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Research article

Averaging principle for space-fractional stochastic partial differential equations driven by Lévy white noise and fractional Brownian motion

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Abstract: This paper's main objective was to obtain an averaging principle for space-fractional stochastic partial differential equations (SFPDEs) driven by Lévy space-time white noise and fractional Brownian motion (fBm). By using the fixed point theorem, we first obtained the existence and uniqueness of mild solutions for the given equation. Subsequently, given some appropriate conditions, we proved that the solution of the original equation converges to that of the averaged equation as the time scale $\epsilon \to 0$. This greatly decreases the complexity since one can focus on the averaged equation rather than the original equation.

Keywords: fractional stochastic partial differential equation; averaging principle; existence of mild solutions; Lévy space-time white noise; fractional Brownian motion
Mathematics Subject Classification: 35R60, 60H15

1. Introduction

In this paper, we consider the stochastic system as follows:

$$\begin{cases} \dfrac{\partial v_\epsilon(\tau,\zeta)}{\partial \tau} = \triangle_\mu v_\epsilon(\tau,\zeta) + \epsilon f(\tau,\zeta,v_\epsilon(\tau,\zeta)) + \sqrt{\epsilon} g(\tau,\zeta,v_\epsilon(\tau,\zeta))\dot{L} + \sqrt{\epsilon}\sigma(\tau,\zeta)\dot{B}^H, \\ v_\epsilon(0,\zeta) = v_0(\zeta), \qquad \tau \in \bar{J} = [0,T], \zeta \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $(\tau,\zeta) \in \bar{J} \times \mathbb{R}$, and $\epsilon > 0$ is a scale parameter. $\dot{L}$ is the Lévy space-time white noise which is encompassed within a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. $\dot{B}^H$ is the fBm defined over $\bar{J} \times \mathbb{R}$ characterized by Hurst parameter $H(> \frac{1}{2})$ which is a central Gaussian process. Let f, g, σ: $\bar{J} \times \mathbb{R} \to \mathbb{R}$ be continuous functions. We employ the Fourier transform to establish the μ-fractional differential operator $\triangle_\mu$, with $1 < \mu < 2$ (details will be provided in Definition 2.1), for each $\mathscr{F}_0$-measurable mapping $v_0 : \mathbb{R} \to \mathbb{R}$ adhering to $\mathbb{E}\left[\|v_0(\cdot)\|_p^p\right] < \infty$.

In recent times, interest in the averaging principle has surged, offering a highly effective tool for simplifying complex systems in reality. The theory of the averaging principle originated in the 18th century through the contributions of Lagrange and others in perturbation theory. Subsequently, Krylov and Bogoliubov conducted more comprehensive and detailed examinations [1]. The averaging principle can either refer to the separation and averaging treatment of slow and fast components in multi-scale analysis or simply involve the overall averaging of temporal terms, depending on the specific application context. Scholars from a multitude of disciplines have also widely employed the averaging principle, with Bodnarchuk, for instance, examining the cable equation's mild form under a general stochastic measure [2]. Gao proposed and improved the averaging principle about multiscale non-autonomous random 2D Navier-Stokes systems [3, 4]. Furthermore, Liu and Cheng examined three distinct averaging principles applicable to stochastic complex Ginzburg-Landau equations [5]. Another growing area of interest is research into averaging principles of fractional stochastic differential equations (FSDEs). A key aspect of this study involves investigating the averaging principle in FSDEs, particularly those with Caputo derivatives, as exemplified in [6–9]. As the investigation of related issues becomes more complex, Yang et al. focused on deriving the averaging principle for the Hilfer fractional stochastic evolution equation (HFSEE) driven by Lévy noise and extended it to Hilbert spaces rather than restricting it to finite-dimensional settings [10]. In a recent publication, Liu et al. presented a standard form for stochastic differential equations of fractional order on natural time scales, and they showed that both the convergence interval and the rate of convergence are contingent upon the fractional order [11]. In conclusion, there has been considerable progress in recent years in developing the theoretical framework for averaging principles in the context of FSDEs. Among these, the analysis of equations driven by Lévy noise produces findings that are very rich. Zhu studied many stochastic equations driven by Lévy noise. Such as the stabilization problem of stochastic delay differential equations and stochastic nonlinear delay systems [12, 13]. It is noteworthy that Ahmed and Zhu investigated the averaging principle for Hilfer fractional stochastic delay differential equations with Poisson jumps [14]. Liu et al. investigated the convergence of the solution to the Caputo-Hadamard fractional stochastic differential equation to the solution of the underlying averaged equation as the time scale parameter approaches zero [15]. Moualkia et al. established novel findings on the averaging principle for a class of Caputo fractional-order stochastic systems with neutral dynamics, subject to Markovian switching, Lévy noise, variable delays, and time-varying order [16]. Kasinathan et al. aimed to present an averaging principle for Hilfer fractional stochastic differential pantograph equations [17].

To date, most research on stochastic partial differential equations (SPDEs) of fractional order has focused on their solution properties. For example, Azerad and Mellouk demonstrated the existence, uniqueness, and regularity of a class of SPDEs with a fractional Laplacian driven by space-time white noise in one dimension [18]. Shi and Wang investigated the existence and uniqueness of global mild solutions for an SFPDE driven by Lévy space-time white noise [19]. Avazzadeh et al. focused on the fractional Rayleigh–Stokes problem for an edge in a viscoelastic fluid [20]. Gunasekar et al. conducted an in-depth investigation into a Volterra-Fredholm integro-differential equation that incorporates Caputo fractional derivatives and is constrained by specific order conditions [21]. Significant prior research has focused on exploring solutions to related problems; however, little discussion has been dedicated to the averaging principles for dynamical systems integrating SPDEs and fractional-order derivatives. Therefore, we are committed to continue researching and addressing this. Building upon the aforementioned work and to address various complex real-world scenarios, this paper presents the following contributions:

(i) The existence and uniqueness of solutions for space-fractional stochastic partial differential equations (SPDEs) driven by combined Lévy noise and fractional Brownian motion are established using the fixed-point theorem.

(ii) The averaging principle has been extended to space-fractional SPDEs driven by a combination of Lévy noise and fractional Brownian motion.

The paper is organized as follows. Section 2 introduces key concepts and notations. In Section 3, we prove the existence and uniqueness of mild solutions for Eq (1.1) in $L^p (p \geq 2)$-space. The paper's primary finding is detailed in Section 4, where it is shown that under suitable conditions, solutions of SFPDEs can be approximated by those of averaged stochastic systems. In Section 5, an example is provided to illustrate our main conclusion.

2. Preliminaries

This section compiles key definitions and theorems in fractional differential operators and noise theory, Additionally. some auxiliary results will be presented to substantiate the proof of our primary theorem.

According to reference [18], let us consider the specific form of $\triangle_\mu$.

Definition 2.1. *[18]* Let $0 < \mu < 2$ be the order of the spatial fractional derivative. The μ-fractional Laplacian $\triangle_\mu$ is defined by

$$\triangle_\mu = -(-\triangle)^{\mu/2} = -(-\partial^2/\partial\zeta^2)^{\mu/2}.$$

This is a non-local operator defined via the Fourier transform $\mathcal{F}$:

$$\mathcal{F}(\triangle_\mu v)(\xi) = -|\xi|^\mu \mathcal{F}(v)(\xi),$$

for $\xi \in \mathbb{R}$ and a function v defined by a given equation.

