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Research article

Modal characteristics and evolutive response of a bar in peridynamics involving a mixed operator

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Abstract: The paper first gives a rigorous proof of existence and highlights proprieties of the eigenvalues and eigenfunctions for a bounded body with peridynamical Dirichlet boundary conditions. In particular, the mechanical behavior of the body is described by mixed local and nonlocal operators where, for the latter, the regional fractional Laplacian is used. The dynamics of the1-dimensional case is thereafter analyzed. More precisely, the previous results are applied to analyze the evolutionary problem which corresponds to free oscillations of a bar taking also into account the damping effects. A peculiar numerical approach is finally proposed to solve both the eigenvalue problem and the time evolution problem. Comparisons with classical local models and super- and sub-critical behaviors are highlighted.

Keywords: peridynamics; regional fractional Laplacian; mixed local and nonlocal operators; variational methods; numerical methods
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1. Introduction

It is well known that the classical local model of elasticity is not suitable to describe a variety of mechanical behaviors with special attention to nano and micro structures, composites and multi scale analysis. Several other models have been proposed to study these materials and one of them is the nonlocal model following the approach proposed by Eringen and Edelen, further details can be found in [1] and the references therein. A very fruitful step is the peridynamics theory proposed by Silling [2], that is, the introduction of a suitable framework for problems where discontinuities appear naturally, such as fractures, dislocations, or, in general, multiscale materials. This approach introduces a length scale and permits it to draw a relation with the classical elasticity [3] and with the hyper-elasticity [4]. With attention to the mechanical behavior of a bar [5, 6], the authors in a recent paper [7], somehow inspired by [8], analyzed the static mixed behavior (local and nonlocal) of finite length elements, where the involved non local operator is the regional fractional Laplacian $(-\Delta)^s_\Omega$. Recent developments of peridynamics include the possibility of defining a double horizon [9] which allows one to reduce the phenomenon of spurious wave reflection and the derivation of the equation of motion starting directly by Navier's displacement equilibrium equations by means of a peridynamics differential operator [10], which has been generalized in [11]. An approach which uses different elements (rod, triangle elements, and tetrahedron elements) depending on the dimensions of the problem (1D, 2D, and 3D) was proposed in [12]. Moreover, the task of estimating the size of the horizon was approached using physics-informed neural networks in a recent paper [13].

In this paper we apply this mixed approach to dynamical problems, and to do that, we first establish the main properties of the eigenvalues and the eigenfunctions of the underlying mixed operator.

For this, we need to introduce the precise setting. Throughout the paper we denote by $B(x_0, r)$ the open ball in $\mathbb{R}^N$ of center x_0 and radius $r > 0$; when $x_0 = 0$ we simply write B_r instead of $B(0, r)$. Here and in what follows $\Omega \subset \mathbb{R}^N$ is a bounded domain. Moreover, Ω is divided into two parts, that is, $\Omega = \Omega_0 \cup \Omega_1$, where Ω_0 is an open set, with smooth boundary $\partial\Omega_0$, and $\Omega_0 \cap \Omega_1 = \emptyset$. Here, Ω_1 represents the peridynamic boundary, or the horizon. Furthermore, the open enlargement $\Omega_\delta = \Omega_0 + B_\delta$, for a suitable small radius $\delta > 0$, is assumed to be a subset of Ω. In this way, the remaining set $\Omega_1 \supset \partial\Omega_0$ can be seen as the nonlocal boundary of Ω_0, see Figure 1.

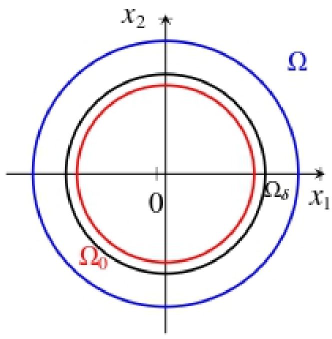

Figure 1. Description of Ω, Ω_0, and Ω_δ.

We are now in the position to consider the eigenvalue problem

$$\begin{cases} -c\Delta u + k(-\Delta)^s_\Omega u + V(x)u = \lambda u & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases} \tag{1.1}$$

where c, $k > 0$ are physical coefficients and usually, but not in this paper, are supposed to satisfy the convex restriction $c + k = 1$. Indeed, letting $c + k \neq 1$ permits it to cover general and more realistic situations, in which the operator is not the convex combination of the local and of the nonlocal one. The additional term $V(x)u$ represents external springs whose stiffness is related to the position of the point along Ω_0, and the potential V is a bounded non–negative continuous function, with $V > 0$ a.e. in Ω_0, while λ is a real parameter. The operator $(-\Delta)^s_\Omega$, defined pointwise for all $x \in \Omega$ along any $\varphi \in C^\infty_c(\Omega)$ by

$$(-\Delta)^s_\Omega \varphi(x) = C_{N,2s} \int_\Omega \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}}\, dy,$$

is the so called *regional fractional Laplacian*. The constant $C_{N,2s}$ plays an important role only in the numerical approach described in Section 4, while in the first part of the paper, for simplicity, $C_{N,2s}$ is normalized to 1.

The natural solution space of (1.1) is

$$\mathbb{H}^s_{0,\Omega} = \left\{ u \in H^1_0(\Omega_0) \cap H^s_0(\Omega) \,:\, u = 0 \text{ a.e. in } \Omega_1 \right\},$$

where $H^1_0(\Omega_0)$ is the completion of $C^\infty_c(\Omega_0)$ with respect to the norm $\|\nabla \cdot\|_{L^2(\Omega_0)}$ and $H^s_0(\Omega)$ is the completion of $C^\infty_c(\Omega)$, with respect to the Gagliardo seminorm

$$[u]_{2,\Omega} = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{1/2}.$$

The canonical Hilbertian norm on $\mathbb{H}^s_{0,\Omega}$ is

$$|||u|||_{\mathbb{H}^s_{0,\Omega}} = \left(\int_{\Omega_0} |\nabla u|^2 dx + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{1/2} = \left(\|\nabla u\|^2_{L^2(\Omega_0)} + [u]^2_{2,\Omega} \right)^{1/2},$$

which, since V is a bounded non–negative continuous function, with $V > 0$ a.e. in Ω_0, is equivalent to the Hilbertian norm

$$\begin{aligned} \|u\|_{\mathbb{H}^s_{0,\Omega}} &= \left(\int_{\Omega_0} V(x)|u|^2 dx + \|\nabla u\|^2_{L^2(\Omega_0)} + [u]^2_{2,\Omega} \right)^{1/2} \\ &= \left(\|u\|^2_{L^2(\Omega_0,V)} + \|\nabla u\|^2_{L^2(\Omega_0)} + [u]^2_{2,\Omega} \right)^{1/2}, \end{aligned}$$

being

$$|||u|||^2_{\mathbb{H}^s_{0,\Omega}} \le \|u\|^2_{\mathbb{H}^s_{0,\Omega}} \le \max\{C_P\|V\|_\infty,\ 1\}|||u|||^2_{\mathbb{H}^s_{0,\Omega}},$$

where C_P is the Poincaré constant. It is convenient for later purposes to endow $\mathbb{H}^s_{0,\Omega}$ with the Hilbertian norm

$$\|u\| = \left(\|u\|^2_{L^2(\Omega_0,V)} + c\|\nabla u\|^2_{L^2(\Omega_0)} + k[u]^2_{2,\Omega} \right)^{1/2},$$

which is equivalent to $\|\cdot\|_{\mathbb{H}^s_{0,\Omega}}$, since $c, k > 0$, being $\kappa\|u\|_{\mathbb{H}^s_{0,\Omega}} \le \|u\| \le K\|u\|_{\mathbb{H}^s_{0,\Omega}}$ for all $u \in \mathbb{H}^s_{0,\Omega}$, where $\kappa = \min\{c,\ k,\ 1\}$ and $K = \max\{c,\ k,\ 1\}$. The inner product inducing $\|\cdot\|$ is denoted by $\langle\cdot,\cdot\rangle$.

Since we consider the weak formulation of problem (1.1), it is convenient to introduce the functional

$$\mathcal{J}(u) = \frac{1}{2}\|u\|^2 \tag{1.2}$$

defined on $\mathbb{H}^s_{0,\Omega}$. Indeed, a function $u \in \mathbb{H}^s_{0,\Omega}$ is a (weak) solution of (1.1) if

$$\langle \mathcal{J}'(u), \varphi\rangle = \lambda\langle u, \varphi\rangle_{L^2(\Omega_0)} \quad \text{for all } \varphi \in \mathbb{H}^s_{0,\Omega}. \tag{1.3}$$

Hence the (weak) solutions of (1.1) are exactly the critical points of $\mathcal{J}$ in $\mathbb{H}^s_{0,\Omega}$.

Following the proof of [14, Proposition 9] by Servadei and Valdinoci and of [15, Theorem A.3], for related but different problems, we prove for (1.1) the next result.

Theorem 1.1.

(a) *Problem* (1.1) *admits an eigenvalue* λ_1 *which is positive and can be characterized by*

$$\lambda_1 = \min_{u \in \mathbb{H}^s_{0,\Omega} \setminus \{0\}} \frac{\|u\|^2}{\|u\|^2_{L^2(\Omega_0)}}. \tag{1.4}$$

(b) *There exists a non–negative function* $e_1 \in \mathbb{H}^s_{0,\Omega}$, *which is a normalized eigenfunction corresponding to* λ_1, *attaining the minimum in* (1.4)*, that is,* $\|e_1\|_{L^2(\Omega_0)} = 1$ *and*

$$\lambda_1 = \|e_1\|^2. \tag{1.5}$$

(c) λ_1 *is simple, that is, if* $u \in \mathbb{H}^s_{0,\Omega}$ *is a solution of*

$$\langle \mathcal{J}'(u), \varphi \rangle = \lambda_1 \langle u, \varphi \rangle_{L^2(\Omega_0)} \text{ for any } \varphi \in \mathbb{H}^s_{0,\Omega}, \tag{1.6}$$

then $u = t\, e_1$, *for some* $t \in \mathbb{R}$.

(d) *The set of the eigenvalues of problem* (1.1) *consists of a sequence* $(\lambda_k)_{k\in\mathbb{N}}$, *with*

$$0 < \lambda_1 < \lambda_2 \le \dots \le \lambda_k \le \lambda_{k+1} \le \dots \tag{1.7}$$

and

$$\lambda_k \to \infty \text{ as } k \to \infty. \tag{1.8}$$

Moreover, for any $k \in \mathbb{N}$ *the eigenvalues can be characterized by*

$$\lambda_{k+1} = \min_{u \in \mathbb{P}_{k+1} \setminus \{0\}} \frac{\|u\|^2}{\|u\|^2_{L^2(\Omega_0)}}, \tag{1.9}$$

where

$$\mathbb{P}_{k+1} = \left\{ u \in \mathbb{H}^s_{0,\Omega} : \langle u, e_j \rangle = 0 \text{ for any } j = 1, \dots, k \right\}. \tag{1.10}$$

(e) *For any* $k \in \mathbb{N}$ *there exists a function* $e_{k+1} \in \mathbb{P}_{k+1}$, *which is a normalized eigenfunction corresponding to* λ_{k+1}, *attaining the minimum in* (1.9)*, that is,* $\|e_{k+1}\|_{L^2(\Omega_0)} = 1$ *and*

$$\lambda_{k+1} = \|e_{k+1}\|^2. \tag{1.11}$$

(f) *The sequence* $(e_k)_{k\in\mathbb{N}}$ *of eigenfunctions* e_k *corresponding to the eigenvalues* λ_k *is an orthogonal basis for both the Hilbert spaces* $\mathbb{H}^s_{0,\Omega}$ *and* $L^2(\Omega_0)$.

