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Inhomogeneous NLS with partial harmonic confinement

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Abstract: We investigate the inhomogeneous nonlinear Schrödinger equation with partial harmonic confinement. First, we present a global well-posedness result for small data in the intercritical regime. Second, we obtain a threshold of global existence versus finite-time blow-up in the mass-critical regime. Finally, we prove the L^2 concentration of the mass-critical non-global solution with minimal mass. The challenge is to address the fact that the standard scale invariance is broken by the partial confinement. We use the associated ground state without potential in order to describe the threshold of global versus non-global existence of solutions.

Keywords: Schrödinger equation; partial harmonic confinement; global existence; blow-up; ground state; nonlinear equations; approximation
Mathematics Subject Classification: 35Q55

1. Introduction

We consider the nonlinear, focusing, inhomogeneous Schrödinger equation with a partial harmonic confinement

$$\begin{cases} i\partial_t u + \Delta u - \sum_{j\in J} |x_j|^2 u + |x|^{-\varrho}|u|^{p-1}u = 0; \\ u(0,\cdot) = u_0. \end{cases} \tag{1.1}$$

Hereafter, the space dimension is $N \geq 2$ and the wave function is denoted by $u := u(t,x) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$. The set of partial confinement components is $\emptyset \neq J := \{i_1,\dots,i_k\}$, where $1 \leq k < N$ and $1 \leq i_1 < \cdots < i_k < N$. Finally, the exponent of the source term is $p > 1$, and the singular inhomogeneity satisfies $0 < \varrho < \tilde{2} := \frac{N}{3}\chi_{[2,3]} + 2\chi_{[4,\infty)}$.

The cubic nonlinear Schrödinger equation, commonly referred to as the Gross–Pitaevskii equation (GPE), plays a crucial role in physics. Specifically, (1.1) with $p = 3$, $\varrho = 0$, and an external trapping potential provides an effective description of Bose–Einstein condensates (BEC). A Bose–Einstein condensate is a macroscopic collection of bosons that, at extremely low temperatures, occupy the same quantum state. This phenomenon was experimentally observed only in the last two decades [1, 2], which has spurred extensive theoretical and numerical research. In experiments, BEC is observed in the presence of a confining potential trap, and its macroscopic behavior is highly dependent on the shape of this trapping potential. When the trap potential is confined along partial directions in the space, [3, 4] showed the same properties as the whole confinement in the space under some assumptions. At low enough temperature, neglecting the thermal and quantum fluctuations, a Bose condensate can be represented by (1.1). Specifically, if we consider a condensate of particles of mass and negative effective scattering length in a partial confining potential using variables rescaled by the natural quantum harmonic oscillator units of time, we get (1.1). Equation (1.1) arises also in the propagation of mutually incoherent wave packets in nonlinear optics. For more details we refer to [5].

Several historic works have addressed the non-linear Schrödinger equation with partial confinement. The existence of orbitally stable ground states was investigated in [4], and the strong instability of standing waves was studied in [6]. A mixed source term was considered in [7]. The energy scattering of global solutions in the focusing intercritical regime was treated in [3, 8–10], while the finite-time blow-up of energy solutions was examined in [11, 12]. Thresholds for global existence versus energy concentration were established in [13, 14]. See also [15, 16] for the mass-critical NLS concentration without any potential. All these works focus on the homogeneous regime, corresponding to (1.1) with $\varrho = 0$. The only paper addressing the inhomogeneous case appears to be [17], which investigated the existence and stability of standing waves. This work aims to extend the existing literature to the inhomogeneous regime, specifically treating the case $\varrho \neq 0$ in (1.1). The challenge lies in addressing the partial confinement, which breaks the scaling invariance, as well as the singular inhomogeneous term $|\cdot|^{-\varrho}$. The method used here does not cover the energy-critical regime, which is investigated in a work in progress. A solid theoretical understanding of the problem motivates us to explore its numerical and practical aspects in future work.
Let us outline the plan of the manuscript. Section 2 proves a global existence result. Section 3 establishes a threshold of global existence versus blowup of mass-critical solutions. Sections 4-5 investigate the finite-time blow-up of mass-critical non-global solutions. In the appendix, some variance-type identities are established.

For simplicity, let us denote the Lebesgue space $L^p := L^p(\mathbb{R}^N)$ and the classical norms $\|\cdot\|_p := \|\cdot\|_{L^p}$, $\|\cdot\| := \|\cdot\|_2$. Let us denote the Sobolev space $H^1 := \{f \in L^2 \text{ and } \nabla f \in L^2\}$. Finally, if A and B are positive real numbers, $A \lesssim B$ means that $A \leq CB$ for an absolute positive constant $C > 0$.

1.1. *Preliminary*

First, let us give some notations. Let the real numbers

$$B_k := \varrho + \frac{(N-k)(p-1)}{2}, \quad B := B_0 \quad \text{and} \quad A := -B + 1 + p.$$

Let us define $p_{k,c} := 1 + \frac{4-2\varrho}{N-k}$, the mass critical exponent $p_c := p_{0,c}$ and the energy critical one

$$p^c := \begin{cases} 1 + \frac{4-2\varrho}{N-2}, & if \quad N \geq 3; \\ +\infty, & if \quad N = 2. \end{cases}$$

Note that, since the partial confinement breaks the scaling invariance, the mass-critical and energy-critical exponents are taken as the same for the INLS without any potential. The so-called energy space is

$$\Sigma = \Sigma_J := \{u \in H^1; \quad x_j u \in L^2, \quad \forall j \in J\},$$

endowed with the norm

$$\|\cdot\|_\Sigma := \Big(\|\cdot\|^2 + \sum_{j\in J} \|x_j \cdot\|^2\Big)^{\frac{1}{2}}.$$

Hereafter, we denote for $u \in \Sigma$, the conserved real quantities under the flow of (1.1), which are respectively referred to as the mass and the energy

$$M(u) := \|u\|^2; \quad \text{(Mass)}$$
$$E(u) := \|\nabla u\|^2 + \sum_{j\in J} \|x_j u\|^2 - \frac{2}{1+p}\int_{\mathbb{R}^N} |u|^{1+p}|x|^{-\varrho}\,dx. \quad \text{(energy)}$$

The problem (1.1) is locally well-posed in the energy space, as demonstrated by the results in [8, 18, 19]. Specifically, by applying the Strichartz estimate from [8], we employ the standard fixed-point method introduced in [18], while incorporating techniques from [19] to address the inhomogeneous term.

