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Research article

Robust stability analysis of large-scale interconnected systems with time-varying delays

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Abstract: The robust stability analysis of large-scale interconnected systems constrained by time-varying delays among subsystems is studied. In practical engineering applications, due to the spatially distributed nature of large-scale interconnected systems, the transfer of information among subsystems is usually affected by communication delays. Based on the integral quadratic constraint theory, the calculationally efficient conditions of robust stability for the large-scale interconnected systems were established by taking advantage of the sparsity of the subsystem connection topology. The derived decoupling robust stability conditions rely solely on the subsystem connection matrix, each subsystem parameter and the chosen integral quadratic constraint multiplier. Finally, the simulation results showed that the obtained conditions are valid for the analysis of large-scale interconnected systems constrained by time-varying delays among subsystems.

Keywords: large-scale systems; interconnected systems; robust stability; integral quadratic constraint; time-varying delays
Mathematics Subject Classification: 15A39, 93A15, 93D09

1. Introduction

In the past decades, with the development of society and the progress of science and technology, large-scale interconnected systems (LSIS) have received extensive attention due to their widespread use in drone formation flight [1], micro-cantilever arrays in optical atomic force microscopy [2], Micro-Electro-Mechanical systems (MEMS) [3], smart power grids [4], large-scale transportation systems [5], and other applications. LSIS consist of numerous subsystems, which have typical spatial distribution characteristics, and there are mutual influences and functions among neighboring subsystems [6]. Due to the mechanical structure characteristics and information interaction characteristics of LSIS, parameter uncertainty [7], nonlinearity [8], delay [9], and other uncertainties inevitably exist in the connection of subsystems. In practical engineering applications, it is needed to guarantee the long-term stable operation [10]. Due to the spatially distributed nature of LSIS, the transmission of information among subsystems is usually affected by communication delays, which are usually not only unanticipated and unestimated, but also highly likely to be time-varying. Such delays may cause performance degradation and system instability.

One of the difficulties in robust stability analysis of LSIS is that the calculation rapidly becomes more complex as the system state dimension grows. Due to the low calculational efficiency of the traditional lumped robust stability analysis method, our aim of this paper is to exploit the sparsity of the subsystem connection topology to provide calculationally efficient conditions for robust stability analysis of LSIS with time-varying delays among subsystems.

Integral quadratic constraint (IQC) is an integral quadratic inequality that characterizes the dynamic properties of certain signals combinations in a particular dynamic system [11, 12]. The appearance of IQC provides a unified framework for abstract description of time-varying, nonlinear, uncertain elements in the system, and plays a pivotal role in robust stability analysis and performance evaluation [13, 14]. Based on the work of Yakubovich [15], IQC-based methods provide the framework for analyzing robustness, where the system is divided into the known linear time-invariant (LTI) system and the feedback connection of disturbances, which depends only on the choice of the IQC describing the uncertainty.

Analyzing system stability effectively involves leveraging the structural characteristics of LSIS. Based on the model described in [16], the reseacher in [17] presented a dynamic system model, where topology relations among subsystems are expressed by a connection matrix. Notably, this model describes a more general networked system, making it highly applicable. Furthermore, [18] provided two sufficient conditions based on linear matrix inequalities (LMIs) for stability analysis of networked systems with the LTI dynamic that rely solely on subsystem parameters and subsystem connection matrix (SCM). Building upon the foundation, a distributed state observer was constructed. In [19], sufficient conditions to ensure the stability and performance of interconnected systems that exhibit spatially invariant properties and depend on network communication are derived by taking advantage of the hybrid system theory. Moreover, the researchers in [20] delved into the stability and stabilization aspects of a LTI networked heterogeneous system, characterized by arbitrarily connected subsystems. Notably, this work provides computational efficiency conditions. However, it is essential to highlight that these interconnected subsystems share identical dynamic properties that can be linked only according to a particular spatial structure. Notably absent from the aforementioned research is an examination of the impact of uncertainties among subsystems on the overall system behavior.

Stability analysis of interconnected systems with time-varying delays has been greatly studied. The researchers in [21] developed decentralized delay stability and stabilization methods for a class of linear interconnected continuous-time systems. The researchers in [22] considered the control problem for a class of time varying nonlinear large-scale systems with time delays in the interconnections and proposed an adaptive state feedback controller that is independent of time delays. In [23, 24], differential equation models with time-varying delays were used to describe the existence and stability of solutions in nonlinear dynamics. Additionally, new sufficient conditions for the global exponential stability of time-delayed inertial neural networks were derived. The researcher in [25] analyzed robust stability of discrete linear time-invariant systems with time-varying delays in the framework of integral quadratic constraint. The researchers in [26] introduced the interconnections among subsystems with IQC and derived efficient computational conditions for robust stability by solving sparse linear matrix inequalities.

According to IQC theory, the robust stability of LSIS with time-varying delays among subsystems is studied. Each subsystem exhibits different dynamic characteristics, and their interactions are arbitrary. By separating time-varying delay operators from interconnects, we reconstruct uncertain large-scale interconnected systems into uncertain LTI systems defined by linear fractional transform (LFT) model representations, and apply the usual robust stability analysis approaches to uncertain linear systems.

The contributions of this paper include the following two areas:

(1) To reduce the computational burden of the lumped model description method, the sparse structure of the subsystem connection matrix is leveraged and combined with the IQC describing the delay difference operator. The two sufficient conditions for the robust stability of LSIS, which can be efficiently solved using sparse solvers;
(2) The structural feature of SCM is utilized to demonstrate the sufficient condition for robust stability of LSIS, which is solely relevant to the subsystem parameter, the selected IQC multiplier, and the SCM Φ. Every subsystem can be verified independently, so that the condition provides a significant calculational advantage. Compared with the lumped formula, the approach avoids the numerical instability arising from the operation of high-dimensional matrix inversion. Additionally it improves the calculational efficiency.

The paper is structured as follows: In Part 2, we introduce the IQC theory and explain the formulation of the problem. In Part 3, we give robust stability conditions of LSIS with time-varying delays among subsystems based on IQC. In Part 4, we give numerical simulation. In Part 5, we summarize this paper.

The symbolic meaning is as follows: $\mathbb{RL}_\infty^{(m+n)\times(m+n)}$ denotes a space of rational transfer matrix with real coefficients and no poles on the unit circle, $\mathbb{RH}_\infty^{(m+n)\times(m+n)}$ represents the subspace of $\mathbb{RL}_\infty^{(m+n)\times(m+n)}$ that is regular and resolved outside the unit disk. $\mathbb{R}^n$ is an n-dimensional real vector space. l_2^n denotes an n-dimensional square additive signal set, l_{2e}^n denotes an n-dimensional local square additive signal expansion set. P^T and P^* denote the transpose and conjugate transpose of matrix P, respectively. $(*)^T MX$ or $X^T M(*)$ is a brief representation of $X^T MX$, respectively. $\mathbb{Z}^+$ is the set of nonnegative integers. $\psi^\sim$ denotes the conjugate transpose of the discrete transfer function matrix ψ. $\mathbf{diag}\left\{X_i|_{i=1}^N\right\}$ denotes a diagonal block matrix. $\mathbf{col}\left\{X_i|_{i=1}^N\right\}$ denotes a column vector consisting of X_i. I denotes n-dimensional identity matrix, the dimension subscript will be omitted if no ambiguity exists, and the same is true for the zero matrix 0. The symbol $\star$ in a matrix refers to the symmetric elements of a symmetric matrix. n_x is total number of states of networked systems, $n_{\hat{\psi}}$ aggregates the number of states of the IQC induction system. $\mathcal{S}_\varepsilon$ means time-varying delay operator, and $\mathcal{T}_\varepsilon$ means time-varying delay difference operator.

2. Problem formulation and preliminaries

IQC theory is an effective method for analyzing robust stability of uncertain systems [12]. The main idea is to substitute the uncertainty Δ in a feedback interconnected system with an integral quadratic form describing the characteristics of that uncertain input and output. The definition of discrete-time frequency domain IQC follows [27].

Definition 1. *For all $q \in l_2^m$ and $p = \Delta(q)$, the bounded causal operator $\Delta : l_{2e}^n \to l_{2e}^m$ satisfies the IQC defined by Π, if*

$$\int_{-\pi}^{\pi} \begin{bmatrix} \hat{q}\left(e^{j\omega}\right) \\ \hat{p}\left(e^{j\omega}\right) \end{bmatrix}^* \Pi\left(e^{j\omega}\right) \begin{bmatrix} \hat{q}\left(e^{j\omega}\right) \\ \hat{p}\left(e^{j\omega}\right) \end{bmatrix} \mathrm{d}\omega \geq 0, \tag{2.1}$$

in which $\Pi = \Pi^* \in \mathbb{RL}_\infty^{(m+n)\times(m+n)}$, $\hat{q}$ and $\hat{p}$ are Fourier transforms of q and p.

Subsequently, the lemma described below elaborates the IQC framework relevant to robust stability analysis of discrete uncertain systems, which corresponds to the discrete-time version of the IQC stability theory elaborated in [13].

Lemma 1. *Suppose a discrete uncertain system,*

$$q = Gp, p = \Delta(q), \tag{2.2}$$

where matrix G represents the LTI transfer function, Δ denotes the bounded causal operator describing the uncertainty in the system. The system model is shown in Figure 1.

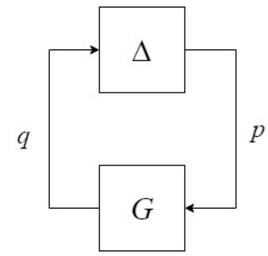

Figure 1. Feedback configuration.

Consequently, the robust stability of the uncertain system is sufficiently guaranteed if the following conditions are met:

(1) For every $\beta \in [0, 1]$, the interconnection specified in (2.2), with $\beta\Delta$, is well-posed;
(2) for every $\beta \in [0, 1]$, $\beta\Delta \in IQC(\Pi)$;
(3) there exists $\theta > 0$ such that

$$\begin{bmatrix} G\left(e^{j\omega}\right) \\ I \end{bmatrix}^* \Pi\left(e^{j\omega}\right) \begin{bmatrix} G\left(e^{j\omega}\right) \\ I \end{bmatrix} \leq -\theta I, \quad \forall|\omega| \leq \pi. \tag{2.3}$$

Condition (3) is a frequency domain inequality with infinite dimension. For the dynamic IQC multiplier $\Pi \in \mathbb{RL}_\infty^{(m+n)\times(m+n)}$, it can be factorized as $\Pi = \psi\left(e^{j\omega}\right)^* M \psi\left(e^{j\omega}\right)$. Subsequently, based on (2.3), the following inequality can be derived.

$$\left(\psi\left(e^{j\omega}\right) \begin{bmatrix} G\left(e^{j\omega}\right) \\ I \end{bmatrix}\right)^* M \left(\psi\left(e^{j\omega}\right) \begin{bmatrix} G\left(e^{j\omega}\right) \\ I \end{bmatrix}\right) \leq -\theta I, \forall|\omega| \leq \pi. \tag{2.4}$$

It is shown that the frequency domain inequality condition (3) is equivalent to the frequency-independent finite-dimensional LMI condition that can be proved by utilizing the Kalman-Yakubovich-Popov(KYP) lemma [28, 29].

