q)$, respectively.

Since $E[\exp(-s\bar{L}(e_q))]$ and $E[\exp(-s(L(e_q) - \bar{L}(e_q)))]$ are analytic for $Re(s) \geq 0$ and $Re(s) \leq 0$, then by (2.2) and partial fraction we can obtain

$$\hat{f}_{-,q}(s) = \sum_{i=1}^{m+1}\frac{a_{-,i,q}}{s + \beta_{i,q}}, \quad \hat{f}_{+,q}(s) = \sum_{j=1}^{n+1}\frac{a_{+,j,q}}{s + \gamma_{j,q}}, \tag{2.3}$$

where $a_{-,i,q}$ $(i = 1, 2, \cdots, m + 1)$, $a_{+,j,q}$ $(j = 1, 2, \cdots, n + 1)$ are some constants.

Inverting the Laplace transforms in (2.3) gives us

$$f_{-,q}(x) = -\sum_{i=1}^{m+1} a_{-,i,q}\exp(-\beta_{i,q}x), x < 0, \tag{2.4}$$

$$f_{+,q}(x) = \sum_{j=1}^{n+1} a_{+,j,q}\exp(-\gamma_{j,q}x), x > 0. \tag{2.5}$$

Let $\tau_u = \inf(t \geq 0 : L(t) > u)$ be the first time when $L(t)$ cross upwards u. In particular, by Yin et al. [16] we have

$$E[e^{-q\tau_u}; L(\tau_u) = u] = \sum_{j=1}^{n+1} B_j A_j \exp(-\gamma_{j,q}u), \tag{2.6}$$

where

$$B_j = \frac{\prod_{k=1}^{n}(1 - \frac{\gamma_{j,q}}{\theta_k})}{\prod_{k=1,k\neq j}^{n+1}(1 - \frac{\gamma_{j,q}}{\gamma_{k,q}})}, \quad A_j = \frac{\prod_{k=1}^{n}\theta_k}{\prod_{k=1,k\neq j}^{n+1}\gamma_{k,q}}.$$

For $q > 0$, we define a q-potential measure as follows:

$$R^{(q)}(u; dx) = \int_0^\infty e^{-qt}P(u - L(t) \in \mathrm{d}x, \tau_u > t)\mathrm{d}t. \tag{2.7}$$

By (23) of Zhang et al. [17], we have

$$R^{(q)}(u; dx) = \frac{1}{q}\int_{y\in[0\vee(u-x),u]} f_{+,q}(y)f_{-,q}(u - y - x)dydx. \tag{2.8}$$

For $0 \le x \le u$, by submitting (2.4) and (2.5) into (2.8) and some straightforward calculations, we obtain

$$R^{(q)}(u; \mathrm{d}x) = \frac{1}{q} \sum_{j_1=1}^{m+1} \sum_{j_2=1}^{n+1} \frac{a_{-,j_1,q} a_{+,j_2,q}}{\gamma_{j_2,q} - \beta_{j_1,q}} [\exp(-\gamma_{j_2,q} u + \beta_{j_1,q} x) - \exp(-\gamma_{j_2,q}(u - x))]\mathrm{d}x. \tag{2.9}$$

Analogously, for $x > u$, we get

$$R^{(q)}(u; \mathrm{d}x) = \frac{1}{q} \sum_{j_1=1}^{m+1} \sum_{j_2=1}^{n+1} \frac{a_{-,j_1,q} a_{+,j_2,q}}{\gamma_{j_2,q} - \beta_{j_1,q}} [\exp(-\gamma_{j_2,q} u + \beta_{j_1,q} x) - \exp(-\beta_{j_1,q}(u - x))]\mathrm{d}x. \tag{2.10}$$

3. Laplace transforms for the EDP functions

3.1. Laplace transform for $\phi_\zeta(u)$

By conditioning on whether bankruptcy occurred by jump-diffusion process before the first claim, we have

$$\begin{aligned} \phi_\zeta(u) &= \int_0^\infty \int_0^\infty \mathbb{E}[e^{-\delta\tau} \omega(U(\tau-), |U(\tau)|) \mathbb{I}_{(\tau<t)} | U(0) = u] f_{W,X}(t, x)\mathrm{d}t\mathrm{d}x \\ &\quad + \int_0^\infty \int_0^\infty e^{-\delta t} \mathbb{E}[\phi_\zeta(u - L(t) - x) \mathbb{I}_{(\bar{L}(t)<u, x+L(t)<u)}] f_{W,X}(t, x)\mathrm{d}t\mathrm{d}x \\ &= I_1(u) + I_2(u). \end{aligned}$$

First, we deal with $I_1(u)$. It is apparent that

$$\begin{aligned} I_1(u) &= \int_0^\infty \int_0^\infty \mathbb{E}[e^{-\delta\tau_u} \omega(u - L(\tau_u-), L(\tau_u) - u) \mathbb{I}_{(\tau_u<t)}] f_{W,X}(t, x)\mathrm{d}t\mathrm{d}x \\ &= \int_0^\infty \int_0^\infty \int_0^t e^{-\delta s} \omega(u - L(s), L(s) - u) f_{\tau_u}(s)\mathrm{d}s f_{W,X}(t, x)\mathrm{d}t\mathrm{d}x. \end{aligned} \tag{3.1}$$

Substituting $f_{W,X}(t, x)$ into (3.1) and noting that $\int_0^\infty f_X(x)\mathrm{d}x = 1$ and $\int_0^\infty h_X(x)\mathrm{d}x = 0$, we have

$$I_1(u) = \mathbb{E}[e^{-q_1\tau_u} \omega(U(\tau_u-), |U(\tau_u)|)],$$

where $q_1 = \lambda + \delta$. For notational convenience, denote by $q_2 = \lambda + q_1$.