Then, using the Fourier transform, we can easily see that $G_\mu(\tau, \zeta)$ is given by:

$$G_\mu(\tau, \zeta) = \mathcal{F}^{-1}(e^{-\tau|\xi|^\mu})(\zeta) = \int_{\mathbb{R}} e^{2i\pi\zeta\xi} e^{-\tau|\xi|^\mu} d\xi = \mathcal{F}(e^{-\tau|\xi|^\mu})(\zeta),$$

where $(\tau, \zeta) \in \bar{J} \times \mathbb{R}$.

We propose some related concepts of fBm with covariance kernel

$$R_H(\tau, \varsigma) = \frac{1}{2}(\tau^{2H} + \varsigma^{2H} - |\tau - \varsigma|^{2H}).$$

For more details, one can see [22].

Moreover, there is the covariance kernel $R_H(\tau, \varsigma)$ that satisfies $R_H(\tau, \varsigma) = \int_0^{\tau\wedge\varsigma} K_H(\tau, r) K_H(\varsigma, r)\, dr$, where the square kernel $K_H(\tau, \varsigma)$ holds for $0 < \varsigma < \tau$, by

$$K_H(\tau, \varsigma) = c_H \left[\left(\frac{\tau}{\varsigma}\right)^{H-\frac{1}{2}} (\tau - \varsigma)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) \varsigma^{\frac{1}{2}-H} \int_\varsigma^\tau v^{H-\frac{3}{2}} (v - \varsigma)^{H-\frac{1}{2}}\, dv \right],$$

and $c_H^2 = \frac{2H}{(1-2H)\beta(1-2H,H+1/2)}$, with the Beta function $\beta(\cdot,\cdot)$.

In particular, for $H > \frac{1}{2}$, the expression of $R_H(\tau,\varsigma)$ is given by $H(2H-1)\int_0^\tau \int_0^\varsigma |v-u|^{2H-2}\,du\,dv$.

Furthermore, we give a linear operator $\mathcal{K}_H^*$ defined by $(\mathcal{K}_H^*\chi)(\varsigma,\zeta) = K_H(T,\varsigma)\chi(\varsigma,\zeta) + \int_\varsigma^T (\chi(\tau,\zeta) - \chi(\varsigma-\zeta))\frac{\partial K_H}{\partial\tau}(\tau,\varsigma)\,d\tau$, where the operator $\mathcal{K}_H^*$ means an isometry from Hilbert space $\mathscr{H}$ to $L^2(\bar{J}\times\mathbb{R})$.

In addition, B^H can be expressed as $B^H([0,\tau]\times\mathbb{R}) = \int_0^\tau \int_{\mathbb{R}} K_H(\tau,\varsigma)W(\,d\varsigma,\,d\varrho)$.

Subsequently, we present several concepts related to Lévy noise. Detailed definitions can be found in [23, 24].

Definition 2.2. *[23, 24]* Consider two σ-finite measurable spaces, $(E_i,\mathcal{E}_i,\varphi_i)$ for $i = 1,2$. There is a Poisson-distributed random variable N map from $(E_1,\mathcal{E}_1,\varphi_1)\times(E_2,\mathcal{E}_2,\varphi_2)\times(\Omega,\mathscr{F},\mathbb{P})$ to $\mathbb{N}\cup\{0\}\cup\{\infty\}$, with associated Poisson noise on $(E_1,\mathcal{E}_1,\varphi_1)$, applicable to every $A\in\mathcal{E}_1$ and $B\in\mathcal{E}_2$,

$$P(N(A,B)=n) = \frac{e^{-\varphi_1(A)\varphi_2(B)[\varphi_1(A)\varphi_2(B)]^n}}{n!},$$

and each $n\in\mathbb{N}\cup\{0\}\cup\{\infty\}$. In particular, $N(A,B)=\infty$, holds almost everywhere whenever $\varphi_1(A)=\infty$ or $\varphi_2(B)=\infty$. If $(E_1,\mathcal{E}_1,\varphi_1) = ([0,\infty)\times\mathbb{R},\mathcal{B}([0,\infty)\times\mathbb{R}),d\tau\times d\zeta)$, then the compensated random martingale measure is defined by

$$M(A,B,\tau) = N(\bar{J}\times A,B) - \varphi_1(\bar{J}\times A)\varphi_2(B).$$

Additionally, the following properties are true.

Lemma 2.1. *[23, 24] For each $(\tau,A,B)\in[0,\infty)\times\mathbb{R}\times\mathcal{E}_2$, $\varphi_1(\bar{J}\times A)\varphi_2(B)<\infty$. In addition, let $\phi: E_1\times E_2\times\Omega\to\mathbb{R}$ be a $\{\mathscr{F}_\tau\}_{\tau\geq 0}$-predictable function such that it satisfies:*

$$\mathbb{E}\left[\int_0^\tau \int_A \int_B |\phi(\tau,\zeta,\varrho)|^2\varphi_2(\,d\varrho)\,d\zeta\,d\varsigma\right] < \infty,\ \ \tau>0,\ \ (A,B)\in\mathcal{E}_1\times\mathcal{E}_2.$$

A stochastic integration procedure follows:

$$R_\tau = \int_A \int_B \phi(\varsigma,\zeta,\varrho)M(\,d\varrho,\,d\zeta,\,d\varsigma),$$

which is a square-integrable $\{\mathscr{F}_\tau\}_{\tau\geq 0}$-martingale.

$\dot{L}$ denotes a Lévy space-time white noise, which includes terms not only controlled by both Poisson and Gaussian space-time white noise. Consequently, the study will focus on noise incorporating a Lévy process:

Lemma 2.2. *[23, 24]*

$$\dot{L}(\zeta,\tau) = \int_{U_0} \omega_1(\tau,\zeta,\varrho)\dot{M}(\,d\varrho,\zeta,\tau) + \int_{E_2\setminus U_0} \omega_2(\tau,\zeta,\varrho)\dot{N}(\,d\varrho,\zeta,\tau),$$

for some $U_0\in E_2$ with $\varphi_2(E_2\setminus U_0)<\infty$ and $\int_{U_0} z^2\varphi_2(\,dz)<+\infty$. In this case, $\omega_1,\omega_2:[0,\infty)\times\mathbb{R}\times E_2\to\mathbb{R}$ are measurable functions; and $\dot{M}$ and $\dot{N}$ are Randon-Nikodym derivatives, also called the Poisson random measure, given by

$$\dot{M}(d\varrho,\zeta,\tau) = \frac{M(\,d\varrho,\,d\zeta,\,d\tau)}{d\tau\times\,d\zeta},$$

$$\dot{N}(d\varrho,\zeta,\tau)=\frac{N(\,d\varrho,\,d\zeta,\,d\tau)}{d\tau\times\,d\zeta},$$

where $(\tau,\zeta,\varrho)\in[0,\infty)\times E_2\times\mathbb{R}$.