Lemma 2. *Let matrices A, B, L be given, assume that for any $\omega \in \mathbb{R}$, $\det(e^{j\omega}I - A) \neq 0$ stands while (A, B) is controllable. The following inequalities are equivalent if $P = P^T \in \mathbb{R}^{n\times n}$ exists,*

$$\begin{bmatrix} \left(e^{j\omega}I - A\right)^{-1} B \\ I \end{bmatrix}^* L \begin{bmatrix} \left(e^{j\omega}I - A\right)^{-1} B \\ I \end{bmatrix} \leq 0, \tag{2.5}$$

$$\begin{bmatrix} A^{\mathrm{T}}PA - P & A^{\mathrm{T}}PB \\ B^{\mathrm{T}}PA & B^{\mathrm{T}}PB \end{bmatrix} + L \leq 0. \tag{2.6}$$

A large-scale interconnected system Σ composed of N linear time-invariant dynamic subsystems is considered. Each subsystem Σ_i is characterized by the following discrete state space equations:

$$\begin{bmatrix} x(t+1,i) \\ z(t,i) \\ e(t,i) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{TT}}(i) & A_{\mathrm{TS}}(i) & B_{\mathrm{T}}(i) \\ A_{\mathrm{ST}}(i) & A_{\mathrm{SS}}(i) & B_{\mathrm{S}}(i) \\ C_{\mathrm{T}}(i) & C_{\mathrm{S}}(i) & D_{\mathrm{T}}(i) \end{bmatrix} \begin{bmatrix} x(t,i) \\ v(t,i) \\ d(t,i) \end{bmatrix}, \quad i = 1, 2, \ldots, N, \tag{2.7}$$

where t denotes the discrete time variable and i denotes the sequence numbers of the subsystems. $x(t, i)$ denotes the state vector of the subsystem Σ_i. Refer to $z(t, i)$ and $v(t, i)$ as internal output and input vectors, respectively, which are used to describe the interactions among Σ_i and other subsystems. Then, $e(t, i)$ and $d(t, i)$ denote the performance output and disturbance input of the subsystem Σ_i, respectively.

Suppose that the dimensions of vectors $x(t, i)$, $e(t, i)$, $z(t, i)$, $d(t, i)$, and $v(t, i)$ are denoted by m_{xi}, m_{ei}, m_{zi}, m_{di} and m_{vi}, respectively. For two different subsystems Σ_i and Σ_j, it is assumed that there exists an interconnection channel between the two subsystems. Partition the internal output $z(t, j)$ of Σ_j denoted by $z(t, j) = \mathbf{col}\{z_k(t, j)|_{k=1}^{m_{zj}}\}$ and the internal input $v(t, i)$ of Σ_i deonted by $v(t, i) = \mathbf{col}\{v_r(t, i)|_{r=1}^{m_{vi}}\}$. $z_k(t, j)$ and $v_r(t, i)$ represent sub-vectors of internal output and input vectors, respectively. Assume that there exist delays in the signal transmission. Hence, the constraint between the two subsystems could be expressed as follows:

$$v_r(t, i) = \left(\mathcal{S}_{\varepsilon_{i,r}} z_k\right)(t, j), \forall i \neq j, 1 \leq i, j \leq N. \tag{2.8}$$

$\mathcal{S}_{\varepsilon_{i,r}}$ describes the delay operator, defined by $v_r(t, i) = z_k(t - \varepsilon_{i,r}(t), j)$, where the delay duration $\varepsilon_{i,r}(t)$ is time-varying and uncertain. In order to streamline the symbol, $\varepsilon_{i,r}(t)$ is written as $\varepsilon_{i,r}$. The bound of $\varepsilon_{i,r}$ is denoted by the upper bound $T_{u_{i,r}}$ and the lower bound $T_{l_{i,r}}$ such that $\varepsilon_{i,r} \in [T_{l_{i,r}}, T_{u_{i,r}}]$. In this way, the connection relationships among subsystems are denoted by

$$v(t) = (\mathcal{S}_\varepsilon \Phi z)(t), \tag{2.9}$$

where $z(t) = \mathbf{col}\left\{z(t, i)|_{i=1}^{N}\right\}$, $v(t) = \mathbf{col}\left\{v(t, i)|_{i=1}^{N}\right\}$. $\mathcal{S}_\varepsilon$ is a diagonal delay operator formed by $\mathcal{S}_{\varepsilon_{i,r}}$,

$$\mathcal{S}_\varepsilon = \mathbf{diag}\left\{\mathbf{diag}\left\{\mathcal{S}_{\varepsilon_{i,r}}\Big|_{r=1}^{m_{vi}}\right\}\Big|_{i=1}^{N}\right\}. \tag{2.10}$$

Assume that each row of the subsystem connection matrix Φ has solely one non-zero element with a value equal to one, which implies that every internal input channel of a subsystem is solely impacted by a single external output channel from another subsystem. On the other hand, the internal output of one subsystem, can affect the internal inputs of multiple other subsystems. Notably, as noted in [17], the assumption would not introduce additional conservatism to the system. Next, we introduce the vector $\bar{v}(t)$ defined as $\bar{v}(t) = \Phi z(t)$. Based on (2.9), we obtain $v(t) = (\mathcal{S}_\varepsilon \bar{v})(t)$. We denote the internal input vector $v(t,i)$ of subsystem Σ_i as $v(t,i) = (\mathcal{S}_{\varepsilon_i}\bar{v})(t,i)$, where $\mathcal{S}_{\varepsilon_i} = \mathbf{diag}\left\{\mathcal{S}_{\varepsilon_{i,r}}\Big|_{r=1}^{m_{vi}}\right\}$. The structural schematic diagram of LSIS with delays is shown in Figure 2.

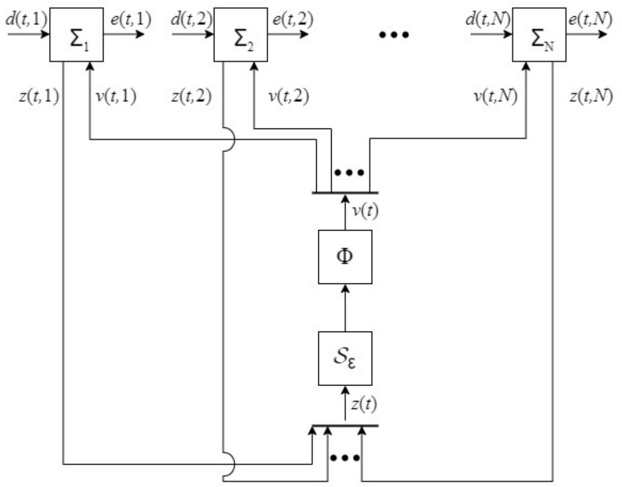

Figure 2. The stucture of LSIS with delays.

Associate the relationship mentioned above with (2.7), the state space expression of the subsystem Σ_i and the interconnections among subsystems could be reformulated as

$$\begin{bmatrix} x(t+1,i) \\ z(t,i) \\ e(t,i) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{TT}}(i) & A_{\mathrm{TS}}(i) & B_{\mathrm{T}}(i) \\ A_{\mathrm{ST}}(i) & A_{\mathrm{SS}}(i) & B_{\mathrm{S}}(i) \\ C_{\mathrm{T}}(i) & C_{\mathrm{S}}(i) & D_{\mathrm{T}}(i) \end{bmatrix} \begin{bmatrix} x(t,i) \\ (\mathcal{S}_{\varepsilon_i}\bar{v})(t,i) \\ d(t,i) \end{bmatrix}, i = 1,2,\ldots,N, \tag{2.11}$$

$$\bar{v}(t) = \Phi z(t). \tag{2.12}$$

3. Robust stability analysis with time-varying delays

The above model is transformed using a traditional robust control approach to facilitate robust stability analysis. This transformation effectively separates uncertainties from the LTI component of the system. As a starting point, two vectors $q(t,i)$ and $p(t,i)$ are imported. Next, let $q(t,i) = \bar{v}(t,i)$ and $p(t,i) = (\mathcal{S}_{\varepsilon_i}\bar{v})(t,i) - \bar{v}(t,i)$. From (2.11), the augmented state space model of the subsystem Σ_i can be derived.

$$\begin{bmatrix} x(t+1,i) \\ q(t,i) \\ z(t,i) \\ e(t,i) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{TT}}(i) & A_{\mathrm{TS}}(i) & A_{\mathrm{TS}}(i) & B_{\mathrm{T}}(i) \\ 0 & 0 & I & 0 \\ A_{\mathrm{ST}}(i) & A_{\mathrm{SS}}(i) & A_{\mathrm{SS}}(i) & B_{\mathrm{S}}(i) \\ C_{\mathrm{T}}(i) & C_{\mathrm{S}}(i) & C_{\mathrm{S}}(i) & D_{\mathrm{T}}(i) \end{bmatrix} \begin{bmatrix} x(t,i) \\ p(t,i) \\ \bar{v}(t,i) \\ d(t,i) \end{bmatrix}, \tag{3.1}$$

$$p(t,i) = (\mathcal{T}_{\varepsilon_i} q)(t,i), \tag{3.2}$$

$\mathcal{T}_{\varepsilon_i}$ denotes the diagonal delay difference operator, which clearly has $\mathcal{T}_{\varepsilon_i} = \mathbf{diag}\left\{\mathcal{T}_{\varepsilon_{i,r}}\Big|_{r=1}^{m_{vi}}\right\}$. Define $\mathcal{T}_\varepsilon(\bar{v}) := (\mathcal{S}_\varepsilon \bar{v})(t) - \bar{v}(t)$, thus $\mathcal{T}_\varepsilon = \mathcal{S}_\varepsilon - I$, where $\mathcal{T}_\varepsilon = \mathbf{diag}\left\{\mathcal{T}_{\varepsilon_i}\Big|_{i=1}^{N}\right\}$. Let $d(t) = \mathbf{diag}\left\{d(t,i)|_{i=1}^{N}\right\}$ and $e(t) = \mathbf{diag}\left\{e(t,i)|_{i=1}^{N}\right\}$.

Under the linear fractional transformation representations (3.1) and (3.2) of each subsystem, the input-output behavior of the delay-difference operator is described using IQC multipliers. Then, IQC theory is applied to demonstrate the robust stability of the system Σ with respect to the delay uncertainties.

