AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(4): 9042–9054.
DOI: 10.3934/math.2025416
Received: 22 December 2024
Revised: 30 March 2025
Accepted: 09 April 2025
Published: 21 April 2025

*Research article*

# Existence of solutions for the fractional $p\&q$-Laplacian equation with nonlocal Choquard reaction

**Liyan Wang[1,2,3], Baocheng Zhang[3], Zhihui Lv[4,*], Kun Chi[5,*] and Bin Ge[6]**

1 School of Computer Science and Technology, Harbin University of Science and Technology, Harbin 150080, China

2 Department of Applied Mathematics, Harbin University of Science and Technology, Harbin 150080, China

3 National Key Laboratory of Coal Coke and Graphite Testing, Harbin Costoms Technical Center, Hegang 154100, China

4 School of Mathematics and Statistics, Guangdong University of Foreign Studies, Guangzhou 510006, China

5 School of Automation, Harbin University of Science and Technology, Harbin 150080, China

6 College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China

* **Correspondence:** Email: Lvzh@gdufs.edu.cn, chikun@hrbust.edu.cn.

**Abstract:** We consider the following class of fractional $p\&q$-Laplacian differential equation with Choquard term:

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V(x)(|u|^{p-2}u + |u|^{q-2}u) + \int_{\mathbb{R}^N} g(x)|u|^r dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{k(u(x))K(u(y))}{|x-y|^\alpha} dxdy, & x \in \mathbb{R}^N, \\ u \in W_V^{s,p}(\mathbb{R}^N) \bigcap W_V^{s,q}(\mathbb{R}^N), & x \in \mathbb{R}^N, \end{cases}$$

where $s \in (0,1), 2 \le p \le r \le q < N/s, 0 < \alpha < N$, $(-\Delta)_m^s$ with $m \in \{p,q\}$ is the fractional $m$-Laplacian operator, $g(x) : \mathbb{R}^N \to \mathbb{R}$, by introducing a potential term function to restore compactness in the corresponding spaces. Using variational techniques and inequalities such as Hardy–Littlewood–Sobolev, we ensure the geometric conditions of the mountain pass theorem in order to show the existence of solutions.

**Keywords:** fractional $p\&q$-Laplacian; Hardy–Littlewood–Sobolev inequality; mountain pass theorem

**Mathematics Subject Classification:** 35J92, 35D05

## 1. Introduction and main results

The purpose of this paper is to investigate the existence of solutions to the following fractional $p\&q$-Laplacian equation with Choquard term:

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(x)(|u|^{p-2}u + |u|^{q-2}u) + \int_{\mathbb{R}^N} g(x)|u|^r dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{k(u(x))K(u(y))}{|x-y|^\alpha} dxdy \tag{1.1}$$

where $(-\Delta)_p^s$ and $(-\Delta)_q^s$ are the fractional $p\&q$-Laplacian operators, $s \in (0,1), 2 \le p \le r \le q < N/s, 0 < \alpha < N$, $g(x) : \mathbb{R}^N \to \mathbb{R}^+$.

The nonlocal operator $(-\Delta)_m^s$ is the fractional $m$-Laplacian operator, defined as

$$(-\Delta)_m^s u(x) = 2\lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{m-2}(u(x)-u(y))}{|x-y|^{N+sm}} dy, \ \ x \in \mathbb{R}^N,$$

where $m \in (p,q)$, $s \in (0,1)$, $2 \le m < \frac{N}{s}$.

The term on the right side of Eq (1.1) is referred to as the Choquard term, for the case $s = 1$ and $K(u) = |u|^2$, Eq (1.1) is called the Choquard–Pekar equation [1]. The nonlinear Choquard equation can be used to describe mathematical physical phenomena. The Choquard nonlinear term in it appears in many mathematical physical models, such as the mean field limit of weakly interacting molecules, the Pekar theory of polarons, and the Schrodinger–Newton system, etc [2–6].

Ambrosio [7] explored the issue of the lack of compactness associated with the critical Sobolev exponents. He addressed this issue by making an asymptotic estimation of the minimum values. Subsequently, he applied the mountain pass theorem. By virtue of this theorem, he proved the existence of weak solutions to Eq (1.2). These weak solutions are both non-negative and non-trivial.

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \mu|u|^{q-2}u + \lambda|u|^{p-2}u + |u|^{p_{s_1}^*-2}u. \tag{1.2}$$

In the local case $p = q \neq 2$, Eq (1.2) becomes a fractional $p$-Laplacian equation of the form:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u), x \in \mathbb{R}^N. \tag{1.3}$$

Cheng and Tang [8] discussed that both potential function and nonlinear term are allowed to change signs, and obtained the existence of solutions to the above problem. Torres [9] uses the mountain pass theorem with $(C)_c$ conditions and other theorems; problem (1.3) is presented as the existence of solutions and radial symmetric solutions when $f$ satisfies the $p$-superlinearity condition. The idea to prove their result, consists in replacing the path in the mountain pass setting by its symmetrization. By applying the fixed-point method, Souza [10] considers the existence of solutions to the above problem with mixed nonlinearity term $f(x,u)$, where the nonlinear $f(x,u) = l(x,u) + \lambda h(x)$.