Proposition 1.1. *Let $N \geq 2$, $0 < \varrho < \tilde{2}$, $1 < p < p^c$ and $u_0 \in \Sigma$. Then, the Cauchy problem* (1.1) *has a unique maximal solution $u \in C([0, T^{\max}), \Sigma)$, in the sense that*

$$T^{\max} < \infty \Longrightarrow \limsup_{T^{\max}} \|u(t)\|_\Sigma = \infty.$$

Moreover, the mass and energy are conserved

$$M(u(t)) = M(u_0), \quad E(u(t)) = E(u_0).$$

In order to investigate the problem (1.1), we take the elliptic problem with no potential

$$\Delta\phi - \phi + |x|^{-\varrho}|\phi|^{p-1}\phi = 0, \quad 0 \neq \phi \in H^1. \tag{1.2}$$

The existence and uniqueness of the ground state hold for $0 < \varrho < \tilde{2}$ and $1 < p < p^c$. Specifically, the existence of the ground state is established in [20–22], while its uniqueness is derived in [23, 24]. Below, we outline the main contributions of this paper.

1.2. Main results

First, we give a global existence result, for small data in the intercritical regime.

Theorem 1.1. *Let $N \geq 2$, $0 < \varrho < \tilde{2}$ and $p_c < p < p^c$. Let ϕ be the positive radial decreasing ground state of* (1.2) *and $0 \neq u_0 \in \Sigma$ satisfying*

$$\|u_0\| \leq \Big(\frac{B-2}{A}\Big)^{\frac{B-2}{2A}} \|\phi\|^{\frac{p-1}{A}} \Big(\|\nabla u_0\|^2 + \sum_{j\in J} \|x_j u_0\|^2\Big)^{-\frac{B-2}{2A}}. \tag{1.3}$$

Then, there exists a unique global solution $u \in C(\mathbb{R}, \Sigma)$ to the Schrödinger problem (1.1)*, which satisfies*

$$\|\nabla u(t)\|^2 + \sum_{j\in J} \|x_j u(t)\|^2 < \frac{2B}{B-2} E(u_0), \quad \forall t \geq 0.$$

Remarks 1.1. *1. Taking $0 < \lambda \ll 1$, one checks that λu_0 satisfies the above condition. This gives an infinite family of global solutions.*

2. When $B \to 2$, equivalently $p \to p_c$, the above conditions read $\|u_0\| < \|\phi\|$.

3. The previous result complements [25] to the inhomogeneous regime, namely, $\varrho \neq 0$.

In order to prove the finite-time blow-up of solutions, one needs the next variance identities.

Proposition 1.2. *Let $N \geq 2$, $0 < \varrho < \tilde{2}$ and $1 < p < p^c$. Let $u \in C([0,T];\Sigma)$ be a local solution to the problem* (1.1)*. Then, for all $t \in [0,T]$, holds*

$$\begin{aligned}\partial_t^2\Big(\sum_{j\notin J} \|x_j u(t)\|^2\Big) = 8\Big(&\sum_{j\notin J} \|\partial_j u(t)\|^2 - (N-k)\Big(\frac{1}{2} - \frac{1}{1+p}\Big) \int_{\mathbb{R}^N} |u(t,x)|^{1+p} |x|^{-\varrho}\, dx \\ &- \frac{\varrho}{2(1+p)} \int_{\mathbb{R}^N} |u(t,x)|^{1+p} \Big(\sum_{j\in J} x_j^2\Big) |x|^{-\varrho-2}\, dx\Big).\end{aligned} \tag{1.4}$$

Moreover, if $xu_0 \in L^2$, then

$$\partial_t^2\Big(\|xu(t)\|^2\Big) = 8\Big(\|\nabla u(t)\|^2 - \sum_{j\in J} \int_{\mathbb{R}^N} |x_j|^2 |u(t,x)|^2\, dx - \frac{B}{1+p} \int_{\mathbb{R}^N} |u(t,x)|^{1+p} |x|^{-\varrho}\, dx\Big). \tag{1.5}$$

Remarks 1.2. *1. In order to use* (1.5)*, we need that the data belongs to the set*

$$\Sigma' := \Big\{u \in H^1, \quad xu \in L^2\Big\} \hookrightarrow \Sigma. \tag{1.6}$$

2. The previous result is proved in the appendix.

Using the variance-type identities in Proposition 1.2, we present the following blow-up result.

Proposition 1.3. *Let $N \geq 2$, $0 < \varrho < \tilde{2}$ and $1 < p < p^c$. Let $u \in C([0,T),\Sigma)$ be a local solution to the problem* (1.1)*. Then, u is non-global if*

1. $p \geq p_c$ and $u_0 \in \Sigma'$ satisfies one of the following:

(a) $E(u_0) < 0$;

(b) $E(u_0) = 0$ and $\Im \int_{\mathbb{R}^N} \bar{u}_0 (x \cdot \nabla u_0)\, dx < 0$.

2. $k = 1$, $0 < \varrho < \frac{2}{N-1}$, $p \geq 1 + \frac{4}{N-k}$ *and one of the following holds:*
   (a) $E(u_0) < 0$;
   (b) $E(u_0) = 0$ *and* $\sum_{j \notin J} \Im \int_{\mathbb{R}^N} \bar{u}_0 (x_j \cdot \nabla u_0)\, dx < 0$.

Remarks 1.3. *1. The identity* (1.5) *reads by use of the energy*

$$\partial_t^2 \Big(\|xu(t)\|^2 \Big) = 8 \Big(E(u_0) - 2 \sum_{j \in J} \int_{\mathbb{R}^N} |x_j|^2 |u(t,x)|^2\, dx - \frac{B-2}{1+p} \int_{\mathbb{R}^N} |u(t,x)|^{1+p} |x|^{-\varrho}\, dx \Big). \tag{1.7}$$

So, by (1.7)*, since $p \geq p_c$ reads $B \geq 2$, the first point of Proposition 1.3 follows by time integration.*

2. *The identity* (1.4) *reads by use of the energy*

$$\begin{aligned} \partial_t^2 \Big(\sum_{j \notin J} \|x_j u(t)\|^2 \Big) &= 8 \Big(E(u_0) - \sum_{j \in J} \int_{\mathbb{R}^N} |x_j|^2 |u(t,x)|^2\, dx - \sum_{j \in J} \|\partial_j u\|^2 \\ &- \Big[(N-k)(\frac{1}{2} - \frac{1}{1+p}) - \frac{2}{1+p} \Big] \int_{\mathbb{R}^N} |u|^{1+p} |x|^{-\varrho}\, dx \\ &- \frac{\varrho}{2(1+p)} \int_{\mathbb{R}^N} (\sum_{j \in J} x_j^2) |x|^{-\varrho-2} |u|^{1+p}\, dx \Big) \\ &\leq 8 \Big(E(u_0) - \Big[(N-k)(\frac{1}{2} - \frac{1}{1+p}) - \frac{2}{1+p} \Big] \int_{\mathbb{R}^N} |u|^{1+p} |x|^{-\varrho}\, dx \Big). \end{aligned} \tag{1.8}$$

So, by (1.8)*, since $p \geq 1 + \frac{4}{N-k}$, the second point of Proposition 1.3 follows by time integration.*

3. *The assumption $k = 1$ in the second case is because $1 + \frac{4}{N-k} \leq p < p^c$.*

The ground state of (1.2) gives a threshold of global existence versus finite-time blow-up of mass-critical solutions to (1.1).