Suppose that the operator $\mathcal{T}_{\varepsilon_{i,r}}$ complies with the IQC defined by $\Pi^{i,r}$ that could be classified as

$$\Pi^{i,r} = \begin{bmatrix} \Pi_{11}^{i,r} & \Pi_{12}^{i,r} \\ \Pi_{12}^{i,r*} & \Pi_{22}^{i,r} \end{bmatrix}. \tag{3.3}$$

Similarly, diagonal difference operators $\mathcal{T}_{\varepsilon_i}$ and $\mathcal{T}_{\varepsilon}$ conform to the IQCs defined by Π^i and $\hat{\Pi}$, respectively.

$$\Pi^{i} = \begin{bmatrix} \Pi_{11}^{i} & \Pi_{12}^{i} \\ \Pi_{12}^{i*} & \Pi_{22}^{i} \end{bmatrix}, \tag{3.4}$$

$$\hat{\Pi} = \begin{bmatrix} \hat{\Pi}_{11} & \hat{\Pi}_{12} \\ \hat{\Pi}_{12}^{*} & \hat{\Pi}_{22} \end{bmatrix}. \tag{3.5}$$

The (h, l) blocks of matrices Π^i and $\hat{\Pi}$ can be expressed as $\Pi_{hl}^i = \mathbf{diag}\left\{\Pi_{hl}^{i,r}\Big|_{r=1}^{m_{vi}}\right\}$ and $\hat{\Pi}_{hl} = \mathbf{diag}\left\{\Pi_{hl}^{i}\Big|_{i=1}^{N}\right\}$, respectively, where $h, l = 1, 2$.

By spectral factorization of Π^i, $\Pi^i = \psi^{i\sim} M^i \psi^i$ is obtained, in which $\psi^i \in \mathbb{RH}_\infty^{n_{\psi^i} \times 2m_{vi}}$, M^i denotes the self-adjoint real symmetric matrix. Suppose that the state space implementation of ψ^i can be represented as

$$\begin{bmatrix} x_\psi(t+1, i) \\ z_\psi(t, i) \end{bmatrix} = \begin{bmatrix} A_\psi(i) & B_{\psi q}(i) & B_{\psi p}(i) \\ C_\psi(i) & D_{\psi q}(i) & D_{\psi p}(i) \end{bmatrix} \begin{bmatrix} x_\psi(t, i) \\ q(t, i) \\ p(t, i) \end{bmatrix}, \tag{3.6}$$

where $x_\psi(t, i) \in \mathbb{R}^{n_{\psi^i}}$ represents the state vector of ψ^i, satisfying the zero initial condition $x_\psi(0, i) = 0$. $z_\psi(t, i) \in \mathbb{R}^{2m_{vi}}$ denotes the output vector of ψ^i.

Similarly, it can be obtained that $\hat{\Pi}$, $\hat{\Pi} = \hat{\psi}^\sim \hat{M} \hat{\psi}$. Let $\hat{\psi}$ has the following state space expression:

$$\begin{bmatrix} x_{\hat{\psi}}(t+1) \\ z_{\hat{\psi}}(t) \end{bmatrix} = \begin{bmatrix} A_{\hat{\psi}} & B_{\hat{\psi} q} & B_{\hat{\psi} p} \\ C_{\hat{\psi}} & D_{\hat{\psi} q} & D_{\hat{\psi} p} \end{bmatrix} \begin{bmatrix} x_{\hat{\psi}}(t) \\ q(t) \\ p(t) \end{bmatrix}, \tag{3.7}$$

in which $x_{\hat{\psi}}(t) \in \mathbb{R}^{n_{\hat{\psi}}}$ denotes the state vector, $n_{\hat{\psi}} = \Sigma_{i=1}^{N} n_{\psi^i}$, $z_\psi(t) \in \mathbb{R}^{2n_v}$ denotes the output vector. According to the relationship of IQC multipliers $\hat{\Pi}$ and Π^i, the matrix parameters in (3.7) are expressed as

$$\begin{aligned} A_{\hat{\psi}} &= \mathbf{diag}\left\{A_\psi(i)\Big|_{i=1}^{N}\right\}, \\ B_{\hat{\psi} q} &= \mathbf{col}\left\{B_{\psi q}(i)\Big|_{i=1}^{N}\right\}, \\ B_{\hat{\psi} p} &= \mathbf{col}\left\{B_{\psi p}(i)\Big|_{i=1}^{N}\right\}, \\ C_{\hat{\psi}} &= \mathbf{diag}\left\{C_\psi(i)\Big|_{i=1}^{N}\right\}, \\ D_{\hat{\psi} q} &= \mathbf{col}\left\{D_{\psi q}(i)\Big|_{i=1}^{N}\right\}, \\ D_{\hat{\psi} p} &= \mathbf{col}\left\{D_{\psi p}(i)\Big|_{i=1}^{N}\right\}. \end{aligned} \tag{3.8}$$

Then, $q(t)$ and $p(t)$ comply with IQC prescribed by $\hat{\Pi} = \hat{\psi}^{\sim}\hat{M}\hat{\psi}$ if the vector $z_{\hat{\psi}}$ complies with the subsequent time domain quadratic constraint:

$$\sum_{t=0}^{\mathrm{T}} z_{\hat{\psi}}{}^{T}(t)\hat{M}z_{\hat{\psi}}(t) \geq 0, \tag{3.9}$$

for all $\mathrm{T} \geq 0$, $q \in l_2^m$, $p = \Delta(q)$ and $z = \hat{\psi}\begin{bmatrix} q \\ p \end{bmatrix}$. Then, the delay difference operator $\mathcal{T}_\varepsilon$ satisfies the hard-IQC defined by $\hat{\Pi} = \hat{\psi}^{\sim}\hat{M}\hat{\psi}$. $(\hat{\psi}, \hat{M})$ is a hard factorization of the IQC multiplier $\hat{\Pi}$.

Each subsystem has internal inputs and internal outputs for information interaction with other subsystems in addition to external inputs and external outputs. According to the input-output structure of the subsystem, the supply rate function can be specified based on the quadratic performance criteria of the system [30]. Since the system Σ is expected to be finite gain l_2 stable that equates to the presence of $\gamma > 0$ such that it is $(-I, 0, \gamma^2 I)$-dissipative. Therefore, the dissipation rate of the subsystem with respect to the external inputs and outputs is defined as

$$s_i(d(t,i), e(t,i)) = \begin{bmatrix} e(t,i) \\ d(t,i) \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e(t,i) \\ d(t,i) \end{bmatrix}. \tag{3.10}$$

The typical method for analysing robust stability of LSIS using dissipativity theory and IQC stability theory is to eliminate the SCM constraints in (2.12) and derive the lumped representation describing the large-scale interconnected system. Define $A_{*\#} = \mathbf{diag}\left\{A_{*\#}(i)|_{i=1}^{N}\right\}$, $B_* = \mathbf{diag}\left\{B_*(i)|_{i=1}^{N}\right\}$, $C_* = \mathbf{diag}\left\{C_*(i)|_{i=1}^{N}\right\}$, where $* = \mathrm{T}, \mathrm{S}$, $\# = \mathrm{T}, \mathrm{S}$, $D_{\mathrm{T}} = \mathbf{diag}\left\{D_{\mathrm{T}}(i)|_{i=1}^{N}\right\}$. First, the lumped model of the interconnected system is established,

$$\begin{aligned} \begin{bmatrix} x(t+1) \\ q(t) \\ e(t) \end{bmatrix} &= \begin{bmatrix} \hat{A}_{\mathrm{XX}} & \hat{A}_{\mathrm{XP}} & \hat{B}_{\mathrm{XD}} \\ \hat{A}_{\mathrm{QX}} & \hat{A}_{\mathrm{QP}} & \hat{B}_{\mathrm{QD}} \\ \hat{C}_{\mathrm{EX}} & \hat{C}_{\mathrm{EP}} & \hat{D}_{\mathrm{ED}} \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \\ d(t) \end{bmatrix}, \\ p(t) &= (\mathcal{T}_\varepsilon q)(t), \end{aligned} \tag{3.11}$$

in which

$$\begin{aligned} \hat{A}_{\mathrm{XX}} &= A_{\mathrm{TT}} + A_{\mathrm{TS}}(I - \Phi A_{\mathrm{SS}})^{-1}\Phi A_{\mathrm{ST}}, \\ \hat{A}_{\mathrm{XP}} &= A_{\mathrm{TS}} + A_{\mathrm{TS}}(I - \Phi A_{\mathrm{SS}})^{-1}\Phi A_{\mathrm{SS}}, \\ \hat{B}_{\mathrm{XD}} &= B_{\mathrm{T}} + A_{\mathrm{TS}}(I - \Phi A_{\mathrm{SS}})^{-1}\Phi B_{\mathrm{S}}, \\ \hat{A}_{\mathrm{QX}} &= (I - \Phi A_{\mathrm{SS}})^{-1}\Phi A_{\mathrm{ST}}, \\ \hat{A}_{\mathrm{QP}} &= (I - \Phi A_{\mathrm{SS}})^{-1}\Phi A_{\mathrm{SS}}, \\ \hat{B}_{\mathrm{QD}} &= (I - \Phi A_{\mathrm{SS}})^{-1}\Phi B_{\mathrm{S}}, \\ \hat{C}_{\mathrm{EX}} &= C_{\mathrm{T}} + C_{\mathrm{S}}(I - \Phi A_{\mathrm{SS}})^{-1}\Phi A_{\mathrm{ST}}, \\ \hat{C}_{\mathrm{EP}} &= C_{\mathrm{S}} + C_{\mathrm{S}}(I - \Phi A_{\mathrm{SS}})^{-1}\Phi A_{\mathrm{SS}}, \\ \hat{D}_{\mathrm{ED}} &= D_{\mathrm{T}} + C_{\mathrm{S}}(I - \Phi A_{\mathrm{SS}})^{-1}\Phi B_{\mathrm{S}}. \end{aligned} \tag{3.12}$$

By applying Condition (3) from Lemma 1, it is established that for $\theta > 0$ and the existence of IQC multipliers, the large-scale interconnected system with time-varying delays among subsystems exhibits robust stability, so that

$$\begin{bmatrix} \hat{G}\left(e^{j\omega}\right) \\ I \end{bmatrix}^{*} \hat{\Pi}\left(e^{j\omega}\right) \begin{bmatrix} \hat{G}\left(e^{j\omega}\right) \\ I \end{bmatrix} \leq -\theta I, \forall|\omega| \leq \pi. \tag{3.13}$$

By associating the lumped model (3.11) of the system Σ with (3.7), the augmented system with extended state variables is obtained. Based on the KYP Lemma, the frequency-dependent, infinite-dimensional inequality (3.13) can be converted to a finite-dimensional LMI equivalently.

The following theorem presents the sufficient conditions for the robust stability analysis of the large-scale interconnected system Σ with respect to time-varying delays and for it to satisfy the given l_2 performance.