In addition, under the continuity and boundedness of potential function $V(x)$ , Xu et al. [11] establish the existence of a solution via the Fountain theorem for the case $f(x,u) = l(x,u) + \lambda|u|^{q-2}u + h(x)|u|^{p-2}u$. Using variational methods, Xiang [12] prove the nonexistence and multiplicity of solutions to problem (1.3) depending on $\lambda$ in the case $f(x,u) = \lambda a(x)|u|^{p-2}u - h(x)|u|^{p-2}u$; their results extend the previous work of the fractional $p$-Laplacian setting. Furthermore, they weaken one of the conditions used in their paper. Hence the results of this paper are new even in the fractional Laplacian case.

$$(-\Delta)^s u + V(\varepsilon x)u + Cu = f(x,u), x \in \mathbb{R}^N. \tag{1.4}$$

Very recently, numerous studies exist concerning the existence, multiplicity, and concentration behavior of solutions for equations of the form (1.4), particularly under varying Sobolev subcritical potential conditions; we also refer to [13–16] and references therein.

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(u^{p-1} + u^{q-1}) = f(u) + u^{q_s^*-1}, x \in \mathbb{R}^N, \tag{1.5}$$

where $\varepsilon > 0$ is a small parameter, $s \in (0,1), 1 < p < q < N/s, V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function. Ambrosio [17] used the topological structure of the set of potential functions $V$ that reach their minimum value, combined the truncation of independent variables with Morse-type iterations, the minimax theorem, and the Ljusternik–Schnirellmann category theory to obtain the multiplicity results of the solutions to Eq (1.5).

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(x)(|u|^{p-2}u + |u|^{q-2}u) + g(x)|u|^{r-2}u = K(x)f(x,u) + h(u), x \in \mathbb{R}^N, \tag{1.6}$$

where $V : \mathbb{R}^N \to \mathbb{R}^+$ is a continuous function, and $h : \mathbb{R} \to \mathbb{R}$ is a perturbation term, Wang [18] studied two cases of Eq (1.6): If $f(x,u)$ is sublinear, by means of Clark's theorem, which takes into account the symmetric condition of the functional, infinitely many solutions can be obtained; if $f(x,u)$ is superlinear, by using the symmetric mountain pass theorem, infinitely many solutions can be derived.

$$\varepsilon^p(-\Delta)_p u + \varepsilon^q(-\Delta)_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = W(x)f(u), x \in \mathbb{R}^N, \tag{1.7}$$

where $1 < p < q < N, \Delta_s u = -\mathrm{div}(|\nabla u|^{s-2}\nabla u)$, $s \in \{p,q\}$, is the $s$-Laplacian operator, and $\varepsilon$ is a small positive parameter. Zhang et al. [19] assumed that the potential energy $V$, $W$, and nonlinear term $f$ satisfy certain conditions. Using topology and variational methods, they established the existence of positive solutions to Eq (1.7) when the function is sufficiently small, and also proved the nonexistence of ground state solutions.

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u),\ x \in \mathbb{R}^N, \tag{1.8}$$

where $\varepsilon > 0$ is a small parameter, $s \in (0,1), 1 < p < q < N/s$, the above equation has a continuous potential energy function and a nonlinear term with subcritical growth. Alves et al. [20] applied theoretical tools, such as the minimax theorem, to establish the existence and multiplicity of solutions to Eq (1.8), provided that the parameter $\varepsilon$ is sufficiently small.

Motivated by the aforementioned papers, in this article, this paper investigates the fractional $p\&q$-Laplacian problem with a Choquard term, characterized by the following features:

i) The problem involves the combined effects of fractional differential operators with unbalanced growth, where the associated energy corresponds to a fractional differential variational functional.

ii) Due to the unbounded nature of the domain, the Palais–Smale sequence fails to exhibit compactness.

iii) The Choquard reaction term on the right-hand side introduces a nonlocal characteristic.

Before presenting our main result, we first introduce the assumptions on the potential $V(x)$ and the nonlinearity $k(u)$.

$(V_1)$ $V(x) \in C(\mathbb{R}^N, \mathbb{R}^+)$: there exists a constant $V_0 > 0$, such that for all $x \in \mathbb{R}^N$, $V(x) \geq V_0$.

$(V_2)$ There exist constants $d, b > 0$ and $\gamma > N(m-1)$ such that

$$\lim_{|y|\to\infty} meas(\{x \in \mathbb{R}^N \mid \frac{V(x)}{|x|^\gamma} \leq b\} \cap B_d(y)) = 0,$$

where $meas(\cdot)$ is the Lebesgue measure in $\mathbb{R}^N$.