By examining if $L(\tau_u) = u$ occurs, we have

$$\begin{aligned} I_1(u) &= \mathbb{E}[e^{-q_1\tau_u}; L(\tau_u) = u] + \mathbb{E}[e^{-q_1\tau_u} \omega(u - L(\tau_u), L(\tau_u) - u); L(\tau_u) > u] \\ &= I_{1,1}(u) + I_{1,2}(u). \end{aligned} \tag{3.2}$$

Replacing q by q_1 in (2.6) gives

$$I_{1,1}(u) = \sum_{j=1}^{n+1} B_j A_j \exp(-\gamma_{j,q_1} u).$$

Solving the Laplace transform of $I_{1,1}(u)$ leads to

$$\begin{aligned}\hat{I}_{1,1}(s) &= \sum_{j=1}^{n+1} B_j A_j \int_0^\infty \exp\big(-(s+\gamma_{j,q_1})u\big)\mathrm{d}u \\ &= \sum_{j=1}^{n+1} \frac{B_j A_j}{s+\gamma_{j,q_1}} \\ &= \frac{\pi(s)}{q_1 - G(s)}, \end{aligned} \tag{3.3}$$

where

$$\pi(s) = \frac{\sigma^2 \prod_{k=1}^{m+1}(s+\beta_{k,q_1}) \sum_{j=1}^{n+1}\Big(B_j A_j \prod_{k=1,k\neq j}^{n+1}(s+\gamma_{k,q_1})\Big)}{2\prod_{k=1}^{m}(\eta_k - s)\prod_{k=1}^{n}(\theta_k + s)}. \tag{3.4}$$

Let N_W be the Poisson random measure associated with the jumps of $L(t)$. The p.d.f. of N_W is denoted by $\kappa f_Y(-y)\mathrm{d}t\mathrm{d}y$. Then we have

$$\begin{aligned}I_{1,2}(u) &= \mathbb{E}\int_0^\infty\int_0^\infty e^{-q_1 t}\omega(u-L(t-), y-u+L(t-))\mathbb{I}_{(\bar{L}(t-)\le u, y\ge u-L(t-))}N_W(\mathrm{d}t,\mathrm{d}x) \\ &= \int_0^\infty\int_0^\infty\int_x^\infty e^{-q_1 t}\omega(x,y-x)\kappa q_d \sum_{j=1}^{n}\theta_j e^{-\theta_j y}\mathrm{d}y P(u-L(t)\in \mathrm{d}x, \tau_u > t)\mathrm{d}t \\ &= \int_0^\infty z(x)R^{(q_1)}(u;\mathrm{d}x), \end{aligned} \tag{3.5}$$

where

$$z(x) = \kappa q_d \sum_{j=1}^{n}\theta_j \int_x^\infty \omega(x, y-x)e^{-\theta_j y}\mathrm{d}y.$$

Substituting $R^{(q_1)}(u;\mathrm{d}x)$ together with (2.9) and (2.10), Eq (3.5) becomes

$$\begin{aligned}I_{1,2}(u) &= \frac{1}{q_1}\sum_{j_1=1}^{m+1}\sum_{j_2=1}^{n+1}\frac{a_{-,j_1,q_1}a_{+,j_2,q_1}}{\gamma_{j_2,q_1}-\beta_{j_1,q_1}} \times \bigg[\int_0^\infty z(x)\exp(-\gamma_{j_2,q_1}u + \beta_{j_1,q_1}x)\mathrm{d}x \\ &\quad - \int_0^u z(x)\exp(-\gamma_{j_2,q_1}(u-x))\mathrm{d}x - \int_u^\infty z(x)\exp(-\beta_{j_1,q_1}(u-x))\mathrm{d}x\bigg]. \end{aligned} \tag{3.6}$$

Based on (3.6), we can obtain

$$\begin{aligned}\hat{I}_{1,2}(s) &= \frac{1}{q_1}\sum_{j_1=1}^{m+1}\sum_{j_2=1}^{n+1}\frac{a_{-,j_1,q_1}a_{+,j_2,q_i}}{\gamma_{j_2,q_1}-\beta_{j_1,q_1}} \times \bigg[\frac{\hat{z}(-\beta_{j_1,q_1})}{s+\gamma_{j_2,q_1}} - \frac{\hat{z}(s)}{s+\gamma_{j_2,q_1}} - \frac{\hat{z}(-\beta_{j_1,q_1})-\hat{z}(s)}{s+\beta_{j_1,q_1}}\bigg] \\ &= \frac{1}{q_1}\sum_{j_1=1}^{m+1}\sum_{j_2=1}^{n+1}\frac{a_{-,j_1,q_1}a_{+,j_2,q_1}}{\beta_{j_1,q_1}-\gamma_{j_2,q_1}} \times \bigg[\frac{(\beta_{j_1,q_1}-\gamma_{j_2,q_1})(\hat{z}(s)-\hat{z}(-\beta_{j_1,q_1}))}{(s+\gamma_{j_2,q_1})(s+\beta_{j_1,q_1})}\bigg] \\ &= \frac{\hat{z}(s)}{q_1 - G(s)} - \hat{f}_{+,q_1}(s)\sum_{j=1}^{m+1}\frac{a_{-,j,q_1}\hat{z}(-\beta_{j,q_1})}{q_1(s+\beta_{j,q_1})}. \end{aligned} \tag{3.7}$$

Now, we deal with $I_2(u)$.

$$\begin{aligned} I_2(u) &= \int_0^\infty \int_0^\infty e^{-\delta t}\mathbb{E}[\phi_\zeta(u-L(t)-x)\mathbb{I}_{(x<u-L(t),\tau_u>t)}]f_{W,X}(t,x)\mathrm{d}t\mathrm{d}x \\ &= \lambda\int_0^\infty (\eta_{\zeta,1}(y)-\eta_{\zeta,2}(y))R^{(q_1)}(u;\mathrm{d}y) + 2\lambda\int_0^\infty \eta_{\zeta,2}(y)R^{(q_2)}(u;\mathrm{d}y), \end{aligned} \tag{3.8}$$

where $\eta_{\zeta,1}(y)=\int_0^y f_X(x)\phi_\zeta(y-x)\mathrm{d}x$, $\eta_{\zeta,2}(y)=\int_0^y \theta h_X(x)\phi_\zeta(y-x)\mathrm{d}x$.