Theorem 1.2. *Let $N \geq 2$, $0 < \varrho < \tilde{2}$ and $p = p_c$. Let $u \in C([0, T^{\max}), \Sigma)$ be a maximal solution to* (1.1) *and ϕ be the positive radial solution to* (1.2)*. Then,*

1. $T^{\max} < \infty$ *if* $\|u_0\| > \|\phi\|$ *and* $u_0 \in \Sigma'$;
2. $T^{\max} = \infty$ *if* $\|u_0\| < \|\phi\|$.

In the mass-critical regime, we give a mass-concentration result of non-global solutions.

Theorem 1.3. *Let $N \geq 2$, $0 < \varrho < \tilde{2}$ and $p = p_c$. Let $u \in C([0, T^{\max}), \Sigma')$ a non-global solution to* (1.1) *and a positive real function $\mu := \mu(t)$ such that*

$$\lim_{t \to T^{\max}} \mu(t) \|\nabla u(t)\| = \infty. \tag{1.9}$$

Thus, there exists $x(t) \in \mathbb{R}^N$ satisfying

$$\|\phi\|^2 \leq \liminf_{t \to T^{\max}} \int_{|x - x(t)| \leq \mu(t)} |u(t,x)|^2 dx, \tag{1.10}$$

where ϕ is the ground state to (1.2).

Remarks 1.4. *1. Thanks to the identity* (1.7)*, the concentration does not occur in the potential quantity, namely* $\limsup_{t\to T^{\max}} \|x_j u(t)\|^2 < \infty$*, for any* $j \notin J$*.*

2. *The assumption* $u_0 \in \Sigma'$ *is imposed because the use of* (1.8) *needs* $p \geq 1 + \frac{4}{N-1}$*, which fails for* $p = p_c$*.*
3. *The mass concentration* (1.10) *implies in particular that the solution has no* L^2 *limit when* $t \to T^{\max}$*.*

Eventually, we study the lower bound for the mass-critical blow-up rate.

Theorem 1.4. *Let* $N \geq 3$*,* $0 < \varrho < \tilde{2}$*,* $p = p_c$ *and* ϕ *be the radial positive solution to* (1.2)*. Let* $u \in C([0, T^{\max}), \Sigma)$ *be a maximal non-global solution to* (1.1) *satisfying* $\|u_0\| = \|\phi\|$*. Then,*

$$\|\nabla u(t)\| \gtrsim \frac{1}{T^{\max} - t}, \quad \forall t \in [0, T^{\max}). \tag{1.11}$$

Remarks 1.5. *1. In the case* $p = p_c$*, in order to study the non-global solutions to* (1.1)*, we use the associated ground state without any potential, namely the solution to* (1.2)*.*

2. *The restriction on space dimensions* $N > 2$ *is needed when using Hardy estimate.*
3. *In the standard NLS case, namely without partial confinement, a classical scaling argument gives* $\|\nabla u(t)\| \gtrsim \frac{1}{\sqrt{T^{\max}-t}}$*, which is better than the lower bound* (1.11)*.*

In the next sub-section, we gather some standard estimates.

1.3. Useful estimates

The next compactness result [26, Theorem 1.3] is adapted to the analysis of the blow-up phenomenon of Schrödinger equations.

Lemma 1.1. *Let* $N \geq 2$*,* $0 < \varrho < 2$*,* m*,* $M > 0$ *and a sequence of* H^1 *satisfying*

$$\sup_n \|u_n\|_{H^1} < \infty, \quad \limsup_{n\to\infty} \|\nabla u_n\| \leq M^2 \quad \textit{and} \quad \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{1+p_c} |x|^{-\varrho}\, dx \geq m^{1+p_c}.$$

Then, there exist $V \in H^1$ *and a sequence* (x_n) *in* $\mathbb{R}^N$ *such that up to a sub-sequence, one has*

$$u_n(\cdot + x_n) \rightharpoonup V \quad \textit{weakly in} \quad H^1;$$
$$\|V\| \geq \Big(\frac{2m}{M(1+p_c)}\Big)^{\frac{1}{p_c-1}} \|\phi\|,$$

where ϕ *is a ground state to* (1.2)*.*

The following Gagliardo-Nirenberg inequality [27, 28] will be useful.

Proposition 1.4. *Let* $N \geq 1$*,* $0 < \varrho < \min\{2, N\}$ *and* $1 < p < p^c$*. Then, for all* $f \in H^1$*,*

$$\int_{\mathbb{R}^N} |f(x)|^{1+p} |x|^{-\varrho}\, dx \leq \mathrm{K}_{opt} \|f\|^A \|\nabla f\|^B \tag{1.12}$$
$$:= \frac{1+p}{A}\Big(\frac{A}{B}\Big)^{\frac{B}{2}} \|\phi\|^{-(p-1)} \|f\|^A \|\nabla f\|^B, \tag{1.13}$$

where ϕ is a the ground state solution to (1.2). *Moreover, we have the Pohozaev type identities*

$$\|\nabla\phi\|^2 = \frac{B}{A}\|\phi\|^2 = \frac{B}{1+p}\int_{\mathbb{R}^N} |\phi(x)|^{1+p}|x|^{-\varrho}dx. \tag{1.14}$$

Finally, one gives an elementary useful result [29].

Lemma 1.2. *Let an open interval $I \subset \mathbb{R}$, $t_0 \in I$, $\theta > 1$, $a, z > 0$ and $g \in C(I, \mathbb{R}_+)$. Let the real function defined on $\mathbb{R}_+$ by $f(x) := a - x + zx^\theta$, $x_* := (z\theta)^{-\frac{1}{\theta-1}}$ and $z_* := \frac{\theta-1}{\theta}x_*$. Then,*

$$g < x_*, \quad on \quad I,$$

provided that

$$a \leq z_* \quad g(t_0) < x_* \quad and \quad f \circ g > 0.$$

Now, let us establish the main results.