(g) *Each eigenvalue* λ_k *has finite multiplicity. More precisely, if* λ_k *is such that*

$$\lambda_{k-1} < \lambda_k = \dots = \lambda_{k+h} < \lambda_{k+h+1} \tag{1.12}$$

for some $h \in \mathbb{N}_0$, *then the set of all the eigenfunctions corresponding to* λ_k *agrees with*

$$\operatorname{span}\{e_k, \dots, e_{k+h}\}.$$

(h) *If* $v \in \mathbb{H}^s_{0,\Omega}$ *is a solution of* (1.1) *such that* $v > 0$ *a.e. in* Ω_0, *then* $\lambda = \lambda_1$, *defined in* (1.4).

In particular, Theorem 1.1 gives that the first eigenvalue of (1.1) is positive and that the eigenfunctions are a basis of the natural solution space $\mathbb{H}^s_{0,\Omega}$ of (1.1), as well as of the underlying Lebesgue space $L^2(\Omega_0)$.

2. Proof of Theorem 1.1

Let us first present some preliminary results.

Proposition 2.1. *If $X_\star$ is a non–trivial weakly closed subspace of $\mathbb{H}^s_{0,\Omega}$ and*

$$\mathcal{M}_\star = \left\{ u \in X_\star \ : \ \|u\|_{L^2(\Omega_0)} = 1 \right\},$$

then there exists $u_\star \in \mathcal{M}_\star$ such that

$$\min_{u\in\mathcal{M}_\star} \mathcal{J}(u) = \mathcal{J}(u_\star). \tag{2.1}$$

Moreover, if $\lambda_\star$ denotes the number $2\mathcal{J}(u_\star)$, then $\lambda_\star > 0$ and

$$\langle \mathcal{J}'(u_\star), \varphi\rangle = \lambda_\star \langle u_\star, \varphi\rangle_{L^2(\Omega_0)} \ \textit{for any} \ \ \varphi \in X_\star. \tag{2.2}$$

Proof. In order to prove (2.1), we use the direct method of minimization. Let us take a minimizing sequence $(u_j)_j$ of $\mathcal{J}$ on $\mathcal{M}_\star$, i.e., a sequence $(u_j)_j \subset \mathcal{M}_\star$ such that

$$\mathcal{J}(u_j) \to \inf_{u\in\mathcal{M}_\star} \mathcal{J}(u) \geq 0 \text{ as } j \to \infty. \tag{2.3}$$

Then the sequence $j \mapsto \mathcal{J}(u_j)$ is bounded in $\mathbb{R}$, and so, by the definition of $\mathcal{J}$, we get that $(\|u_j\|)_j$ is also bounded.

Since $\mathbb{H}^s_{0,\Omega}$ is a reflexive space, up to a subsequence still denoted by $(u_j)_j$, we have that $(u_j)_j$ converges weakly in $\mathbb{H}^s_{0,\Omega}$ to some $u_\star \in X_\star$, being $X_\star$ weakly closed. The weak convergence gives that

$$\langle \mathcal{J}'(u_j), \varphi\rangle \to \langle \mathcal{J}'(u_\star), \varphi\rangle \text{ for any } \varphi \in \mathbb{H}^s_{0,\Omega}$$

as $j \to \infty$. Now, the weak convergence of $(u_j)_j$ to $u_\star$ in $\mathbb{H}^s_{0,\Omega}$ gives at once that

$$u_j \to u_\star \quad \text{in } L^2(\Omega_0)$$

as $j \to \infty$, since

$$\text{the embedding } \mathbb{H}^s_{0,\Omega} \hookrightarrow L^2(\Omega_0) \text{ is compact.} \tag{2.4}$$

Hence, $\|u_\star\|_{L^2(\Omega_0)} = 1$, that is, $u_\star \in \mathcal{M}_\star$.

Hence, (2.3) and the weak lower semicontinuity of $\|\cdot\|$ in $\mathbb{H}^s_{0,\Omega}$ imply that

$$\begin{aligned}\inf_{u\in\mathcal{M}_\star} \mathcal{J}(u) &= \lim_{j\to\infty} \mathcal{J}(u_j) = \frac{1}{2}\lim_{j\to\infty} \left(c\|\nabla u_j\|^2_{L^2(\Omega_0)} + k[u_j]^2_{2,\Omega} + \|u_j\|^2_{L^2(\Omega_0,V)} \right) \\ &\geq \frac{1}{2} \left(c\|\nabla u_\star\|^2_{L^2(\Omega_0)} + k[u_\star]^2_{2,\Omega} + \|u_\star\|^2_{L^2(\Omega_0,V)} \right) = \mathcal{J}(u_\star) \geq \inf_{u\in\mathcal{M}_\star} \mathcal{J}(u).\end{aligned}$$

Thus, $u_\star$ is a minimizer of $\mathcal{J}$ in $\mathcal{M}_\star$. This gives (2.1).

To prove (2.2), fix $\epsilon \in (0,1)$, $\varphi \in X_\star \setminus \{0\}$, $c_\epsilon = \|u_\star + \epsilon\varphi\|_{L^2(\Omega_0)} > 0$ and $u_\epsilon = (u_\star + \epsilon\varphi)/c_\epsilon$. Next, (2.1) yields that as $\epsilon \to 0^+$

$$\begin{gathered} c_\epsilon^2 = \|u_\star\|^2_{L^2(\Omega_0)} + 2\epsilon \int_{\Omega_0} u_\star(x)\varphi(x)dx + o(\epsilon), \\ \|\nabla u_\star + \epsilon\nabla\varphi\|^2_{L^2(\Omega_0)} = \|\nabla u_\star\|^2_{L^2(\Omega_0)} + 2\epsilon \int_{\Omega_0} \nabla u_\star(x) \cdot \nabla\varphi(x)dx + o(\epsilon), \\ [u_\star + \epsilon\varphi]^2_s = [u_\star]^2_s + 2\epsilon\langle u_\star, \varphi\rangle_s + o(\epsilon). \end{gathered}$$

Consequently, the fact that $u_\star \in \mathcal{M}_\star$ gives that $\|u_\star\|_{L^2(\Omega_0)} = 1$, so that as $\epsilon \to 0^+$

$$\begin{aligned} 2\mathcal{J}(u_\epsilon) &= \frac{\|u_\star\|^2 + 2\epsilon\langle u_\star, \varphi\rangle + o(\epsilon)}{1 + 2\epsilon \int_{\Omega_0} u_\star \varphi dx + o(\epsilon)} \\ &= \Big(2\mathcal{J}(u_\star) + 2\epsilon\langle \mathcal{J}'(u_\star), \varphi\rangle + o(\epsilon)\Big) \cdot \Big(1 - 2\epsilon \int_{\Omega_0} u_\star \varphi dx + o(\epsilon)\Big) \\ &= 2\mathcal{J}(u_\star) + 2\epsilon \left(\langle \mathcal{J}'(u_\star), \varphi\rangle - 2\mathcal{J}(u_\star) \int_{\Omega_0} u_\star \varphi dx\right) + o(\epsilon). \end{aligned}$$

Clearly, $\mathcal{J}(u_\star) > 0$, because otherwise we would have $u_\star \equiv 0$, while $0 \notin \mathcal{M}_\star$. Hence $\lambda_\star$ is positive. This and the minimality of $u_\star$ imply (2.2). □

Proposition 2.2. *If $\lambda \neq \tilde{\lambda}$ are two different eigenvalues of problem* (1.1)*, with eigenfunctions e and $\tilde{e}$ in $\mathbb{H}^s_{0,\Omega}$, respectively, then $e \perp_{\mathbb{H}^s_{0,\Omega}} \tilde{e}$ and $e \perp_{L^2(\Omega_0)} \tilde{e}$, that is,*

$$\langle e, \tilde{e}\rangle = 0 = \langle e, \tilde{e}\rangle_{L^2(\Omega_0)}. \tag{2.5}$$

Moreover, if e is an eigenfunction of problem (1.1) *corresponding to an eigenvalue λ, then*

$$\|e\|^2 = \lambda \|e\|^2_{L^2(\Omega_0)}. \tag{2.6}$$

Proof. Clearly $e \not\equiv 0$ and $\tilde{e} \not\equiv 0$, since they are eigenfunctions. Put $g = e/\|e\|_{L^2(\Omega_0)}$ and $\tilde{g} = \tilde{e}/\|\tilde{e}\|_{L^2(\Omega_0)}$, which are eigenfunctions, as well. The weak formulation of problem (1.1) for g with test function $\tilde{g}$ and viceversa gives

$$\lambda \int_{\Omega_0} g(x)\tilde{g}(x)dx = \langle \mathcal{J}'(g), \tilde{g}\rangle = \langle \mathcal{J}'(\tilde{g}), g\rangle = \tilde{\lambda} \int_{\Omega_0} g(x)\tilde{g}(x)dx, \tag{2.7}$$

that is,

$$(\lambda - \tilde{\lambda}) \int_{\Omega_0} g(x)\tilde{g}(x)dx = 0.$$

Thus, since $\lambda \neq \tilde{\lambda}$,

$$\int_{\Omega_0} g(x)\tilde{g}(x)dx = 0. \tag{2.8}$$

By plugging (2.8) into (2.7), we obtain

$$\langle g, \tilde{g}\rangle = \langle \mathcal{J}'(g), \tilde{g}\rangle = 0.$$

This and (2.8) complete the proof of (2.5). Finally, (2.6) can be easily proved by choosing $\varphi = e$ in the weak formulation (1.3) of (1.1). □

We are now able to prove the first main result of the paper.

Proof of Theorem 1.1. We adapt the proofs of [14, Proposition 9] and [15, Theorem A.3].

(*a*) For this, we note that the minimum defining λ_1 exists and that λ_1 is an eigenvalue, thanks to (2.1) and (2.2), applied here with $X_\star = \mathbb{H}^s_{0,\Omega}$.

(b) Again by (2.1), the minimum defining λ_1 is attained at some $e_1 \in \mathbb{H}^s_{0,\Omega}$, with $\|e_1\|_{L^2(\Omega_0)} = 1$. The fact that e_1 is an eigenfunction corresponding to λ_1 and that formula (1.4) holds are consequences of (2.2), again with $X_\star = \mathbb{H}^s_{0,\Omega}$.