Further, we give the following remark.

Remark 2.1.

$$\psi(\tau,\varrho)=\int_{E_2\setminus U_0}\omega_2(\tau,\varrho,z)\varphi_2(\,dz),$$
$$\omega(\tau,\varrho,z)=\omega_1(\tau,\varrho,z)I_{U_0}(z)+\omega_2(\tau,\varrho,z)I_{E_2\setminus U_0}(z),$$

with the set $A\in\mathcal{E}_2$ and its indicator $I_A(\cdot)$.

Next, we review some well-known characteristics of $G_\mu(\tau,\zeta)$.

Lemma 2.3. *[25, 26] Let* $\mu\in[0,2]$. *The transition density of a Lévy stable process is the function* $G_\mu(\tau,\zeta)$, *which satisfies:*

(a) *For all* $(\tau,\zeta)\in[0,\infty)\times\mathbb{R}$, $G_\mu(\tau,\zeta)\geq 0$, $\int_{\mathbb{R}}G_\mu(\tau,\zeta)\,d\zeta=1$.

(b) *For all* $(\tau,\zeta)\in[0,\infty)\times\mathbb{R}$, $G_\mu(\tau,\zeta)=\tau^{-\frac{1}{\mu}}G_\mu(1,\tau^{-\frac{1}{\mu}}\zeta)$.

(c) $\int_0^T\int_{\mathbb{R}}|G_\mu(\tau,\zeta)|^\gamma\,d\zeta\,d\tau<\infty$, $1/2<\gamma<1+\mu$.

Finally, we present the definition of the Burkholder-Davis-Gundy (B-D-G) inequality.

Lemma 2.4. *[24] Let* $\phi:[0,\infty)\times\mathbb{R}\times E_2\times\Omega\to\mathbb{R}$ *be a measurable function that satisfies Lemma 2.2. Define the integral process as follows:*

$$\left\{X_\tau=\int_0^{\tau^+}\int_{\mathbb{R}}\int_{E_2}\phi(\varsigma,\zeta,\varrho)M(d\varrho,d\zeta,d\varsigma),\tau\geq 0\right\},$$

and then, for any $T>0$, $p>0$, *there is a positive constant* $C_{p,T}$ *such that*

$$\sup_{\tau\in\bar{J}}\mathbb{E}\left(|X_\tau|^p\right)\leq C_{p,T}\left[\int_0^T\int_{\mathbb{R}}\int_{E_2}\left(\mathbb{E}\left(|\phi(\varsigma,\zeta,\varrho)|^p\right)\right)^{\frac{p}{2}}\mu_2(d\varrho)d\zeta d\varsigma\right]^{\frac{p}{2}}.$$

3. Existence and uniqueness results

This section will proof the existence and uniqueness of the mild solutions.

The following equation is a reformulation of Eq (1.1), which makes sense according to Walsh [27]:

$$\begin{aligned}v_\epsilon(\tau,\zeta)=&\int_{\mathbb{R}}G_\mu(\tau,\zeta-\varrho)v_0(\varrho)\,d\varrho\\&+\sqrt{\epsilon}\int_0^\tau\int_{\mathbb{R}}G_\mu(\tau-\varsigma,\zeta-\varrho)\sigma(\varsigma,\varrho)B^H(\,d\varsigma,\,d\varrho)\\&+\epsilon\int_0^\tau\int_{\mathbb{R}}G_\mu(\tau-\varsigma,\zeta-\varrho)f(\varsigma,\varrho,v_\epsilon)\,d\varrho\,d\varsigma\\&+\sqrt{\epsilon}\int_0^\tau\int_{\mathbb{R}}G_\mu(\tau-\varsigma,\zeta-\varrho)g(\varsigma,\varrho,v_\epsilon)\dot{L}(\varrho,\varsigma)\,d\varrho\,d\varsigma.\end{aligned}\tag{3.1}$$

Then, by Lemma 2.2 and Remark 2.1, we can further get

$$\begin{aligned} v_\epsilon(\tau,\zeta) = &\int_{\mathbb{R}} G_\mu(\tau,\zeta-\varrho)v_0(\varrho)\,d\varrho \\ &+ \sqrt{\epsilon}\int_0^\tau\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\sigma(\varsigma,\varrho)B^H(\,d\varsigma,\,d\varrho) \\ &+ \epsilon\int_0^\tau\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)f(\varsigma,\varrho,v_\epsilon)\,d\varrho\,d\varsigma \\ &+ \sqrt{\epsilon}\int_0^\tau\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)g(\varsigma,\varrho,v_\epsilon)\psi(\varsigma,\varrho)\,d\varrho\,d\varsigma \\ &+ \sqrt{\epsilon}\int_0^{\tau^+}\int_{\mathbb{R}}\int_{E_2} G_\mu(\tau-\varsigma,\zeta-\varrho)g(\varsigma,\varrho,v_\epsilon)\omega(\varsigma,\varrho,z)M(\,dz,\,d\varrho,\,d\varsigma). \end{aligned} \tag{3.2}$$

In Eq (3.2), the term with B^H can be rearranged as follows:

$$\int_0^\tau\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\sigma(\varsigma,\varrho)B^H(\,d\varsigma,\,d\varrho) = \int_0^\tau\int_{\mathbb{R}}(\mathcal{K}_H^* G_\mu)(\tau-\varsigma,\zeta-\varrho)\sigma(\varsigma,\varrho)W(\,d\varsigma,\,d\varrho), \tag{3.3}$$

utilizing the space-time white noise that is discussed in Section 2.

We also assume:

(H_1) For $(\tau,\zeta)\in\bar{J}\times\mathbb{R}$ and $v_\epsilon,u_\epsilon\in\mathbb{R}$, a positive constant L_1 exists such that:

$$|f(\tau,\zeta,v_\epsilon)-f(\tau,\zeta,u_\epsilon)|^p \vee |g(\tau,\zeta,v_\epsilon)-g(\tau,\zeta,u_\epsilon)|^p \le L_1|v_\epsilon-u_\epsilon|^p.$$

(H_2) There is a constant C_p such that:

$$|f(\tau,\zeta,v_\epsilon)|^p \vee |g(\tau,\zeta,v_\epsilon)|^p \le C_p(1+|v_\epsilon|^p),$$

for any $v_\epsilon\in\mathbb{R}$.

(H_3) For $p\in\left(\frac{2(\mu+1)}{\mu-1},+\infty\right)$ with $\mu\in(1,2)$, we have

$$\begin{aligned} &\sup_{\tau\in\bar{J}}\|\psi(\tau,\varsigma)\|_p^p<\infty, \\ &\sup_{\tau\in\bar{J}}\|\sigma(\tau,\varsigma)\|_p^p<\infty, \\ &\sup_{\tau\in\bar{J}}\left\|\int_{E_2}|\omega(\tau,\varrho,z)|^2\varphi_2(\,dz)\right\|_{\frac{p}{2}}^{\frac{p}{2}}<\infty, \\ &\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^p\,d\varrho \le C_{\mu,H}(\tau-\varsigma)^{-\frac{p}{\mu}+\frac{1}{\mu}}. \end{aligned}$$

The following lemmas and propositions are necessary in order to demonstrate Theorem 3.1.

