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Research article

On iterated function systems with inverses

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Abstract: In this article we consider iterated function systems that contain inverse maps, which we call IFS with inverses. We show that the invariant measures for IFS with inverses agree with the invariant measures for associated graph-directed IFS under the suitable choice of weight.

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1. Introduction

A finite collection of strictly contractive maps on the real line is called an *iterated function system* (IFS). Let $\Phi = \{\varphi_a\}_{a\in\Lambda}$ be an IFS and $p = (p_a)_{a\in\Lambda}$ be a probability vector. It is well-known that there exists a unique Borel probability measure ν, called the *invariant measure*, such that

$$\nu = \sum_{a\in\Lambda} p_a \cdot \varphi_a \nu,$$

where $\varphi_a\nu$ is the push-forward of ν under the map $\varphi_a : \mathbb{R} \to \mathbb{R}$.

When the construction does not involve complicated overlaps (say, under the strong separation condition), the invariant measures are relatively easy to understand. For example, if the strong separation condition holds, then the invariant measure ν is supported on a Cantor set and is singular, and the dimension of ν is given by

$$\dim \nu = \frac{h}{\chi},$$

where $h = h(p)$ is the *entropy* and $\chi = \chi(\Phi, p)$ is the *Lyapunov exponent*.

In this paper we consider IFS with inverses (i.e., IFS that contain inverse maps). IFS with inverses were first introduced by the author in [5], motivated by the Furstenberg measure. See also [6]. We show that the invariant measures for IFS with inverses agree with the invariant measures for associated graph-directed IFS under the suitable choice of weight. The main results of [5] and [6] follow directly from our result.

The paper is organized as follows: In section 2, we recall IFS with inverses and state the main result. Section 3 is devoted to preliminary lemmas. In section 4 we prove the main result.

2. Definitions and the main results

2.1. Random walks on the free groups

Let G be the free group of rank $r \geq 2$, and let W be a free generating set of G. Let Λ be a set that satisfies

$$W \subset \Lambda \subset W \cup W^{-1},$$

where $W^{-1} = \{a^{-1}\}_{a \in W}$. Let $\mathcal{E}^* = \bigcup_{n \geq 1} \Lambda^n$ and $\mathcal{E} = \Lambda^{\mathbb{N}}$. For $\omega = \omega_0\omega_1 \cdots$ we denote $\omega|_n = \omega_0 \cdots \omega_{n-1}$. For $\omega, \xi \in \mathcal{E} \cup \mathcal{E}^*$ we denote by $\omega \wedge \xi$ their common initial segment. For $\omega \in \mathcal{E}^*$ and $\xi \in \mathcal{E} \cup \mathcal{E}^*$, we say that ω *precedes* ξ if $\omega \wedge \xi = \omega$.

Let $p = (p_a)_{a \in \Lambda}$ be a non-degenerate probability vector, and let μ be the associated Bernoulli measure on $\mathcal{E}$. We say that a (finite or infinite) sequence $\omega \in \mathcal{E}^* \cup \mathcal{E}$ is *reduced* if $\omega_i\omega_{i+1} \neq aa^{-1}$ for all $i \geq 0$ and $a \in \Lambda$. Let Γ^* (resp. Γ) be the set of all finite (resp. infinite) reduced sequences. For $\omega \in \Gamma^*$ we denote the associated cylinder set in Γ by $[\omega]$. Define the map

$$\mathrm{red} : \mathcal{E}^* \to \Gamma^*$$

in the obvious way, i.e., $\mathrm{red}(\omega)$ is the sequence derived from ω by deleting all occurrences of consecutive pairs aa^{-1} ($a \in \Lambda$). Let $\bar{\mathcal{E}} \subset \mathcal{E}$ be the set of all ω such that the limit

$$\lim_{n\to\infty} \mathrm{red}(\omega|_n) \tag{2.1}$$

exists. For example, for any $a \in \Lambda$ we have $aaa \cdots \in \bar{\mathcal{E}}$ and $aa^{-1}aa^{-1} \cdots \notin \bar{\mathcal{E}}$. By abuse of notation, for $\omega \in \bar{\mathcal{E}}$ we denote the limit (2.1) by $\mathrm{red}(\omega)$. The following is well-known (see, e.g., chapter 14 in [3]):

Lemma 2.1. *There exists* $0 < \ell \leq 1$ *(*drift *or* speed*) such that*

$$\lim_{n\to\infty} \frac{1}{n} |\mathrm{red}(\omega|_n)| = \ell$$

for μ-a.e. $\omega \in \mathcal{E}$. In particular, $\bar{\mathcal{E}}$ has full measure.