2. Global well-posedness

In this section, we prove Theorem 1.1. Let us define the quantities

$$g(t) := \|\nabla u(t)\|^2 + \sum_{i\in J}\|x_i u(t)\|^2; \tag{2.1}$$

$$a := \|\nabla u_0\|^2 + \sum_{i\in J}\|x_i u_0\|^2; \tag{2.2}$$

$$z := \frac{2\mathrm{K}_{opt}}{1+p}\|u_0\|^A, \tag{2.3}$$

where $t \in [0, T^{\max})$ and K_{opt} is given in Proposition 1.4. Let also the real function defined on $(0, \infty)$, by

$$h : s \mapsto \Big(\frac{B-2}{A}\Big)^{\frac{B-2}{2A}} \|\phi\|^{\frac{p-1}{A}} s^{-\frac{B-2}{2A}}. \tag{2.4}$$

With the conservation laws, we write

$$\begin{aligned} g(t) &= \|\nabla u(t)\|^2 + \sum_{i\in J}\|x_i u(t)\|^2 \\ &= E(u_0) + \frac{2}{1+p}\int_{\mathbb{R}^N} |u(t,x)|^{1+p}|x|^{-\varrho}\,dx \\ &< \|\nabla u_0\|^2 + \sum_{i\in J}\|x_i u_0\|^2 + \frac{2}{1+p}\int_{\mathbb{R}^N} |u(t,x)|^{1+p}|x|^{-\varrho}\,dx. \end{aligned} \tag{2.5}$$

So, (2.5) via Proposition 1.4 and the mass conservation law implies that

$$\begin{aligned}g(t) &<\|\nabla u_0\|^2+\sum_{i\in J}\|x_iu_0\|^2+\frac{2\mathrm{K}_{opt}}{1+p}\|u_0\|^A\|\nabla u(t)\|^B\\&<\|\nabla u_0\|^2+\sum_{i\in J}\|x_iu_0\|^2+\frac{2\mathrm{K}_{opt}}{1+p}\|u_0\|^A\Big(g(t)\Big)^{\frac{B}{2}}\\&=a+z\Big(g(t)\Big)^{\frac{B}{2}}.\end{aligned}\tag{2.6}$$

By (2.6), the real function $f: x\mapsto a-x+zx^{\frac{B}{2}}$ satisfies $f\Big(g(t)\Big)>0$, for any $t<T^{\max}$. Now, the assumption (1.3) reads $\|u_0\|\le\Big(\frac{B-2}{A}\Big)^{\frac{B-2}{2A}}\|\phi\|^{\frac{p-1}{A}}a^{-\frac{B-2}{2A}}$, rewritten as

$$a\le\frac{B-2}{A}\|\phi\|^{\frac{2(p-1)}{B-2}}\|u_0\|^{-\frac{2A}{B-2}}.\tag{2.7}$$

Let us keep the notations of Lemma 1.2, namely

$$\theta:=\frac{B}{2};\tag{2.8}$$

$$x_*:=(z\theta)^{-\frac{1}{\theta-1}};\tag{2.9}$$

$$z_*:=\frac{B-2}{B}x_*.\tag{2.10}$$

Taking into account of Proposition 1.4, yields

$$\begin{aligned}z_* &=\frac{B-2}{B}\Big(\frac{2\mathrm{K}_{opt}}{1+p}\|u_0\|^A\frac{B}{2}\Big)^{-\frac{2}{B-2}}\\&=\frac{B-2}{B}\Big(\frac{1}{A}(\frac{A}{B})^{\frac{B}{2}}\|\phi\|^{-(p-1)}\|u_0\|^AB\Big)^{-\frac{2}{B-2}}\\&=\frac{B-2}{B}\Big((\frac{A}{B})^{\frac{B}{2}-1}\|\phi\|^{-(p-1)}\|u_0\|^A\Big)^{-\frac{2}{B-2}}\\&=\frac{B-2}{A}\|\phi\|^{\frac{2(p-1)}{B-2}}\|u_0\|^{-\frac{2A}{B-2}}.\end{aligned}\tag{2.11}$$

So, (2.7) and (2.11), imply that $a\le z_*<x_*$. Applying Lemma 1.2, it follows that $\sup_{t\in[0,T^{\max})}g(t)<x_*$. Then, u is global, namely $T^{\max}=\infty$. Moreover, the energy reads via Proposition 1.4,

$$\begin{aligned}E(u_0) &=\|\nabla u\|^2+\sum_{i\in J}\|x_iu\|^2-\frac{2}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^{-\varrho}\,dx\\&\ge\|\nabla u\|^2+\sum_{i\in J}\|x_iu\|^2-\frac{2\mathrm{K}_{opt}}{1+p}\|u_0\|^A\|\nabla u\|^B\\&\ge\|\nabla u\|^2+\sum_{i\in J}\|x_iu\|^2-\frac{2\mathrm{K}_{opt}}{1+p}\|u_0\|^A\Big(\sum_{i\in J}\|x_iu\|^2+\|\nabla u\|^2\Big)^{\frac{B}{2}}.\end{aligned}\tag{2.12}$$

So, by (2.12), we write

$$E(u_0) \geq \Big(\|\nabla u\|^2 + \sum_{i\in J}\|x_i u\|^2\Big)\Big(1 - \frac{2\mathrm{K}_{opt}}{1+p}\|u_0\|^A(\sum_{i\in J}\|x_i u\|^2 + \|\nabla u\|^2)^{\frac{B}{2}-1}\Big)$$
$$\geq \Big(1 - \frac{2\mathrm{K}_{opt}}{1+p}\|u_0\|^A g^{\frac{B}{2}-1}\Big)g. \tag{2.13}$$

Since $g < x_*$, we get by (2.13),

$$E(u_0) > \Big(1 - \frac{2}{B}\Big)g. \tag{2.14}$$

Finally, (2.14) implies that

$$\sup_{t\in[0,\infty)} g(t) < \frac{B}{B-2}E(u_0). \tag{2.15}$$

The proof of Theorem 1.1 is closed by (2.15).