$(k_1)$ $k(x) \in C(\mathbb{R}, \mathbb{R})$, there exist $p \leq p_1 \leq p_2$ and a positive constant $C_0$ such that

$$|k(t)| \leq C_0(|t|^{p_1-1} + |t|^{p_2-1}), \forall t \in \mathbb{R},$$

where $p_2 \in \left(\frac{(2N-\alpha)q}{N}, \frac{(2N-\alpha)q^*}{N}\right)$, $q^* = \frac{Nq}{N-q}$.

$(k_2)$ There exists $\varsigma > q$ such that

$$0 < \varsigma K(t) \leq 2k(t)t, \forall t \in \mathbb{R},$$

where $K(t) = \int_0^t k(x)dx$.

**Theorem 1.1.** Assuming that conditions $(V_1)$, $(V_2)$, $(k_1)$, and $(k_2)$ are satisfied, problem (1.1) has at least one nontrivial solution in $E$.

This paper is organized as follows: In Section 2, we state the notations and main Lemma. Section 3 will be devoted to the proofs of the main results of the paper.

## 2. Notations and main lemma

There may be multiple selection methods for some spaces, but they should be considered as suitable for discussion. In order to obtain a solution to the problem (1.1), the space should be smaller, which can make the direction of the discussion clearer and eliminate some uncertainty. As for its smoothness, it is also advisable to request a lower level as much as possible to avoid increasing the complexity of argumentation and calculation. Next we introduce some relevant space and variational setting.

The fractional Sobolev space $W^{s,m}(\mathbb{R}^N)$ is defined by

$$W^{s,m}(\mathbb{R}^N) = \{u \in L^m(\mathbb{R}^N),\ [u]_{s,m} < \infty\},$$

with the corresponding norm

$$\|u\|_{s,m} = (\|u\|_m^m + [u]_{s,m}^m)^{\frac{1}{m}},$$

where $[u]_{s,m}$ denotes the Gagliardo seminorm and $\|u\|_s$ is given by

$$[u]_{s,m} = (\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^m}{|x-y|^{N+sm}} dxdy)^{\frac{1}{m}}, m \in \{p, q\}, \|u\|_s = (\int_{\mathbb{R}^N} |u|^s ds)^{\frac{1}{s}},$$

for $u, v \in W^{s,m}(\mathbb{R}^N)$, we have

$$\langle u, v\rangle_{s,m} = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{m-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sm}} dxdy,$$

let us introduce the space

$$W_V^{s,m}(\mathbb{R}^N) = \{u \in W^{s,m}(\mathbb{R}^N), \int_{\mathbb{R}^N} V(x)|u|^m dx < +\infty\},$$

and it is equipped with the norm

$$\|u\|_{W_V^{s,m}} = \left([u]_{s,m}^m + \|u\|_{m,V}^m\right)^{\frac{1}{m}},\ \|u\|_{m,V}^m = \int_{\mathbb{R}^N} V(x)|u|^m dx,$$

after consideration, we define the working space:

$$E = W_V^{s,p}(\mathbb{R}^N) \bigcap W_V^{s,q}(\mathbb{R}^N),$$

which is a uniformly convex Banach space (similar to [7]), endowed with the norm

$$\|u\|_E = \sum_{m=p,q} \|u\|_{W_V^{s,m}} = \|u\|_{W_V^{s,p}} + \|u\|_{W_V^{s,q}}.$$

We define the energy functional associated with problem (1.1)

$$\begin{aligned} J(u) =& \frac{1}{p}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dydx + \int_{\mathbb{R}^N} V(x)|u|^p dx\right) + \frac{1}{q}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} dydx + \int_{\mathbb{R}^N} V(x)|u|^q dx\right) \\ &+ \frac{1}{r}\int_{\mathbb{R}^N} g(x)|u|^r dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(u(x))K(u(y))}{|x-y|^{\alpha}} dydx, \end{aligned}$$

applying some standard arguments, we can see that $J$ with the derivative given by

$$\begin{aligned} \langle J'(u), v\rangle = &\sum_{m=p,q}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{m-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sm}} dxdy + \int_{\mathbb{R}^N} V(x)|u|^{m-2}uv(x)dx\right) \\ &+ \int_{\mathbb{R}^N} g(x)|u|^{r-2}uv(x)dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(u(y))k(u(x))v(x)}{|x-y|^{\alpha}} dydx. \end{aligned}$$

- "$\rightharpoonup$" means weak convergence, "$\to$" means strong convergence.
- "$\hookrightarrow$" means continuous embedding, "$\hookrightarrow\hookrightarrow$" means compact embedding.

**Definition 2.1.** [21] Let $(E, \|\cdot\|)$ be a real Banach space, $\varphi \in C^1(E, \mathbb{R})$. We say that $\varphi \in C^1(E, \mathbb{R})$ satisfies the $(PS)_c$-condition if any sequence $\{u_n\} \subset E$ such that

$$\varphi(u_n) \to c \text{ and } \varphi'(u_n) \to 0, \tag{2.1}$$

admits a strongly convergent subsequence. Such a sequence is called a $(PS)$ sequence on level $c$, or a $(PS)_c$-sequence for short.