2.2. Iterated Function Systems with inverses

Denote

$$\Lambda^\star = \left\{(a, b) \in \Lambda^2 : a \neq b^{-1}\right\}.$$

For $a \in \Lambda$, write $\mathbb{R}_a = \mathbb{R} \times \{a\}$. We freely identify $\mathbb{R}_a$ with $\mathbb{R}$ below. Let $X_a \subset \mathbb{R}_a$ ($a \in \Lambda$) be open intervals and write $X = \bigcup_{a \in \Lambda} X_a$. Assume that there exist $0 < \gamma < 1$ and $0 < \theta \leq 1$ such that for all $(a, b) \in \Lambda^\star$, the map $\varphi_{ab} : X_b \to X_a$ is $C^{1+\theta}$ and satisfies

(i) $\overline{\varphi_{ab}(X_b)} \subset X_a$;
(ii) $0 < |\varphi'_{ab}(x)| < \gamma$ for all $x \in X_b$;
(iii) $\varphi_{ab}^{-1} : \varphi_{ab}(X_b) \to X_b$ is $C^{1+\theta}$.

We say that $\Phi = \{\varphi_{ab}\}_{(a,b)\in\Lambda^\star}$ is an *IFS with inverses*. For $\omega = \omega_0 \cdots \omega_n \in \Gamma^*$, we denote

$$\varphi_\omega = \varphi_{\omega_0\omega_1} \circ \cdots \circ \varphi_{\omega_{n-1}\omega_n}.$$

Let $\Pi : \Gamma \to X$ be the natural projection map, i.e.,

$$\Pi(\omega) = \bigcap_{n\geq 1} \varphi_{\omega|_{n+1}}(\overline{X_{\omega_n}}).$$

Define $\Pi^{\mathcal{E}} : \bar{\mathcal{E}} \to X$ by $\Pi^{\mathcal{E}} = \Pi \circ \mathrm{red}$. Define the measure $\nu = \nu(\Phi, p)$ by $\nu = \Pi^{\mathcal{E}}\mu$ (i.e., the push-forward of the measure μ under the map $\Pi^{\mathcal{E}} : \bar{\mathcal{E}} \to X$). We call ν an *invariant measure*. It is easy to see that if $\Lambda = W$, then the measure ν is an invariant measure of an IFS. Let $\chi = \chi(\Phi, p)$ be the *Lyapunov exponent*, and $h_{RW} = h_{RW}(p)$ be the *random walk entropy*. See section 3 in [5] for the precise definition. Fix $x_a \in X_a$ for each $a \in \Lambda$. For $\omega \in \bar{\mathcal{E}}$ and $n \in \mathbb{N}$ we denote $x_{\omega,n} = x_j$, where $j = j(\omega, n) \in \Lambda$ is the last letter of $\mathrm{red}(\omega|_n)$.

Proposition 2.1 (Proposition 3.1 in [5])**.** *We have*

$$\chi = -\lim_{n\to\infty} \frac{1}{n} \log \left|\varphi'_{\mathrm{red}(\omega|_n)}(x_{\omega,n})\right|$$

for μ-a.e. ω.

Notice that an IFS with inverses $\Phi = \{\varphi_{ab}\}_{(a,b)\in\Lambda^\star}$ does not have any explicit inverse map. The next example illustrates why we call Φ an IFS with inverses. For more detail, see Example 2.1 and Appendix in [5].

Example 2.1. *Let $r = 2$, $W = \{0, 1\}$ and $\Lambda = \{0, 1, 1^{-1}\}$. For $0 < k, l < 1$, define*

$$f_0(x) = kx,\ \ f_1(x) = \frac{(1 + l)x + 1 - l}{(1 - l)x + 1 + l}.$$

Let $f_{1^{-1}} = f_1^{-1}$. It is easy to see that we have $f_0(0) = 0$, $f_1(-1) = -1$, $f_1(1) = 1$ and $f_0'(0) = k$, $f_1'(1) = l$. It is well-known that there exists a unique Borel probability measure ν that satisfies

$$\nu = \sum_{a\in\Lambda} p_a f_a \nu.$$

The above measure is called a Fustenberg measure. See., e.g., [2]. Let

$$Y_0 = (-k, k),\ \ Y_1 = (f_1(-k), 1)\ \ \text{and}\ \ Y_{-1} = (-1, f_{-1}(k)).$$