Similar to (3.7), taking Laplace transforms on both sides of (3.8) leads to

$$\begin{aligned} \hat{I}_2(s) &= \lambda\Big[\frac{\hat{\eta}_{\zeta,1}(s)-\hat{\eta}_{\zeta,2}(s)}{q_1-G(s)} - \hat{f}_{+,q_1}(s)\sum_{j=1}^{m+1}\frac{a_{-,j,q_1}[\hat{\eta}_{\zeta,1}(-\beta_{j,q_1})-\hat{\eta}_{\zeta,2}(-\beta_{j,q_1})]}{q_1(s+\beta_{j,q_1})}\Big] \\ &\quad +2\lambda\Big[\frac{\hat{\eta}_{\zeta,2}(s)}{q_2-G(s)} - \hat{f}_{+,q_2}(s)\sum_{j=1}^{m+1}\frac{a_{-,j,q_2}\hat{\eta}_{\zeta,2}(-\beta_{j,q_2})}{q_2(s+\beta_{j,q_2})}\Big]. \end{aligned} \tag{3.9}$$

By noting that $\hat{\eta}_{\zeta,1}(s)=\hat{f}_X(s)\hat{\phi}_\zeta(s)$ and $\hat{\eta}_{\zeta,2}(s)=\theta\hat{h}_X(s)\hat{\phi}_\zeta(s)$, we have

$$\hat{\phi}_\zeta(s) = \frac{\frac{\pi(s)+\hat{z}(s)}{q_1-G(s)} - \sum_{i=1}^{2}\sum_{j=1}^{m+1}\frac{l_i(j)\hat{f}_{+,q_i}(s)}{s+\beta_{j,q_i}}}{1-\lambda\frac{\hat{f}_X(s)-\theta\hat{h}_X(s)}{q_1-G(s)} - 2\lambda\frac{\theta\hat{h}_X(s)}{q_2-G(s)}}, \tag{3.10}$$

where

$$l_1(j) = \frac{a_{-,j,q_1}}{q_1}[\hat{z}(-\beta_{j,q_1}) + \lambda(\hat{\eta}_{\zeta,1}(-\beta_{j,q_1})-\hat{\eta}_{\zeta,2}(-\beta_{j,q_1}))],$$

$$l_2(j) = \frac{a_{-,j,q_2}}{q_2}[2\lambda\hat{\eta}_{\zeta,2}(-\beta_{j,q_2})].$$

3.2. Laplace transform for $\phi_\varepsilon(u)$

Now, we condition on the arrival of the first claim. Thus,

$$\begin{aligned} \phi_\varepsilon(u) &= \int_0^\infty\int_0^\infty e^{-\delta t}\mathbb{E}[\phi_\varepsilon(u-L(t)-x)\mathbb{I}_{(\bar{L}(t)<u,x+L(t)<u)}]f_{W,X}(t,x)\mathrm{d}t\mathrm{d}x \\ &\quad +\int_0^\infty\int_0^\infty \mathbb{E}[e^{-\delta\tau}\omega(u-L(t),x+L(t)-u)\mathbb{I}_{(\bar{L}(t)<u,x+L(t)>u)}]f_{W,X}(t,x)\mathrm{d}t\mathrm{d}x \\ &= J_1(u)+J_2(u). \end{aligned} \tag{3.11}$$

Substituting $f_{W,X}(x,t)$ together with (2.7), we derive

$$J_1(u) = \lambda\int_0^\infty (\eta_{\varepsilon,1}(y)-\eta_{\varepsilon,2}(y))R^{(q_1)}(u;\mathrm{d}y) + 2\lambda\int_0^\infty \eta_{\varepsilon,2}(y)R^{(q_2)}(u;\mathrm{d}y), \tag{3.12}$$

$$J_2(u) = \lambda\int_0^\infty (\omega_1(y)-\omega_2(y))R^{(q_1)}(u;\mathrm{d}y) + 2\lambda\int_0^\infty \omega_2(y)R^{(q_2)}(u;\mathrm{d}y), \tag{3.13}$$

where

$$\eta_{\varepsilon,1}(y) = \int_0^y f_X(x)\phi_\varepsilon(y-x)\mathrm{d}x,\ \eta_{\varepsilon,2}(y) = \int_0^y \theta h_X(x)\phi_\varepsilon(y-x)\mathrm{d}x,$$

$$\omega_1(y) = \int_y^\infty f_X(x)\omega(y, x-y)\mathrm{d}x,\ \omega_2(y) = \int_y^\infty \theta h_X(x)\omega(y, x-y)\mathrm{d}x.$$

Taking Laplace transforms on both sides of (3.12) and (3.13) yields

$$\begin{aligned}\hat{J}_1(s) &= \lambda\Big[\frac{\hat{\eta}_{\varepsilon,1}(s)-\hat{\eta}_{\varepsilon,2}(s)}{q_1-G(s)} - \hat{f}_{+,q_1}(s)\sum_{j=1}^{m+1}\frac{a_{-,j,q_1}[\hat{\eta}_{\varepsilon,1}(-\beta_{j,q_1})-\hat{\eta}_{\varepsilon,2}(-\beta_{j,q_1})]}{q_1(s+\beta_{j,q_1})}\Big]\\ &\quad +2\lambda\Big[\frac{\hat{\eta}_{\varepsilon,2}(s)}{q_2-G(s)} - \hat{f}_{+,q_2}(s)\sum_{j=1}^{m+1}\frac{a_{-,j,q_2}\hat{\eta}_{\varepsilon,2}(-\beta_{j,q_2})}{q_2(s+\beta_{j,q_2})}\Big],\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\hat{J}_2(s) &= \lambda\Big[\frac{\hat{\omega}_1(s)-\hat{\omega}_2(s)}{q_1-G(s)} - \hat{f}_{+,q_1}(s)\sum_{j=1}^{m+1}\frac{a_{-,j,q_1}[\hat{\omega}_1(-\beta_{j,q_1})-\hat{\omega}_2(-\beta_{j,q_1})]}{q_1(s+\beta_{j,q_1})}\Big]\\ &\quad +2\lambda\Big[\frac{\hat{\omega}_2(s)}{q_2-G(s)} - \hat{f}_{+,q_2}(s)\sum_{j=1}^{m+1}\frac{a_{-,j,q_2}\hat{\omega}_2(-\beta_{j,q_2})}{q_2(s+\beta_{j,q_2})}\Big],\end{aligned} \tag{3.15}$$

where $\hat{\eta}_{\varepsilon,1}(s) = \hat{f}_X(s)\hat{\phi}_\varepsilon(s)$, $\hat{\eta}_{\varepsilon,2}(s) = \theta\hat{h}_X(s)\hat{\phi}_\varepsilon(s)$. Substituting (3.14) and (3.15) into the Laplace transforms $\hat{\phi}_\varepsilon(s)$, some rearrangements yield