Let us now show that $e_1 \geq 0$ in Ω. First, we claim that if e is an eigenfunction associated to λ_1, with $\|e\|_{L^2(\Omega_0)} = 1$, then both e and $|e|$ attain the minimum in (1.4) so that either $e \geq 0$ or $e \leq 0$ a.e. in Ω. To check this, Proposition 2.2 and (1.5) give

$$2\mathcal{J}(e) = \lambda_1 = 2\mathcal{J}(e_1). \tag{2.9}$$

Clearly, $\|e\|_{L^2(\Omega_0,V)} = \||e|\|_{L^2(\Omega_0,V)}$ and also $\|\nabla e\|_{L^2(\Omega_0)} = \|\nabla|e|\|_{L^2(\Omega_0)}$, being $|\nabla e| = |\nabla|e||$, while $[|e|]^2_{2,\Omega} \leq [e]^2_{2,\Omega}$ by direct calculation. Consequently, $\mathcal{J}(|e|) \leq \mathcal{J}(e)$. Moreover, if both sets $E^+ = \{x \in \Omega : e(x) > 0\}$ and $E^- = \{x \in \Omega : e(x) < 0\}$ have positive measure, then

$$\mathcal{J}(|e|) < \mathcal{J}(e). \tag{2.10}$$

Furthermore, $|e| \in \mathbb{H}^s_{0,\Omega}$ and $\||e|\|_{L^2(\Omega_0)} = \|e\|_{L^2(\Omega_0)} = 1$. Hence, (2.9), (2.10), and the minimality of e_1 imply that $\mathcal{J}(|e|) = \mathcal{J}(e) = \mathcal{J}(e_1)$ and that either E^+ or E^- has zero measure. This is impossible and proves the claim.

Thanks to the claim, by possibly replacing e_1 with $|e_1|$, we may and do suppose that $e_1 \geq 0$ in Ω. This completes the proof of (b).

(c) Assume by contradiction that there exists another eigenfunction f_1 in $\mathbb{H}^s_{0,\Omega}$, corresponding to λ_1, with $f_1 \not\equiv e_1$. Clearly, $f_1 \not\equiv 0$, since it is an eigenfunction. By the proof of (b), we know that either $f_1 \geq 0$ or $f_1 \leq 0$ a.e. in Ω. Let us consider only the case

$$f_1 \geq 0 \quad \text{a.e. in } \Omega, \tag{2.11}$$

since the proof in the other case is similar. Set $\tilde{f}_1 = f_1/\|f_1\|_{L^2(\Omega_0)}$ and $g_1 = e_1 - \tilde{f}_1$. We claim that

$$g_1 = 0 \quad \text{a.e. in } \Omega. \tag{2.12}$$

To this aim we argue by contradiction. Hence,

$$g_1 \neq 0 \text{ in a subset of positive measure of } \Omega. \tag{2.13}$$

Then, also g_1 is an eigenfunction associated to λ_1 and so either $g_1 \geq 0$ or $g_1 \leq 0$ a.e. in Ω by (b). Hence, either $e_1 \geq \tilde{f}_1$ or $e_1 \leq \tilde{f}_1$. Thus, by (2.11) and the non-negativity of e_1 either

$$e_1^2 \geq \tilde{f}_1^2 \quad \text{or} \quad e_1^2 \leq \tilde{f}_1^2 \quad \text{a.e. in } \Omega. \tag{2.14}$$

In both cases,

$$\int_\Omega \left(e_1^2(x) - \tilde{f}_1^2(x)\right)dx = \|e_1\|^2_{L^2(\Omega_0)} - \|\tilde{f}_1\|^2_{L^2(\Omega_0)} = 1 - 1 = 0.$$

This and (2.14) give at once that $e_1 = \tilde{f}_1$ a.e. in Ω, that is, $g_1 = 0$ a.e. in Ω. This is in contradiction with (2.13) and proves the claim (2.12).

Then, as a consequence of (2.12), we obtain that f_1 is proportional to e_1, and this proves (c).

(*d*) We define λ_{k+1} as in (1.9). Clearly, the minimum in (1.9) exists and it is attained at some $e_{k+1} \in \mathbb{P}_{k+1}$, thanks to (2.1) and (2.2), applied here with $X_\star = \mathbb{P}_{k+1}$, which, by construction, is weakly closed.

Moreover, since $\mathbb{P}_{k+1} \subseteq \mathbb{P}_k \subset \mathbb{H}^s_{0,\Omega}$, we get that

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_k \leq \lambda_{k+1} \leq \ldots .$$

Also, (2.2) with $X_\star = \mathbb{P}_{k+1}$ says that

$$\langle \mathcal{J}'(e_{k+1}), \varphi \rangle = \lambda_{k+1} \langle e_{k+1}, \varphi \rangle_{L^2(\Omega_0)} \qquad \text{for any } \varphi \in \mathbb{P}_{k+1}. \tag{2.15}$$

In order to prove that λ_{k+1} is an eigenvalue with eigenfunction e_{k+1}, it is enough to show that formula (2.15) holds for any $\varphi \in \mathbb{H}^s_{0,\Omega}$, not only in $\mathbb{P}_{k+1}$. To this aim, we argue recursively, assuming that the claim holds at each step $1, \ldots, k$ and proving (2.15) at $k+1$. The base of induction is given by the fact that λ_1 is an eigenvalue, as shown in assertion (*a*). We use the direct sum decomposition

$$\mathbb{H}^s_{0,\Omega} = \operatorname{span}\{e_1, \ldots, e_k\} \oplus \left(\operatorname{span}\{e_1, \ldots, e_k\}\right)^\perp = \operatorname{span}\{e_1, \ldots, e_k\} \oplus \mathbb{P}_{k+1},$$

where the orthogonal $\perp$ is intended with respect to the scalar product of $\mathbb{H}^s_{0,\Omega}$, namely $\langle \cdot, \cdot \rangle$. Thus, given any $\varphi \in \mathbb{H}^s_{0,\Omega}$, we write $\varphi = \varphi_1 + \varphi_2$, with $\varphi_2 \in \mathbb{P}_{k+1}$ and $\varphi_1 = \sum_{i=1}^k c_i e_i$, for some $c_1, \ldots, c_k \in \mathbb{R}$. Then, from (2.15) tested with $\varphi_2 = \varphi - \varphi_1$, we know that

$$\begin{aligned} \langle \mathcal{J}'(e_{k+1}), \varphi \rangle - \lambda_{k+1} \langle e_{k+1}, \varphi \rangle_{L^2(\Omega_0)} &= \langle \mathcal{J}'(e_{k+1}), \varphi_1 \rangle - \lambda_{k+1} \langle e_{k+1}, \varphi_1 \rangle_{L^2(\Omega_0)} \\ &= \sum_{i=1}^k c_i \left[\langle \mathcal{J}'(e_{k+1}), e_i \rangle - \lambda_{k+1} \langle e_{k+1}, e_i \rangle_{L^2(\Omega_0)} \right]. \end{aligned} \tag{2.16}$$

Furthermore, testing the weak formulation of problem (1.1) for e_i against e_{k+1} for $i = 1, \ldots, k$, allowed by the inductive assumption, and recalling that $e_{k+1} \in \mathbb{P}_{k+1}$, we see that

$$0 = \langle e_{k+1}, e_i \rangle = \langle \mathcal{J}'(e_{k+1}), e_i \rangle = \lambda_{k+1} \langle e_{k+1}, e_i \rangle_{L^2(\Omega_0)},$$

so that, by (1.7),

$$\langle \mathcal{J}'(e_{k+1}), e_i \rangle = 0 = \langle e_{k+1}, e_i \rangle_{L^2(\Omega_0)}$$

for any $i = 1, \ldots, k$. By plugging this into (2.16), we conclude that (2.15) holds true for any $\varphi \in \mathbb{H}^s_{0,\Omega}$, that is, λ_{k+1} is an eigenvalue with eigenfunction e_{k+1}.

Now we prove (1.8). Let us start by showing that if $k, h \in \mathbb{N}$, $k \neq h$, then

$$\langle e_k, e_h \rangle = 0 = \int_{\Omega_0} e_k(x) e_h(x) dx.$$

Indeed, if $k > h$, then $k - 1 \geq h$. Thus,

$$e_k \in \mathbb{P}_k = \left(\operatorname{span}\{e_1, \ldots, e_{k-1}\}\right)^\perp \subseteq \left(\operatorname{span}\{e_h\}\right)^\perp,$$

and therefore $\langle e_k, e_h \rangle = 0$. But e_k is an eigenfunction and so, using the weak formulation of problem (1.1) for e_k tested with $\varphi = e_h$, we get

$$0 = \langle e_k, e_h \rangle = \langle \mathcal{J}'(e_k), e_h \rangle = \lambda_k \int_{\Omega_0} e_k(x) e_h(x) dx,$$

as claimed.
To complete the proof of (1.8), suppose by contradiction that $\lambda_k \to C$ for some constant $C \in \mathbb{R}$. Then $(\lambda_k)_k$ is bounded in $\mathbb{R}$. Since $\|e_k\|^2 = \lambda_k$ by (2.6), we deduce by (2.4) that there is a subsequence and some $e_\infty \in \mathbb{H}^s_{0,\Omega}$ for which $e_{k_j} \rightharpoonup e_\infty$ in $\mathbb{H}^s_{0,\Omega}$, so that (2.4) implies that

$$e_{k_j} \to e_\infty \qquad \text{in } L^2(\Omega_0)$$

as $j \to \infty$. In particular, $(e_{k_j})_j$ is a Cauchy sequence in $L^2(\Omega_0)$. But this is in contradiction with the fact that if e_{k_i} and e_{k_j} are orthogonal in $L^2(\Omega_0)$, then

$$\|e_{k_j} - e_{k_i}\|^2_{L^2(\Omega_0)} = \|e_{k_j}\|^2_{L^2(\Omega_0)} + \|e_{k_i}\|^2_{L^2(\Omega_0)} = 2.$$

Now, to complete the proof of (d), we need to show that the sequence of eigenvalues constructed in (1.9) exhausts all the eigenvalues of the problem, i.e., that any eigenvalue of problem (1.1) can be written in the form (1.9). We show this by arguing, once more, by contradiction. Let us suppose that there exists an eigenvalue

$$\lambda \notin \{\lambda_k\}_{k\in\mathbb{N}}, \tag{2.17}$$

and let $e \in \mathbb{H}^s_{0,\Omega}$ be a normalized eigenfunction relative to λ, that is, $\|e\|_{L^2(\Omega_0)} = 1$. Then, by (2.6) we have

$$2\mathcal{J}(e) = \|e\|^2 = \lambda. \tag{2.18}$$

Thus, by the minimality of λ_1 given in (1.4) and (1.5), we get that

$$\lambda = 2\mathcal{J}(e) \geq 2\mathcal{J}(e_1) = \lambda_1.$$

This, (2.17) and (1.8) imply that there exists $k \in \mathbb{N}$ such that

$$\lambda_k < \lambda < \lambda_{k+1}. \tag{2.19}$$

We claim that $e \notin \mathbb{P}_{k+1}$. Indeed, if $e \in \mathbb{P}_{k+1}$, from (2.18) and (1.9) we deduce that $\lambda = 2\mathcal{J}(e) \geq \lambda_{k+1}$, which contradicts (2.19). As a consequence, there exists $i \in \{1, \dots, k\}$ such that $\langle e, e_i \rangle \neq 0$. But this is in contradiction with (2.5), which can be easily adapted to our problem. This shows that (2.17) is false and completes the proof of (d).

(e) Again using (2.1) with $X_\star = \mathbb{P}_{k+1}$, the minimum defining λ_{k+1} is attained in some $e_{k+1} \in \mathbb{P}_{k+1}$. The fact that e_{k+1} is an eigenfunction corresponding to λ_{k+1} was checked in (d) and (1.11) follows from (2.2).