Then we have

$$f_a(Y \setminus Y_{a^{-1}}) \subset Y_a,$$

for all $a \in \Lambda$, where $Y = \bigcup_{a\in\Lambda} Y_a$ and $Y_{0^{-1}} = \emptyset$. Notice that the sets $\{Y_a\}_{a\in\Lambda}$ are not mutually disjoint if and only if $k > f_1(-k)$, which is equivalent to

$$\sqrt{l} > \frac{1 - k}{1 + k}.$$

It is easy to see that there exist open intervals $X_0, X_1, X_{1^{-1}} \subset \mathbb{R}$ such that

$$Y_a \subset X_a\ \ \text{and}\ \ \overline{f_a(X \setminus X_{a^{-1}})} \subset X_a$$

for all $a \in \Lambda$, where $X = \bigcup_{a\in\Lambda} X_a$ and $X_{0^{-1}} = \emptyset$. Then $\{f_a|_{X_b}\}_{(a,b)\in\Lambda^\star}$ is an IFS with inverses, and the associated invariant measure agrees with ν. For the proof, see the Appendix in [5].

Denote

$$\dim \nu = \inf \{\dim_H Y : \nu(\mathbb{R} \setminus Y) = 0\}.$$

Proposition 2.2 (Proposition 3.3 in [5])**.** *Assume that for all $a \in \Lambda$, the sets $\{\overline{\varphi_{ab}(X_b)}\}_{b \in \Lambda_a^\star}$ are mutually disjoint, where*

$$\Lambda_a^\star = \{b \in \Lambda : (a, b) \in \Lambda^\star\}.$$

Then we have

$$\dim \nu_a = \frac{h_{RW}}{\chi}$$

for all $a \in \Lambda$.

2.3. Graph-directed IFS and the main result

Given an IFS with inverses $\Phi = \{\varphi_{ab}\}_{(a,b) \in \Lambda^\star}$, one can naturally associate a graph-directed IFS by restricting transitions from a to a^{-1} for all $a \in W$. For the precise definitions of graph-directed IFS, see section 1.7 in [1].

Let $\widetilde{P} = (\widetilde{p}_{ab})$ be a $|\Lambda| \times |\Lambda|$ stochastic matrix that satisfies $\widetilde{p}_{ab} > 0$ $(a \neq b^{-1})$ and $\widetilde{p}_{ab} = 0$ $(a = b^{-1})$. Let $\widetilde{p} = (\widetilde{p}_1, \cdots, \widetilde{p}_N)$ be the unique row vector satisfying $\widetilde{p}\widetilde{P} = \widetilde{p}$. Let $\widetilde{\mu}$ be the probability measure on Γ associated with $\widetilde{P}$ and $\widetilde{p}$. Define the measure $\widetilde{\nu} = \widetilde{\nu}(\Phi, \widetilde{P})$ by $\widetilde{\nu} = \Pi \widetilde{\mu}$. For $a \in \Lambda$, denote $\widetilde{\nu}_a = \widetilde{\nu}|_{X_a}$. It is easy to see that

$$\widetilde{\nu}_a = \sum_{(a,b) \in \Lambda^\star} \widetilde{p}_{ab} \cdot \varphi_{ab} \widetilde{\nu}_b.$$

Let $\widetilde{h} = \widetilde{h}(\widetilde{P})$ be the *entropy* and $\widetilde{\chi} = \widetilde{\chi}(\Phi, \widetilde{P})$ be the *Lyapunov exponent*, i.e.,

$$\widetilde{h} = -\sum_{(a,b) \in \Lambda^\star} \widetilde{p}_a \widetilde{p}_{ab} \log \widetilde{p}_{ab},$$

and

$$\widetilde{\chi} = -\lim_{n \to \infty} \frac{1}{n} \log \left| \varphi'_{\omega|_n}(x_{\omega_n}) \right|$$

for $\widetilde{\mu}$-a.e. ω. Under the separation condition, we obtain the following. The argument is classical, so we omit the proof. See, e.g., the proof of (2.6) in [4].

Proposition 2.3. *For every $a \in \Lambda$, assume that the sets $\{\overline{\varphi_{ab}(X_b)}\}_{b \in \Lambda_a^\star}$ are mutually disjoint. Then we have*

$$\dim \widetilde{\nu}_a = \frac{\widetilde{h}}{\widetilde{\chi}}$$

for all $a \in \Lambda$.