$$\hat{\phi}_\varepsilon(s) = \frac{\lambda[\frac{\hat{\omega}_1(s)-\hat{\omega}_2(s)}{q_1-G(s)}] + 2\lambda[\frac{\hat{\omega}_2(s)}{q_2-G(s)}] - \sum_{i=1}^{2}\sum_{j=1}^{m+1}\frac{d_i(j)\hat{f}_{+,q_i}(s)}{s+\beta_{j,q_i}}}{1-\lambda\frac{\hat{f}_X(s)-\theta\hat{h}_X(s)}{q_1-G(s)} - 2\lambda\frac{\theta\hat{h}_X(s)}{q_2-G(s)}}, \tag{3.16}$$

where

$$d_1(j) = \frac{\lambda a_{-,j,q_1}}{q_1}[\hat{\eta}_{\varepsilon,1}(-\beta_{j,q_1}) + \hat{\omega}_1(-\beta_{j,q_1}) - \hat{\eta}_{\varepsilon,2}(-\beta_{j,q_1}) - \hat{\omega}_2(-\beta_{j,q_1})],$$

$$d_2(j) = \frac{2\lambda a_{-,j,q_2}}{q_2}[\hat{\eta}_{\varepsilon,2}(-\beta_{j,q_2}) + \hat{\omega}_2(-\beta_{j,q_2})].$$

To identify the analytical expressions fo $\hat{\phi}_\zeta(s)$ and $\hat{\phi}_\varepsilon(s)$, the constants $l_i(j)$ and $d_i(j)$ should be determined. Consider the following generalized Lundberg equation:

$$1 = \lambda\frac{\hat{f}_X(s)+\theta\hat{h}_X(s)}{q_1-G(s)} + 2\lambda\frac{\theta\hat{h}_X(s)}{q_2-G(s)}. \tag{3.17}$$

Lemma 3.1. *For $\delta > 0$ the generalized Lundberg equation (3.17) has $2m+2$ roots, say $\rho_1(\delta)\cdots\rho_{2m+2}(\delta)$ such that $Re(\rho_i(\delta)) > 0$ for i=1,...,2m+2.*

Proof. Equation (3.17) can be rewritten as

$$\lambda(\hat{f}_X(s)+\theta\hat{h}_X(s))(q_2-G(s)) + 2\lambda\theta\hat{h}_X(s)(q_1-G(s)) = (q_1-G(s))(q_2-G(s)).$$

The discussions in Section 2 imply that $(q_1 - G(s))(q_2 - G(s)) = 0$ has exactly $2m + 2$ roots in the right half complex plane. By the Rouch$\acute{e}$ theorem, it is sufficient to show that

$$|\lambda(\hat{f}_X(s) + \theta\hat{h}_X(s))(q_2 - G(s)) + 2\lambda\theta\hat{h}_X(s)(q_1 - G(s))| < |(q_1 - G(s))(q_2 - G(s))|,$$

which can be completed by imitating the same steps as Proposition 1 of Chadjiconstantinidis and Vrontos [19]. □

Without loss of generality, let $\rho_1(\delta)$ be the root with the smallest real part, it tends to zero as $\delta \to 0$. Furthermore, it is a simple root due to the net profit condition (1.3). For the sake of symbol simplicity, we denote these $2m + 2$ roots by $\rho_1 \cdots \rho_{2m+2}$ and suppose that the roots are distinct.

Since $\hat{\phi}_\zeta(s)$ and $\hat{\phi}_\varepsilon(s)$ are analytic for $Re(s) \geq 0$, by (3.10), (3.16), and Lemma 3.1, we obtain

$$\frac{\pi(\rho_k) + \hat{z}(\rho_k)}{q_1 - G(\rho_k)} = \sum_{i=1}^{2}\sum_{j=1}^{m+1} \frac{l_i(j)\hat{f}_{+,q_i}(\rho_k)}{\rho_k + \beta_{j,q_i}}, \tag{3.18}$$

$$\lambda\left[\frac{\hat{\omega}_1(\rho_k) - \hat{\omega}_2(\rho_k)}{q_1 - G(\rho_k)}\right] + 2\lambda\left[\frac{\hat{\omega}_2(\rho_k)}{q_2 - G(\rho_k)}\right] = \sum_{i=1}^{2}\sum_{j=1}^{m+1} \frac{d_i(j)\hat{f}_{+,q_i}(\rho_k)}{\rho_k + \beta_{j,q_i}}, \tag{3.19}$$

for $k = 1, ..., 2m + 2$. Therefore, the unknown constants $l_i(j)$ and $d_i(j)$ can be given explicitly through solving the linear Eqs (3.18) and (3.19).

4. Defective renewal equations

In this section, we need an operator T_s introduced by Dickson and Hipp [23] as follows. For a measurable function f, define

$$T_s f(x) = \int_x^\infty \exp(-s(y - x)) f(y) \mathrm{d}y,$$

see Li and Garrido [24] for more properties of this operator.

We first investigate the common denominator in (3.10) and (3.16). Define

$$A(s) = \prod_{j=1}^{m+1}(s + \beta_{j,q_1})(s + \beta_{j,q_2}),\ A_i(s) = \frac{A(s)}{\prod_{j=1}^{m+1}(s + \beta_{j,q_i})}, i = 1, 2,$$

$$\hat{g}_1(s) = \frac{2\lambda}{\sigma^2}(\hat{f}_X(s) - \theta\hat{h}_X(s))\prod_{i=1}^{m}(\eta_i - s)\hat{f}_{+,q_1}(s),\ \hat{g}_2(s) = \frac{4\lambda}{\sigma^2}(\theta\hat{h}_X(s))\prod_{i=1}^{m}(\eta_i - s)\hat{f}_{+,q_2}(s),$$

we assume that $\hat{g}_1(s)$ and $\hat{g}_2(s)$ are the Laplace transforms of functions $g_1(x)$ and $g_2(x)$.