3. Global/ non-global mass-critical solutions

This section proves Theorem 1.2. So, we fix $p = p_c$ and taking account of Proposition 1.4, we denote the quantities

$$B_c = 2, \quad A_c = \frac{4-2\varrho}{N}, \quad \mathrm{K}_{opt} = \Big(1 + \frac{2-\varrho}{N}\Big)\|\phi\|^{-\frac{4-2\varrho}{N}}. \tag{3.1}$$

Thus, by Proposition 1.4, we write

$$E(u_0) \geq \|\nabla u(t)\|^2 - \frac{2}{1+p_c}\mathrm{K}_{opt}\|\nabla u(t)\|^{B_c}\|u(t)\|^{A_c} + \sum_{j\in J}|x_j|^2|u(t,x)|^2$$
$$\geq \|\nabla u(t)\|^2\Big(1 - \frac{2}{1+p_c}\mathrm{K}_{opt}\|u_0\|^{\frac{4-2\varrho}{N}}\Big) + \sum_{j\in J}|x_j|^2|u(t,x)|^2$$
$$\geq \|\nabla u(t)\|^2\Big(1 - [\frac{\|u_0\|}{\|\phi\|}]^{\frac{4-2\varrho}{N}}\Big) + \sum_{j\in J}|x_j|^2|u(t,x)|^2. \tag{3.2}$$

By (3.2), if $\frac{\|u_0\|}{\|\phi\|} < 1$, it follows that $T^{\max} = \infty$.

Now, we take for $\lambda, \mu > 0$ the scaling $u_0 := \lambda\phi(\frac{\cdot}{\mu})$ and we compute

$$\|u_0\|^2 = \lambda^2\mu^N\|\phi\|^2; \tag{3.3}$$
$$\|x_j u_0\|^2 = \lambda^2\mu^{N+2}\|x_j\phi\|^2; \tag{3.4}$$

$$\|\nabla u_0\|^2 = \lambda^2\mu^{N-2}\|\nabla\phi\|^2; \tag{3.5}$$

$$\||x|^{-\frac{\varrho}{1+p}}u_0\|_{1+p_c}^{1+p_c} = \lambda^{1+p}\mu^{N-\varrho}\||x|^{-\frac{\varrho}{1+p_c}}\phi\|_{1+p_c}^{1+p_c}. \tag{3.6}$$

Let us pick $0 < \varepsilon \ll 1$ and

$$\mu^4\sum_{j\in J}\|x_j\phi\|^2 < \frac{N}{2-\varrho}\|\phi\|^2\Big([(\varepsilon+\|\phi\|^2)\|\phi\|^{-2}]^{\frac{2-\varrho}{N}} - 1\Big); \tag{3.7}$$

$$\lambda^2 := (\varepsilon+\|\phi\|^2)\|\phi\|^{-2}\mu^{-N}. \tag{3.8}$$

Taking account of the Pohozaev identities, namely (1.14),

$$\|\phi\|^2 = \frac{A_c}{B_c}\|\nabla\phi\|^2 = \frac{A_c}{1+p_c}\||x|^{-\frac{\varrho}{1+p_c}}\phi\|_{1+p_c}^{1+p_c},$$

we write by (3.3) to (3.6),

$$\begin{aligned}
E(u_0) &= \|\nabla u_0\|^2 + \sum_{j\in J}\|x_j^2u_0\|^2 - \frac{2}{1+p}\||x|^{-\frac{\varrho}{1+p}}u_0\|_{1+p_c}^{1+p_c} \\
&= \lambda^2\mu^N\Big(\mu^{-2}\|\nabla\phi\|^2 + \mu^2\sum_{j\in J}\|x_j\phi\|^2 - \frac{2}{1+p_c}\lambda^{-1+p_c}\mu^{-\varrho}\||x|^{-\frac{\varrho}{1+p}}\phi\|_{1+p_c}^{1+p_c}\Big) \\
&= \lambda^2\mu^N\Big((\mu^{-2} - \lambda^{\frac{4-2\varrho}{N}}\mu^{-\varrho})\|\nabla\phi\|^2 + \mu^2\sum_{j\in J}\|x_j\phi\|^2\Big).
\end{aligned} \tag{3.9}$$

Thus, by (3.9) via (3.7) and (3.8), we write

$$\begin{aligned}
E(u_0) &= \lambda^2\mu^{N-2}\Big((1 - \lambda^{\frac{4-2\varrho}{N}}\mu^{2-\varrho})\|\nabla\phi\|^2 + \mu^4\sum_{j\in J}\|x_j\phi\|^2\Big) \\
&= \lambda^2\mu^{N-2}\Big((1 - [(\varepsilon+\|\phi\|^2)\|\phi\|^{-2}]^{\frac{2-\varrho}{N}})\|\nabla\phi\|^2 + \mu^4\sum_{j\in J}\|x_j\phi\|^2\Big) \\
&= \lambda^2\mu^{N-2}\Big(\frac{N}{2-\varrho}(1 - [(\varepsilon+\|\phi\|^2)\|\phi\|^{-2}]^{\frac{2-\varrho}{N}})\|\phi\|^2 + \mu^4\sum_{j\in J}\|x_j\phi\|^2\Big) \\
&< 0.
\end{aligned} \tag{3.10}$$

The proof is achieved via Remark 1.3.

4. Mass-critical concentration

This section proves Theorem 1.3. Let us pick the sequences

$$t_n \to T^{\max}, \quad \text{as} \quad n \to \infty; \tag{4.1}$$

$$\lambda_n := \frac{\|\nabla \phi\|}{\|\nabla u(t_n)\|}; \tag{4.2}$$

$$v_n := \lambda_n^{\frac{N}{2}} u(t_n, \lambda_n \cdot). \tag{4.3}$$

Thus, by (3.3) and (3.6), we write

$$\|v_n\| = \|u_0\| \quad \text{and} \quad \|\nabla v_n\| = \|\nabla \phi\|. \tag{4.4}$$

Moreover, by (4.2) because $p = p_c$, we have

$$\begin{aligned} H(v_n) &:= E(v_n) - \sum_{j \in J} \|x_j v_n\|^2 \\ &= \lambda_n^2 H(u(t_n)) \\ &= \lambda_n^2 \Big(E(u_0) - \sum_{j \in J} \|x_j u(t_n)\|^2 \Big). \end{aligned} \tag{4.5}$$

Applying (1.7), it follows that $\partial_t^2 \Big(\|xu(t)\|^2 \Big) \leq 8E(u_0)$, which implies that

$$\sup_n \sum_{j \in J} \|x_j u(t_n)\| \lesssim 1. \tag{4.6}$$

So, (4.5) via (4.6) and the fact that λ_n vanishes at infinity, implies that

$$\int_{\mathbb{R}^N} |v_n|^{1+p_c} |x|^{-\varrho} \, dx \to \frac{1+p_c}{2} \|\nabla \phi\|^2 \quad \text{when} \quad n \to \infty. \tag{4.7}$$