Our main result is the following:

Theorem 2.1. *Let $\Phi = \{\varphi_{ab}\}_{(a,b) \in \Lambda^\star}$ be an IFS with inverses. Then there exists a stochastic matrix $\widetilde{P}$ such that*

$$\nu = \widetilde{\nu}, \ h_{RW} = \ell \widetilde{h} \ \text{and} \ \chi = \ell \widetilde{\chi}.$$

Since the graph directed IFS has essentially the same structure as IFS, by the above theorem most of the results of IFS can be immediately extended to IFS with inverses. For example, the main results of [5] and [6] follow directly from the above result.

3. Preliminaries

Define μ_{red} by $\mathrm{red}_*\mu$, i.e., the push-forward of the measure μ under the map $\mathrm{red} : \mathcal{E} \to \Gamma$. From below, for $n > 0$, which is not necessarily an integer, we interpret $\omega|_n$ to be $\omega|_{\lfloor n \rfloor}$. The following lemma is immediate.

Lemma 3.1. *We have*

$$\chi = -\lim_{n\to\infty} \frac{1}{n} \log \left|\varphi'_{\omega|_{\ell n}}(x_{\omega,\ell n})\right|$$

for μ_{red}-a.e. $\omega \in \Gamma$.

Proof. Let $\omega \in \Gamma$, and let $\eta \in \mathcal{E}$ be such that $\omega = \mathrm{red}(\eta)$. We can assume that η satisfies

$$\lim_{n\to\infty} \frac{1}{n} |\mathrm{red}(\eta|_n)| = \ell$$

and

$$\chi = -\lim_{n\to\infty} \frac{1}{n} \log \left|\varphi'_{\mathrm{red}(\eta|_n)}(x_{\eta,n})\right|.$$

Let $\epsilon > 0$, and let $n \in \mathbb{N}$ be sufficiently large. Then, since $\omega|_{(\ell-\epsilon)n}$ precedes $\mathrm{red}(\eta|_n)$ and $|\mathrm{red}(\eta|_n)| < (\ell + \epsilon)n$, we have

$$-\log \left|\varphi'_{\omega|_{(\ell-\epsilon)n}}(x_{\omega(\ell-\epsilon)n})\right| < n\chi < -\log \left|\varphi'_{\omega|_{(\ell-\epsilon)n}}(x_{\omega(\ell-\epsilon)n})\right| - 2\epsilon n \cdot \log \lambda_{\min},$$

where

$$\lambda_{\min} = \min\{|\varphi'_{ab}(x)| : (a,b) \in \Lambda^\star,\ x \in X_b\}.$$

The result follows from this. $\square$

For $\omega \in \Gamma^*$, we denote

$$\mathcal{E}_\omega = \left\{\upsilon \in \bar{\mathcal{E}} : \omega \text{ precedes } \mathrm{red}(\upsilon)\right\}$$

and

$$\hat{\mathcal{E}}_\omega = \left\{\upsilon \in \bar{\mathcal{E}} : \text{there exists } n \in \mathbb{N} \text{ s.t. } \mathrm{red}(\upsilon|_n) = \omega\right\}.$$

Notice that $\mathcal{E}_\omega \subset \hat{\mathcal{E}}_\omega$. For $a \in \Lambda$, write

$$q_a = \begin{cases} 1 - \mu(\mathcal{E}_{a^{-1}}) & (a^{-1} \in \Lambda) \\ 1 & (a^{-1} \notin \Lambda) \end{cases}$$

and $p_a = \mu(\hat{\mathcal{E}}_a)$. We next prove the following crucial lemma.

Lemma 3.2. *Let $\omega \in \Gamma^*$ and $a \in \Lambda$ be such that $\omega a \in \Gamma^*$. Then we have*

$$p_a = \frac{\mu(\hat{\mathcal{E}}_{\omega a})}{\mu(\hat{\mathcal{E}}_\omega)}.$$

Proof. Fix such $\omega \in \Gamma^*$ and $a \in \Lambda$. Notice that

$$\hat{\mathcal{E}}_\omega = \bigsqcup_{i=|\omega|}^{\infty} \hat{\mathcal{E}}_\omega^{(i)},$$

where

$$\hat{\mathcal{E}}_\omega^{(i)} = \left\{\upsilon \in \bar{\mathcal{E}} : \mathrm{red}(\upsilon|_i) = \omega,\ \mathrm{red}(\upsilon|_k) \neq \omega\ (|\omega| \leq k < i)\right\}.$$