Now, we present the following lemmas which will play a crucial role in the proof of the main theorems.

**Lemma 2.1.** (Hardy–Littlewood–Sobolev inequality [22]) Let $1 < r, s < +\infty$ and $0 < \alpha < N$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\alpha}{N} = 2$. If $\phi \in L^r(\mathbb{R}^N)$ and $\psi \in L^s(\mathbb{R}^N)$, then there exists a sharp constant $C(N, \alpha, r, s) > 0$, independent of $\phi$ and $\psi$, such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\phi(x)\psi(y)}{|x-y|^\alpha}dydx \le C(N,\alpha,r,s)|\phi|_r|\psi|_s.$$

**Lemma 2.2.** [18] Suppose that $(V_1)$ and $(V_2)$ are valid. If $\{u_j\}$ is a bounded sequence in $E$, then there exists $u \in E \cap L^\vartheta(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_j \to u \textit{ strongly in } L^\vartheta(\mathbb{R}^N),$$

as $j \to \infty$, for any $\vartheta \in [1, m_s^*)$, $m_s^* = \frac{Nm}{N-sm}$.

**Lemma 2.3.** [21] Let $E$ be a real Banach space with its dual space $E^*$, $\varphi \in C^1(E,\mathbb{R})$, for some $\mu < \eta$, $\rho > 0$, and $e \in E$ with $\|e\| > \rho$, such that

$$\max\{\varphi(0),\varphi(e)\} \le \mu < \eta \le \inf_{\|u\|=\rho}\varphi(u),$$

under the above assumption, if the functional $\varphi$ satisfies the $(PS)_c$-condition, then we can see that $c$ is a critical point of $\varphi$, where $c$ is the mountain pass level of $\varphi$, defined as

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi(\gamma(t)),$$

and $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$, $\Gamma$ is the set of continuous paths joining 0 and $e$.

Based on self-argumentation or utilizing existing abstract critical point theorems, provide reasonable and specific conditions for the studied problem to ensure the existence of differentiable functional critical points. Starting from the given specific conditions, verify the requirements of the Mountain Pass theorem, and obtain the existence of the solution to the problem.

## 3. Proof of the Theorem 1.1

To prove Theorem 1.1, we will apply a mountain pass-type argument to find critical the point of $J$. Firstly, we need to verify that the functional $J$ satisfies the geometrical assumptions of the mountain pass theorem.

**Lemma 3.1.** Assume that $(V_1)$, $(V_2)$, $(k_1)$, and $(k_2)$ hold; then the functional $J(u)$ satisfies the following conditions:
(a) There exist $\sigma, \varrho > 0$, such that $J(u) \ge \sigma$ with $\|u\| = \varrho$.
(b) There exist $\delta > 0$ and $u_0 \in E$ such that $\|u_0\| > \delta \Rightarrow J(u_0) < 0$.

*Proof.* (a) Using the Hardy–Littlewood–Sobolev inequality and $(k_1)$, we immediately obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(u(x))K(u(y))}{|x-y|^\alpha}dydx \le C_1\left(\|u\|_{rp_1}^{2p_1} + \|u\|_{rp_2}^{2p_2}\right),$$

where $C_1$ is a constant, let $u \in E$; we can choose $\|u\| = \varrho \in (0,1)$, then we have $\|u\|^q_{W_V^{s,q}} \le \|u\|^p_{W_V^{s,p}} \le 1$.

Combining the above inequality, we can infer

$$
\begin{aligned}
J(u) &= \frac{\|u\|_{W_V^{s,p}}^p}{p} + \frac{\|u\|_{W_V^{s,q}}^q}{q} + \frac{\|u\|_{r,g}^r}{r} - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(u(x))K(u(y))}{|x-y|^\alpha}dydx \\
&\geq \frac{\|u\|_{W_V^{s,p}}^p}{p} + \frac{\|u\|_{W_V^{s,q}}^q}{q} + \frac{\|u\|_{r,g}^r}{r} - C_1\left(\|u\|_{rp_1}^{2p_1} + \|u\|_{rp_2}^{2p_2}\right) \\
&\geq \frac{\|u\|_{W_V^{s,p}}^q}{q} + \frac{\|u\|_{W_V^{s,q}}^q}{q} - C_1\left(\|u\|_{rp_1}^{2p_1} + \|u\|_{rp_2}^{2p_2}\right) \\
&\geq \frac{C_2}{q}\left(\|u\|_{W_V^{s,p}} + \|u\|_{W_V^{s,q}}\right)^q - C_1\left(\|u\|_E^{2p_1} + \|u\|_E^{2p_2}\right) \\
&= \frac{C_2\|u\|_E^q}{q} - C_1\left(\|u\|_E^{2p_1} + \|u\|_E^{2p_2}\right),
\end{aligned}
$$

where $C_2$ is a constant and

$$
\|u\|_{r,g} = \left(\int_{\mathbb{R}^N} g(x)|u|^r dx\right)^{\frac{1}{r}}.
$$

Since $p_2 \in ((2N-\alpha)q/N, (2N-\alpha)q^*/N)$, then $2p_2 > q$; consequently, $\varrho$ is sufficiently small and can find $\sigma > 0$ such that $J(u) \geq \sigma$.