Applying Lemma 1.1, with $\|\nabla \phi\|^2 = M$ and $\frac{1+p_c}{2} \|\nabla \phi\|^2 = m^{1+p_c}$, there exist $x_n \in \mathbb{R}^N$ and $V \in H^1$ such that $\|\phi\| \leq \|V\|$ and

$$v_n(\cdot + x_n) \rightharpoonup V, \quad \text{in} \quad H^1. \tag{4.8}$$

Thus, for any real number $R > 0$, yields

$$\liminf_n \int_{|x - x_n| \leq R\lambda_n} |u(t_n, x)|^2 \, dx \geq \int_{|x| \leq R} |V(x)|^2 \, dx. \tag{4.9}$$

Now, since

$$\mu(t_n) \frac{\|\nabla u(t_n)\|}{\|\nabla \phi\|} = \frac{\mu(t_n)}{\lambda_n} \to \infty,$$

taking $n \gg 1$, by (4.9), we write

$$\liminf_{n} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq\mu(t_n)} |u(t_n,x)|^2\,dx \geq \|V\|^2 \geq \|\phi\|^2.$$

Then,

$$\|\phi\|^2 \leq \liminf_{t\to T^{\max}} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq\mu(t)} |u(t,x)|^2\,dx.$$

With a continuity argument, there exists $x(t) \in \mathbb{R}^N$ satisfying

$$\|\phi\|^2 \leq \liminf_{t\to T^{\max}} \int_{|x-x(t)|\leq\mu(t)} |u(t,x)|^2\,dx.$$

This concludes the proof of Theorem 1.3.

5. Blow-up rate

This section proves Theorem 1.4.

5.1. Weak convergence

We start with the next auxiliary result.

Proposition 5.1. *Let $N \geq 2$, $p = p_c$ and ϕ a ground state of* (1.2). *Let $u \in C([0,T^{\max}),\Sigma')$ be a blowing-up solution to* (1.1) *satisfying $\|u_0\| = \|\phi\|$. Then, there exists $x_0 \in \mathbb{R}^N$ such that*

$$\|u(t)\|^2 \rightharpoonup \|\phi\|^2\delta_{x_0}, \quad as \quad t \to T^{\max},$$

in the sense of distribution.

Proof. By Theorem 1.3, namely (1.10), we have for any $R > 0$,

$$\|\phi\|^2 \leq \liminf_{t\to T^{\max}} \int_{|x-x(t)|\leq R} |u(t,x)|^2\,dx. \tag{5.1}$$

So, (5.1) via the identity $\|u_0\| = \|u(t)\| = \|\phi\|$ implies that for any $R > 0$,

$$\|\phi\|^2 = \liminf_{t\to T^{\max}} \int_{|x-x(t)|\leq R} |u(t,x)|^2\,dx. \tag{5.2}$$

Now, we take $\psi \in C_0^\infty(\mathbb{R}^N)$, $\lambda_n \to 0$ and $t_n \to T^{\max}$, when $n \to \infty$. We define $w(t) := u(t,\cdot + x(t))$ and $z_n := \lambda_n^{\frac{N}{2}} w_n(\lambda_n\cdot)$, so by the dominated convergence theorem, we get

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} \psi(x)|w(t_n)|^2\,dx - \psi(0)\|\phi\|^2\Big| &= \Big|\int_{\mathbb{R}^N} \psi(\lambda_n y)\lambda_n^N|w(t_n,\lambda_n y)|^2\,dy - \psi(0)\|\phi\|^2\Big| \\ &= \Big|\int_{\mathbb{R}^N} \psi(\lambda_n y)|z_n|^2\,dy - \psi(0)\|\phi\|^2\Big|\end{aligned}$$

$$\lesssim \Big| \|z_n\|^2 - \|\phi\|^2 \Big| + \int_{\mathbb{R}^N} \Big|\psi(\lambda_n y) - \psi(0)\Big| |\phi|^2 \, dy$$
$$\to 0. \tag{5.3}$$

Thus, (5.3) implies that in the sense of distribution, when $t \to T^{\max}$,

$$|u(t, \cdot + x(t))|^2 \rightharpoonup \|\phi\|^2 \delta_0. \tag{5.4}$$

Now, for a real-valued function $\theta(x)$, we compute

$$|\nabla(ue^{i\tau\theta(x)})|^2 = |\nabla u|^2 + \tau^2 |\nabla\theta(x)|^2 |u|^2 + 2\tau \nabla\theta(x) \cdot \mathrm{Im}(\bar{u}\nabla u). \tag{5.5}$$

Hence, by (5.5), we get

$$\begin{aligned} H(ue^{i\tau\theta}) &= \|\nabla(ue^{i\tau\theta})\|^2 - \frac{2}{1+p_c} \int_{\mathbb{R}^N} |u|^{1+p_c} |x|^{-\varrho} \, dx \\ &= \Big(\|\nabla u\|^2 + \tau^2 \int_{\mathbb{R}^N} |\nabla\theta|^2 |u|^2 \, dx + 2\tau \int_{\mathbb{R}^N} \nabla\theta \cdot \mathrm{Im}(\bar{u}\nabla u) \, dx \Big) - \frac{2}{1+p_c} \int_{\mathbb{R}^N} |u|^{1+p_c} |x|^{-\varrho} \, dx \\ &= H(u) + \tau^2 \|u\nabla\theta\|^2 + 2\tau \int_{\mathbb{R}^N} \nabla\theta \cdot \mathrm{Im}(\bar{u}\nabla u) \, dx. \end{aligned} \tag{5.6}$$

Moreover, by Proposition 1.4 for any $\tau \geq 0$,

$$\begin{aligned} H(ue^{i\tau\theta}) &\geq \|\nabla(ue^{i\tau\theta})\|^2 \left(1 - \Big(\frac{\|ue^{i\tau\theta}\|}{\|\phi\|} \Big)^{p-1} \right) \\ &= \|\nabla(ue^{i\tau\theta})\|^2 \left(1 - \Big(\frac{\|u_0\|}{\|\phi\|} \Big)^{p-1} \right) \\ &= 0. \end{aligned} \tag{5.7}$$

Now, (5.7) and (5.6) give a negative discriminant of the polynomial $\tau \mapsto H(ue^{i\tau\theta})$, namely

$$\Big| \int_{\mathbb{R}^N} \nabla\theta \cdot \mathrm{Im}(\bar{u}\nabla u) \, dx \Big| \leq \sqrt{H(u_0)} \|u\nabla\theta\|. \tag{5.8}$$