The denominator in (3.10) can be rewritten as

$$D(s) = \frac{1}{A(s)}[A(s) - A_1(s)\hat{g}_1(s) - A_2(s)\hat{g}_2(s)].$$

Note that $A(s)$ is a polynomial with leading coefficient 1 and degree $2m+2$. Taylor's expansion formula implies that

$$A(s) = A(\rho_k) + A^{'}(\rho_k)(s-\rho_k) + \cdots + \frac{A^{(2m+1)}}{(2m+1)!}(s-\rho_k)^{2m+1} + (s-\rho_k)^{2m+2}.$$

By Lemma 3.1 in Li and Garrido [24], we obtain

$$\sum_{k=1}^{2m+2} \frac{\frac{A(s)-A(\rho_k)}{s-\rho_k}}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)} = \sum_{k=1}^{2m+2} \frac{A^{'}(\rho_k) + \cdots + \frac{A^{(2m+1)}}{(2m+1)!}(s-\rho_k)^{2m} + (s-\rho_k)^{2m+1}}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)} = -1.$$

Similarly, for $i = 1, 2$, we have

$$\sum_{k=1}^{2m+2} \frac{\frac{A_i(s)-A_i(\rho_k)}{s-\rho_k}}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)} = \sum_{k=1}^{2m+2} \frac{A_i^{'}(\rho_k) + \cdots + \frac{A_i^{(m-1)}}{(m-1)!}(s-\rho_k)^{m-1} + (s-\rho_k)^{m}}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)} = 0.$$

Let

$$\tau(s) = \prod_{i=1}^{2m+2}(s-\rho_i).$$

Then, $A(s) - \tau(s)$ is a polynomial with degree $2m+1$. Lagrange interpolation formula leads to

$$A(s) - \tau(s) = \sum_{k=1}^{2m+2} \prod_{l=1,l\neq k}^{2m+2} \frac{s-\rho_l}{\rho_k-\rho_l}[A_1(\rho_k)\hat{g}_1(\rho_k) + A_2(\rho_k)\hat{g}_2(\rho_k)].$$

Thus, the denominator in (3.10) can be calculated as

$$\begin{aligned} D(s) &= \frac{1}{A(s)}\Big[\tau(s) + \sum_{k=1}^{2m+2} \prod_{l=1,l\neq k}^{2m+2} \frac{s-\rho_l}{\rho_k-\rho_l}[A_1(\rho_k)\hat{g}_1(\rho_k) + A_2(\rho_k)\hat{g}_2(\rho_k)] - A_1(s)\hat{g}_1(s) - A_2(s)\hat{g}_2(s)\Big] \\ &= \frac{\tau(s)}{A(s)}\Big[1 - \sum_{i=1}^{2}\Big(\frac{A_i(s)\hat{g}_i(s)}{\tau(s)} - \sum_{k=1}^{2m+2} \prod_{l=1,l\neq k}^{2m+2} \frac{A_i(\rho_k)\hat{g}_i(\rho_k)}{(s-\rho_k)(\rho_k-\rho_l)}\Big)\Big] \\ &= \frac{\tau(s)}{A(s)}\Big[1 - \sum_{i=1}^{2}\Big(\sum_{k=1}^{2m+2} \frac{\frac{A_i(s)-A_i(\rho_k)}{s-\rho_k}\hat{g}_i(s) + A_i(\rho_k)\frac{\hat{g}_i(s)-\hat{g}_i(\rho_k)}{s-\rho_k}}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)}\Big)\Big] \\ &= \frac{\tau(s)}{A(s)}\Big[1 + \sum_{i=1}^{2}\sum_{k=1}^{2m+2} C_i(k) T_s T_{\rho_k} g_i(0)\Big], \end{aligned} \tag{4.1}$$

where

$$C_i(k) = \frac{A_i(\rho_k)}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)}.$$

We continue to study the numerator in (3.10), let

$$\hat{h}_\zeta(s) = \frac{2}{\sigma^2}(\pi(s) + \hat{z}(s))\prod_{i=1}^{m}(\eta_i - s)\hat{f}_{+,q_1}(s), \ B_{\zeta,i}(s) = A(s)\sum_{j=1}^{m+1}\frac{l_i(j)}{s+\beta_{j,q_i}}.$$

Thus, we can rewrite the numerator of (3.10) as

$$N_\zeta(s) = \frac{1}{A(s)} \sum_{i=1}^{2} [A_1(s)\hat{h}_\zeta(s) - B_{\zeta,i}(s)\hat{f}_{+,q_i}(s)].$$

By imitating the same steps discussed above, we can derive that

$$\sum_{k=1}^{m+1} \frac{\frac{B_{\zeta,1}(s)-B_{\zeta,1}(\rho_k)}{s-\rho_k}}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)} = 0, \quad \sum_{k=1}^{m+1} \frac{\frac{B_{\zeta,2}(s)-B_{\zeta,2}(\rho_k)}{s-\rho_k}}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)} = 0.$$

Then, we have

$$N_\zeta(s) = \frac{\tau(s)}{A(s)} \sum_{i=1}^{2} \sum_{k=1}^{2m+2} [C_{\zeta,i}(k)T_sT_{\rho_k}f_{+,q_i}(0) - C_1(k)T_sT_{\rho_k}h_\zeta(0)], \tag{4.2}$$

where

$$C_{\zeta,i}(k) = \frac{B_{\zeta,i}(\rho_k)}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)},$$

Therefore, submitting (4.1) and (4.2) into (3.10) implies that the Laplace transform $\hat{\phi}_\zeta(s)$ can be simplified to

$$\hat{\phi}_\zeta(s) = \frac{\sum_{i=1}^{2}\sum_{k=1}^{2m+2} [C_{\zeta,i}(k)T_sT_{\rho_k}f_{+,q_i}(0) - C_1(k)T_sT_{\rho_k}h_\zeta(0)]}{1 + \sum_{i=1}^{2}\sum_{k=1}^{2m+2} C_i(k)T_sT_{\rho_k}g_i(0)}. \tag{4.3}$$

Finally, we consider the numerator in Eq (3.16). Define

$$\hat{h}_{\varepsilon,1}(s) = \frac{2\lambda}{\sigma^2}\prod_{i=1}^{m}(\eta_i - s)\hat{f}_{+,q_1}(s)[\hat{\omega}_1(s) - \hat{\omega}_2(s)],\ \hat{h}_{\varepsilon,2}(s) = \frac{4\lambda}{\sigma^2}\prod_{i=1}^{m}(\eta_i - s)\hat{f}_{+,q_2}(s)\hat{\omega}_2(s), \tag{4.4}$$

$$B_{\varepsilon,i}(s) = A(s)\sum_{j=1}^{m+1}\frac{d_i(j)}{s+\beta_{j,q_i}},\ C_{\varepsilon,i}(k) = \frac{B_{\varepsilon,i}(\rho_k)}{\prod_{l=1,l\neq k}^{2m+2}(\rho_k-\rho_l)},\ \text{‘}i = 1,2,; k = 1,\cdots,2m+2. \tag{4.5}$$