(f) The orthogonality has been already showed in (d), so we need to prove that the sequence of eigenfunctions $\{e_k\}_k$ is a basis for $\mathbb{H}^s_{0,\Omega}$. Let us start to prove that $\{e_k\}_k$ is a basis for $\mathbb{H}^s_{0,\Omega}$. For this, we show that if $v \in \mathbb{H}^s_{0,\Omega}$ is such that $\langle v, e_k \rangle = 0$ for any $k \in \mathbb{N}$, then $v \equiv 0$. For this, we argue by contradiction and suppose that there exists a nontrivial $v \in \mathbb{H}^s_{0,\Omega}$ satisfying

$$\langle v, e_k \rangle = 0 \text{ for any } k \in \mathbb{N}. \tag{2.20}$$

Then, up to normalization, we assume that $\|v\|_{L^2(\Omega_0)} = 1$. Hence, from (1.8), there exists $k \in \mathbb{N}$ such that

$$2\mathcal{J}(v) < \lambda_{k+1} = \min_{\substack{u \in \mathbb{P}_{k+1} \\ \|u\|_{L^2(\Omega_0)}=1}} \|u\|.$$

Hence, $v \notin \mathbb{P}_{k+1}$ and so there exists $j \in \mathbb{N}$ for which $\langle v, e_j \rangle \neq 0$. This contradicts (2.20), as claimed. To prove that $\{e_k\}_{k\in\mathbb{N}}$ is also a basis for $L^2(\Omega_0)$, let us introduce a standard Fourier analysis technique which shows again that $\{e_k\}_{k\in\mathbb{N}}$ is a basis for $\mathbb{H}^s_{0,\Omega}$. Define $E_i = e_i/\|e_i\|$. Given $g \in \mathbb{H}^s_{0,\Omega}$,

$$g_j = \sum_{i=1}^{j} \langle g, E_i \rangle E_i.$$

We point out that g_j belongs to span $\{e_1, \ldots, e_j\}$ for any $j \in \mathbb{N}$. Let $v_j = g - g_j$. By the orthogonality of $\{e_k\}_{k\in\mathbb{N}}$ in $\mathbb{H}^s_{0,\Omega}$, we get

$$\begin{aligned} 0 \le \|v_j\| = \langle v_j, v_j \rangle &= \|g\|^2 + \|g_j\|^2 - 2\langle g, g_j \rangle \\ &= \|g\|^2 + \langle g_j, g_j \rangle^2 - 2\sum_{i=1}^{j} \langle g, E_i \rangle^2 = \|g\|^2 - \sum_{i=1}^{j} \langle g, E_i \rangle^2. \end{aligned}$$

Therefore, for any $j \in \mathbb{N}$

$$\sum_{i=1}^{j} \langle g, E_i \rangle^2 \le \|g\|^2$$

and so $\sum_{i=1}^{\infty} \langle g, E_i \rangle^2$ is a convergent series. Thus, setting

$$\tau_j = \sum_{i=1}^{j} \langle g, E_i \rangle^2,$$

we get that $(\tau_j)_j$ is a Cauchy sequence in $\mathbb{R}$. Moreover, using again the orthogonality of $\{e_k\}_{k\in\mathbb{N}}$ in $\mathbb{H}^s_{0,\Omega}$, we see that, if $h > j$,

$$\|v_h - v_j\|^2 = \left\| \sum_{i=j+1}^{h} \langle g, E_i \rangle E_i \right\|^2 = \sum_{i=j+1}^{h} \langle g, E_i \rangle^2 = \tau_h - \tau_j.$$

Therefore, $(v_j)_j$ is a Cauchy sequence in the Hilbert space $\mathbb{H}^s_{0,\Omega}$. Hence, there exists $v \in \mathbb{H}^s_{0,\Omega}$ such that

$$v_j \to v \text{ in } \mathbb{H}^s_{0,\Omega} \text{ as } j \to \infty. \tag{2.21}$$

Now, we observe that if $j \ge k$,

$$\langle v_j, E_k \rangle = \langle g, E_k \rangle - \langle g_j, E_k \rangle = \langle g, E_k \rangle - \langle g, E_k \rangle = 0.$$

Hence, by (2.21), it easily follows that $\langle v, E_k\rangle = 0$ for any $k \in \mathbb{N}$, so that $v \equiv 0$. Consequently,

$$g_j = g - v_j \to g - v = g \quad \text{in } \mathbb{H}^s_{0,\Omega} \quad \text{as } j \to \infty.$$

This and the fact that g_j belongs to span $\{e_1, \dots, e_j\}$ for any $j \in \mathbb{N}$ yield that $\{e_k\}_k$ is a basis in $\mathbb{H}^s_{0,\Omega}$, as claimed.

To complete the proof of (f), we have to show that $\{e_k\}_k$ is also a basis in $L^2(\Omega_0)$. To this aim, fix $v \in L^2(\Omega_0)$ and let $(v_j)_j$ be a sequence in $C^\infty_c(\Omega_0)$ such that $\|v_j - v\|_{L^2(\Omega_0)} \le 1/j$ for each j. Clearly, $(v_j)_j \subset \mathbb{H}^s_{0,\Omega}$. Since $\{e_k\}_k$ is a basis in $\mathbb{H}^s_{0,\Omega}$, as shown above, for each j there exists $k_j \in \mathbb{N}$ and a function $w_j \in E_{1,k_j}$, where $E_{1,k_j} = \text{span}\{e_1, \dots, e_{k_j}\}$, such that $\|v_j - w_j\| \le 1/j$. Thus, since $V \in L^\infty(\Omega_0)$ and is nontrivial,

$$\begin{aligned}\|v_j - w_j\|_{L^2(\Omega_0)} &\le \frac{1}{\|V\|_{L^\infty(\Omega_0)}}\|v_j - w_j\|_{L^2(\Omega_0,V)} \le \frac{1}{\|V\|_{L^\infty(\Omega_0)}}\|v_j - w_j\| \\ &\le \frac{\|V\|^{-1}_{L^\infty(\Omega_0)}}{j}.\end{aligned}$$

Hence,

$$\|v - w_j\|_{L^2(\Omega_0)} \le \|v - v_j\|_{L^2(\Omega_0)} + \|v_j - w_j\|_{L^2(\Omega_0)} \le \left(1 + \|V\|^{-1}_{L^\infty(\Omega_0)}\right)\frac{1}{j}.$$

This shows that the sequence $\{e_k\}_k$ of eigenfunctions of (1.1) is a basis in $L^2(\Omega_0)$. Thus, the proof of (f) is complete.

(g) Let $h \in \mathbb{N}_0$ be such that (1.12) holds true. Thanks to (e), we already know that each element of the space $E_{k,h} = \text{span}\{e_k, \dots, e_{k+h}\}$ is an eigenfunction of problem (1.1) corresponding to $\lambda_k = \cdots = \lambda_{k+h}$. Thus, it is enough to show that any eigenfunction $\psi \not\equiv 0$ corresponding to λ_k belongs to $E_{k,h}$. For this we write $\mathbb{H}^s_{0,\Omega} = E_{k,h} \bigoplus E^\perp_{k,h}$ and so $\psi = \psi_1 + \psi_2$, with

$$\psi_1 \in E_{k,h} \quad \text{and} \quad \psi_2 \in E^\perp_{k,h}. \tag{2.22}$$

In particular,

$$\langle \psi_1, \psi_2\rangle = 0. \tag{2.23}$$

Since ψ is an eigenfunction corresponding to λ_k, taking ψ itself as a test function, we obtain by (2.23)

$$\lambda_k\|\psi\|^2_{L^2(\Omega_0)} = \|\psi\|^2 = \|\psi_1\|^2 + \|\psi_2\|^2. \tag{2.24}$$

Moreover, from (e) we know that the functions $e_k, \dots, e_{k+h}$ are eigenfunctions corresponding to $\lambda_k = \cdots = \lambda_{k+h}$, and so

$$\psi_1 \quad \text{is also an eigenfunction corresponding to } \lambda_k. \tag{2.25}$$

As a consequence, taking ψ_2 as a test function for ψ_1, by (2.23), we have

$$\lambda_k \int_{\Omega_0} \psi_1(x)\psi_2(x)dx = \langle \psi_1, \psi_2\rangle = 0,$$

that is,

$$\int_{\Omega_0} \psi_1(x)\psi_2(x)dx = 0.$$

Hence

$$\|\psi\|^2_{L^2(\Omega_0)} = \|\psi\|^2_{L^2(\Omega)} = \|\psi_1\|^2_{L^2(\Omega)} + \|\psi_2\|^2_{L^2(\Omega)}. \tag{2.26}$$

Write

$$\psi_1 = \sum_{i=k}^{k+h} c_i e_i,$$

for some $c_i \in \mathbb{R}$. Now, we use the orthogonality in (f) and (1.11) to obtain

$$\|\psi_1\|^2 = \sum_{i=k}^{k+h} c_i^2 \|e_i\|^2 = \sum_{i=k}^{k+h} c_i^2 \lambda_i = \lambda_k \sum_{i=k}^{k+h} c_i^2 = \lambda_k \|\psi_1\|^2_{L^2(\Omega_0)}. \tag{2.27}$$

By (2.25) and by the fact that ψ is an eigenfunction corresponding to λ_k, we deduce that also ψ_2 is an eigenfunction corresponding to λ_k. Therefore, recalling (1.12) and Proposition 2.2, we conclude that

$$\langle \psi_2, e_1 \rangle = \cdots = \langle \psi_2, e_{k-1} \rangle = 0.$$

This and (2.22) imply that

$$\psi_2 \in E^{\perp}_{1,h} = \mathbb{P}_{k+h+1}. \tag{2.28}$$

We claim that

$$\psi_2 \equiv 0. \tag{2.29}$$

Otherwise, by (1.9) and (2.28),

$$\lambda_k < \lambda_{k+h+1} = \min_{u \in \mathbb{P}_{k+h+1}\setminus\{0\}} \frac{\|u\|^2}{\|u\|^2_{L^2(\Omega_0)}} \leq \frac{\|\psi_2\|^2}{\|\psi_2\|^2_{L^2(\Omega_0)}}. \tag{2.30}$$

Thus, (2.24), (2.26), (2.27), and (2.30) imply that

$$\begin{aligned} \lambda_k \|\psi\|^2_{L^2(\Omega)} &= \lambda_k \|\psi\|^2_{L^2(\Omega_0)} = \|\psi_1\|^2 + \|\psi_2\|^2 \\ &> \lambda_k \|\psi_1\|^2_{L^2(\Omega)} + \lambda_k \|\psi_2\|^2_{L^2(\Omega)} = \lambda_k \|\psi\|^2_{L^2(\Omega)}. \end{aligned}$$

This obvious contradiction proves (2.29).
From (2.22) and (2.29), we obtain that

$$\psi = \psi_1 \in E_{k,h},$$

as desired. This completes the proof of (g).