Lemma 3.1. *[23, 28] If $H>\frac{1}{2}$, then*

$$L^{\frac{1}{H}}(\bar{J}\times\mathbb{R})\subset\mathscr{H},$$

where $\mathscr{H}$ is a Hilbert space.

Let $\mathbb{B}$ denote the space of all $\mathscr{F}_\tau$-adapted processes $\{v_\epsilon(\tau,\cdot)\}_{\tau\in\bar{J}}$, valued in $L^p(\mathbb{R})$. The norm in this domain is defined as

$$\|v_\epsilon\|_{\mathbb{B}} := \left[\sup_{\tau\in\bar{J}} e^{-\eta\tau}\mathbb{E}\left[\|v_\epsilon(\tau,\cdot)\|_p^p\right]\right]^{\frac{1}{p}}, \eta > 0, \tag{3.4}$$

where $\|\cdot\|_p$ is the standard norm on $L^p(\mathbb{R})$. Obviously, $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a Banach space. Presently, we give a bounded, closed, and convex subset $\hat{\mathbb{B}} \subseteq \mathbb{B}$. Additionally, for each $v_\epsilon \in \mathbb{B}$, we present an operator represented by S_μ:

$$S_\mu(v_\epsilon)(\tau,\zeta) = \sum_{i=1}^{5} \mathfrak{I}_\mu^i(v_\epsilon)(\tau,\zeta),$$

where

$$\begin{aligned}
\mathfrak{I}_\mu^1(v_\epsilon)(\tau,\zeta) &:= \int_{\mathbb{R}} G_\mu(\tau,\zeta-\varrho)u_0(\varrho)\,d\varrho,\\
\mathfrak{I}_\mu^2(v_\epsilon)(\tau,\zeta) &:= \epsilon\int_0^\tau\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)f(\varsigma,\varrho,v_\epsilon)\,d\varrho\,d\varsigma,\\
\mathfrak{I}_\mu^3(v_\epsilon)(\tau,\zeta) &:= \sqrt{\epsilon}\int_0^\tau\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\sigma(\varsigma,\varrho)B^H(\,d\varsigma,\,d\varrho),\\
\mathfrak{I}_\mu^4(v_\epsilon)(\tau,\zeta) &:= \sqrt{\epsilon}\int_0^\tau\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)g(\varsigma,\varrho,v_\epsilon)\psi(\varsigma,\varrho)\,d\varrho\,d\varsigma,\\
\mathfrak{I}_\mu^5(v_\epsilon)(\tau,\zeta) &:= \sqrt{\epsilon}\int_0^{\tau^+}\int_{\mathbb{R}}\int_{E_2} G_\mu(\tau-\varsigma,\zeta-\varrho)g(\varsigma,\varrho,v_\epsilon)\omega(\varsigma,\varrho,z)M(\,dz,\,d\varrho,\,d\varsigma).
\end{aligned}$$

Proposition 3.1. *Assume* (H_1)–(H_3) *are satisfied. Let* $\sigma \in L^{\frac{2}{2H-1}}(\bar{J}\times\mathbb{R}) \subset L^2(\bar{J}\times\mathbb{R})$ *when* $\frac{1}{2} < H < 1$. *Then for each* $p > \frac{2(\mu+1)}{\mu-1}$ *and* $v_\epsilon \in \hat{\mathbb{B}}$, *it holds that* $S_\mu(v_\epsilon) \in \hat{\mathbb{B}}$.

Proof. Applying Lemma 2.3 and Young's inequality, we have

$$\begin{aligned}
\|\mathfrak{I}_\mu^1(v_\epsilon)(\tau,\zeta)\|_p &\le \tau^{-\frac{1}{\mu}}\left\|\int_{\mathbb{R}} G_\mu(1,\tau^{-\frac{1}{\mu}}(\cdot,-\varrho))v_0(\varrho)\,d\varrho\right\|_p\\
&\le \tau^{-\frac{1}{\mu}}\left\|\left[G_\mu(1,\tau^{-\frac{1}{\mu}}\cdot)*v_0(\cdot)\right](\cdot)\right\|_p\\
&\le \tau^{-\frac{1}{\mu}}\left\|G_\mu(1,\tau^{-\frac{1}{\mu}}\cdot)\right\|_1\cdot\|v_0(\cdot)\|_p\\
&\le C\,\|v_0(\cdot)\|_p < \infty,
\end{aligned}$$

for $\mathbb{E}\left[\|v_0(\cdot)\|_p^p\right] < \infty$.

In view of Lemma 3.1 in [19], $\frac{1}{r} = \frac{1}{p} - \frac{1}{p} + 1 = 1$, and hypothesis (H_2), we have

$$\begin{aligned}
\mathbb{E}\left[\|\mathfrak{I}_\mu^2(v_\epsilon)(\tau,\zeta)\|_p^p\right] &\le C\epsilon^p\mathbb{E}\left[\int_0^\tau(\tau-\varsigma)^{-\frac{1-r}{\mu r}}\|f(\varsigma,\varrho,v_\epsilon(\varsigma,\varrho))\|_p\,d\varsigma\right]^p\\
&\le C_{p,T}\epsilon^p\mathbb{E}\left[\int_0^\tau(1+\|v_\epsilon(\varsigma,\varrho)\|_p)\,d\varsigma\right]^p
\end{aligned}$$

$$\leq C_{p,T}\epsilon^p \left[1 + \sup_{\tau\in\bar{J}} \mathbb{E}\|v_\epsilon(\varsigma,\varrho)\|_p^p\right]$$
$$\leq C_{p,T}\epsilon^p \left[1 + \|v_\epsilon(\varsigma,\cdot)\|_{\mathbb{B}}^p\right] < \infty,$$

since $v_\epsilon \in \mathbb{B}$.

We shall subsequently analyze $\mathfrak{I}_\mu^3(v_\epsilon)$. It can be deduced from Eq (3.3) that

$$\begin{aligned}
&\mathbb{E}\left[\|\mathfrak{I}_\mu^3(v_\epsilon)(\tau,\zeta)\|_p^p\right] \\
&= \epsilon^{\frac{p}{2}} \int_{\mathbb{R}} \mathbb{E}\left|\int_0^\tau \int_{\mathbb{R}} (\mathcal{K}_H^* G_\mu(\tau-\varsigma,\zeta-\varrho))\sigma(\varsigma,\varrho)W(\,d\varrho,\,d\varsigma)\right|^p d\zeta \\
&\leq C_p\epsilon^{\frac{p}{2}} \int_{\mathbb{R}} \left\langle \mathcal{K}_H^* G_\mu\sigma(\tau-\varsigma,\zeta-\varrho), \mathcal{K}_H^* G_\mu\sigma(\tau-\varsigma,\zeta-\varrho)\right\rangle_{L^2(\bar{J}\times\mathbb{R})}^{\frac{p}{2}} d\zeta \\
&\leq C_p\epsilon^{\frac{p}{2}} \int_{\mathbb{R}} \left\|G_\mu\sigma(\tau-\varsigma,\zeta-\varrho)\right\|_{L^{\frac{1}{H}}(\bar{J}\times\mathbb{R})}^p d\zeta,
\end{aligned}$$