Moreover, (1.1) gives for any $1 \leq j \leq N$,

$$\begin{aligned} \Big| \partial_t \int_{\mathbb{R}^N} x_j |u(t,x)|^2 \, dx \Big| &= 2 \Big| \int_{\mathbb{R}^N} x_j \Re(\bar{u}\partial_t u) \, dx \Big| \\ &= 2 \Big| \int_{\mathbb{R}^N} x_j \Im(\bar{u}\Delta u) \, dx \Big|. \end{aligned} \tag{5.9}$$

since $H(u) \leq E(u_0)$, an integration by parts via (5.8), (5.9) and the mass conservation law, implies that

$$\begin{aligned}\Big|\partial_t \int_{\mathbb{R}^N} x_j|u(t,x)|^2\,dx\Big| &= 2\Big|\int_{\mathbb{R}^N} \nabla x_j \cdot \Im(\bar{u}\nabla u)\,dx\Big| \\ &\lesssim \|u\nabla x_j\| \\ &\lesssim 1. \end{aligned} \tag{5.10}$$

Taking $t_n \to T^{\max}$, with (5.10) via the Cauchy criteria, it follows that

$$\Big|\int_{\mathbb{R}^N} x_j|u(t_n,x)|^2 - x_j|u(t_m,x)|^2\,dx\Big| \lesssim |t_n - t_m| \to 0, \quad \text{when} \quad n,m \to \infty. \tag{5.11}$$

So, (5.11) implies that the next limit exists

$$x^* := \|\phi\|^{-1} \lim_{t\to T^{\max}} \int_{\mathbb{R}^N} x|u(t,x)|^2\,dx. \tag{5.12}$$

Moreover, since

$$|x(t)|^2 \int_{\mathbb{R}^N} |u(t)|^2\,dx \lesssim \int_{\mathbb{R}^N} |u(t,x+x(t))|^2|x+x(t)|^2\,dx = \|xu(t)\|^2, \tag{5.13}$$

keeping in mind (1.5), it follows that

$$\limsup_{t\to T^{\max}} |x(t)| \lesssim 1. \tag{5.14}$$

Furthermore, using (5.4), via the equality

$$\int_{|x|<R} |u(t)|^2 x\,dx = \int_{|x+x(t)|<R} |u(t,x+x(t))|^2 x\,dx + \int_{|x+x(t)|<R} |u(t,x+x(t))|^2 x(t)\,dx,$$

we write for $R \gg 1$ and $t \to T^{\max}$,

$$\int_{|x|<R} |u(t)|^2 x\,dx - \int_{|x+x(t)|<R} |u(t,x+x(t))|^2 x(t)\,dx \to 0. \tag{5.15}$$

Additionally, by Hölder estimate via (5.10), we have

$$\int_{|x|>R} |u(t)|^2 x\,dx \leq R^{-1} \int_{|x|<R} |u(t)|^2|x|^2\,dx$$

$$\lesssim R^{-1}. \tag{5.16}$$

Hence, by (5.15) and (5.16), it follows that

$$\int_{\mathbb{R}^N} |u(t)|^2 x\, dx - x(t)\|\phi\|^2 \ \to 0. \tag{5.17}$$

Thus, by (5.12) and (5.17), we write when $t \to T^{\max}$,

$$x(t) \ \to x^*. \tag{5.18}$$

Finally, with (5.4) via (5.18), yields when $t \to T^{\max}$,

$$|u(t,x)|^2 \rightharpoonup \|\phi\|^2 \delta_{x^*}. \tag{5.19}$$

The proof of Proposition 5.1 is achieved by (5.19).

5.2. Proof of Theorem 1.4

Let us take a nonnegative smooth radial function denoted by $\Theta \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$\Theta(x) := |x|^2, \text{ if } |x| < 1, \text{ and } |\nabla\Theta|^2 \lesssim \Theta. \tag{5.20}$$

Using the above function, we define, for $R > 0$ and x^* from Proposition 5.1,

$$\Theta_R := R^2 \Theta(\frac{\cdot}{R}); \tag{5.21}$$

$$\Upsilon_R(t) := \int_{\mathbb{R}^N} \Theta_R(x - x^*)|u(t,x)|^2\, dx. \tag{5.22}$$

We compute using (1.1) via (5.20) and (5.8),

$$\begin{aligned} |\Upsilon_R'(t)| &= 2\Big|\int_{\mathbb{R}^N} \nabla\Theta_R(\cdot - x^*)\cdot \operatorname{Im}(\bar{u}\nabla u)\, dx\Big| \\ &\lesssim \sqrt{H(u_0)}\|u\nabla\Theta_R(\cdot - x^*)\| \\ &\lesssim \sqrt{\Upsilon_R(t)}. \end{aligned} \tag{5.23}$$

We integrate in time the identity (5.23) on $[t, t_n]$, where $t_n \to T^{\max}$, to get via (5.19),

$$\begin{aligned} \sqrt{\Upsilon_R(t)} &= \lim_{n\to\infty} |\sqrt{\Upsilon_R(t)} - \sqrt{\Upsilon_R(t_n)}| \\ &\lesssim |t - T^{\max}|. \end{aligned} \tag{5.24}$$

We rewrite (5.24) as follows

$$\Upsilon_R \lesssim (T^{\max} - \cdot)^2. \tag{5.25}$$

Letting $R \to \infty$ in (5.25), yields

$$\Upsilon(t) := \|(\cdot - x^*)u(t)\|^2 \lesssim (T^{\max} - t)^2. \tag{5.26}$$

Using (5.26) via Hölder and Hardy estimates, we write

$$\begin{aligned}\|u(t)\|^2 &= \int_{\mathbb{R}^N} \left|(x - x^*)u(t)|(x - x^*)|^{-1}u(t)\right| dx \\ &\lesssim \|(\cdot - x^*)u(t)\|\||\cdot - x^*|^{-1}u(t)\| \\ &\lesssim \|(\cdot - x^*)u(t)\|\|\nabla u(t)\|. \end{aligned} \tag{5.27}$$

We collect (5.26) and (5.27) to get

$$\begin{aligned}\|u(t)\|^2 &\lesssim \|u(t)(\cdot - x^*)\|\|\nabla u(t)\| \\ &\lesssim (T^{\max} - t)\|\nabla u(t)\|. \end{aligned} \tag{5.28}$$

Finally, (5.28) via the mass conservation law gives (1.11). The proof of Theorem 1.4 is achieved.