(h) Assume that $v \in \mathbb{H}^s_{0,\Omega}$ as in the statement. Without loss of generality, we suppose that the function v is normalized in $L^2(\Omega_0)$. Thanks to (a) and (b) there exists a non–negative eigenfunction $e_1 \in \mathbb{H}^s_{0,\Omega}$ corresponding to λ_1, with $\|e_1\|_{L^2(\Omega_0)} = 1$. Then,

$$\langle \mathcal{J}'(v), e_1 \rangle = \lambda \langle v, e_1 \rangle_{L^2(\Omega_0)} \quad \text{and} \quad \langle \mathcal{J}'(e_1), v \rangle = \lambda_1 \langle e_1, v \rangle_{L^2(\Omega_0)}.$$

Hence,

$$(\lambda_1 - \lambda) \int_{\Omega_0} e_1 v \, dx = 0.$$

In conclusion, the fact that $v > 0$ a.e. in Ω_0 implies that $\lambda = \lambda_1$, as required.

The proof of the theorem is thus complete. □

3. An application

Here, following [15, 16, Section 5], we consider the next evolutionary problem

$$\begin{cases} u_{tt} - c\Delta u + k(-\Delta)^s_\Omega u + a(t)t^\alpha u_t + V(t,x)u = 0 & \text{in } I \times \Omega_0, \\ u(t,x) = 0 & \text{on } I \times \Omega_1, \end{cases} \tag{3.1}$$

where $I = [1, \infty)$, the continuous function a satisfies

$$1/C \le a(t) \le C \ \text{ in } I \tag{3.2}$$

for some $C > 0$, $\alpha \in \mathbb{R}$, and V is a non–negative bounded continuous function in $I \times \Omega_0$, with continuous partial derivative V_t in $I \times \Omega_0$ such that $V_t(t, x) \le 0$ in $I \times \Omega_0$.

A natural solution space for (3.1) is

$$X = \left\{\phi \in C(I \to \mathbb{H}^s_{0,\Omega}) \cap C^1(I \to L^2(\Omega_0)) \ : \ E\phi \text{ is locally bounded on } I\right\},$$

where $E\phi$ is the total energy of the function $\phi \in X$, that is,

$$E\phi = E\phi(t) = \frac{1}{2}\left(\|\phi_t\|^2_{L^2(\Omega_0)} + \|\phi\|^2\right).$$

Remark 3.1. Actually, $\|\phi\|$ depends on t, since $V = V(t, x)$. However, for the sake of simplicity, we still use the same notation as the one for the case $V = V(x)$.

A strong solution of (3.1) is a function $u \in X$ satisfying the following two conditions:

(A) Distributional identity

$$\langle u_t, \phi\rangle]_1^t = \int_1^t \left\{\langle u_t, \phi_t\rangle_{L^2(\Omega_0)} - \langle u, \phi\rangle - \langle a(t)t^\alpha u_t + V(t,\cdot)u, \phi\rangle_{L^2(\Omega_0)}\right\} d\tau$$

for all $t \in I$ and $\phi \in X$.

(B) Conservation law

(i) $\mathscr{D}u(t) = a(t)t^\alpha \|u_t\|^2_{L^2(\Omega_0)} - \frac{1}{2}\int_{\Omega_0} V_t(t,x)u^2 dx \in L^1_{\text{loc}}(I)$,

(ii) $t \mapsto Eu(t) + \int_1^t \mathscr{D}u(\tau)d\tau$ is non–increasing in I.

We emphasize that condition (B) is an essential attribute of the solution. Indeed, standard existence theorems for (3.1) in the literature always yield solutions satisfying both (A) and (B) in the stronger form in which the function in (B)–(ii) is assumed to be constant. On the other hand, (A) alone does not imply (B), even if the integrability condition (B)–(i) is assumed a priori, see [17]. Condition (B)–(ii) implies, however, that Eu is non–increasing in I.

Let u be a fixed strong solution of (3.1). Then the non–increasing energy function Eu verifies in I that

$$Eu(1) \ge Eu(t) \ge \frac{1}{2}\|u_t\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|u\|^2 \ge 0. \tag{3.3}$$

Moreover

$$\|u\|_{L^2(\Omega_0)},\ \|u_t\|_{L^2(\Omega_0)},\ \|u\|,\ \|u\|_{L^{2^*}(\Omega_0)} \in L^\infty(I),$$
$$\mathscr{D}u(t) = a(t)t^\alpha \|u_t\|^2_{L^2(\Omega_0)} - \frac{1}{2}\int_{\Omega_0} V_t(t,x)u^2 dx \in L^1(I). \tag{3.4}$$

The fact that $\mathscr{D}u \in L^1(I)$ is a direct consequence of (B). Indeed, $Eu \geq 0$ by (3.3) and $\mathscr{D}u \geq 0$, giving

$$0 \leq \int_1^t \mathscr{D}u(\tau)d\tau \leq Eu(1) - Eu(t) \leq Eu(1).$$

By (B)–(ii) and (3.3) it is clear that there exists $\ell \geq 0$ such that

$$\lim_{t\to\infty} Eu(t) = \ell. \tag{3.5}$$

Put

$$r = 2^* = \begin{cases} \dfrac{2N}{N-2}, & \text{if } N \geq 3, \\ \infty, & \text{if } N = 1,\ 2, \end{cases} \qquad r' = \begin{cases} \dfrac{2N}{N+2}, & \text{if } N \geq 3, \\ 1, & \text{if } N = 1,\ 2, \end{cases}$$

for simplicity. Then, in I, thanks to the Hölder inequality, we get

$$\begin{aligned}\int_{\Omega_0} [a(t)t^\alpha u_t^2]^{r'} dx &= [a(t)t^\alpha]^{r'/2} \int_{\Omega_0} [a(t)t^\alpha u_t^2]^{r'/2} dx \\ &\leq [a(t)t^\alpha]^{r'/2} |\Omega_0|^{(r-2)/2(r-1)} \left(\int_{\Omega_0} [a(t)t^\alpha u_t^2]^{\frac{r'}{2}\cdot\frac{2(r-1)}{r}} dx \right)^{\frac{r}{2(r-1)}} \\ &= [a(t)t^\alpha]^{r'/2} |\Omega_0|^{(r-2)/2(r-1)} \|a(t)t^\alpha u_t^2\|^{r'/2}_{L^1(\Omega_0)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|a(t)t^\alpha u_t^2\|_{L^{r'}(\Omega_0)} &\leq |\Omega_0|^{(r-2)/2r} [a(t)t^\alpha]^{1/2} \|a(t)t^\alpha u_t^2\|^{1/2}_{L^1(\Omega_0)} \\ &\leq |\Omega_0|^{(r-2)/2r} [a(t)t^\alpha]^{1/2} \mathcal{D}u(t)^{1/2}.\end{aligned}$$

Hence, for all $T \geq 1$ and for all $t \geq T$ we obtain by the Hölder inequality

$$\begin{aligned}\int_T^t a(\tau)\tau^\alpha \int_{\Omega_0} |u_t^2 u| dx d\tau &\leq \int_T^t \|a(\tau)\tau^\alpha u_t^2\|_{L^{r'}(\Omega_0)} \|u\|_{L^r(\Omega_0)} d\tau \\ &\leq |\Omega_0|^{(r-2)/2r} \sup_{\tau\in I} \|u(\tau,\cdot)\|_{L^r(\Omega_0)} \cdot \\ &\quad \times \left(\int_T^t a(\tau)\tau^\alpha d\tau \right)^{1/2} \left(\int_T^t \mathcal{D}u(\tau) d\tau \right)^{1/2}.\end{aligned}$$

Finally, recalling that $r = 2^*$, we get

$$\int_T^t a(\tau)\tau^\alpha \int_{\Omega_0} |u_t^2 u| dx d\tau \leq \varepsilon(T) \left(\int_T^t a(\tau)\tau^\alpha d\tau \right)^{1/2},$$
$$\varepsilon(T) = |\Omega_0|^{1/N} \sup_{\tau\in I} \|u(\tau,\cdot)\|_{L^{2^*}(\Omega_0)} \left(\int_T^\infty \mathcal{D}u(t) dt \right)^{1/2} \longrightarrow 0 \quad \text{as } T \to \infty, \tag{3.6}$$

thanks to (3.4).

Similarly, for all $T \geq 1$ and for all $t \geq T$, we obtain by the Hölder inequality

$$\begin{aligned}\int_T^t \|u_t(\tau,\cdot)\|_{L^2(\Omega_0)} d\tau &\leq \sup_{\tau\in I} \|u_t(\tau,\cdot)\|_{L^2(\Omega_0)}^{1/2} \int_T^t \|u_t(\tau,\cdot)\|_{L^2(\Omega_0)}^{1/2} d\tau \\ &\leq \sup_{\tau\in I} \|u_t(\tau,\cdot)\|_{L^2(\Omega_0)}^{1/2} \int_T^t [a(\tau)\tau^\alpha]^{-1/2} \left(\int_{\Omega_0} a(\tau)\tau^\alpha u_t^2 dx\right)^{1/2} d\tau \\ &\leq \sup_{\tau\in I} \|u_t(\tau,\cdot)\|_{L^2(\Omega_0)}^{1/2} \left(\int_T^t [a(\tau)\tau^\alpha]^{-1} d\tau\right)^{1/2} \left(\int_T^t \mathcal{D}u(\tau) d\tau\right)^{1/2}.\end{aligned}$$

Hence, by (3.4), we have

$$\begin{gathered}\int_T^t \|u_t(\tau,\cdot)\|_{L^2(\Omega_0)} d\tau \leq \widehat{\varepsilon}(T) \left(\int_T^t [a(\tau)\tau^\alpha]^{-1} d\tau\right)^{1/2}, \\ \widehat{\varepsilon}(T) = \sup_{\tau\in I} \|u_t(\tau,\cdot)\|_{L^2(\Omega_0)}^{1/2} \left(\int_T^\infty \mathcal{D}u(t) dt\right)^{1/2} \longrightarrow 0 \quad \text{as } T \to \infty.\end{gathered} \tag{3.7}$$

Theorem 3.1. *Let the assumptions listed in this section hold. If* $|\alpha| \leq 1$*, then all the strong solutions of problem* (3.1) *have the property*

$$\lim_{t\to\infty} Eu(t) = 0 \quad \textit{and so} \quad \lim_{t\to\infty} (\|u_t\|_{L^2(\Omega_0)} + \|u\|) = 0. \tag{3.8}$$

Proof. The argument will show (3.8) follows the main steps of the proofs of [15, Theorem 5.1]. We present all the details since the framework is fairly different.