Theorem 1. *Consider the large-scale interconnected system Σ based on the lumped model. Given the upper bound vector $T_u = \mathbf{col}\left\{T_{u_i}\big|_{i=1}^{N}\right\}$ and the lower bound vector $T_l = \mathbf{col}\left\{T_{l_i}\big|_{i=1}^{N}\right\}$ of the time-varying delay $\varepsilon = \mathbf{col}\left\{\mathbf{col}\left\{\varepsilon_{i,r}\big|_{r=1}^{m_{vi}}\right\}\Big|_{i=1}^{N}\right\}$, where $T_{u_i} = \mathbf{col}\left\{T_{u_{i,r}}\big|_{r=1}^{m_{vi}}\right\}$ and $T_{l_i} = \mathbf{col}\left\{T_{l_{i,r}}\big|_{r=1}^{m_{vi}}\right\}$, $T_{u_{i,r}} \in \mathbb{Z}^+$, $T_{l_{i,r}} \in \mathbb{Z}^+$. The delay difference operator $\mathcal{T}_\varepsilon$ satisfies the hard-IQC defined by $\hat{\Pi} = \hat{\psi}^{\sim}\hat{M}_\varepsilon\hat{\psi}$. $(\hat{\psi}, \hat{M}_\varepsilon)$ is a hard factorization of the IQC multiplier $\hat{\Pi}$, if a positive definite matrix $\hat{P} \in R_S^{n_x+n_{\hat{\psi}}}$ exists such that the LMI (3.14) holds, the large-scale interconnected system Σ is robustly stable for the time-varying delay $\varepsilon \in [T_l, T_u]$, and the induced l_2 gain from d to e will not exceed the pre-determined $\gamma > 0$.*

$$\begin{bmatrix} \hat{A}^T\hat{P}\hat{A} - \hat{P} & \hat{A}^T\hat{P}\hat{B} \\ \hat{B}^T\hat{P}\hat{A} & \hat{B}^T\hat{P}\hat{B} \end{bmatrix} + \begin{bmatrix} \hat{C}_{\mathrm{M}}^T \\ \hat{D}_{\mathrm{M}}^T \end{bmatrix} \hat{M}_\varepsilon \begin{bmatrix} \hat{C}_{\mathrm{M}} & \hat{D}_{\mathrm{M}} \end{bmatrix} + \begin{bmatrix} \hat{C}_{\mathrm{L}}^T \\ \hat{D}_{\mathrm{L}}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \hat{C}_{\mathrm{L}} & \hat{D}_{\mathrm{L}} \end{bmatrix} < 0, \tag{3.14}$$

in which

$$\begin{aligned} \hat{A} &= \begin{bmatrix} A_{\hat{\psi}} & B_{\hat{\psi}q}\hat{A}_{\mathrm{QX}} \\ 0 & \hat{A}_{\mathrm{XX}} \end{bmatrix}, \\ \hat{B} &= \begin{bmatrix} B_{\hat{\psi}q}\hat{A}_{\mathrm{QP}} + B_{\hat{\psi}p} & B_{\hat{\psi}q}\hat{B}_{\mathrm{QD}} \\ \hat{A}_{\mathrm{XP}} & \hat{B}_{\mathrm{XD}} \end{bmatrix}, \\ \hat{C}_{\mathrm{M}} &= \begin{bmatrix} C_{\hat{\psi}} & D_{\hat{\psi}q}\hat{A}_{\mathrm{QX}} \end{bmatrix}, \\ \hat{D}_{\mathrm{M}} &= \begin{bmatrix} D_{\hat{\psi}q}\hat{A}_{\mathrm{QP}} + D_{\hat{\psi}p} & D_{\hat{\psi}q}\hat{B}_{\mathrm{QD}} \end{bmatrix}, \\ \hat{C}_{\mathrm{L}} &= \begin{bmatrix} 0 & \hat{C}_{\mathrm{EX}} \\ 0 & 0 \end{bmatrix}, \\ \hat{D}_{\mathrm{L}} &= \begin{bmatrix} \hat{C}_{\mathrm{EP}} & \hat{D}_{\mathrm{ED}} \\ 0 & I \end{bmatrix}. \end{aligned} \tag{3.15}$$

Proof. According to (3.11), $q(t) = \hat{A}_{\mathrm{QX}}x(t) + \hat{A}_{\mathrm{QP}}p(t) + \hat{B}_{\mathrm{QD}}d(t)$ can be obtained. Then, (3.7) can be rewritten as

$$\begin{bmatrix} x_{\hat{\psi}}(t+1) \\ z_{\hat{\psi}}(t) \end{bmatrix} = \begin{bmatrix} A_{\hat{\psi}} & B_{\hat{\psi}q} & B_{\hat{\psi}p} \\ C_{\hat{\psi}} & D_{\hat{\psi}q} & D_{\hat{\psi}p} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \hat{A}_{\mathrm{QX}} & \hat{A}_{\mathrm{QP}} & \hat{B}_{\mathrm{QD}} \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{\hat{\psi}}(t) \\ x(t) \\ p(t) \\ d(t) \end{bmatrix}. \tag{3.16}$$

Similarly, according to (3.11), $x(t+1) = \hat{A}_{\mathrm{XX}}x(t) + \hat{A}_{\mathrm{XP}}p(t) + \hat{B}_{\mathrm{XD}}d(t)$ and $e(t) = \hat{C}_{\mathrm{EX}}x(t) + \hat{C}_{\mathrm{EP}}p(t) + \hat{D}_{\mathrm{ED}}d(t)$ can be obtained. Substitute the vector $x(t+1)$ and $e(t)$ into (3.16), the augmented system with extended state variables is obtained.

$$\left[\begin{array}{c} x_{\hat{\psi}}(t+1) \\ x(t+1) \\ \hline z_{\hat{\psi}}(t) \\ e(t) \end{array}\right] = \left[\begin{array}{cc|cc} A_{\hat{\psi}} & B_{\hat{\psi}q}\hat{A}_{\mathrm{QX}} & B_{\hat{\psi}q}\hat{A}_{\mathrm{QP}} + B_{\hat{\psi}p} & B_{\hat{\psi}q}\hat{B}_{\mathrm{QD}} \\ 0 & \hat{A}_{\mathrm{XX}} & \hat{A}_{\mathrm{XP}} & \hat{B}_{\mathrm{XD}} \\ \hline C_{\hat{\psi}} & D_{\hat{\psi}q}\hat{A}_{\mathrm{QX}} & D_{\hat{\psi}q}\hat{A}_{\mathrm{QP}} + D_{\hat{\psi}p} & D_{\hat{\psi}q}\hat{B}_{\mathrm{QD}} \\ 0 & \hat{C}_{\mathrm{EX}} & \hat{C}_{\mathrm{EP}} & \hat{D}_{\mathrm{ED}} \end{array}\right] \left[\begin{array}{c} x_{\hat{\psi}}(t) \\ x(t) \\ \hline p(t) \\ d(t) \end{array}\right]. \tag{3.17}$$

Based on Lemma 2 and the system parameters in (3.17), the frequency-dependent, infinite-dimensional inequality (3.13) can be equivalently transformed into a finite- dimensional LMI. □

Remark 1. *The matrices* $A_{*\#} = \textbf{\textit{diag}}\left\{A_{*\#}(i)|_{i=1}^{N}\right\}$, $B_{*} = \textbf{\textit{diag}}\left\{B_{*}(i)|_{i=1}^{N}\right\}$, $C_{*} = \textbf{\textit{diag}}\left\{C_{*}(i)|_{i=1}^{N}\right\}$, *where* $* = \mathrm{T}, \mathrm{S}$, $\# = \mathrm{T}, \mathrm{S}$, $D_{\mathrm{T}} = \textbf{\textit{diag}}\left\{D_{\mathrm{T}}(i)|_{i=1}^{N}\right\}$, *are usually dense, the finite-dimensional LMI transformed by (3.13) can be expected to be dense. The matrix parameters stated in (3.12) all incorporate* $(I - \Phi A_{\mathrm{SS}})^{-1}$. *While the interconnections of subsystems typically exhibit sparsity, this characteristic is also evident in the SCM* Φ, *resulting in* $(I - \Phi A_{\mathrm{SS}})$ *being predominantly sparse. (3.15) gives the matrix parameters of the augmented system after the lumped interconnected system is combined with the IQC system, which describes the uncertainty of time-varying delays. According to the Cayley-Hamilton theorem [31], the system parameters in (3.17) is dense. The lumped formula conceals the connection relationships among subsystems within the parameters, rendering their structural information underutilized. Consequently, in the process of validating the robust stability condition of the lumped model, the calculational burden grows as the system scale increases. Notably, for large-scale interconnected systems, solving the robust stability problem using the lumped formula becomes computationally challenging. In order to mitigate the issues arising from the scaling up of the system, we propose the following robust stability condition.*

Lemma 3. *([32]) If matrices* $E_1 \in \mathbb{R}^{n_1 \times n_2}$, $E_2 \in \mathbb{R}^{n_2 \times n_3}$, $E_3 \in \mathbb{R}^{n_2 \times n_1}$ *exist, such that* $\det(I - E_3 E_1) \neq 0$, *the following inequalities are equivalent if a scalar* μ *exists,*

$$\begin{bmatrix} I \\ E_1(I - E_3E_1)^{-1}E_2 \end{bmatrix}^T Q \begin{bmatrix} I \\ E_1(I - E_3E_1)^{-1}E_2 \end{bmatrix} < 0, \tag{3.18}$$

$$Q - \mu \begin{bmatrix} E_1E_2 & E_1E_3 - I \end{bmatrix}^T \begin{bmatrix} E_1E_2 & E_1E_3 - I \end{bmatrix} < 0. \tag{3.19}$$

Proof. Note that $det(I - E_3E_1) \neq 0$, therefore, $det(I - E_1E_3) \neq 0$. It follows from the fact $(I - E_1E_3)E_1 = E_1(I - E_3E_1)$ that $E_1(I - E_3E_1)^{-1} = (I - E_1E_3)^{-1}E_1$, which implies that (3.18) is equivalent to

$$\begin{bmatrix} I \\ (I - E_1E_3)^{-1}E_1E_2 \end{bmatrix}^T Q \begin{bmatrix} I \\ (I - E_1E_3)^{-1}E_1E_2 \end{bmatrix} < 0. \tag{3.20}$$

According to direct matrix manipulations, it can be verified for any vector $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \neq 0$ that

$$\begin{bmatrix} E_1E_2 & E_1E_3 - I \end{bmatrix} \begin{bmatrix} I \\ (I - E_1E_3)^{-1}E_1E_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \equiv 0. \tag{3.21}$$

Denote that $D = \begin{bmatrix} E_1E_2 & E_1E_3 - I \end{bmatrix}$, $s = \begin{bmatrix} I \\ (I - E_1E_3)^{-1}E_1E_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

It can be observed that $D \in \mathbb{R}^{n_1 \times (n_1 + n_3)}$, which implies that $rank(D) < n_1 + n_3$. Thus, by Finsler's lemma [33], it follows that (3.20) is equivalent to

$$Q - \mu \begin{bmatrix} E_1E_2 & E_1E_3 - I \end{bmatrix}^T \begin{bmatrix} E_1E_2 & E_1E_3 - I \end{bmatrix} < 0, \tag{3.22}$$

which is precisely (3.19).