By (4.4) and (4.5), the numerator of Eq (3.16) can be rewritten as

$$N_\varepsilon(s) = \frac{1}{A(s)} \sum_{i=1}^{2} [A_i(s)\hat{h}_{\varepsilon,i}(s) - B_{\varepsilon,i}(s)\hat{f}_{+,q_i}(s)].$$

Then, by imitating the same procedure to derive (4.2), we have

$$N_\varepsilon(s) = \frac{\tau(s)}{A(s)} \sum_{i=1}^{2} \sum_{k=1}^{2m+2} [C_{\varepsilon,i}(k)T_sT_{\rho_k}f_{+,q_i}(0) - C_i(k)T_sT_{\rho_k}h_{\varepsilon,i}(0)]. \tag{4.6}$$

Therefore, submitting (4.1) and (4.6) into (3.16) implies that the Laplace transform $\hat{\phi}_\varepsilon(s)$ can be simplified to

$$\hat{\phi}_\varepsilon(s) = \frac{\sum_{i=1}^{2}\sum_{k=1}^{2m+2} [C_{\varepsilon,i}(k)T_sT_{\rho_k}f_{+,q_i}(0) - C_i(k)T_sT_{\rho_k}h_{\varepsilon,i}(0)]}{1 + \sum_{i=1}^{2}\sum_{k=1}^{2m+2} C_i(k)T_sT_{\rho_k}g_i(0)}. \tag{4.7}$$

Theorem 4.1. *The EDP functions $\phi_\zeta(u)$ and $\phi_\varepsilon(u)$ satisfy the following defective renewal equations:*

$$\phi_\zeta(u) = \int_0^u \phi_\zeta(u-x)g(x)\mathrm{d}x + H_\zeta(u), \tag{4.8}$$

$$\phi_\varepsilon(u) = \int_0^u \phi_\varepsilon(u-x)g(x)\mathrm{d}x + H_\varepsilon(u), \tag{4.9}$$

where

$$g(x) = -\sum_{i=1}^{2}\sum_{k=1}^{2m+2} C_i(k)T_{\rho_k}g_i(x),$$

$$H_\zeta(u) = \sum_{i=1}^{2}\sum_{k=1}^{2m+2}[C_{\zeta,i}(k)T_{\rho_k}f_{+,q_i}(u) - C_1(k)T_{\rho_k}h_\zeta(u)],$$

$$H_\varepsilon(u) = \sum_{i=1}^{2}\sum_{k=1}^{2m+2}[C_{\varepsilon,i}(k)T_{\rho_k}f_{+,q_i}(u) - C_i(k)T_{\rho_k}h_{\varepsilon,i}(u)].$$

Proof. Equations (4.8) and (4.9) can be obtained by inverting (4.3) and (4.7) directly. To complete the proof, we only need to show that $\int_0^\infty g(x)\mathrm{d}x < 1$, that is, $\hat{g}(0) < 1$. Since

$$\frac{\tau(s)}{A(s)}[1-\hat{g}(s)] = 1 - \lambda\frac{\hat{f}_X(s) - \theta\hat{h}_X(s)}{q_1 - G(s)} - 2\lambda\frac{\theta\hat{h}_X(s)}{q_2 - G(s)}.$$

Then, we have

$$\hat{g}(s) = 1 - \frac{A(s)}{\tau(s)}\Big[1 - \lambda\frac{\hat{f}_X(s) - \theta\hat{h}_X(s)}{q_1 - G(s)} - 2\lambda\frac{\theta\hat{h}_X(s)}{q_2 - G(s)}\Big], \tag{4.10}$$

setting $s = 0$ in (4.10) leads to

$$\begin{aligned}\hat{g}(0) &= 1 - \frac{\prod_{j=1}^{m+1}\beta_{j,q_1}\beta_{j,q_2}}{\prod_{k=1}^{2m+2}\rho_k}\Big[1 - \lambda\frac{\hat{f}_X(0) - \theta\hat{h}_X(0)}{q_1 - G(0)} - 2\lambda\frac{\theta\hat{h}_X(0)}{q_2 - G(0)}\Big] \\ &= 1 - \frac{\delta}{\lambda+\delta}\frac{\prod_{j=1}^{m+1}\beta_{j,q_1}\beta_{j,q_2}}{\prod_{k=1}^{2m+2}\rho_k} \\ &< 1.\end{aligned}$$

For the case of $\delta = 0$, we obtain

$$\begin{aligned}\hat{g}(0) &= 1 - \frac{\prod_{j=1}^{m+1}\beta_{j,q_1}\beta_{j,q_2}}{\prod_{k=2}^{2m+2}\rho_k}\lim_{\delta\to 0}\frac{1 - \int_0^\infty\int_0^\infty e^{-\delta t}f_{W,X}(t,x)\mathrm{d}t\mathrm{d}x}{\rho_1(\delta)} \\ &= 1 - \frac{\prod_{j=1}^{m+1}\beta_{j,q_1}\beta_{j,q_2}}{\prod_{k=2}^{2m+2}\rho_k}\frac{\mathbb{E}W}{\rho_1'(0)} \\ &= 1 - \frac{\prod_{j=1}^{m+1}\beta_{j,q_1}\beta_{j,q_2}}{\prod_{k=2}^{2m+2}\rho_k}\Big[c + \kappa\Big(p_u\sum_{i=1}^{m}\frac{p_i}{\eta_i} + q_d\sum_{i=1}^{n}\frac{-\vartheta_j}{\theta_j}\Big) - \lambda\mathbb{E}X\Big] \\ &< 1,\end{aligned}$$

where the L'Hôpital's rule has been used in the second equality and the last step is guaranteed by inequality (1.3). □