Suppose by contradiction that $\ell > 0$ in (3.5). Define a Lyapunov function by

$$L(t) = \int_{\Omega_0} u(t,x) u_t(t,x) dx.$$

Hence, by the distributional identity (A), with $\phi = u$, and setting $\mathcal{L}u = 2Eu$, we have $\mathcal{L}u \geq 2\ell$ in I and for any $t \geq T \geq 1$

$$L(\tau)\big]_T^t = \int_T^t \left\{2\|u_t\|_{L^2(\Omega_0)}^2 - \mathcal{L}u(\tau) d\tau\right\} - \int_T^t a(\tau)\tau^\alpha \int_{\Omega_0} |u_t u| dx d\tau. \tag{3.9}$$

Let us now estimate the right-hand side of (3.9). The definition of $\mathcal{L}u$ gives at once

$$-\int_T^t \mathcal{L}u(\tau) d\tau \leq -2\ell(t-T). \tag{3.10}$$

Now, applying (3.6), (3.7), and (3.10), from (3.9) we obtain

$$L(\tau)\big]_T^t \leq 2\widehat{\varepsilon}(T) \left(\int_T^t [a(\tau)\tau^\alpha]^{-1} d\tau\right)^{1/2} - 2\ell(t-T) + \varepsilon(T) \left(\int_T^t a(\tau)\tau^\alpha d\tau\right)^{1/2}, \tag{3.11}$$

where $\widehat{\varepsilon}(T)$ is defined in (3.7) and $\varepsilon(T)$ in (3.6). Moreover, (3.2) implies that

$$\liminf_{t\to\infty} \frac{1}{t} \left\{ \left(\int_T^t [a(\tau)\tau^\alpha]^{-1} d\tau\right)^{1/2} + \left(\int_T^t a(\tau)\tau^\alpha d\tau\right)^{1/2} \right\}$$

$$\leq \text{Const.} \lim_{t\to\infty} t^{(|\alpha|-1)/2} < \infty,$$

since $|\alpha| \leq 1$. Hence, there exists a sequence $(t_i)_i$ tending to ∞ as $i \to \infty$ and a number $l > 0$ such that

$$\left(\int_T^{t_i} [a(\tau)\tau^\alpha]^{-1} d\tau\right)^{1/2} + \left(\int_T^{t_i} a(\tau)\tau^\alpha d\tau\right)^{1/2} \leq l\, t_i.$$

Fix $T \geq 1$ so large that

$$\widetilde{\varepsilon}(T) l \leq \ell,$$

where $\widetilde{\varepsilon}(T) = \max\{\varepsilon(T), \widehat{\varepsilon}(T)\} = o(1)$ as $T \to \infty$. Consequently, for $t_i \geq T$,

$$L(t_i) \leq S(T) - \ell t_i,$$

where $S(T) = L(T) + T\varepsilon(T)\ell$. Thus, we get

$$\lim_{i\to\infty} L(t_i) = -\infty. \tag{3.12}$$

On the other hand,

$$\sup_{t\in I} \left|L(t)\right| \leq \sup_{t\in I} \|u(t,\cdot)\|_{L^2(\Omega_0)} \cdot \|u_t(t,\cdot)\|_{L^2(\Omega_0)} = U < \infty \tag{3.13}$$

by (3.4). Clearly, (3.13) contradicts (3.12). The proof is then completed. □

When $|\alpha| > 1$, we cannot apply Theorem 3.1. If $|\alpha| > 1$, we consider the special case

$$\begin{cases} u_{tt} - c\Delta u + k(-\Delta)^s_\Omega u + a(t)t^\alpha u_t + V(x)u = 0 & \text{in } I \times \Omega_0, \\ u(t,x) = 0 & \text{on } I \times \Omega_1, \end{cases} \tag{3.14}$$

of (3.1), that is, the case in which $V(t,x) = V(x)$, $V(x) > 0$ a.e. in Ω_0.

Now consider solutions of (3.14) having the separated form

$$u(t,x) = w(t)e(x), \tag{3.15}$$

where $e = e_k$ is an eigenfunction of $-c\Delta + k(-\Delta)^s_\Omega + V(x)$ in $\mathbb{H}^s_{0,\Omega}$, with peridynamical Dirichlet boundary conditions. The corresponding eigenvalue λ_k is positive, by Theorem 1.1–(d). An easy calculation shows that w is a solution of the ordinary differential equation

$$w'' + a(t)t^\alpha w' + \lambda w = 0, \qquad t \in I. \tag{3.16}$$

We recall now [18, Theorem 5.1′] related to the scalar case, which provides necessary conditions for global asymptotic stability of the rest state $u \equiv 0$ for the quasi–variational ordinary differential equation

$$(\partial_v G(u,u'))' - \partial_u G(u,u') + f(t,u) + Q(t,u,u') = 0, \qquad J = [R,\infty), \tag{3.17}$$

where $G = G(u,v)$ and $f(t,u) = \partial_u F(t,u)$. We suppose that

$$G \in C^1(\mathbb{R}\times\mathbb{R}), \qquad F \in C^1(J\times\mathbb{R}), \qquad Q \in C(J\times\mathbb{R}\times\mathbb{R}),$$

and that $G(u,0) = F(u,0) = 0$. Denote with $H(u,v) = \partial_v G(u,v)v - G(u,v)$, so in particular $H(u,0) = 0$ for all u.

Theorem 3.2. *[18, Theorem 5.1′] Assume that the function* $H(0, v)$ *is strictly increasing for* $v > 0$ *and strictly decreasing for* $v < 0$*. Let* u *be a solution of* (3.17) *on* J *such that* $u(t) \to 0$ *as* $t \to \infty$*. Suppose that for every* $t \in J$ *and for all* u, v *sufficiently small*

$$H(u, v) > 0, \qquad v \neq 0, \tag{3.18}$$

$$F(t, u) \geq 0, \qquad 0 \leq F_t(t, u) \leq \psi(t) \qquad \text{with } \psi \in L^1(J), \tag{3.19}$$

$$0 \leq Q(t, u, v)v \leq \hat{\delta}(t)H(u, v), \tag{3.20}$$

where

$$\hat{\delta} \in L^1(J). \tag{3.21}$$

Then $u \equiv 0$ *in* J*.*

Observe now that, as in [16], Eq (3.16) satisfies hypotheses (3.18)–(3.21), with $J = I$, $H(u, v) = v^2/2$, $F(t, u) = \lambda u^2/2$, and $\hat{\delta}(t) = 2a(t)t^\alpha$. Therefore, if $\alpha < -1$, the only solution of (3.16) that approaches zero at infinity is $w \equiv 0$.

This being shown, if $\alpha < -1$, it is easy to argue that all nontrivial solutions of (3.16) are oscillatory, with amplitude approaching a nonzero limit as $t \to \infty$. It may be noted that the behavior of solutions of (3.16) when $\alpha < -1$ is then essentially the same as for the wave equation itself in a bounded domain with zero boundary data.

When $\alpha > 1$, solutions of (3.14) again do not in general approach zero as $t \to \infty$, though their behavior is quite different from the case where $\alpha < -1$. We say that a function $\psi \in L^2(\Omega_0)$ is *attainable*, if there exists a strong solution u of (3.14) such that

$$\lim_{t\to\infty} \|u(t, \cdot) - \psi\|_{L^2(\Omega_0)} = 0. \tag{3.22}$$

Theorem 3.3. *If* $V(t, x) = V(x)$ *in* $I \times \Omega_0$, $V > 0$ *a.e. in* Ω_0 *and* $\alpha > 1$*, then every function* $\psi \in Y$*,*

$$Y = \operatorname{span}\{e_k\}_{k=1}^{\infty},$$

is attainable for problem (3.14) *and the set of attainable functions is dense in* $L^2(\Omega_0)$*.*

Proof. The proof is based on that of [16, Theorem 5.1]. We first show that every eigenfunction e_k of $-c\Delta + k(-\Delta)^s_\Omega + V(x)$ in $\mathbb{H}^s_{0,\Omega}$, with eigenvalue $\lambda_k > 0$ by Theorem 1.1–(*d*), is attainable. For this purpose, consider the function

$$u_k(t, x) = w_k(t)e_k(x),$$

which satisfies (3.14) if and only if w_k is a solution of (3.16), with $\lambda = \lambda_k$. Moreover, we have

$$\frac{1}{a(t)t^\alpha} \in L^1(I),$$

since $\alpha > 1$ and a verifies (3.2). Hence, by [19, Theorem 4.4], it follows that the set of attainable limits at ∞ of the solutions of (3.16) is dense in $\mathbb{R}$. On the other hand, since (3.16) is linear, the set of attainable limits for (3.16) must in fact be all of $\mathbb{R}$. Hence for an appropriate solution $w_k \neq 0$ of (3.16), corresponding to λ_k, we get

$$\lim_{t\to\infty} \|u_k(t, \cdot) - e_k\|_{L^2(\Omega_0)} = 0. \tag{3.23}$$

Finally, again from the linearity of (3.14), we obtain (3.22) for every $\psi \in Y$. Indeed, given $\psi \in Y$, there exists $e_{k_1}, \dots, e_{k_j}$ and real coefficients $\beta_1, \dots, \beta_j$ such that $\psi = \sum_{i=1}^{j} \beta_i e_{k_i}$. Consider the function

$$u(t,x) = \sum_{i=1}^{j} \beta_i u_{k_i}(t,x) = \sum_{i=1}^{j} \beta_i w_{k_i}(t) e_{k_i}(x),$$

where the functions w_{k_i} are the appropriate solutions of (3.16), corresponding to λ_{k_i}. Then, by (3.23)

$$\lim_{t\to\infty} \|u_{k_i}(t,\cdot) - e_{k_i}\|_{L^2(\Omega_0)} = 0 \quad \text{for any } i = 1, \dots, j.$$

Since (3.14) is linear, we get that u is a solution. Moreover, we have

$$\begin{aligned} \|u(t,\cdot) - \psi\|_{L^2(\Omega_0)} &= \left\| \sum_{i=1}^{j} \beta_i u_{k_i}(t,\cdot) - \sum_{i=1}^{j} \beta_i e_{k_i} \right\|_{L^2(\Omega_0)} \\ &\le \sum_{i=1}^{j} \beta_i \|u_{k_i}(t,\cdot) - e_{k_i}\|_{L^2(\Omega_0)} \to 0 \end{aligned}$$

as $t \to \infty$. Hence, (3.22) holds for every $\psi \in Y$, as claimed. □

4. Numerical application

In general it is not possible to achieve in a convenient way a closed form solution for both problems (1.1) and (3.1). Therefore we look for a numerical approximation of their solution.

In this case we consider a rod, that is, the case $N = 1$. Thus, $\Omega = (-L, L) = \Omega_0 \cup \Omega_1$ and $\Omega_0 = (-L_0, L_0)$. Let us discretize the intervals $(-L, L)$ and $(-L_0, L_0)$, in a finite number, n, of points denoted by x_i, with $i = 0, 2, \dots, n-1$, as shown in Figure 2. From the definition we have

$$x_i = -L + ih, \quad i = 0, 1, \dots, n-1,$$

with

$$h = \frac{2L}{n-1}.$$

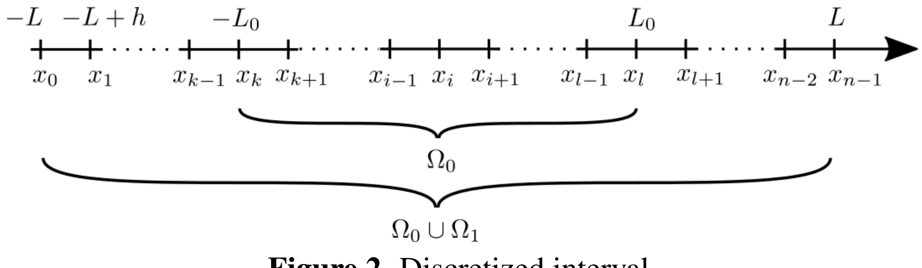

Figure 2. Discretized interval.