(b) Fix $u_0 \in W_V^{1,\xi}(\mathbb{R}^N)\backslash\{0\}$, with $u_0 > 0$; let us define

$$
h(\tau) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(\frac{\tau u_0}{\|u_0\|})K(\frac{\tau u_0}{\|u_0\|})}{|x-y|^\alpha}dxdy,
$$

according to the $(k_2)$ condition, we immediately obtain

$$
\begin{aligned}
h'(\tau) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{2K(\frac{\tau u_0}{\|u_0\|})k(\frac{\tau u_0}{\|u_0\|})\frac{u_0}{\|u_0\|}}{|x-y|^\alpha}dxdy \\
&\geq \frac{\lambda}{\tau}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(\frac{\tau u_0}{\|u_0\|})K(\frac{\tau u_0}{\|u_0\|})}{|x-y|^\alpha}dxdy \\
&\geq \frac{\lambda}{\tau}h(\tau),
\end{aligned}
\tag{3.1}
$$

in terms of interval $[1, \tau\|u_0\|]$, for $\tau > \max\{1/2, 1/\|u_0\|\}$, we deduce that

$$
h(\tau\|u_0\|) \geq h(1)(\tau\|u_0\|)^\lambda,
$$

and therefore

$$
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(\tau u_0)K(\tau u_0)}{|x-y|^\alpha}dxdy \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(\frac{u_0}{\|u_0\|})K(\frac{u_0}{\|u_0\|})}{|x-y|^\alpha}dxdy(\tau\|u_0\|)^\lambda.
$$

Moreover, for $\tau > 1/\|u_0\|$, it follows from the above conclusion that

$$\begin{aligned}J(\tau u_0) =&\frac{\tau^p}{p}\left(\int_{\mathbb{R}^{2N}}\frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+sp}}dydx+\int_{\mathbb{R}^N}V(x)|u_0|^p dx\right)\\&+\frac{\tau^q}{q}\left(\int_{\mathbb{R}^{2N}}\frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+sq}}dydx+\int_{\mathbb{R}^N}V(x)|u_0|^q dx\right)\\&+\frac{\tau^r}{r}\int_{\mathbb{R}^N}g(x)|u_0|^r dx-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(\tau u_0(x))K(\tau u_0(y))}{|x-y|^\alpha}dydx\\\le& C_3(\tau^p+\tau^q+\tau^r)-C_4\tau^{\varsigma},\end{aligned}$$

where $C_3$ and $C_4$ are constants. Taking $e=\tau u_0$, for $\tau$ sufficiently large, we deduce that (b) holds since $\varsigma > q$.

**Proof of Theorem 1.1.** First, under the assumptions in Theorem 1.1, we need to prove that any $(PS)_c$ sequence $\{u_n\}$ is bounded in $E$. For every $c\in\mathbb{R}$, let $\{u_n\}\subset E$ be a $(PS)_c$-sequence; that is, $J(u_n)\to c$ and $J'(u_n)\to 0$. Then we deduce that

$$\begin{aligned}c+\|u_n\|_E \ge& J(u_n)-\frac{1}{\varsigma}\langle J'(u_n),u_n\rangle\\=&(\frac{1}{p}-\frac{1}{\varsigma})\left(\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}dydx+\int_{\mathbb{R}^N}V(x)|u_n(x)|^p dx\right)\\&+(\frac{1}{q}-\frac{1}{\varsigma})\left(\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}dydx+\int_{\mathbb{R}^N}V(x)|u_n|^q dx\right)\\&+(\frac{1}{r}-\frac{1}{\varsigma})\int_{\mathbb{R}^N}g(x)|u_n|^r dx+\int_{R^N}\int_{R^N}\frac{K(u_n)}{|x-y|^\alpha}[\frac{1}{\varsigma}k(u_n)u_n-\frac{1}{2}K(u_n)]dydx\\\ge&(\frac{1}{q}-\frac{1}{\varsigma})\sum_{m=p,q}[\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^m}{|x-y|^{N+sm}}dxdy+\int_{\mathbb{R}^N}V(x)|u_n|^{m-2}u_n v(x)dx]\\=&(\frac{1}{q}-\frac{1}{\varsigma})\left(\|u_n\|^p_{W_V^{s,p}}+\|u_n\|^q_{W_V^{s,q}}\right).\end{aligned}$$

Arguing by contradiction, we assume that $\|u_n\|\to\infty$, and then we have the following two cases:

(1) $\|u_n\|_{W_V^{s,p}}\to\infty$.
(2) $\|u_n\|_{W_V^{s,p}}$ is bounded, $\|u_n\|_{W_V^{s,q}}\to\infty$.

We divide into two cases to prove the conclusion.