since when $H > \frac{1}{2}$, $L^{\frac{1}{H}}(\bar{J}\times\mathbb{R}) \in \mathscr{H}$. The following may be obtained from hypothesis (H_3) and the Hölder inequality:

$$\begin{aligned}
&\left\|G_\mu\sigma(\tau-\varsigma,\zeta-\varrho)\right\|_{L^{\frac{1}{H}}(\bar{J}\times\mathbb{R})}^p \\
&= \left[\int_0^\tau \int_{\mathbb{R}} |G_\mu(\tau-\varsigma,\zeta-\varrho)\sigma(\varsigma,\varrho)|^{\frac{1}{H}}\,d\varrho\,d\varsigma\right]^{pH} \\
&\leq \left[\int_0^\tau \int_{\mathbb{R}} |G_\mu(\tau-\varsigma,\zeta-\varrho)|^{\frac{1}{H}}|\sigma(\varsigma,\varrho)|^{\frac{1}{H}}\,d\varrho\,d\varsigma\right]^{pH} \\
&\leq \left[\int_0^\tau \left(\int_{\mathbb{R}} |G_\mu(\tau-\varsigma,\zeta-\varrho)|^{\frac{1}{H}\cdot 2H}\,d\varrho\right)^{\frac{1}{2H}} \left(\int_{\mathbb{R}} |\sigma(\tau-\varsigma,\zeta-\varrho)|^{\frac{2}{2H-1}}\,d\varrho\right)^{\frac{2H-1}{2H}} d\varsigma\right]^{pH} \\
&\leq \left[C_{\mu,p,H} \int_0^\tau (\tau-\varsigma)^{-\frac{1}{\mu H}} \left(\int_{\mathbb{R}} |\sigma(\tau-\varsigma,\zeta-\varrho)|^{\frac{2}{2H-1}}\,d\varrho\right)^{\frac{2H-1}{2H}} d\varsigma\right]^{pH} \\
&\leq C_{\mu,p,H} \left(\int_0^\tau (\tau-\varsigma)^{-\frac{1}{\mu H}\cdot 2H}\,d\varsigma\right)^{\frac{p}{2}} \left(\int_0^\tau \int_{\mathbb{R}} |\sigma(\tau-\varsigma,\zeta-\varrho)|^{\frac{2}{2H-1}}\,d\varrho\,d\varsigma\right)^{\frac{2H-1}{2}\cdot p} \\
&\leq C_{\mu,p,T,H}\,\|\sigma(\varsigma,\varrho)\|_{L^2(\bar{J}\times\mathbb{R})}^p < \infty.
\end{aligned}$$

The following fact can be used: $\sigma \in L^{\frac{2}{2H-1}}(\bar{J}\times\mathbb{R}) \subset L^2(\bar{J}\times\mathbb{R})$ for $H > \frac{1}{2}$. Assuming $1 - \frac{1}{\mu} > 0$, we conclude that $\mathfrak{I}_\mu^3(v_\epsilon)(\tau,\zeta) \in \hat{\mathbb{B}}$ when $p \in [2,\infty)$.

As for $\mathfrak{I}_\mu^4(v_\epsilon)$, when $\frac{1}{r} = \frac{1}{p} - \frac{2}{p} + 1 = 1 - \frac{1}{p} \in (0,1]$, and for $p \in [2,\infty)$, hypothesis (H_3) and Lemma 3.1 in [19] imply

$$\begin{aligned}
&\mathbb{E}\left[\|\mathfrak{I}_\mu^4(v_\epsilon)(\tau,\zeta)\|_p^p\right] \\
&\leq C\epsilon^{\frac{p}{2}}\mathbb{E}\left[\int_0^\tau (\tau-\varsigma)^{-\frac{1}{\mu}(1-\frac{1}{r})}\|1+|v_\epsilon(\varsigma,\varrho)|\|_p \cdot \|\psi\|_p\,d\varsigma\right]^p
\end{aligned}$$

$$\leq C_p\epsilon^{\frac{p}{2}}\left[1+\|v_\epsilon(\cdot)\|_{\mathbb{B}}^p\right]\sup_{\tau\in\bar{J}}\|\psi(\tau,\cdot)\|_p^p\left[\int_0^\tau(\tau-\varsigma)^{-\frac{1}{\mu}(1-\frac{1}{r})\frac{p}{p-1}}\,d\varsigma\right]^{p-1}$$
$$\leq C_{p,T}\epsilon^{\frac{p}{2}}\sup_{\tau\in\bar{J}}\|\psi(\tau,\varsigma)\|_p^p\left(1+\|v_\epsilon(\cdot)\|_{\mathbb{B}}^p\right)<\infty,$$

where $1-\frac{1}{\mu(p-1)}>0$, and $p>1+\frac{1}{\mu}$.

For $\mathfrak{I}_\mu^5(v_\epsilon)$, by virtue of Lemma 3.1 in [19] and the hypotheses (H_2)–(H_3) with $\frac{1}{r}=\frac{1}{p}-\frac{2}{p}+1=1-\frac{1}{p}\in(0,1]$ and the B-D-G inequality, for $p>\frac{2(\mu+1)}{\mu-1}$,

$$\mathbb{E}\left[\|\mathfrak{I}_\mu^5(v_\epsilon)(\tau,\zeta)\|_p^p\right]$$
$$\leq C_p\epsilon^{\frac{p}{2}}\int_{\mathbb{R}}\left[\int_0^\tau\int_{\mathbb{R}}\int_{E_2}|G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)|^2\left(\mathbb{E}[1+|v_\epsilon(\varsigma,\varrho)|^p]\right)^{\frac{2}{p}}\varphi_2(dz)\,d\varrho\,d\varsigma\right]^{\frac{p}{2}}d\zeta$$
$$\leq C_p\epsilon^{\frac{p}{2}}\left[\int_0^\tau(\tau-\varsigma)^{-\frac{p+2}{\mu p}}\left\|\left[\int_{E_2}|\omega(\varsigma,\varrho,z)|^2\varphi_2(dz)\right][1+\mathbb{E}|v_\epsilon(\varsigma,\varrho)|^p]^{\frac{2}{p}}\right\|_{\frac{p}{4}}\right]^{\frac{p}{2}}$$
$$\leq C_p\epsilon^{\frac{p}{2}}\left[\int_0^\tau(\tau-\varsigma)^{-\frac{p+2}{\mu p}}\left\|\int_{E_2}|\omega(\varsigma,\varrho,z)|^2\varphi_2(dz)\right\|_{\frac{p}{2}}\left\|1+\mathbb{E}|v_\epsilon(\varsigma,\varrho)|^p]^{\frac{2}{p}}\right\|_{\frac{p}{2}}d\varsigma\right]^{\frac{p}{2}}$$
$$\leq C_p\epsilon^{\frac{p}{2}}\left[\sup_{\tau\in\bar{J}}\left\|\int_{E_2}|\omega(\varsigma,\varrho,z)|^2\varphi_2(dz)\right\|_{\frac{p}{2}}^{\frac{p}{2}}\cdot\sup_{\tau\in\bar{J}}\left\|1+\mathbb{E}|v_\epsilon(\varsigma,\varrho)|^p]^{\frac{2}{p}}\right\|_{\frac{p}{2}}^{\frac{p}{2}}\right]\left[\int_0^\tau(\tau-\varsigma)^{-\frac{p+2}{\mu(p-2)}}d\varsigma\right]^{\frac{p-2}{p}}$$
$$\leq C_{p,T}\epsilon^{\frac{p}{2}}\left[\sup_{\tau\in\bar{J}}\left\|\int_{E_2}|\omega(\varsigma,\varrho,z)|^2\varphi_2(dz)\right\|_{\frac{p}{2}}^{\frac{p}{2}}\right]\left(1+\|v_\epsilon(\cdot)\|_{\mathbb{B}}^p\right)<\infty.$$