Note that the points of $\Omega_0 = (-L_0, L_0)$ are given by $i = k, k+1, \dots, l-1, l$.

The values of a function u defined in Ω at x_i will be denoted by $u_i = u(x_i)$.

We approximate the Laplacian Δu using the central difference formula

$$(\Delta u)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2},$$

in all the interior points, while when $i = 0$ and $i = n - 1$ we resort to forward difference and backward difference formulas, respectively.

Thereafter, we approximate $(-\Delta)^s_\Omega u$ using the approach proposed in [20] and already used in [7], which we recall briefly. The regional fractional Laplacian can be defined as

$$(-\Delta)^s_\Omega u(x) = C_{1,2s} \int_{-L}^{L} \frac{u(x) - u(y)}{|x - y|^{1+2s}}\, dy$$

with

$$C_{1,2s} = \frac{2s2^{2s-1}\Gamma(\frac{2s+1}{2})}{\pi^{1/2}\Gamma(\frac{2-2s}{2})}.$$

The integral over Ω is split in two contributions: the sum of the integrals in the intervals $(-L, x_i - h)$ and $(x_i + h, L)$ and the (improper) integral in the interval $(x_i - h, x_i + h)$.

$$\begin{aligned}
(-\Delta)^s_\Omega u(x_i) &= C_{1,2s}\left[\int_{-L}^{x_i-h} \frac{u(x_i) - u(y)}{(x_i - y)^{1+2s}} dy + \int_{x_i+h}^{L} \frac{u(x_i) - u(y)}{(y - x_i)^{1+2s}} dy\right] \\
&\quad + C_{1,2s}\int_{x_i-h}^{x_i+h} \frac{u(x_i) - u(y)}{|x_i - y|^{1+2s}} dy \\
&= C_{1,2s}\left[\int_{h}^{x_i-(-L)} \frac{u(x_i) - u(x_i - t)}{t^{1+2s}}\, dt + \int_{h}^{L-x_i} \frac{u(x_i) - u(x_i + t)}{t^{1+2s}}\, dt\right] \\
&\quad + C_{1,2s}\int_{-h}^{h} \frac{u(x_i) - u(x_i - t)}{|t|^{1+2s}} dt.
\end{aligned}$$

We note that the second contribution is more problematic since it contains the singularity. In order to achieve a numerical approximation, we employ the result in [20, Section 2.2] so that

$$C_{1,2s}\int_{-h}^{h} \frac{u(x_i) - u(x_i - t)}{|t|^{1+2s}}\, dt = -C_{1,2s}\frac{u(x_i + h) - 2u(x_i) + u(x_i + h)}{(2 - 2s)h^{2s}}.$$

The first contribution is easier to handle. Indeed, following the main argument in [20], we perform an exact integration using the interpolant of the terms $u(x_i) - u(x_i - y)$ as follows

$$u(x_i) - u(x_i \pm t) = \sum_{j\in\mathbb{N}} \left[u(x_i) - u(x_{i\pm j})\right] T_h(t - x_j),$$

with

$$T_h(t) = \begin{cases} 1 - \dfrac{|t|}{h} & |t| \le h, \\ 0 & \text{otherwise.} \end{cases}$$

The numerical approximation of the regional fractional Laplacian can therefore be evaluated numerically as

$$(-\Delta)^s_\Omega u_i = \sum_{j=1}^{i} \left(u_i - u_{i-j}\right) w_j + \sum_{j=1}^{n-1-i} \left(u_i - u_{i+j}\right) w_j$$

with

$$w_j = h^{-2s}\begin{cases} \dfrac{C_{1,2s}}{2-2s} - F'(1) + F(2) - F(1,) & j = 1, \\ F(j+1) - 2F(j) + F(j-1), & j = 2, 3, \dots, \end{cases}$$

where

$$F(t) = \begin{cases} \dfrac{C_{1,2s}}{(2s-1)2s}|t|^{1-2s}, & 2s \neq 1, \\ -C_{1,2s}\log|t|, & 2s = 1. \end{cases}$$

As highlighted in [20] the function $F = F(t)$ has the second derivative

$$F''(t) = C_{1,2s}\frac{1}{|t|^{1+2s}}.$$

The discretized form of (1.1) is therefore

$$\begin{cases} -c(\Delta u)_i + k((-\Delta)^s_\Omega u)_i + V_i\, u_i = \lambda\, u_i, & i = k, k+1, \dots, l-1, l, \\ u_i = 0, & i = 0, 1, \dots, k-1, l+1, \dots, n-1, \end{cases} \tag{4.1}$$

where V_i is $V(x)$ evaluated at $x = x_i$ and the points in Ω_0 are x_i with $i = j, j+1, \dots, k$.

Problem (4.1) can now be interpreted as a root-finding problem, where we have to find the values of u_i and λ so that the following discretized function g is zero:

$$g(x_i) = \begin{cases} -c(\Delta u)_i + k((-\Delta)^s_\Omega u)_i + V_i\, u_i - \lambda\, u_i, & i = k, k+1, \dots, l-1, l, \\ u_i, & i = 0, 1, \dots, k-1, l+1, \dots, n-1. \end{cases}$$

There are numerical tools to achieve the solution, and we use the Python programming language and the procedure "fsolve" from the Scipy package, which is based on the Powell hybrid method, as implemented in MINPACK [21].

In the case of problem (3.14), we must resort to numerical tools for integrating in the time domain. In particular, the time interval $(0, T]$ is divided in $m-1$ time steps equally spaced:

$$t_j = j\Delta T, \quad j = 1, \dots, m-1,$$

with $\Delta T = T/(m-1)$. The time $t = 0$ is not considered to allow for negative values of α, therefore the initial conditions are set for $t = \Delta T$.

In the present case, Newmark's method is used [22]. Within this method, the velocity and displacement at time t_{j+1} are evaluated as follows:

$$\begin{cases} {}^{j+1}\dot{u}_i = {}^{j}\dot{u}_i + \left[(1-\gamma)\,\Delta t\right]\, {}^{j}\ddot{u}_i + (\gamma\Delta t)\, {}^{j+1}\ddot{u}_i, \\ {}^{j+1}u_i = {}^{j}u_i + (\Delta t)^{j}\dot{u}_i + \left[(0.5-\beta)\,(\Delta t)^2\right]\, {}^{j}\ddot{u}_i + \left[\beta(\Delta t)^2\right]\, {}^{j+1}\ddot{u}_i, \end{cases} \tag{4.2}$$

where ${}^{j}u_i = u(x_i, t_j)$, ${}^{j}\dot{u}_i = \dot{u}(x_i, t_j)$, and ${}^{j}\ddot{u}_i = \ddot{u}(x_i, t_j)$. We recall that Newmark's method is considered an implicit method since the acceleration at time t_{j+1} is not known. The advantage of using Newmark's method is that it is well known that it is stable for a suitable choice of the coefficients γ and β: In the present case, in order to achieve numerical stability, we have used $\gamma = 1/2$ and $\beta = 1/4$, which correspond to assuming a constant average acceleration between t_j and t_{j+1}.

By means of (4.2) the discrete form of (3.14) is

$$\begin{cases} {}^{j+1}\ddot{u}_i - c\,{}^{j+1}(\Delta u)_i + k\,{}^{j+1}((-\Delta)^s_\Omega u)_i + V_i\,u_i + {}^{j+1}at^\alpha_{j+1}\,\dot{u}_i = 0, \\ \qquad i = k, k+1, \dots, l-1, l \text{ and } j = 0, 1, \dots, \\ {}^{j+1}u_0 = {}^{j+1}u_{n-1} = 0, \quad i = 0, 1, \dots, k-1, l+1, \dots, n-1 \text{ and } j = 1, 2, \dots, \end{cases} \tag{4.3}$$

where

$${}^{j+1}(\Delta u)_i = \frac{{}^{j+1}u_{i+1} - 2^{j+1}u_i + {}^{j+1}u_{i-1}}{h^2},$$

$${}^{j+1}(-\Delta)^s_\Omega u_i = \sum_{r=1}^{i} \left({}^{j+1}u_i - {}^{j+1}u_{i-r}\right) w_r + \sum_{r=1}^{n-1-i} \left({}^{j+1}u_i - {}^{j+1}u_{i+r}\right) w_r.$$

Again, the left-hand side of (4.3) can be interpreted as the definition of the vector function for which we search the zeros as done previously for the eigenproblem.

In order to perform the integration, initial condition are needed. In the present case, it is assumed that the initial displacements are given by one of the eigenvalues and the initial velocities are zero.

4.1. Eigenvalue problem

In order to find a solution for problem (4.1), the following values have been taken:

$$c = 50{,}000\,\text{N/mm}^2, \quad \kappa = 15{,}000\,\text{N/mm}^{4-2s}, \quad V = 20\,\text{N/mm}^4,$$
$$L = 100\,\text{mm}, \quad L_0 = 80\,\text{mm}, \quad h = 1\,\text{mm}.$$

The values of the parameters are chosen so that the contribution due to the ordinary Laplacian $-c(\Delta u)$, the regional fractional Laplacian $\kappa((-\Delta)^s_\Omega u)$ and the linear restoring term $V\,u$ are approximately of equal magnitude at $x = 0$. We stress that, even if the model is able to account for a nonlinear expression in place of $V(x)u$, we choose the linear form to analyze the eigenvalue problem.

We note that choosing $h = 1$ mm is a compromise between precision and computational efficiency. In fact, with smaller values of h we obtain the same results. For the same reason, we opted for a uniform spacing of points. We plan in the future to explore the possibility to use adaptive mesh refinement or non-uniform grids, particularly near the boundaries, in order to achieve better accuracy.

Three different values of s are used in order to analyze its influence on the response: 0.65, 0.75, and 0.90. In what follows, the eigenfunctions are normalized so that $\int_{-L}^{L} u^2(x)dx = 1$.

It is worth noting that as $s \to 1$ the purely local case is recovered, since the regional fractional Laplacian tends to the usual Laplacian and so the problem at hand becomes the following Sturm-Liouville problem:

$$-(c + \kappa)\Delta u + V\,u - \lambda u = 0,$$

for which the eigenvalues are given by

$$\lambda^{loc}_m = \frac{m^2\,\pi^2}{4\,L_0^2}\,(c + \kappa) + V,$$

with the values shown in Table 1 for the first modes, and the eigenfunctions are

$$u_m^{loc}(x) = \frac{1}{\sqrt{L_0}} \sin\left(\frac{m\pi}{2\,L_0}(x + L_0)\right),$$

where $m = 1, 2, \ldots$.

Table 1. First three eigenvalues for the purely local case.

λ_1^{loc}	λ_2^{loc}	λ_3^{loc}
45.06	120.24	245.54

We note that in order to find the nontrivial solution a starting guess is needed for the numerical procedure: In the present case, the initial guess is the eigenfunction for the corresponding local case ($m = 1$ to find the first eigenmode, $m = 2$ to find the second eigenmode, and so on). The results for the first eigenvalue are shown in Figure 3 for the first eigenmode and in Figure 4 for the second eigenmode.