Case 1. Since $p<q$, we can see that $|u_n\|^{q-p}_{W_V^{s,q}}\ge 1$, then $\|u_n\|^q_{W_V^{s,q}}\ge\|u_n\|^p_{W_V^{s,q}}>1$, for $n$ large enough

$$\begin{aligned}c+\|u_n\|_E \ge& (\frac{1}{q}-\frac{1}{\varsigma})\left(\|u_n\|^p_{W_V^{s,p}}+\|u_n\|^q_{W_V^{s,q}}\right)\\\ge&(\frac{1}{q}-\frac{1}{\varsigma})\left(\|u_n\|^p_{W_V^{s,p}}+\|u_n\|^p_{W_V^{s,q}}\right)\\\ge& C_5\left(\|u_n\|_{W_V^{s,p}}+\|u_n\|_{W_V^{s,q}}\right)^p\\=&C_5\|u_n\|^p_E,\end{aligned}$$

clearly, this is impossible since $p > 1$, and we get a contradiction.

Case 2. In view of the above relation, we immediately obtain

$$c + \|u_n\|_{W_V^{s,p}} + \|u_n\|_{W_V^{s,q}} \geq (\frac{1}{q} - \frac{1}{\varsigma})\left(\|u_n\|_{W_V^{s,p}}^p + \|u_n\|_{W_V^{s,q}}^q\right) \geq (\frac{1}{q} - \frac{1}{\varsigma})\|u_n\|_{W_V^{s,q}}^p,$$

moreover, we have

$$\frac{c}{\|u_n\|_{W_V^{s,q}}^p} + \frac{\|u_n\|_{W_V^{s,p}}}{\|u_n\|_{W_V^{s,q}}^p} + \frac{\|u_n\|_{W_V^{s,q}}}{\|u_n\|_{W_V^{s,q}}^p} \geq (\frac{1}{q} - \frac{1}{\varsigma}),$$

which is a contradiction.

Hence, the $(PS)_c$ sequence $\{u_n\}$ of $J$ is bounded in $E$. According to Lemma 2.2, we can extract a subsequence (still denoted $\{u_n\}$) and $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } E, \\ u_n \to u \text{ in } L^{\vartheta}(\mathbb{R}^N), \\ u_n \to u \text{ a.e. in } \mathbb{R}^N, \end{cases}$$

where $\vartheta \in [m, m_s^*)$, $m_s^* = \frac{Nm}{N-sm}$.

Observe that, by a direct calculation, we have

$$\begin{aligned} \langle J'(u_n) - J'(u), u_n - u\rangle = & \sum_{m=p,q} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{m-2}(u_n(x) - u_n(y))((u_n - u)(x) - (u_n - u)(y))}{|x - y|^{N+sm}} dydx \\ & - \sum_{m=p,q} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{m-2}(u(x) - u(y))((u_n - u)(x) - (u_n - u)(y))}{|x - y|^{N+sm}} dydx \\ & + \sum_{m=p,q} \int_{\mathbb{R}^N} V(x)(|u_n|^{m-2}u_n - |u|^{m-2}u)(u_n - u)dx \\ & + \int_{\mathbb{R}^N} g(x)(|u_n|^{r-2}u_n - |u|^{r-2}u)(u_n - u)dx \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(u_n(y))k(u_n(x))(u_n(x) - u(x))}{|x - y|^{\alpha}} dydx \\ & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(u(y))k(u(x))(u_n(x) - u(x))}{|x - y|^{\alpha}} dydx \\ = & \; o_n(1). \end{aligned} \tag{3.2}$$

Next, we estimate the above each term. Applying the Minkowski inequality and Holder inequality, we can conclude that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(u_n(y))k(u_n(x))(u_n(x)-u(x))}{|x-y|^{\alpha}}dydx\right|\\
\le&\left[\int_{\mathbb{R}^N}|K(u_n(y))|^r dy\right]^{\frac{1}{r}}\left[\int_{\mathbb{R}^N}|k(u_n(x))(u_n(x)-u(x))|^r dx\right]^{\frac{1}{r}}\\
\le& C_6\left[\int_{\mathbb{R}^N}(|u_n|^{p_1}+|u_n|^{p_2})^r dy\right]^{\frac{1}{r}}\left[\int_{\mathbb{R}^N}[(|u_n|^{p_1-1}+|u_n|^{p_2-1})\cdot|u_n(x)-u(x)|]^r dx\right]^{\frac{1}{r}}\\
\le& C_7\left[\|u_n\|_{rp_1}^{p_1}+\|u_n\|_{rp_2}^{p_2}\right)\left(\|u_n\|_{rp_1}^{p_1-1}\|u_n(x)-u(x)\|_{rp_1}+\|u_n\|_{rp_2}^{p_2-1}\|u_n(x)-u(x)\|_{rp_2}\right)\\
\le& C_8\left[\|u_n(x)-u(x)\|_{rp_1}+\|u_n(x)-u(x)\|_{rp_2}\right]\\
=& o_n(1).
\end{aligned}
\tag{3.3}
$$