Up to this point, we have demonstrated that (3.18) ⇔ (3.20) ⇔ (3.22) ⇔ (3.19). Consequently, the proof is complete. □

Theorem 2. *The system Σ is robustly stable for the time-varying delay $\varepsilon \in [T_l, T_u]$, and the induced l_2 gain from d to e will not exceed the pre-determined $\gamma > 0$, if there exist a real number $\lambda > 0$ and a positive definite matrix $P \in R_S^{n_x+n_{\hat{\psi}}}$, the upper bound vector $T_u = \boldsymbol{col}\left\{T_{u_i}\big|_{i=1}^{N}\right\}$ and the lower bound vector $T_l = \boldsymbol{col}\left\{T_{l_i}\big|_{i=1}^{N}\right\}$ of the time-varying delay $\varepsilon = \mathbf{col}\left\{\mathbf{col}\left\{\varepsilon_{i,r}\big|_{r=1}^{m_{vi}}\right\}\Big|_{i=1}^{N}\right\}$, where $T_{u_i} = \boldsymbol{col}\left\{T_{u_{i,r}}\big|_{r=1}^{m_{vi}}\right\}$ and $T_{l_i} = \boldsymbol{col}\left\{T_{l_{i,r}}\big|_{r=1}^{m_{vi}}\right\}$, $T_{u_{i,r}} \in \mathbb{Z}^+$, $T_{l_{i,r}} \in \mathbb{Z}^+$, such that*

$$\begin{bmatrix} \Theta_{11} - \lambda L_2^T L_1^T L_1 L_2 & \Theta_{12} - \lambda L_2^T L_1^T (L_1 L_3 - I) \\ \star & \Theta_{22} - \lambda (L_1 L_3 - I)^T (L_1 L_3 - I) \end{bmatrix} < 0, \tag{3.23}$$

in which

$$\begin{aligned}
\Theta_{11} &= \begin{bmatrix} \begin{bmatrix} A_{\hat{\psi}} & 0 \\ 0 & A_{\mathrm{TT}} \end{bmatrix}^T P \begin{bmatrix} A_{\hat{\psi}} & 0 \\ 0 & A_{\mathrm{TT}} \end{bmatrix} - P & \begin{bmatrix} A_{\hat{\psi}} & 0 \\ 0 & A_{\mathrm{TT}} \end{bmatrix}^T P \begin{bmatrix} B_{\hat{\psi}p} & 0 \\ A_{\mathrm{TS}} & B_{\mathrm{T}} \end{bmatrix} \\ \begin{bmatrix} B_{\hat{\psi}p} & 0 \\ A_{\mathrm{TS}} & B_{\mathrm{T}} \end{bmatrix}^T P \begin{bmatrix} A_{\hat{\psi}} & 0 \\ 0 & A_{\mathrm{TT}} \end{bmatrix} & \begin{bmatrix} B_{\hat{\psi}p} & 0 \\ A_{\mathrm{TS}} & B_{\mathrm{T}} \end{bmatrix}^T P \begin{bmatrix} B_{\hat{\psi}p} & 0 \\ A_{\mathrm{TS}} & B_{\mathrm{T}} \end{bmatrix} \end{bmatrix} \\
&+ \begin{bmatrix} C_{\hat{\psi}}^T \hat{M}_\varepsilon C_{\hat{\psi}} & 0 & C_{\hat{\psi}}^T \hat{M}_\varepsilon D_{\hat{\psi}p} & 0 \\ 0 & C_{\mathrm{T}}^T C_{\mathrm{T}} & C_{\mathrm{T}}^T C_{\mathrm{S}} & C_{\mathrm{T}}^T D_{\mathrm{T}} \\ D_{\hat{\psi}p}^T \hat{M}_\varepsilon C_{\hat{\psi}} & C_{\mathrm{S}}^T C_{\mathrm{T}} & D_{\hat{\psi}p}^T \hat{M}_\varepsilon D_{\hat{\psi}p} + C_{\mathrm{S}}^T C_{\mathrm{S}} & C_{\mathrm{S}}^T D_{\mathrm{T}} \\ 0 & D_{\mathrm{T}}^T C_{\mathrm{T}} & D_{\mathrm{T}}^T C_{\mathrm{S}} & D_{\mathrm{T}}^T D_{\mathrm{T}} - \gamma^2 I \end{bmatrix}, \\
\Theta_{12} &= \begin{bmatrix} \begin{bmatrix} A_{\hat{\psi}} & 0 \\ 0 & A_{\mathrm{TT}} \end{bmatrix}^T P \begin{bmatrix} 0 & B_{\hat{\psi}q} \\ 0 & A_{\mathrm{TS}} \end{bmatrix} & \begin{bmatrix} A_{\hat{\psi}} & 0 \\ 0 & A_{\mathrm{TT}} \end{bmatrix}^T P \begin{bmatrix} B_{\hat{\psi}q} & B_{\hat{\psi}q} \\ A_{\mathrm{TS}} & A_{\mathrm{TS}} \end{bmatrix} \\ \begin{bmatrix} B_{\hat{\psi}p} & 0 \\ A_{\mathrm{TS}} & B_{\mathrm{T}} \end{bmatrix}^T P \begin{bmatrix} 0 & B_{\hat{\psi}q} \\ 0 & A_{\mathrm{TS}} \end{bmatrix} & \begin{bmatrix} B_{\hat{\psi}p} & 0 \\ A_{\mathrm{TS}} & B_{\mathrm{T}} \end{bmatrix}^T P \begin{bmatrix} B_{\hat{\psi}q} & B_{\hat{\psi}q} \\ A_{\mathrm{TS}} & A_{\mathrm{TS}} \end{bmatrix} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & C_{\hat{\psi}}^T \hat{M}_\varepsilon D_{\hat{\psi}q} & C_{\hat{\psi}}^T \hat{M}_\varepsilon D_{\hat{\psi}q} & C_{\hat{\psi}}^T \hat{M}_\varepsilon D_{\hat{\psi}q} \\ 0 & C_{\mathrm{T}}^T C_{\mathrm{S}} & C_{\mathrm{T}}^T C_{\mathrm{S}} & C_{\mathrm{T}}^T C_{\mathrm{S}} \\ 0 & D_{\hat{\psi}p}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} & D_{\hat{\psi}p}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} & D_{\hat{\psi}p}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} \\ 0 & D_{\mathrm{T}}^T C_{\mathrm{S}} & D_{\mathrm{T}}^T C_{\mathrm{S}} & D_{\mathrm{T}}^T C_{\mathrm{S}} \end{bmatrix}, \\
\Theta_{22} &= \begin{bmatrix} \begin{bmatrix} 0 & B_{\hat{\psi}q} \\ 0 & A_{\mathrm{TS}} \end{bmatrix}^T P \begin{bmatrix} 0 & B_{\hat{\psi}q} \\ 0 & A_{\mathrm{TS}} \end{bmatrix} & \begin{bmatrix} 0 & B_{\hat{\psi}q} \\ 0 & A_{\mathrm{TS}} \end{bmatrix}^T P \begin{bmatrix} B_{\hat{\psi}q} & B_{\hat{\psi}q} \\ A_{\mathrm{TS}} & A_{\mathrm{TS}} \end{bmatrix} \\ \begin{bmatrix} B_{\hat{\psi}q} & B_{\hat{\psi}q} \\ A_{\mathrm{TS}} & A_{\mathrm{TS}} \end{bmatrix}^T P \begin{bmatrix} 0 & B_{\hat{\psi}q} \\ 0 & A_{\mathrm{TS}} \end{bmatrix} & \begin{bmatrix} B_{\hat{\psi}q} & B_{\hat{\psi}q} \\ A_{\mathrm{TS}} & A_{\mathrm{TS}} \end{bmatrix}^T P \begin{bmatrix} B_{\hat{\psi}q} & B_{\hat{\psi}q} \\ A_{\mathrm{TS}} & A_{\mathrm{TS}} \end{bmatrix} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & D_{\hat{\psi}q}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} & D_{\hat{\psi}q}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} & D_{\hat{\psi}q}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} \\ 0 & D_{\hat{\psi}q}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} & D_{\hat{\psi}q}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} & D_{\hat{\psi}q}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} \\ 0 & D_{\hat{\psi}q}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} & D_{\hat{\psi}q}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} & D_{\hat{\psi}q}^T \hat{M}_\varepsilon D_{\hat{\psi}q} + C_{\mathrm{S}}^T C_{\mathrm{S}} - \gamma^2 I \end{bmatrix},
\end{aligned} \tag{3.24}$$

$$\begin{aligned} L_1 &= \boldsymbol{diag}\{\Phi, \Phi, \Phi, \Phi\}, \\ L_2 &= \boldsymbol{diag}\{0, A_{\mathrm{ST}}, A_{\mathrm{SS}}, B_{\mathrm{S}}\}, \\ L_3 &= \boldsymbol{diag}\{A_{\mathrm{SS}}, A_{\mathrm{SS}}, A_{\mathrm{SS}}, A_{\mathrm{SS}}\}. \end{aligned} \tag{3.25}$$

Proof. In addition, (3.23) can be written as

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \star & \Theta_{22} \end{bmatrix} - \lambda \begin{bmatrix} L_1L_2 & L_1L_3 - I \end{bmatrix}^T \begin{bmatrix} L_1L_2 & L_1L_3 - I \end{bmatrix} < 0. \tag{3.26}$$

From Lemma 3, (3.26) is equivalent to

$$\begin{bmatrix} I \\ L_1(I - L_3L_1)^{-1}L_2 \end{bmatrix}^T \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \star & \Theta_{22} \end{bmatrix} \begin{bmatrix} I \\ L_1(I - L_3L_1)^{-1}L_2 \end{bmatrix} < 0. \tag{3.27}$$

By algebraic operation, (3.27) can be restated in the form of

$$\begin{aligned} &\Theta_{11} + \Theta_{12}L_1(I - L_3L_1)^{-1}L_2 + \left(\Theta_{12}L_1(I - L_3L_1)^{-1}L_2\right)^T \\ &+\left(L_1(I - L_3L_1)^{-1}L_2\right)^T \Theta_{22} \left(L_1(I - L_3L_1)^{-1}L_2\right) < 0. \end{aligned} \tag{3.28}$$

Substituting $L_1, L_2, L_3, \Theta_{11}, \Theta_{12}$, and Θ_{22} into (3.28) implies that (3.28) is equivalent to

$$\begin{bmatrix} \hat{A}^T\hat{P}\hat{A} - \hat{P} & \hat{A}^T\hat{P}\hat{B} \\ \hat{B}^T\hat{P}\hat{A} & \hat{B}^T\hat{P}\hat{B} \end{bmatrix} + \begin{bmatrix} \hat{C}_{\mathrm{M}}^T \\ \hat{D}_{\mathrm{M}}^T \end{bmatrix} \hat{M}(\varepsilon_i) \begin{bmatrix} \hat{C}_{\mathrm{M}} & \hat{D}_{\mathrm{M}} \end{bmatrix} + \begin{bmatrix} \hat{C}_{\mathrm{L}}^T \\ \hat{D}_{\mathrm{L}}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \hat{C}_{\mathrm{L}} & \hat{D}_{\mathrm{L}} \end{bmatrix} < 0, \tag{3.29}$$

where $\hat{A}, \hat{B}, \hat{C}_{\mathrm{M}}, \hat{D}_{\mathrm{M}}, \hat{C}_{\mathrm{L}}, \hat{D}_{\mathrm{L}}$ are defined in (3.15).