5. Explicit expressions for the EDP functions

In this section, the p.d.f. of individual claim amount is assumed to be $f_X(x) = \alpha e^{-\alpha x}$. By (4.10), we have

$$\begin{aligned} 1-\hat{g}(s) &= \frac{A(s)}{\tau(s)}\Big[1+\frac{2}{\sigma^2}\frac{(\theta-1)\lambda\alpha s-2\lambda\alpha^2}{(s+\alpha)(s+2\alpha)}\frac{\prod_{i=1}^{m}(\eta_i-s)\prod_{j=1}^{n}(\theta_j+s)}{\prod_{i=1}^{m+1}(s+\beta_{i,q_1})\prod_{j=1}^{n+1}(s+\gamma_{j,q_1})} \\ &\quad +\frac{2}{\sigma^2}\frac{-2\lambda\alpha\theta s}{(s+\alpha)(s+2\alpha)}\frac{\prod_{i=1}^{m}(\eta_i-s)\prod_{j=1}^{n}(\theta_j+s)}{\prod_{i=1}^{m+1}(s+\beta_{i,q_2})\prod_{j=1}^{n+1}(s+\gamma_{j,q_2})}\Big] \\ &= \frac{K(s)}{\tau(s)(s+\alpha)(s+2\alpha)\prod_{j=1}^{n+1}(s+\gamma_{j,q_1})(s+\gamma_{j,q_2})}, \end{aligned}$$

where

$$\begin{aligned} K(s) &= A(s)(s+\alpha)(s+2\alpha)\prod_{j_2=1}^{n+1}(s+\gamma_{j_2,q_1})(s+\gamma_{j_2,q_2}) \\ &\quad \Big[1+\frac{2}{\sigma^2}\frac{(\theta-1)\lambda\alpha s-2\lambda\alpha^2}{(s+\alpha)(s+2\alpha)}\frac{\prod_{i=1}^{m}(\eta_i-s)\prod_{j=1}^{n}(\theta_j+s)}{\prod_{i=1}^{m+1}(s+\beta_{i,q_1})\prod_{j=1}^{n+1}(s+\gamma_{j,q_1})} \\ &\quad +\frac{2}{\sigma^2}\frac{-2\lambda\alpha\theta s}{(s+\alpha)(s+2\alpha)}\frac{\prod_{i=1}^{m}(\eta_i-s)\prod_{j=1}^{n}(\theta_j+s)}{\prod_{i=1}^{m+1}(s+\beta_{i,q_2})\prod_{j=1}^{n+1}(s+\gamma_{j,q_2})}\Big]. \end{aligned}$$

Since $K(s)$ is a polynomial of degree 2m+2n+6, Lemma 3.1 implies that

$$K(s) = \tau(s)\prod_{i=1}^{2n+4}(s+\xi_i).$$

Assume that $\xi_1, \cdots, \xi_{2n+4}$ are distinct. Then, we have

$$\begin{aligned} \frac{1}{1-\hat{g}(s)} &= \frac{(s+\alpha)(s+2\alpha)\prod_{j=1}^{n+1}(s+\gamma_{j,q_1})(s+\gamma_{j,q_2})}{\prod_{i=1}^{2n+4}(s+\xi_i)} \\ &= 1+\sum_{k=1}^{2n+4}\frac{\frac{(\alpha-\xi_k)(2\alpha-\xi_k)\prod_{j=1}^{n+1}(\gamma_{j,q_1}-\xi_k)(\gamma_{j,q_2}-\xi_k)}{\prod_{i=1,i\neq k}^{2n+4}(\xi_i-\xi_k)}}{s+\xi_k} \\ &= \sum_{k=1}^{2n+4}\frac{Z_k}{s+\xi_k}+1, \end{aligned} \tag{5.1}$$

where

$$Z_k = \frac{(\alpha-\xi_k)(2\alpha-\xi_k)\prod_{j=1}^{n+1}(\gamma_{j,q_1}-\xi_k)(\gamma_{j,q_2}-\xi_k)}{\prod_{i=1,i\neq k}^{2n+4}(\xi_i-\xi_k)}.$$

Plugging (5.1) into (4.3) implies that

$$\hat{\phi}_\zeta(s) = \hat{H}_\zeta(s) + \sum_{k=1}^{2n+4}\frac{Z_k\hat{H}_\zeta(s)}{s+\xi_k}.$$

Similarly, we have

$$\hat{\phi}_\varepsilon(s) = \hat{H}_\varepsilon(s) + \sum_{k=1}^{2n+4} \frac{Z_k \hat{H}_\varepsilon(s)}{s + \xi_k}.$$

Inverting the above Lapalce transforms gives us

$$\phi_\zeta(u) = H_\zeta(u) + \sum_{k=1}^{2n+4} Z_k \int_0^u \exp(-\xi_k(u-x))H_\zeta(x)\mathrm{d}x, \tag{5.2}$$

$$\phi_\varepsilon(u) = H_\varepsilon(u) + \sum_{k=1}^{2n+4} Z_k \int_0^u \exp(-\xi_k(u-x))H_\varepsilon(x)\mathrm{d}x. \tag{5.3}$$

To explain the specific solution procedure, we provide a numerical example. By setting $\omega = 1$ and $\delta = 0$ in (5.2) and (5.3), respectively, the ruin probabilities $\phi_\zeta(u)$ and $\phi_\varepsilon(u)$ can be expressed as

$$\phi_\zeta(u) = P(\tau < \infty, U(\tau) = 0|U(0) = u), \quad \phi_\varepsilon(u) = P(\tau < \infty, U(\tau) < 0|U(0) = u).$$

Example. *Suppose $c = 1$, $\lambda = 0.5$, $\sigma = 2$, $p_u = 0.6$, $q_d = 0.4$, $p_1 = 1$, $\vartheta_1 = 1$, $m = 1$, $n = 1$,$\eta_1 = 1.2$, $\theta_1 = 0.5$, $\alpha = 1.5$, $\theta = -0.5$, $q_1 = 0.5$, $q_2 = 1.5$,$\kappa = 1$. It is easy to check that the safety loading condition (1.3) holds. The corresponding parameters can be calculated and we list them in Table 1. The ruin probabilities $\phi_\zeta(u)$ and $\phi_\varepsilon(u)$ can be obtained as follows:*

$$\begin{aligned}\phi_\zeta(u) \;=\;& 0.6973e^{-2.6712u} - 1.8490e^{-2.3215u} - 0.0811e^{-1.4150u} - 1.8548e^{-0.8365u} \\ & +65.2873e^{-0.4219u} + 22.1026e^{-0.2897u} - 43.0094e^{-0.3248u} + 3.7807e^{-1.1516u} \\ & -14.7828e^{-0.4060u} + 2.1406e^{-1.3841u} - 31.4134e^{-0.5u},\end{aligned}$$

$$\begin{aligned}\phi_\varepsilon(u) \;=\;& -0.2975e^{-2.6712u} + 0.3793e^{-2.3215u} - 0.2403e^{-1.4150u} - 1.3090e^{-0.8365u} \\ & +21.7508e^{-0.4219u} + 0.7284e^{-0.2897u} - 0.2473e^{-0.3248u} - 0.1556e^{-1.1516u} \\ & -20.0873e^{-0.4060u} + 1.4737e^{-1.3841u} + 0.0876e^{-3u} - 2.0828e^{-1.5u}.\end{aligned}$$

Table 1. Parameters needed in solving $\phi_\zeta(u)$ and $\phi_\varepsilon(u)$.