Similarly, we can obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(u(y))k(u(x))(u_n(x)-u(x))}{|x-y|^{\alpha}}dydx\right|\\
\le& C_9\left[\|u_n(x)-u(x)\|_{\beta p_1}+\|u_n(x)-u(x)\|_{\beta p_2}\right]\\
=&\mathrm{o}_n(1),
\end{aligned}
\tag{3.4}
$$

where $C_i(i=6,...9)$ is constant, $\beta=\frac{2N}{2N-\alpha}$. On the other hand, taking into account that $m\ge 2$, applying the following inequality

$$
|d-b|^m\le c_m(|d|^{m-2}d-|b|^{m-2}b)(d-b), d, b\in\mathbb{R}^N.
$$

We infer that

$$
\begin{aligned}
&[|u_n(x)-u_n(y)|^{m-2}(u_n(x)-u_n(y))-|u(x)-u(y)|^{m-2}(u(x)-u(y))]\\
&\times(u_n(x)-u(x)-u_n(y)+u(y))|x-y|^{-(N+sm)}dxdy\\
\ge& c_m\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)-u(x)+u(y)|^m}{|x-y|^{N+sm}}dydx\ge 0,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&\sum_{m=p,q}\int_{\mathbb{R}^N}V(x)(|u_n|^{m-2}u_n-|u|^{m-2}u)(u_n-u)dx\\
\ge& c_m\sum_{m=p,q}\int_{\mathbb{R}^N}V(x)|u_n-u|^m dx\\
=&\sum_{m=p,q}c_m\|u_n-u\|_{m,V}^m\ge 0.
\end{aligned}
\tag{3.6}
$$

Similarly, we obtain

$$
\int_{\mathbb{R}^N}g(x)(|u_n|^{r-2}u_n-|u|^{r-2}u)(u_n-u)dx\ge 0.
\tag{3.7}
$$

Therefore, it is easy to see that

$$\begin{aligned}&[|u_n(x)-u_n(y)|^{m-2}(u_n(x)-u_n(y))-|u(x)-u(y)|^{m-2}(u(x)-u(y))]\\&\times(u_n(x)-u(x)-u_n(y)+u(y))|x-y|^{-(N+sm)}dxdy\\=&o_n(1),\end{aligned}\tag{3.8}$$

we also have

$$\sum_{m=p,q}\int_{\mathbb{R}^N}V(x)(|u_n|^{m-2}u_n-|u|^{m-2}u)(u_n-u)dx=o_n(1),\tag{3.9}$$

and

$$\int_{\mathbb{R}^N}g(x)(|u_n|^{r-2}u_n-|u|^{r-2}u)(u_n-u)dx=o_n(1).\tag{3.10}$$

Now, we obtain

$$\sum_{m=p,q}\|u_n-u\|_{m,V}^m=o_n(1),\tag{3.11}$$

moreover, there holds

$$\sum_{m=p,q}\int_{\mathbb{R}^{2N}}\int_{\mathbb{R}^{2N}}\frac{|(u_n-u)(x)-(u_n-u)(y)|^m}{|x-y|^{N+sm}}dxdy=o_n(1).\tag{3.12}$$

Putting together (3.11) and (3.12), we have

$$\|u_n-u\|_E=\sum_{m=p,q}\left(\int_{\mathbb{R}^{2N}}\frac{|(u_n-u)(x)-(u_n-u)(y)|^m}{|x-y|^{N+sm}}dxdy+\int_{\mathbb{R}^N}V(x)|u_n-u|^mdx\right)^{\frac{1}{m}}=o_n(1).$$

We can see that the functional $J(u)$ satisfies the $(PS)_c$-condition. Thus, problem (1.1) possesses at least one solution. The proof is completed.

## 4. Conclusions

We point out that the main novelty of the paper is the combination of both double phase fractional differential operators and nonlocal Choquard reaction. Using employing variational techniques and inequalities such as Hardy-Littlewood-Sobolev, ensuring the geometric conditions of the Mountain Pass theorem for the energy functional. We demonstrate the existence of solution for the type of problem.

## Author contributions

Every author made an equal contribution to this study. Conceptualization and formal analysis by Baocheng Zhang and Zhihui Lv; validation and visualization by Kun Chi and Bin Ge; and Writing–original draft, review and editing by Liyan Wang. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This work was supported by the Guangdong Basic and Applied Basic Research Foundation (2022A1515110084), the Guangzhou Municipal Science and Technology Bureau (202201010246), and the Fundamental Research Foundation for Universities of Heilongjiang Province (2024-KYYWF-0921).


## Conflict of interest

The authors declare there are no conflicts of interest.