Therefore, in accordance with Theorem 1, it can be concluded that the system Σ is robustly stable for the time-varying delay $\varepsilon \in [T_l, T_u]$, and the induced l_2 gain from d to e will not exceed the pre-determined $\gamma > 0$ if and only if (3.23) holds.

The proof is completed. □

Remark 2. *Comparing with Theorem 1, it is noted that the left side of Inequality (3.23) depends only on λ. Given that the dimensionality of matrix blocks in Theorem 1 remains substantial, calculational efficiency is impacted. The derivation of robust stability conditions for LSIS with time-varying delays is continued to improve the calculational efficiency.*

Consider that the dimensionality of the matrix blocks is still large in Theorem 2, which affects the calculational efficiency. We continue to deduce sufficient conditions for more calculationally efficient robust stability of LSIS with time-varying delays among subsystems.

Lemma 4. *([34]) Given symmetric matrices F and G of suitable dimensions, if for every nonzero vector v satisfying $v^TGv = 0$, $v^TFv > 0$ can be obtained, then $y \in R$ must exist such that $F + yG$ is positive definite.*

Theorem 3. *Given the upper bound vector $T_u = \boldsymbol{col}\left\{T_{u_i}\big|_{i=1}^{N}\right\}$ and the lower bound vector $T_l = \boldsymbol{col}\left\{T_{l_i}\big|_{i=1}^{N}\right\}$ of the time-varying delay $\varepsilon = \mathbf{col}\left\{\mathbf{col}\left\{\varepsilon_{i,r}\big|_{r=1}^{m_{vi}}\right\}\Big|_{i=1}^{N}\right\}$, where $T_{u_i} = \boldsymbol{col}\left\{T_{u_{i,r}}\big|_{r=1}^{m_{vi}}\right\}$ and $T_{l_i} = \boldsymbol{col}\left\{T_{l_{i,r}}\big|_{r=1}^{m_{vi}}\right\}$, $T_{u_{i,r}} \in \mathbb{Z}^+$, $T_{l_{i,r}} \in \mathbb{Z}^+$, if there is a positive definite matrix $P \in R_S^{n_x+n_{\hat{\psi}}}$ and $y > 0$ such that the LMI (3.30) holds, the system Σ is robustly stable for the time-varying delay $\varepsilon \in [T_l, T_u]$, and the induced l_2 gain from d to e will not exceed the pre-determined $\gamma > 0$.*

$$\begin{aligned} &\begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} + \begin{bmatrix} C_{\mathrm{M}}^T \\ D_{\mathrm{M}}^T \end{bmatrix} \hat{M}_\varepsilon \begin{bmatrix} C_{\mathrm{M}} & D_{\mathrm{M}} \end{bmatrix} \\ &+ \begin{bmatrix} C_{\mathrm{L}}^T \\ D_{\mathrm{L}}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C_{\mathrm{L}} & D_{\mathrm{L}} \end{bmatrix} + y \times \begin{bmatrix} C_\Phi^T \\ D_\Phi^T \end{bmatrix} \begin{bmatrix} -\Phi^T\Phi & \Phi^T \\ \Phi & -I \end{bmatrix} \begin{bmatrix} C_\Phi & D_\Phi \end{bmatrix} < 0, \end{aligned} \tag{3.30}$$

in which

$$
\begin{aligned}
A &= \boldsymbol{diag}\left\{ \begin{bmatrix} A_{\hat{\psi}}(i) & 0 \\ 0 & A_{\mathrm{TT}}(i) \end{bmatrix}\Bigg|_{i=1}^{N} \right\}, \\
B &= \left[\boldsymbol{diag}\left\{ \begin{bmatrix} B_{\hat{\psi}p}(i) \\ A_{\mathrm{TS}}(i) \end{bmatrix}\Bigg|_{i=1}^{N} \right\} \quad \boldsymbol{diag}\left\{ \begin{bmatrix} 0 \\ B_{\mathrm{T}}(i) \end{bmatrix}\Bigg|_{i=1}^{N} \right\} \quad \boldsymbol{diag}\left\{ \begin{bmatrix} B_{\hat{\psi}q}(i) \\ A_{\mathrm{TS}}(i) \end{bmatrix}\Bigg|_{i=1}^{N} \right\} \right], \\
C_{\mathrm{M}} &= \left[\boldsymbol{diag}\left\{ \begin{bmatrix} C_{\hat{\psi}}(i) & 0 \end{bmatrix}\Big|_{i=1}^{N} \right\} \right], \\
D_{\mathrm{M}} &= \begin{bmatrix} D_{\hat{\psi}p} & 0 & D_{\hat{\psi}q} \end{bmatrix}, \\
C_{\mathrm{L}} &= \left[\boldsymbol{diag}\left\{ \begin{bmatrix} 0 & C_{\mathrm{T}}(i) \\ 0 & 0 \end{bmatrix}\Bigg|_{i=1}^{N} \right\} \right], \\
D_{\mathrm{L}} &= \begin{bmatrix} \boldsymbol{diag}\left\{ C_{\mathrm{S}}(i)|_{i=1}^{N} \right\} & \boldsymbol{diag}\left\{ D_{\mathrm{T}}(i)|_{i=1}^{N} \right\} & \boldsymbol{diag}\left\{ C_{\mathrm{S}}(i)|_{i=1}^{N} \right\} \\ 0 & I & 0 \end{bmatrix}, \\
C_{\Phi} &= \left[\boldsymbol{diag}\left\{ \begin{bmatrix} 0 & A_{\mathrm{ST}}(i) \\ 0 & 0 \end{bmatrix}\Bigg|_{i=1}^{N} \right\} \right], \\
D_{\Phi} &= \begin{bmatrix} \boldsymbol{diag}\left\{ A_{\mathrm{SS}}(i)|_{i=1}^{N} \right\} & \boldsymbol{diag}\left\{ B_{\mathrm{S}}(i)|_{i=1}^{N} \right\} & \boldsymbol{diag}\left\{ A_{\mathrm{SS}}(i)|_{i=1}^{N} \right\} \\ 0 & 0 & I \end{bmatrix}.
\end{aligned} \tag{3.31}
$$

Proof. Let $L = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, then (3.14) can be reformulated as

$$
\begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \\ \hat{C}_{\mathrm{M}} & \hat{D}_{\mathrm{M}} \\ \hat{C}_{\mathrm{L}} & \hat{D}_{\mathrm{L}} \end{bmatrix}^T \begin{bmatrix} P & & & \\ & -P & & \\ & & \hat{M}_{\varepsilon} & \\ & & & L \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \\ \hat{C}_{\mathrm{M}} & \hat{D}_{\mathrm{M}} \\ \hat{C}_{\mathrm{L}} & \hat{D}_{\mathrm{L}} \end{bmatrix} < 0. \tag{3.32}
$$

Substitute the parameters shown in (3.15),

$$
(*)^T \begin{bmatrix} P & & & \\ & -P & & \\ & & \hat{M}_{\varepsilon} & \\ & & & L \end{bmatrix} \begin{bmatrix} A_{\hat{\psi}} & 0 & B_{\hat{\psi}p} & 0 & B_{\hat{\psi}q} \\ 0 & A_{\mathrm{TT}} & A_{\mathrm{TS}} & B_{\mathrm{T}} & A_{\mathrm{TS}} \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ C_{\hat{\psi}} & 0 & D_{\hat{\psi}p} & 0 & D_{\hat{\psi}q} \\ 0 & C_{\mathrm{T}} & C_{\mathrm{S}} & D_{\mathrm{T}} & C_{\mathrm{S}} \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \\ (I - \Phi A_{\mathrm{SS}})^{-1} \Phi \begin{bmatrix} 0 & A_{\mathrm{ST}} & A_{\mathrm{SS}} & B_{\mathrm{S}} \end{bmatrix} \end{bmatrix} < 0. \tag{3.33}
$$

Let F_1, P_2 and H_1 be expressed as follows,

$$F_1 = (*)^T \begin{bmatrix} -\begin{bmatrix} P & \\ & -P \end{bmatrix} & & \\ & -\hat{M}_\varepsilon & \\ & & -L \end{bmatrix} \begin{bmatrix} A_{\hat{\psi}} & 0 & B_{\hat{\psi}p} & 0 & B_{\hat{\psi}q} \\ 0 & A_{\text{TT}} & A_{\text{TS}} & B_{\text{T}} & A_{\text{TS}} \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ C_{\hat{\psi}} & 0 & D_{\hat{\psi}p} & 0 & D_{\hat{\psi}q} \\ 0 & C_{\text{T}} & C_{\text{S}} & D_{\text{T}} & C_{\text{S}} \\ 0 & 0 & 0 & I & 0 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & (I-\Phi A_{\text{SS}})^{-1}\Phi A_{\text{ST}} & (I-\Phi A_{\text{SS}})^{-1}\Phi A_{\text{SS}} & (I-\Phi A_{\text{SS}})^{-1}\Phi B_{\text{S}} \end{bmatrix},$$
$$H_1 = \begin{bmatrix} -\Phi & I \end{bmatrix} \begin{bmatrix} 0 & A_{\text{ST}} & A_{\text{SS}} & B_{\text{S}} & A_{\text{SS}} \\ 0 & 0 & 0 & 0 & I \end{bmatrix}. \tag{3.34}$$