γ_{1,q_1}	γ_{2,q_1}	β_{1,q_1}	β_{2,q_1}	γ_{1,q_2}	γ_{2,q_2}	β_{1,q_2}	β_{2,q_2}
0.3248	1.1516	-0.2890	-1.3874	0.4060	1.3841	-0.5598	-1.4303
$a_{-,1,q_1}$	$a_{-,2,q_1}$	$a_{+,1,q_1}$	$a_{+,2,q_1}$	$a_{-,1,q_2}$	$a_{-,2,q_2}$	$a_{+,1,q_2}$	$a_{+,2,q_2}$
-0.2073	-0.0427	0.9976	-1.9976	-0.5516	-0.1984	0.9263	-1.9263
$l_1(1)$	$l_1(2)$	$l_2(1)$	$l_2(2)$	$d_1(1)$	$d_1(2)$	$d_2(1)$	$d_2(2)$
0.0721	-0.0005	1.0965	-0.0062	-0.0999	0.0000	1.3400	0.0504
ρ_1	ρ_2	ρ_3	ρ_4	ξ_1	ξ_2	ξ_3	ξ_4
0	0.5685	1.3773	1.4336	2.6712	2.3215	1.4150	0.8365
ξ_5	ξ_6						
0.4219	0.2897						

Figure 1 illustrates the behavior of the ultimate ruin probability and its two decompositions. We note that the probabilities caused by a claim and jump-diffusion process are not monotonic functions of the initial capital. However, the ultimate ruin probability for the risk process $\{U(t)\}$ decreases as the initial capital increases. This is consistent with the conclusion in Zhang et al. [17] and confirms our expectations.

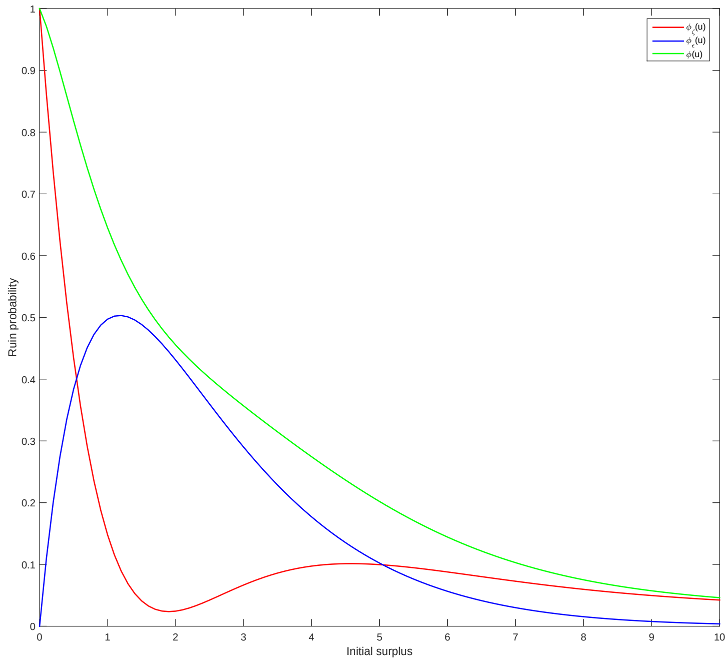

Figure 1. Ultimate ruin probability and its decompositions.

6. Conclusions

In the present paper, we investigate a dependent risk model perturbed by a mixed-exponential jump-diffusion process, in which the joint p.d.f. of the interclaim time and claim size is introduced through a FGM copula. The EDP function and its two decompositions are studied deeply. In terms of a q-potential measure, the Laplace transforms and defective renewal equations satisfied by the EDP functions are obtained. A numerical example is presented to illustrate the ruin probability and its decompositions caused by claims and the jump-diffusion process.

Compared with the traditional insurance model, we use a q-potential measure method instead of deriving the integro-differential equations to obtain the main results. This technique is developed by Zhang et al. [17], and it is very effective in dealing with some perturbed risk models. We generalize the double-exponential jump-diffusion process in Zhang et al. [17] to the case of mixed-exponential jump-diffusion process, and the FGM copula is adopted to model the dependence structure. Although the calculation is tedious, similar closed-form solutions for the EDP function can still be obtained.

From the practical perspective, the proposed jump-diffusion risk model can be used to capture the sudden fluctuations in the real insurance market. The corresponding results obtained may be used to help managers protect the insurance company against possible bankruptcy by informing the minimum capital levels. On the other hand, the assumption of independence between the claim amount random variable and the interclaim time is restrictive from practical contexts, and it is popular to use copulas to model the dependence structure between random variables in financial risk management and actuarial science. However, due to the complexity of Lévy process, it involves a relatively large amount of computation and parameter estimation may be more difficult in practice. Nonetheless, the Lévy risk model has received widespread attention due to its excellent ability to describe the risk characteristics of insurance and financial markets.

The model considered in this paper can be further extended to more general framework. For example, we can consider a general Lévy risk model with hyper-exponential jumps or the jump process having rational Laplace transforms. Finally, one can use generalized FGM copulas to generalize the risk model proposed in this paper.

Author contributions

Zhipeng Liu: Conceptualization, methodology, writing-original draft preparation, writing-review suggestions and editing, formal analysis; Cailing Li: Conceptualization, methodology, software, writing-original draft preparation, writing-review suggestions and editing, formal analysis; Zhenhua Bao: Conceptualization, methodology, writing-original draft preparation, writing-review suggestions and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

The authors would like to thank the anonymous referees for constructive comments that improved the contents and presentation of this paper. This research was funded by Basic Scientific Research Project of the Education Department of Liaoning Province (JYTMS20231043, LJ112410165061, LS2024W003).

Conflict of interest

The authors declare that they have no conflicts of interest.

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