□

A. Appendix: Variance identity

Let us give a proof of the first variance identity in Proposition 1.2. The second identity follows similarly. Let a local solution to (1.1) denoted by $u \in C([0, T^{\max}), \Sigma)$ and the real function

$$V : [0, T^{\max}) \to \mathbb{R}, \quad t \mapsto \sum_{j \notin J}^{m} \|x_j u(t)\|^2. \tag{A.1}$$

Multiplying the equation (1.1) by $2u$ and examining the imaginary parts, we get

$$\partial_t(|u|^2) = -2\Im(\bar{u}\Delta u). \tag{A.2}$$

We denote by $a(x) := \sum_{j \notin J} |x_j|^2$, $b(x) := \sum_{j \in J} |x_j|^2$. By (A.1) and (A.2), we compute using the convention of sum to repeated index

$$\partial_t V = -2 \sum_{j \notin J} \int_{\mathbb{R}^N} |x_j|^2 \Im(\bar{u}\Delta u)\, dx$$

$$= 2\Im \int_{\mathbb{R}^N} (\partial_k a \partial_k u)\bar{u}\, dx. \tag{A.3}$$

Denoting the source term by $N := |x|^{-\varrho}|u|^{p-1}u$ and using the equation (1.1), we write

$$\begin{aligned}
\partial_t \Im(\partial_k u \bar{u}) =& \Im(\partial_k \dot{u}\bar{u}) + \Im(\partial_k u \bar{\dot{u}}) \\
=& \Re(i\dot{u}\partial_k \bar{u}) - \Re(i\partial_k \dot{u}\bar{u}) \\
=& \Re\Big(\partial_k \bar{u}(- \Delta u + \sum_{j\in J} |x_j|^2 u - N)\Big) - \Re\Big(\bar{u}\partial_k(- \Delta u + \sum_{j\in J} |x_j|^2 u - N)\Big) \\
=& \Re\Big(\bar{u}\partial_k \Delta u - \partial_k \bar{u}\Delta u\Big) - \Re\Big(\bar{u}\partial_k(\sum_{j\in J} |x_j|^2 u) - \partial_k \bar{u} \sum_{j\in J} |x_j|^2 u\Big) + \Re\Big(\bar{u}\partial_k N - \partial_k \bar{u} N\Big).
\end{aligned} \tag{A.4}$$

Using the identity

$$\frac{1}{2}\partial_k \Delta(|u|^2) - 2\partial_l \Re(\partial_k u \partial_l \bar{u}) = \Re\Big(\bar{u}\partial_k \Delta u - \partial_k \bar{u}\Delta u\Big), \tag{A.5}$$

it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} \partial_k a \Re\Big(\bar{u}\partial_k \Delta u - \partial_k \bar{u}\Delta u\Big)\, dx =& \int_{\mathbb{R}^N} \partial_k a\Big(\frac{1}{2}\partial_k \Delta(|u|^2) - 2\partial_l \Re(\partial_k u \partial_l \bar{u})\Big)\, dx \\
=& 2\int_{\mathbb{R}^N} \partial_l \partial_k a \Re(\partial_k u \partial_l \bar{u})\, dx \\
=& 4\sum_{j\notin J} \|\partial_j u\|^2.
\end{aligned} \tag{A.6}$$

Moreover,

$$\int_{\mathbb{R}^N} \partial_k a \Re\Big(\bar{u}\partial_k(bu) - \partial_k \bar{u} b u\Big)\, dx = \int_{\mathbb{R}^N} (\partial_k a \partial_k b)|u|^2\, dx = 0. \tag{A.7}$$

Furthermore

$$\begin{aligned}
\int_{\mathbb{R}^N} \partial_k a \Re\Big(\bar{u}\partial_k N - \partial_k \bar{u} N\Big)\, dx =& \int_{\mathbb{R}^N} \partial_k a \Re(\partial_k[\bar{u}N] - 2\partial_k \bar{u} N)\, dx \\
=& -\int_{\mathbb{R}^N} \Big(\Delta a \bar{u} N - 2\Re(\partial_k a \partial_k \bar{u} N)\Big)\, dx \\
=& -2(N-k)\int_{\mathbb{R}^N} |x|^{-\varrho}|u|^{1+p}\, dx - 2\int_{\mathbb{R}^N} \partial_k a \Re(\partial_k \bar{u} N)\, dx.
\end{aligned} \tag{A.8}$$

Using integration by parts, we get

$$\int_{\mathbb{R}^N} \partial_k a \Re\Big(\partial_k \bar{u} N\Big)\, dx = \int_{\mathbb{R}^N} \partial_k a \Re\Big(\partial_k \bar{u}|u|^{p-1}u\Big)|x|^{-\varrho}\, dx$$

$$
\begin{aligned}
&=\frac{1}{1+p}\int_{\mathbb{R}^N}\partial_k a\partial_k(|u|^{1+p})|x|^{-\varrho}\,dx\\
&=-\frac{1}{1+p}\int_{\mathbb{R}^N}\Delta a|u|^{1+p}|x|^{-\varrho}\,dx-\frac{1}{1+p}\int_{\mathbb{R}^N}\partial_k a\partial_k(|x|^{-\varrho})|u|^{1+p}\,dx\\
&=-2\frac{N-k}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^{-\varrho}\,dx-\frac{1}{1+p}\int_{\mathbb{R}^N}\nabla a\cdot\nabla(|x|^{-\varrho})|u|^{1+p}\,dx. \qquad (A.9)
\end{aligned}
$$

Collecting (A.8) and (A.9), we have

$$
\begin{aligned}
\int_{\mathbb{R}^N}\partial_k a\Re\Big(\bar{u}\partial_k N-\partial_k\bar{u}N\Big)\,dx=2(N-k)\Big(-1+\frac{2}{1+p}\Big)\int_{\mathbb{R}^N}|u|^{1+p}|x|^{-\varrho}\,dx\\
+\frac{2}{1+p}\int_{\mathbb{R}^N}\nabla a\cdot\nabla(|x|^{-\varrho})|u|^{1+p}\,dx. \qquad (A.10)
\end{aligned}
$$

Finally, plugging (A.10), (A.7) and (A.6) in (A.3), we get

$$
\frac{1}{2}\partial_t^2 V=4\sum_{j\notin J}\|\partial_j u\|^2-2(N-k)(1-\frac{2}{1+p})\int_{\mathbb{R}^N}|u|^{1+p}|x|^{-\varrho}\,dx+\frac{2}{1+p}\int_{\mathbb{R}^N}\nabla a\cdot\nabla(|x|^{-\varrho})|u|^{1+p}\,dx. \qquad (A.11)
$$

This proves (1.4). The proof of (1.5) follows similarly by taking account of changing (A.7).

Author contributions

The first author performed the analysis and collected the data. The second author wrote the paper and supervised the work. Both authors investigated equally the paper.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Use of Generative-AI tools declaration

The author(s) declare(s) they have not used Artificial Intelligence (AI) tools in the creation of this article.

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