When $v = P\zeta$, $\zeta \in R^{\#}$, for any $v \neq 0$, it can be obtained that $(*)^T(H_1P_2) = 0$, which implies that $v^TF_1v > 0$. Based on Lemma 4, there must exist a real number y such that

$$(*)^T \begin{bmatrix} -\begin{bmatrix} P & \\ & -P \end{bmatrix} & & \\ & -\hat{M}_\varepsilon & \\ & & -L \end{bmatrix} \begin{bmatrix} A_{\hat{\psi}} & 0 & B_{\hat{\psi}p} & 0 & B_{\hat{\psi}q} \\ 0 & A_{\text{TT}} & A_{\text{TS}} & B_{\text{T}} & A_{\text{TS}} \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ C_{\hat{\psi}} & 0 & D_{\hat{\psi}p} & 0 & D_{\hat{\psi}q} \\ 0 & C_{\text{T}} & C_{\text{S}} & D_{\text{T}} & C_{\text{S}} \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$
$$+y \times (*)^T \begin{bmatrix} -\Phi & I \end{bmatrix} \begin{bmatrix} 0 & A_{\text{ST}} & A_{\text{SS}} & B_{\text{S}} & A_{\text{SS}} \\ 0 & 0 & 0 & 0 & I \end{bmatrix} > 0. \tag{3.35}$$

If there is a solution to inequality (3.35), then $y > 0$ must exist. By multiplying both sides of the inequality by -1, the following inequality can be obtained.

$$(*)^T \begin{bmatrix} \begin{bmatrix} P & \\ & -P \end{bmatrix} & & \\ & \hat{M}_\varepsilon & \\ & & L \end{bmatrix} \begin{bmatrix} A_{\hat{\psi}} & 0 & B_{\hat{\psi}p} & 0 & B_{\hat{\psi}q} \\ 0 & A_{\text{TT}} & A_{\text{TS}} & B_{\text{T}} & A_{\text{TS}} \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ C_{\hat{\psi}} & 0 & D_{\hat{\psi}p} & 0 & D_{\hat{\psi}q} \\ 0 & C_{\text{T}} & C_{\text{S}} & D_{\text{T}} & C_{\text{S}} \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$
$$-y \times (*)^T \begin{bmatrix} -\Phi & I \end{bmatrix} \begin{bmatrix} 0 & A_{\text{ST}} & A_{\text{SS}} & B_{\text{S}} & A_{\text{SS}} \\ 0 & 0 & 0 & 0 & I \end{bmatrix} < 0. \tag{3.36}$$

Then, both sides of inequality (3.36) are multiplied by P_2^T and P_2, respectively, and the proof of sufficiency is accomplished by direct algebraic manipulations. □

Remark 3. *In contrast to the condition of Theorem 2, Theorem 3 also circumvents the need for inverse operations involving high-dimensional matrices, and clearly reflects the structure of LSIS. In the analysis of LSIS, the reduction of the system matrix dimension will improve the efficiency of the calculation. However, the dimensions of the matrix parameters in Theorem 3 are reduced from 8 to 5 compared with Theorem 2, which causes Theorem 3 to have better calculational efficiency than Theorem 2. Moreover, it should be noticed that all sub-blocks of the matrix parameters in (3.31) exhibit a block-diagonal structure, while the SCM Φ typically demonstrates sparsity. The left-hand side of Inequality (3.30) exhibits linear dependence on the matrix P. The choice of the IQC multiplier can be made using an efficient solver such as the sparse solver DSDP to solve such a sparse semidefinite programming problem [35]. It is reasonable to anticipate that the condition established in Theorem 3 remains effective for medium scale.*

If the scale of the interconnected system becomes large, proving the condition in Theorem 3 may remain challenging. To address these challenges, in this section, we investigate the structural characteristics of the SCM Φ and analyze the robust stability of LSIS subsequently. Define $n_\phi = \sum_{i=1}^{N} m_{\star_i}$, in which $\star = v, z$. Define $l_{\star,i}$ as $l_{\star,i} = \sum_{k=1}^{i} m_{\star_k}$ where $l_{\star,0} = 0$. The row vector e_k has dimension n_z and consists of 1 in the k-th column and 0 elsewhere. Further, the position of the non-zero element in the i-th row of the SCM Φ is denoted by $j(i), i = 1, 2, \ldots, n_v$. Hence, assuming that each row of Φ contains only one 1, then $\Phi = \mathbf{col}\left\{ e_{j(i)}\Big|_{i=1}^{n_v} \right\}$. Let $l(i)$ denote the number of subsystems directly influenced by the i-th element of the vector $z(t)$. Define Ξ_j, $j = 1, 2, \ldots, N$, so that

$$\Xi_j = \mathbf{diag}\left\{ \sqrt{l(i)}\Big|_{i=l_{z,j-1}+1}^{l_{z,j}} \right\}. \tag{3.37}$$

The diagonal elements of Ξ_j denote the amount of subsystems directly influenced by the respective components of the internal output signal $z(t, j)$ from subsystem j. Additionally, observe that $e_k^T e_k = \mathbf{diag}\left\{0_{k-1}^T, 1, 0_{n_z-k}^T\right\}$, then

$$\begin{aligned} \Phi^T\Phi &= \left\{\mathbf{col}\left\{ e_{j(i)}\Big|_{i=1}^{n_v}\right\}\right\}^T \mathbf{col}\left\{ e_{j(i)}\Big|_{i=1}^{n_v}\right\} \\ &= \mathbf{diag}\left\{ l(i)\big|_{i=1}^{n_z}\right\} \\ &= \mathbf{diag}\left\{ \Xi_j^2\Big|_{j=1}^{N}\right\}. \end{aligned} \tag{3.38}$$

Lemma 5. *([36]) It is assumed that diagonal matrices X and Y have appropriate dimensions. There exists a scalar $\alpha > 0$ such that*

$$XY + Y^T X^T \le \alpha X X^T + \alpha^{-1} Y^T Y. \tag{3.39}$$

Lemma 6. *([36]) Consider a blocked LMI with N×N(N>1) dimensional partitions: $G(P)<0$, matrix P is the independent variable matrix is symmetric, and the other coefficient matrices are diagonal block matrices with L(L>1) partitions. If full block feasible solution P exists for the LMI, there must exist feasible block diagonal solutions of appropriate dimensions.*

According to the structural characteristics of the SCM Φ, leveraging Lemmas 5 and 6, we derive the following sufficient condition for robust stability that rely solely on individual subsystem parameter, the SCM Φ, and the chosen IQC multiplier.

Theorem 4. *Given the upper bound vector* $T_{u_i} = \boldsymbol{col}\left\{T_{u_{i,r}}\Big|_{r=1}^{m_{vi}}\right\}$ *and the lower bound vector* $T_{l_i} = \boldsymbol{col}\left\{T_{l_{i,r}}\Big|_{r=1}^{m_{vi}}\right\}$ *of the time-varying delay* $\varepsilon_i = \boldsymbol{col}\left\{\varepsilon_{i,r}\Big|_{r=1}^{m_{vi}}\right\}$, $T_{u_{i,r}} \in \mathbb{Z}^+$, $T_{l_{i,r}} \in \mathbb{Z}^+$, *The delay difference operator* $\mathcal{T}_{\varepsilon_i}$ *satisfies the hard-IQC defined by* $\Pi^i = \psi^{i\sim} M^i_{\varepsilon_i}\psi^i$, $(\psi^i, M^i_{\varepsilon_i})$ *is a hard factorization of the IQC multiplier* Π^i, *if a positive definite matrix* $P_i \in R_{\mathbf{S}}^{m_{xi}+n_{\psi^i}}$ *and* $k_1 \geq k_2 \geq 0$ $(k_1 < k_2 < 0)$ *exist such that (3.40) holds, the system* Σ *is robustly stable for the time-varying delay* $\varepsilon_i \in [T_{l_i}, T_{u_i}]$, *and the induced* l_2 *gain from d to e will not exceed the pre-determined* $\gamma > 0$.

$$
\begin{aligned}
&\begin{bmatrix} A(i)^T P_i A(i) - P_i & A(i)^T P_i B(i) \\ B(i)^T P_i A(i) & B(i)^T P_i B(i) \end{bmatrix} + \begin{bmatrix} C_{\mathrm{M}}(i)^T \\ D_{\mathrm{M}}(i)^T \end{bmatrix} M^i_{\varepsilon_i} \begin{bmatrix} C_{\mathrm{M}}(i) & D_{\mathrm{M}}(i) \end{bmatrix} \\
&+ \begin{bmatrix} C_{\mathrm{L}}(i)^T \\ D_{\mathrm{L}}(i)^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C_{\mathrm{L}}(i) & D_{\mathrm{L}}(i) \end{bmatrix} \\
&+ \begin{bmatrix} C_{\Phi}(i)^T \\ D_{\Phi}(i)^T \end{bmatrix} \begin{bmatrix} -k_1 \Xi_i^2 & 0 \\ 0 & k_2 I \end{bmatrix} \begin{bmatrix} C_{\Phi}(i) & D_{\Phi}(i) \end{bmatrix} < 0,
\end{aligned} \tag{3.40}
$$

in which

$$
\begin{aligned}
A(i) &= \begin{bmatrix} A_{\psi}(i) & 0 \\ 0 & A_{\mathrm{TT}}(i) \end{bmatrix}, \\
B(i) &= \begin{bmatrix} B_{\psi p}(i) & 0 & B_{\psi q}(i) \\ A_{\mathrm{TS}}(i) & B_{\mathrm{T}}(i) & A_{\mathrm{TS}}(i) \end{bmatrix}, \\
C_{\mathrm{M}}(i) &= \begin{bmatrix} C_{\psi}(i) & 0 \end{bmatrix}, \\
D_{\mathrm{M}}(i) &= \begin{bmatrix} D_{\psi p}(i) & 0 & D_{\psi q}(i) \end{bmatrix}, \\
C_{\mathrm{L}}(i) &= \begin{bmatrix} 0 & C_{\mathrm{T}}(i) \\ 0 & 0 \end{bmatrix}, \\
D_{\mathrm{L}}(i) &= \begin{bmatrix} C_{\mathrm{S}}(i) & D_{\mathrm{T}}(i) & C_{\mathrm{S}}(i) \\ 0 & I & 0 \end{bmatrix}, \\
C_{\Phi}(i) &= \begin{bmatrix} 0 & A_{\mathrm{ST}}(i) \\ 0 & 0 \end{bmatrix}, \\
D_{\Phi}(i) &= \begin{bmatrix} A_{\mathrm{SS}}(i) & B_{\mathrm{S}}(i) & A_{\mathrm{SS}}(i) \\ 0 & 0 & I_{n_{v_i}} \end{bmatrix}.
\end{aligned} \tag{3.41}
$$

Proof. It is clear that the following relationship holds,

$$
\begin{bmatrix} \Phi^T \Phi & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Phi^T \\ 0 \end{bmatrix} \begin{bmatrix} \Phi & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix}. \tag{3.42}
$$

Utilizing Lemma 5, for the scalar $\alpha > 0$, the formula is obtained as follow,

$$
\begin{bmatrix} 0 & -\Phi^T \\ -\Phi & 0 \end{bmatrix} \geq -\alpha \begin{bmatrix} \Phi^T \\ 0 \end{bmatrix} \begin{bmatrix} \Phi & 0 \end{bmatrix} - \frac{1}{\alpha} \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix}. \tag{3.43}
$$