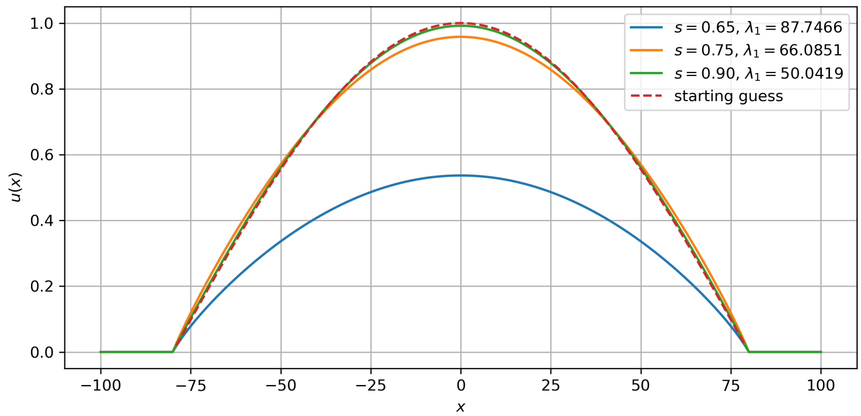

Figure 3. First eigenmode $\tilde{u}_1(x)$.

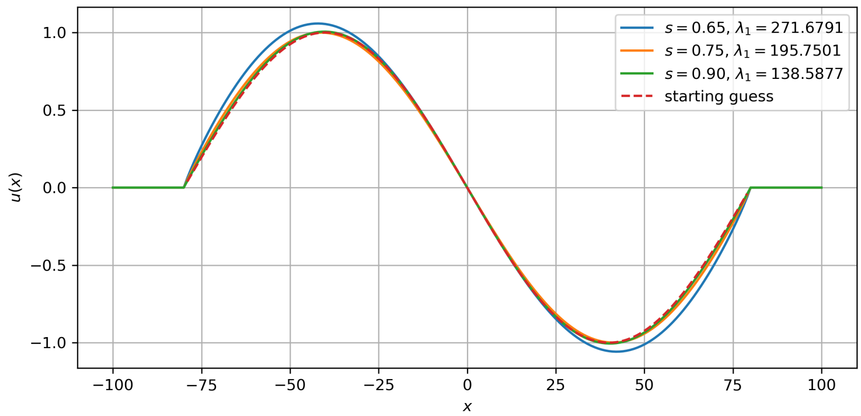

Figure 4. Second eigenmode $\tilde{u}_2(x)$.

We note that as s increases both the eigenvalues and the eigenfunctions tend to the purely local case.

4.2. Time evolution problem

In order to find a solution for problem (4.3), the same values used previously have been taken:

$$c = 50,000\,\mathrm{N/mm^2}, \quad \kappa = 15,000\,\mathrm{N/mm^{4-2s}}, \quad V = 20\,\mathrm{N/mm^4},$$
$$L = 100\,\mathrm{mm}, \quad L_0 = 80\,\mathrm{mm}, \quad h = 1\,\mathrm{mm}.$$

It was assumed that $a(t) = C$, with C variable, and different values of α have been considered. Since there are no forcing terms, the initial conditions have been assumed to be equal to one of the eigenfunctions for displacements $u(x, 0)$ and zero for velocities $\dot{u}(x, 0)$. It is worth noting that when $\alpha = 0$ we are in the case of "classical" damping, and we can define a critical value of C, C^*, which divide the regime of periodic motion (for $C < C^*$) and the regime of aperiodic motion (for $C > C^*$). When considering as the initial condition the configuration of the m-th mode, $u(x, 0) = \tilde{u}_m(x)$, considering the eigenvalue λ_m, the critical value for the mode, C^*_m is given by

$$C^*_m = 2\sqrt{\lambda_m}.$$

As an example, in Figure 5 the response for a super-critical and a sub-critical value of C are shown.

The result, when considering an initial configuration equal to the first eigenfunction, is shown in Figure 6 at the midspan, while using the second eigenfunction, the solution is shown in Figure 7 at $x = L_0/2$ since, obviously, the displacements at the midspan are null.

As expected, since C is quite smaller than C^* the motion is aperiodic, and the oscillation tends to decrease faster as C increases and α increases.

If a larger value of C is used, it is clear from Figure 8 that a transition between the periodic and aperiodic motion occurs.

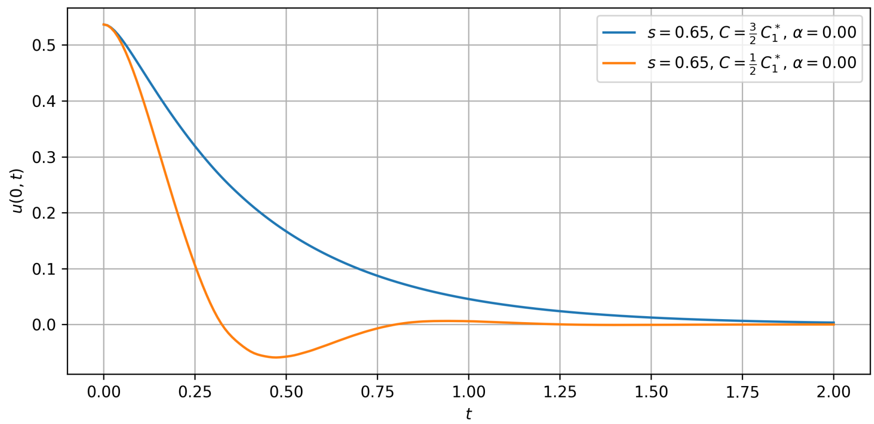

Figure 5. The solution at the midspan with $u(x, 0) = \tilde{u}_1(x)$ and super- and sub-critical values of C.

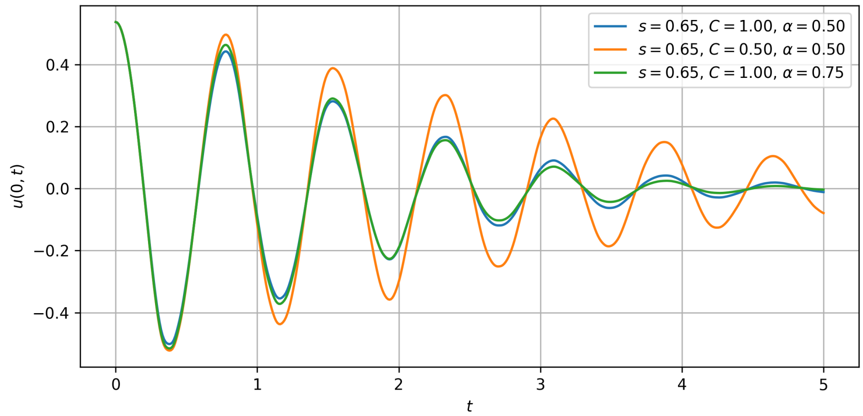

Figure 6. The solution at the midspan with $u(x, 0) = \tilde{u}_1(x)$ and different combinations of C and α.

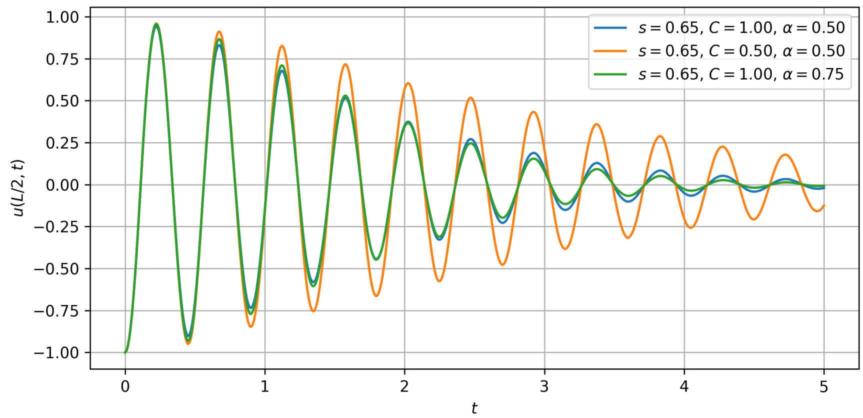

Figure 7. The solution at $x = L_0/2$ with $u(x, 0) = \tilde{u}_2(x)$ and different combinations of C and α.

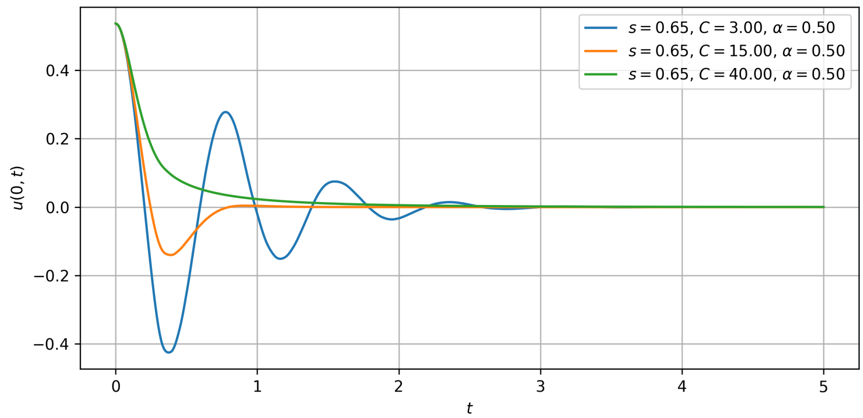

Figure 8. The solution at $x = 0$ with $u(x, 0) = \tilde{u}_1(x)$ and increasing values of C.

In the case where $\alpha < -1$ the results are shown in Figure 9, confirming that the solution is oscillatory.

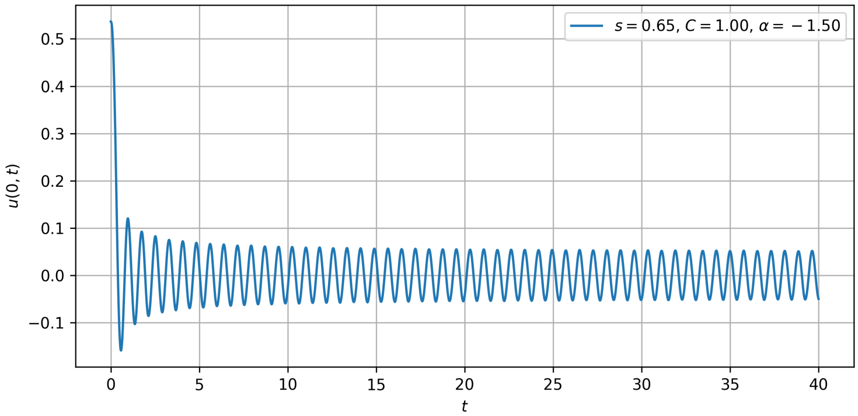

Figure 9. The solution at $x = 0$ with $u(x, 0) = \tilde{u}_1(x)$ and $\alpha < -1$.

We note that the obtained results are satisfactory and we plan to extend the model to take into account complex geometries or non-standard boundary conditions. Moreover, we aim to extend the model to other dimensions, 2D and ideally 3D, and to evaluate the response accounting for stochastic effects and uncertainties.

5. Conclusions

In this paper we have studied existence and properties of the eigenvalues and of the eigenfunctions of a bounded body with peridynamical Dirichlet boundary conditions. Then, we have analyzed the dynamics of the 1-dimensional case, studying the dynamical behavior of a rod, also from a numerical point of view.

Author contributions

All authors contributed equally to this research. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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