Then, since

$$\hat{\mathcal{E}}_{\omega a} \cap \hat{\mathcal{E}}_\omega^{(i)} = \{\upsilon \in \bar{\mathcal{E}} : \sigma^i(\upsilon) \in \hat{\mathcal{E}}_a\} \cap \hat{\mathcal{E}}_\omega^{(i)},$$

we have

$$\mu(\hat{\mathcal{E}}_{\omega a} \cap \hat{\mathcal{E}}_\omega^{(i)}) = p_a \cdot \mu(\hat{\mathcal{E}}_\omega^{(i)}).$$

Therefore,

$$\begin{aligned}
\mu(\hat{\mathcal{E}}_{\omega a}) &= \mu\Big(\bigsqcup_{i=|\omega|}^{\infty} \hat{\mathcal{E}}_{\omega a} \cap \hat{\mathcal{E}}_\omega^{(i)}\Big) \\
&= \sum_{i=|\omega|}^{\infty} p_a \cdot \mu(\hat{\mathcal{E}}_\omega^{(i)}) \\
&= p_a \cdot \mu(\hat{\mathcal{E}}_\omega).
\end{aligned}$$

□

Similarly, we have the following:

Lemma 3.3. *For $a \in \Lambda$ and $\omega = \omega_0 \cdots \omega_n \in \Gamma^*$ with $\omega_n = a$, we have*

$$\mu(\mathcal{E}_\omega) = q_a \cdot \mu(\hat{\mathcal{E}}_\omega).$$

Proof. Fix such $a \in \Lambda$ and $\omega \in \Gamma^*$. Notice that

$$\hat{\mathcal{E}}_\omega = \bigsqcup_{i=|\omega|}^{\infty} \hat{\mathcal{E}}_\omega^{(i)},$$

where

$$\hat{\mathcal{E}}_\omega^{(i)} = \left\{\upsilon \in \bar{\mathcal{E}} : \mathrm{red}(\upsilon|_i) = \omega,\ \mathrm{red}(\upsilon|_k) \neq \omega\ (|\omega| \leq k < i)\right\}.$$

Then, since

$$\mathcal{E}_\omega \cap \hat{\mathcal{E}}_\omega^{(i)} = \{\upsilon \in \bar{\mathcal{E}} : \sigma^i(\upsilon) \notin \bigcup_{b \in \Lambda \setminus \{a^{-1}\}} \mathcal{E}_b\} \cap \hat{\mathcal{E}}_\omega^{(i)},$$

we have

$$\mu(\mathcal{E}_\omega \cap \hat{\mathcal{E}}_\omega^{(i)}) = q_a \cdot \mu(\hat{\mathcal{E}}_\omega^{(i)}).$$

Therefore,

$$\begin{aligned}
\mu(\mathcal{E}_\omega) &= \mu\Big(\bigsqcup_{i=|\omega|}^{\infty} \mathcal{E}_\omega \cap \hat{\mathcal{E}}_\omega^{(i)}\Big) \\
&= \sum_{i=|\omega|}^{\infty} q_a \cdot \mu(\hat{\mathcal{E}}_\omega^{(i)}) \\
&= q_a \cdot \mu(\hat{\mathcal{E}}_\omega).
\end{aligned}$$

□

4. Proof of the main results

In this section we prove Theorem 2.1. Denote

$$\widetilde{p}_a = p_a q_a \text{ and } \widetilde{p}_{ab} = \frac{p_b q_b}{q_a}.$$

Let $\widetilde{P} = (\widetilde{p}_{ab})$. By Lemma 3.2 and Lemma 3.3, we obtain the following.

Proposition 4.1. *For all* $\omega = \omega_0 \cdots \omega_n \in \Gamma^*$*, we have*

$$\mu_{\mathrm{red}}([\omega]) = \widetilde{p}_{\omega_0}\widetilde{p}_{\omega_0\omega_1} \cdots \widetilde{p}_{\omega_{n-1}\omega_n}.$$

The above proposition implies that $\mu_{\mathrm{red}} = \widetilde{\mu}$. Therefore, we obtain $\nu = \widetilde{\nu}$. By Lemma 3.1, we have $\chi = \ell\widetilde{\chi}$. It remains to show the following lemma. Notice that h_{RW}, ℓ and $\widetilde{h}$ all depend only on Λ and p.

Lemma 4.1. *We have*

$$h_{RW} = \ell\widetilde{h}.$$

Proof. Let $\Phi' = \{\phi_{ab}\}_{(a,b)\in\Lambda^\star}$ be an IFS with inverses that satisfies the separation condition. Then, by Proposition 2.2 and Proposition 2.3 we obtain $h_{RW} = \ell\widetilde{h}$. □

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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