## References


1. S. I. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie Verlag: Berlin, Germany, 1954. https://dx.doi.org/10.1515/9783112649305
2. E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, *Stud. Appl. Math.*, **57** (1977), 93–105. https://dx.doi.org/10.1002/sapm197757293
3. E. Lenzmann, Uniqueness of ground states for pseudorelativistic Hartree equations, *Anal. PDE*, **2** (2009), 1–27. https://dx.doi.org/10.2140/apde.2009.2.1
4. V. Moroz, J. V. Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties, and decay asymptotics, *J. Funct. Anal.*, **265** (2013), 153–184. https://dx.doi.org/10.1016/j.jfa.2013.04.007
5. V. Moroz, J. Van Schaftingen, Semi-classical states for the Choquard equations, *Calc. Var.*, **52** (2015), 199–235. https://dx.doi.org/10.1007/s00526-014-0709-x
6. J. Frölich, B. L. G. Jonsson, E. Lenzmann, Boson stars as solitary waves, *Commun. Math. Phys.*, **274** (2007), 1–30. https://dx.doi.org/10.1007/s00220-007-0272-9
7. V. Ambrosio, Fractional $p\&q$ Laplacian problems in $\mathbb{R}^N$ with critical growth, *Z. Anal. Anwend.*, **39** (2020), 289–314. http://dx.doi.org/10.4171/ZAA/1661
8. B. T. Cheng, X. H. Tang, New existence of solutions for the fractional $p$-Laplacian equations with sign-changing potential and nonlinearity, *Mediterr. J. Math.*, **13** (2016), 3373–3387. http://dx.doi.org/10.1007/s00009-016-0691-y
9. C. E. Torres, Existence and symmetry result for fractional $p$-Laplacian in $\mathbb{R}^N$, *Commun. Pure Appl. Anal.*, **16** (2017), 99–114. http://dx.doi.org/10.3934/cpaa.2017004
10. M. de Souza, On a class of nonhomogeneous fractional quasilinear equations in $\mathbb{R}^N$ with exponential growth, *Nonlinear Differ. Equ. Appl.*, **22** (2015), 499–511. http://dx.doi.org/10.1007/s00030-014-0293-y
11. J. F. Xu, Z. L. We, W. Dong, Weak solutions for a fractional $p$-Laplacian equation with sign-changing potential, *Complex Var. Elliptic Equ.*, **61** (2016), 284–296. http://dx.doi.org/10.1080/17476933.2015.1076808

12. M. Q. Xiang, B. L. Zhang, V. D. Radulescu, Existence of solutions for perturbed fractional $p$-Laplacian equations, *J. Differ. Equations*, **260** (2016), 1392–1413. http://dx.doi.org/10.1016/j.jde.2015.09.028
13. N. S. Papageorgiou, V. D. Radulescu, W. Zhang, Global existence and multiplicity for nonlinear Robin eigenvalue problems, *Results Math.*, **78** (2023), 133. https://dx.doi.org/10.1007/s00025-023-01912-8
14. N. S. Papageorgiou, V. D. Radulescu, W. Zhang, Multiple solutions with signinformation for double-phase problems with unbalanced growth, *Bull. London Math. Soc.*, **57** (2025), 638–656. https://dx.doi.org/10.1112/blms.13218
15. N. S. Papageorgiou, J. Zhang, W. Zhang, Global existence and multiplicity of solutions for nonlinear singular eigenvalue problems, *Discrete Cont. Dyn. Syst.-Ser. S*, 18 (2025), 1440–1456. https://dx.doi.org/10.3934/dcdss.2024018
16. J. Zhang, Y. Zhang, An infinite sequence of localized semiclassical states for nonlinear Maxwell-Dirac system, *J. Geom. Anal.*, **34** (2024), 277. https://dx.doi.org/10.1007/s12220-024-01724-4
17. V. Ambrosio, Fractional $(p, q)$-Schrodinger equations with critical and supercritical growth, *Appl. Math. Optim.*, **86** (2022), 31. http://dx.doi.org/10.1007/s00245-022-09893-w
18. L. Y. Wang, K. Chi, J. H. Shen, B. Ge, Infinitely many solutions for the fractional $p\&q$-Laplacian problems in $\mathbb{R}^N$, *Symmetry*, **14** (2022), 2486. https://dx.doi.org/10.3390/sym14122486
19. J. Zhang, W Zhang, V. D. Radulescu, Double phase problems with competing potentials: concentration and multiplication of ground states, *Math. Z.*, **301** (2022), 4037–4078. http://dx.doi.org/10.1007/s00209-022-03052-1
20. C. O. Alves, V. Ambrosio, T. Isernia, Existence, multiplicity and concentration for a class of fractional $p\&q$ Laplacian problems in $\mathbb{R}^N$, *Commun. Pure Appl. Anal.*, **18** (2019), 2009–2045. http://dx.doi.org/10.3934/cpaa.2019091
21. M. Willem, *Minimax theorems*, Basel: Birkhauser, 1996. https://dx.doi.org/10.1007/978-1-4612-4146-1
22. E. H. Lieb, M. P. Loss, *Analysis*, 2 Eds., Providence: American Mathematical Society, 2001. https://dx.doi.org/10.1090/gsm/014


AIMS Press