At this point, the concept that S_μ is an operator from $\hat{\mathbb{B}}$ to itself has been proved. This establishes the result. □

The next step is to demonstrate that $S_\mu:\hat{\mathbb{B}}\mapsto\hat{\mathbb{B}}$ is a contract operator.

Proposition 3.2. *In accordance with hypotheses* (H_1)–(H_3)*, the operator S_μ represents a contraction on $\hat{\mathbb{B}}$. Accordingly, there must be a constant $\delta\in(0,1)$ such that*

$$\|S_\mu(v_\epsilon)-S_\mu(u_\epsilon)\|_{\mathbb{B}}\leq\delta\|v_\epsilon-u_\epsilon\|_{\mathbb{B}},\quad for\quad v_\epsilon,u_\epsilon\in\hat{\mathbb{B}}.$$

Proof. We will address each component of the operator S_μ individually.

First, let v_0,u_0 be the initial values of the $\{\mathscr{F}_\tau\}_{\tau\geq0}$-adapted random fields $v_\epsilon,u_\epsilon\in\hat{\mathbb{B}}$ with the condition that $v_0=u_0$, and it is easy to get that $\left\|\mathfrak{I}_\mu^1(v_\epsilon)(\tau,\varsigma)-\mathfrak{I}_\mu^1(u_\epsilon)(\tau,\varsigma)\right\|_{\mathbb{B}}^p\leq0$.

Consider $\mathfrak{I}_\mu^2(v_\epsilon)$, assuming that the hypotheses (H_1)–(H_3) are fulfilled. Through Lemma 3.1 in [19], note $\frac{1}{r}=\frac{1}{p}-\frac{3}{p}+1=1-\frac{2}{p}$, and we first have

$$\mathbb{E}\left[\left\|\mathfrak{I}_\mu^2(v_\epsilon)(\tau,\varsigma)-\mathfrak{I}_\mu^2(u_\epsilon)(\tau,\varsigma)\right\|_p^p\right]$$
$$\leq C\mathbb{E}\left[\int_0^\tau(\tau-\varsigma)^{-\frac{1-r}{\mu}}\|f(\varsigma,\varrho,v_\epsilon)-f(\varsigma,\varrho,u_\epsilon)\|_p\,d\varsigma\right]^p$$

$$\leq C_p\left[\int_0^\tau (\tau-\varsigma)^{-\frac{1-r}{\mu}}\mathbb{E}\|v_\epsilon-u_\epsilon\|_p\,d\varsigma\right]^p.$$

Then, consider the norm on $\mathbb{B}$, and we can get

$$\begin{aligned}
&\left\|\mathfrak{I}_\mu^2(v_\epsilon)(\tau,\varsigma)-\mathfrak{I}_\mu^2(u_\epsilon)(\tau,\varsigma)\right\|_{\mathbb{B}}^p\\
&=\sup_{\tau\in\bar{J}} e^{-\eta\tau}\mathbb{E}\left[\left\|\mathfrak{I}_\mu^2(v_\epsilon)(\tau,\varsigma)-\mathfrak{I}_\mu^2(u_\epsilon)(\tau,\varsigma)\right\|_p^p\right]\\
&\leq C\sup_{\tau\in\bar{J}}\mathbb{E}\left[\int_0^\tau e^{-\frac{\eta(\tau-\varsigma)}{p}}(\tau-\varsigma)^{-\frac{1-r}{\mu}}e^{-\frac{\eta\varsigma}{p}}\|v_\epsilon(\varsigma,\varrho)-u_\epsilon(\varsigma,\varrho)\|_p\,d\varsigma\right]^p\\
&\leq C\sup_{\tau\in\bar{J}}\left[\int_0^\tau e^{-\eta\varsigma}\mathbb{E}\|v_\epsilon(\varsigma,\cdot)-u_\epsilon(\varsigma,\cdot)\|_p^p\,d\varsigma\right]\left[\int_0^\tau\left(e^{-\frac{\eta(\tau-\varsigma)}{p}}(\tau-\varsigma)^{-\frac{1-r}{\mu}}\right)^{\frac{p}{p-1}}\,d\varsigma\right]^{p-1}\\
&=C_pT\vartheta(p,\tau)\|v_\epsilon-u_\epsilon\|_{\mathbb{B}}^p,
\end{aligned}$$

where

$$\vartheta(p,\tau)=\left[\int_0^\tau\left(e^{-\frac{\eta(\tau-\varsigma)}{p}}(\tau-\varsigma)^{-\frac{1-r}{\mu}}\right)^{\frac{p}{p-1}}\,d\varsigma\right]^{p-1},$$

which yields from $m=\frac{p(1-r)}{\mu(p-1)}$ and $p>1$,

$$\begin{aligned}
\vartheta(p,\tau)&\leq\left[\int_0^\tau\left(e^{-\frac{\eta(\tau-\varsigma)}{p}}(\tau-\varsigma)^{-\frac{1-r}{\mu}}\right)^{\frac{p}{p-1}}\,d\varsigma\right]^{p-1}\\
&=\left[\frac{(p-1)^{m+1}}{\eta^{m+1}}\int_0^{+\infty}e^{-\zeta}\zeta^m\,d\zeta\right]^{p-1}\\
&=\left[\frac{(p-1)^{m+1}}{\eta^{m+1}}\Gamma(m+1)\right]^{p-1}.
\end{aligned}$$

Then, one finds

$$\begin{aligned}
&\left\|\mathfrak{I}_\mu^2(v_\epsilon)(\tau,\varsigma)-\mathfrak{I}_\mu^2(u_\epsilon)(\tau,\varsigma)\right\|_{\mathbb{B}}\\
&\quad\leq C_pT^{\frac{1}{p}}\left[\frac{(p-1)^{m+1}}{\eta^{m+1}}\Gamma(m+1)\right]^{p-1}\|v_\epsilon-u_\epsilon\|_{\mathbb{B}}\\
&\quad\leq\delta\|v_\epsilon-u_\epsilon\|_{\mathbb{B}},
\end{aligned}$$

where $\delta\in(0,1)$ is achieved by choosing $\eta>0$ that is large enough.