According to (3.42) and (3.43), the following formula can be obtained,

$$
\begin{bmatrix} \Phi^T \Phi & -\Phi^T \\ -\Phi & I \end{bmatrix} \geq (1-\alpha) \begin{bmatrix} \Phi^T \\ 0 \end{bmatrix} \begin{bmatrix} \Phi & 0 \end{bmatrix} + (1 - \tfrac{1}{\alpha}) \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix}. \tag{3.44}
$$

Based on the conclusion of Theorem 3 and Formula (3.44), the robust stability of the large-scale interconnected system Σ is ensured by the following sufficient condition,

$$\begin{aligned}&\begin{bmatrix} A^TPA-P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} + \begin{bmatrix} C_{\mathrm{M}}^T \\ D_{\mathrm{M}}^T \end{bmatrix} \hat{M}_{\varepsilon} \begin{bmatrix} C_{\mathrm{M}} & D_{\mathrm{M}} \end{bmatrix} + \begin{bmatrix} C_{\mathrm{L}}^T \\ D_{\mathrm{L}}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} C_{\mathrm{L}} & D_{\mathrm{L}} \end{bmatrix} \\ &+ y \times \begin{bmatrix} C_{\Phi}^T \\ D_{\Phi}^T \end{bmatrix} \left((\alpha-1) \begin{bmatrix} \Phi^T \\ 0 \end{bmatrix} \begin{bmatrix} \Phi & 0 \end{bmatrix} + (\alpha^{-1}-1) \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} \right) \begin{bmatrix} C_{\Phi} & D_{\Phi} \end{bmatrix} < 0. \end{aligned} \tag{3.45}$$

Let $k_1 = y \times (\alpha - 1)$ and $k_2 = -y \times (\alpha^{-1} - 1)$, then $k_2 = \alpha^{-1} k_1$. Of course, when $\alpha \geq 1$, $k_1 \geq k_2 \geq 0$; when $0 < \alpha < 1$, $k_2 < k_1 < 0$. The proof is completed. □

Remark 4. *The stability condition derived from Theorem 4 pertains exclusively to the state space model parameters of subsystem Σ_i, the SCM Φ, and the chosen IQC multiplier. Consequently, the condition could be validated for each subsystem in turn. Building upon the consequence in Theorem 3, a condition based on decoupled LMI can be established. This condition serves to prove robust stability of LSIS characterized by time-varying delays among subsystems. Notably, the calculational burden associated with this approach is lower than that of Theorem 3. However, it is essential to recognize that this reduction in calculational complexity is achieved at the expense of heightened conservatism. Specifically, Theorem 4 exhibits greater conservatism compared to Theorem 3.*

Compared with the lumped condition in Theorem 1, the conditions derived from Theorems 2–4 avoid the inversion of high-dimensional matrices. To validate the effectiveness of the obtained conditions, the LMI toolbox and DSDP will be used in Part 4 to compare the computation time and standard deviation of several conditions.

4. Numerical simulations

In this section, we perform several numerical simulations to compare the calculational efficiency of the obtained conditions and thus demonstrate the validty of the conditions. For all IQC-based robust stability conditions, the delay operator in [13, 25] is selected as follows,

$$\Pi = \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix}^* \begin{bmatrix} (T_u - T_l + 1)X_1 & 0 \\ 0 & -X_1 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix}, \tag{4.1}$$

in which X_1 is any positive semi-definite matrix. T_u and T_l denote the upper and lower bounds of time-varying delays, respectively.

The numerical computations were conducted on the following platforms: An Intel(R) Core(TM) i5-8300H CPU and 16G RAM. In these simulations, assume that $m_{xi} = m_{vi} = m_{zi} = 2, m_{di} = m_{ei} = 1$. Each subsystem parameter is mutually independent as well as continuously and uniformly distributed in the interval $[-0.2, 0.2]$ randomly generated. Additionally, the elements in each row of the SCM Φ are independently generated randomly. The upper bound T_u and the lower bound T_l of each time-varying delay are independently and randomly generated from a discrete uniform distribution on the set $[0, 10]$. Let the l_2 gain from d to e as $\gamma = 1$. The non-zero elements obey the discrete uniform distribution which is uniformly distributed across all potential positions. As the number of subsystems N changes, each approach is computed 100 times. Using the LMI toolbox to evaluate the conditions in

Theorems 1 and 4. The conditions in Theorems 2 and 3 are computed by DSDP. Tables 1 and 2 give numerical simulation results. Figures 3 and 4 indicate the trends in the average and standard deviation of the calculation time of the four approaches when the number of subsystems N increased from 2 to 40.

Table 1. Average of calculation time.

Number of subsystems	Avg. CPU time(s)			
N	Theorem 1	Theorem 2	Theorem 3	Theorem 4
2	0.0078	0.0614	0.0436	0.0064
4	0.0909	0.0940	0.0769	0.0193
6	0.3958	0.2711	0.1523	0.0601
8	2.2764	0.4812	0.3246	0.1349
10	5.0372	1.1305	0.4884	0.2565
12	15.6062	1.6306	1.0071	0.5060
14	33.1044	2.7625	1.7646	0.5711
16	94.0201	5.2836	2.7876	0.7311
18	185.2520	7.6153	3.8383	1.0152
20	315.8718	11.5991	6.1117	1.6457
30	3904.2816	66.3737	40.2154	3.3744
40	-	377.5641	187.2381	11.1487

Table 2. Standard deviation of calculation time.

Number of subsystems	Std. deviation(s)			
N	Theorem 1	Theorem 2	Theorem 3	Theorem 4
2	0.0037	0.0099	0.0046	0.0036
4	0.0083	0.0179	0.0151	0.0041
6	0.0178	0.0225	0.0174	0.0046
8	0.0970	0.0295	0.0295	0.0052
10	0.1835	0.0395	0.0314	0.0085
12	0.3430	0.0497	0.0328	0.0106
14	0.4100	0.0652	0.0332	0.0135
16	0.8901	0.0972	0.0414	0.0143
18	1.2677	0.1109	0.0572	0.0180
20	2.0082	0.2775	0.1102	0.0191
30	8.6759	1.3684	0.8218	0.1037
40	-	5.6962	4.5129	0.3024

The numerical simulations indicate that when the system scale is small, the four approaches have similar calculational efficiency. However, as the number of subsystems increases, the calculational efficiency of Theorems 2–4 is gradually improved compared with Theorem 1. The three conditions avoid the high dimension matrix inverse operation. As the system scale expands, when the number of subsystems is 8, 14, 20, and 30, the ratios of the calculation time based on the lumped formula to that of Theorem 2 are 4.7306733, 11.9834932, 27.2324404, and 58.8227204. Evidently, the calculational efficiency of the method in Theorem 2 is increasing steadily. When the number of subsystems exceeds 30, the computer fails to calculate the time required for Theorem 1 due to insufficient memory. In contrast, the calculations for Theorems 2–4 can still be carried out. Since the matrix dimension of Theorem 3 is lower than that of Theorem 2, it can be seen from Tables 1 and 2 that the calculational efficiency of Theorem 3 has been improved. Notably, when the quantity of subsystems remains limited, the lumped condition has more efficiency than Theorems 2 and 3, because Inequalities (3.23) and (3.30) have larger dimensions than Inequality (3.14). The standard deviations of Theorems 2 and 3 increase with the number of subsystems, but the rate of increase is smaller than Theorem 1. Since Theorem 4 is emulated independently for each subsystem, it can be computed when the system scale continues to grow. The standard deviation of Theorem 4 is the smallest and grows the slowest, demonstrating the highest numerical stability. Therefore, the condition of Theorem 4 shows greater computational advantage in the robust stability analysis of LSIS.

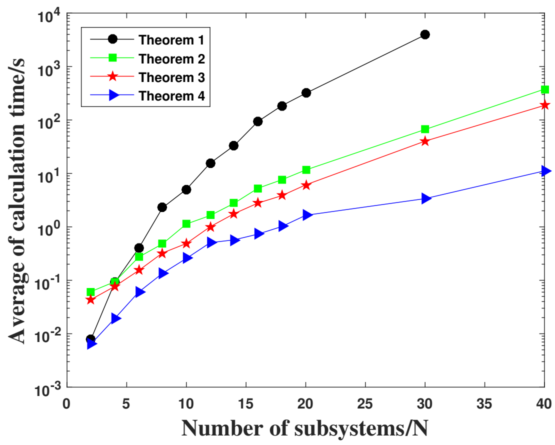

Figure 3. Average of calculation time.

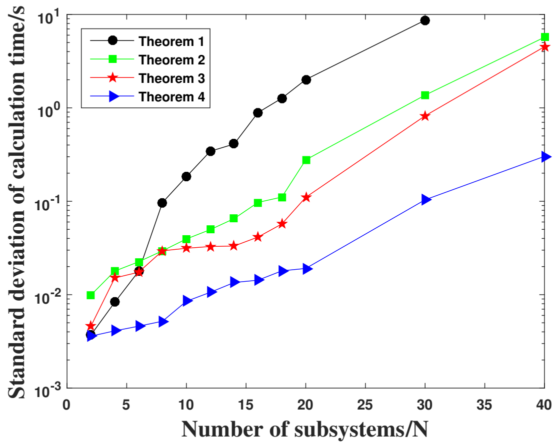

Figure 4. Standard deviation of calculation time.

5. Conclusions

We investigate robust stability of LSIS constrained by time-varying delays among subsystems. Each subsystem exhibits different dynamic characteristics, and their interactions are arbitrary. Given that the system parameters involve matrix inversion terms, calculational difficulties or infeasibility may arise when performing inversions on high-dimensional matrices. Consequently, the lumped description method becomes inappropriate for analyzing and synthesizing LSIS. Based on IQC theory, the calculationally efficient conditions of robust stability for the large-scale interconnected systems are established by taking advantage of the sparsity of the subsystem connection topology. The derived decoupling robust stability conditions rely solely on the SCM Φ, each subsystem parameter, and the chosen IQC multiplier. Finally, the simulation results show that the obtained conditions are valid for the analysis of LSIS constrained by time-varying delays among subsystems. While references provide theoretical analysis of IQC descriptions, there remains a necessity to integrate engineering practices in order to derive additional IQC descriptions. This integration can help mitigate the conservatism associated with robust stability analysis. In future work, we will focus on extending the robust stability analysis of LSIS with time-varying coefficients, addressing more complex dynamic behaviors and uncertainties.

Author contributions

Xiaoyu Sun: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Software, Validation, Writing-original draft preparation, Writing-review and editing; Huabo Liu: Conceptualization, Funding acquisition, Project administration, Resources, Supervision, Validation, Visualization; Bin Wang: Data curation, Investigation, Project administration, Resources, Supervision, Validation. All authors have read and approved the final version of the manuscript for publication.

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Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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