In the sequel, consider $\mathfrak{I}_\mu^5(v_\epsilon)$. Similarly, use the B-D-G inequality again, and we have the following conclusion:

$$\begin{aligned}
&\left\|\mathfrak{I}_\mu^5(v_\epsilon)(\tau,\varsigma)-\mathfrak{I}_\mu^5(u_\epsilon)(\tau,\varsigma)\right\|_{\mathbb{B}}^p\\
&=\sup_{\tau\in\bar{J}} e^{-\eta\tau}\mathbb{E}\left[\left\|\mathfrak{I}_\mu^5(v_\epsilon)(\tau,\cdot)-\mathfrak{I}_\mu^5(u_\epsilon)(\tau,\cdot)\right\|_p^p\right]\\
&\leq C_p\epsilon^{\frac{p}{2}}\sup_{\tau\in\bar{J}} e^{-\eta t}\int_{\mathbb{R}}\left(\int_0^\tau\int_{\mathbb{R}}\int_{E_2}(\mathbb{E}|G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)\right.
\end{aligned}$$

$$
\begin{aligned}
&\times [g(\varsigma,\varrho,v_\epsilon)-g(\varsigma,\varrho,u_\epsilon)]|^p)^{\frac{2}{p}}\varphi_2(dz)\,d\varrho\,d\varsigma\Big)^{\frac{p}{2}}\ d\zeta\\
&\le C_p\epsilon^{\frac{p}{2}}\sup_{\tau\in\bar J}e^{-\eta\tau}\int_{\mathbb{R}}\Big(\int_0^\tau\int_{\mathbb{R}}\int_{E_2}|G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)|^2e^{\frac{2\eta\varsigma}{p}}\\
&\quad\times(e^{-\eta\varsigma}\mathbb{E}|v_\epsilon-u_\epsilon|^p)^{\frac{2}{p}}\varphi_2(dz)\,d\varrho\,d\varsigma\Big)^{\frac{p}{2}}\ d\zeta\\
&\le C_p\epsilon^{\frac{p}{2}}\sup_{\tau\in\bar J}\int_{\mathbb{R}}\Big(\int_0^\tau\int_{\mathbb{R}}\int_{E_2}e^{-\eta\varsigma}\mathbb{E}|v_\epsilon-u_\epsilon|^p\varphi_2(dz)\,d\varrho\,d\varsigma\Big)\\
&\quad\times\Big(\int_0^\tau\int_{\mathbb{R}}\int_{E_2}|G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)|^{\frac{2p}{p-2}}e^{-\frac{2\eta(\tau-\varsigma)}{p-2}}\varphi_2(dz)\,d\varrho\,d\varsigma\Big)^{\frac{p-2}{p}}\ d\zeta\\
&\le C_{p,T}\epsilon^{\frac{p}{2}}\varphi_2(E_2)\|v_\epsilon-u_\epsilon\|_{\mathbb{B}}^p\\
&\quad\times\sup_{\tau\in\bar J}\int_{\mathbb{R}}\int_0^\tau\int_{\mathbb{R}}\int_{E_2}|G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)|^pe^{-\eta(\tau-\varsigma)}\varphi_2(dz)\,d\varrho\,d\varsigma\,d\zeta\\
&\le\delta\|v_\epsilon-u_\epsilon\|_{\mathbb{B}}^p,
\end{aligned}
$$

where $\delta\in(0,1)$ is obtained by selecting a large enough $\eta>0$.

Since $\mathfrak{I}_\mu^3$ and $\mathfrak{I}_\mu^4$ are both contractions on $\hat{\mathbb{B}}$, it is evident that they can similarly arrive at the same conclusion. □

Based on the previously mentioned analysis, we deduce the following conclusion.

Theorem 3.1. *Assuming conditions* (H$_1$)–(H$_3$) *hold. Eq (1.1) has a unique mild solution* $(v_\epsilon(\tau,\zeta))_{(\tau,\zeta)\in\bar J\times\mathbb{R}}$. *For each* $p\in\left(\frac{2(\mu+1)}{\mu-1},+\infty\right)$,

$$\sup_{\tau\in\bar J}\|v_\epsilon(\tau,\cdot)\|_p^p<\infty.$$

Proof. Because Propositions 3.1 and 3.2 hold on set $\{v_\epsilon\in\hat{\mathbb{B}}: v(0)=v_0\}$, we can determine that Eq (1.1) has a unique solution $v_\epsilon\in\hat{\mathbb{B}}$ by applying the fixed point theorem. □

4. The averaging principle

This section provides a detailed derivation demonstrating that the process weakly converges to its limiting behavior as the scale parameter ϵ tends to zero.

We aim to develop an averaging equation for approximating purposes. The solution $\overline{v}$ can be written as

$$
\begin{aligned}
\overline{v}(\tau,\zeta)&=\int_{\mathbb{R}}G_\mu(\tau,\zeta-\varrho)v_0(\varrho)\,d\varrho\\
&\quad+\sqrt{\epsilon}\int_0^\tau\int_{\mathbb{R}}G_\mu(\tau-\varsigma,\zeta-\varrho)\overline{\sigma}(\varrho)B^H(d\varsigma,\,d\varrho)\\
&\quad+\epsilon\int_0^\tau\int_{\mathbb{R}}G_\mu(\tau-\varsigma,\zeta-\varrho)\overline{f}(\varrho,\overline{v})\,d\varrho\,d\varsigma\\
&\quad+\sqrt{\epsilon}\int_0^\tau\int_{\mathbb{R}}\overline{g}(\varrho,\overline{v})\dot{L}(\varrho,\varsigma)\,d\varrho\,d\varsigma,
\end{aligned}
\tag{4.1}
$$

where

$$\overline{\sigma}(\varrho) = \frac{1}{T}\int_0^T \sigma(\varsigma,\varrho)\, d\varsigma,$$
$$\overline{f}(\varrho,\overline{v}) = \frac{1}{T}\int_0^T f(\varsigma,\varrho,\overline{v})\, d\varsigma,$$
$$\overline{g}(\varrho,\overline{v}) = \frac{1}{T}\int_0^T g(\varsigma,\varrho,\overline{v})\, d\varsigma.$$

(H_4) There exist positive bounded functions $K_i(t)$ for any $\tau \in \bar{J}, \zeta \in \mathbb{R}$, with $i = 1, 2, 3$, ensuring that the functions $\overline{\sigma}, \overline{f}, \overline{g}$ exhibit specific properties:

$$\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}} |\sigma(\varsigma,\varrho) - \overline{\sigma}(\varrho)|^p\, d\varrho\, d\varsigma \le K_1(\tau),$$
$$\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}} |f(\varsigma,\varrho,\zeta_1) - \overline{f}(\varrho,\zeta_1)|^p\, d\varrho\, d\varsigma \le K_2(\tau)(1 + |\zeta_1|^p),$$
$$\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}} |g(\varsigma,\varrho,\zeta_1) - \overline{g}(\varrho,\zeta_1)|^p\, d\varrho\, d\varsigma \le K_3(\tau)(1 + |\zeta_1|^p),$$

when $\tau \to \infty$, $K_i(\tau) \to 0$.

Now, we present our main result.

Theorem 4.1. *Given that* (H_1)–(H_4) *hold, for each* $p \in \left(\frac{2(\mu+1)}{\mu-1}, +\infty\right)$, *there is*

$$\lim_{\epsilon\to 0} \sup_{\tau\in\bar{J}} \mathbb{E}|v_\epsilon(\tau,\zeta) - \overline{v}(\tau,\zeta)|^p = 0.$$

Proof. Making use of Minkowski's inequality, one can get

$$\begin{aligned}
&\sup_{\tau\in\bar{J}}\mathbb{E}|v_\epsilon(\tau,\zeta) - \overline{v}(\tau,\zeta)|^p \\
&= \sup_{\tau\in\bar{J}} \mathbb{E}\Big| \sqrt{\epsilon}\int_0^\tau \int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho) - \overline{\sigma}(\varrho))\, B^H(\,d\varsigma,\, d\varrho) \\
&\quad + \epsilon \int_0^\tau \int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\left(f(\varsigma,\varrho,v_\epsilon) - \overline{f}(\varsigma,\varrho,\overline{v})\right) d\varrho\, d\varsigma \\
&\quad + \sqrt{\epsilon}\int_0^\tau \int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\,(g(\varsigma,\varrho,v_\epsilon) - \overline{g}(\varsigma,\varrho,\overline{v}))\,\psi(\varsigma,\varrho)\, d\varrho\, d\varsigma \\
&\quad + \sqrt{\epsilon}\int_0^{\tau^+} \int_{\mathbb{R}}\int_{E_2} G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)\,(g(\varsigma,\varrho,v_\epsilon) - \overline{g}(\varsigma,\varrho,\overline{v}))\, M(\,dz,\, d\varrho,\, d\varsigma)|^p \\
&\le 4^{p-1}\epsilon^{\frac{p}{2}} \sup_{\tau\in\bar{J}} \mathbb{E}\left|\int_0^\tau \int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho) - \overline{\sigma}(\varrho))\, B^H(\,d\varsigma,\, d\varrho)\right|^p \\
&\quad + 4^{p-1}\epsilon^p \sup_{\tau\in\bar{J}} \mathbb{E}\left|\int_0^\tau \int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\left(f(\varsigma,\varrho,v_\epsilon) - \overline{f}(\varsigma,\varrho,\overline{v})\right) d\varrho\, d\varsigma\right|^p \\
&\quad + 4^{p-1}\epsilon^{\frac{p}{2}} \sup_{\tau\in\bar{J}} \mathbb{E}\left|\int_0^\tau \int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\,(g(\varsigma,\varrho,v_\epsilon) - \overline{g}(\varsigma,\varrho,\overline{v}))\,\psi(\varsigma,\varrho)\, d\varrho\, d\varsigma\right|^p
\end{aligned}$$

$$
\begin{aligned}
&+ 4^{p-1}\epsilon^{\frac{p}{2}} \sup_{\tau\in\bar{J}} \mathbb{E}\left|\int_0^{\tau^+}\int_{\mathbb{R}}\int_{E_2} G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)\right.\\
&\times (g(\varsigma,\varrho,v_\epsilon)-\overline{g}(\varsigma,\varrho,\overline{v}))\, M(\,dz,\,d\varrho,\,d\varsigma)|^p\\
&=: 4^{p-1}\epsilon^{\frac{p}{2}} I_1 + 4^{p-1}\epsilon^p I_2 + 4^{p-1}\epsilon^{\frac{p}{2}} I_3 + 4^{p-1}\epsilon^{\frac{p}{2}} I_4.
\end{aligned}
$$

Now, we calculate every term in the aforementioned equation independently.

For the term I_1, adopting the method in Proposition 3.1, and using Hölder's inequality, the following estimation can be obtained:

$$
\begin{aligned}
I_1 &= \sup_{\tau\in\bar{J}} \mathbb{E}\left|\int_0^{\tau}\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))\, B^H(\,d\varsigma,\,d\varrho)\right|^p\\
&= \sup_{\tau\in\bar{J}} \mathbb{E}\left|\int_0^{\tau}\int_{\mathbb{R}} \mathcal{K}^* G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))\, W(\,d\varsigma,\,d\varrho)\right|^p\\
&\leq \sup_{\tau\in\bar{J}} C_p \left\langle \mathcal{K}^* G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))\,,\mathcal{K}^* G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))\right\rangle^{\frac{p}{2}}_{L^2(\bar{J}\times\mathbb{R})}\\
&= \sup_{\tau\in\bar{J}} C_p \left\langle G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))\,, G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))\right\rangle^{\frac{p}{2}}_{\mathcal{H}}\\
&\leq \sup_{\tau\in\bar{J}} C_p \left\| G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))\right\|^p_{L^{\frac{1}{H}}(\bar{J}\times\mathbb{R})}\\
&= \sup_{\tau\in\bar{J}} C_p \left(\int_0^{\tau}\int_{\mathbb{R}} \left|G_\mu(\tau-\varsigma,\zeta-\varrho)\,(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))\right|^{\frac{1}{H}}\,d\varrho\,d\varsigma\right)^{pH}\\
&\leq \sup_{\tau\in\bar{J}} C_p \left(\int_0^{\tau}\int_{\mathbb{R}} \left|G_\mu(\tau-\varsigma,\zeta-\varrho)\right|^{\frac{1}{H}} |(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))|^{\frac{1}{H}}\,d\varrho\,d\varsigma\right)^{pH}\\
&\leq C_p T^{1-pH} \sup_{\tau\in\bar{J}} \int_0^{\tau}\left(\int_{\mathbb{R}} \left|G_\mu(\tau-\varsigma,\zeta-\varrho)\right|^{\frac{1}{H}} |(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))|^{\frac{1}{H}}\,d\varrho\right)^{pH}\,d\varsigma\\
&\leq C_p T^{1-pH} \sup_{\tau\in\bar{J}} \int_0^{\tau}\left(\int_{\mathbb{R}} |(\sigma(\varsigma,\varrho)-\overline{\sigma}(\varrho))|^p\,d\varrho\right)\left(\int_{\mathbb{R}} \left|G_\mu(\tau-\varsigma,\zeta-\varrho)\right|^{\frac{p}{pH-1}}\,d\varrho\right)^{pH-1}\,d\varsigma.
\end{aligned}
$$

Under the assumption $1-\frac{1}{\mu H}+\frac{1}{\mu}>0$, let $S=\frac{\tau-\varsigma}{\epsilon}$, and using hypothesis ($H_3$), we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|G_\mu(\tau-\varsigma,\zeta-\varrho)\right|^{\frac{p}{pH-1}}\,d\varrho\\
&\quad\leq \epsilon \int_{\mathbb{R}} \left|G_\mu(S\epsilon,\zeta-\varrho)\right|^{\frac{p}{pH-1}}\,d\varrho\\
&\quad\leq C_{\mu,H}\cdot \epsilon^{-\frac{p}{\mu(pH-1)}+\frac{1}{\mu}+1}.
\end{aligned}
$$

Then, applying Lemma 2.1 (c) and (H_4), there is

$$
I_1 \leq C\cdot T^{2-pH}\cdot \epsilon^{(pH-1)\left(-\frac{p}{\mu(pH-1)}+\frac{1}{\mu}+1\right)} K_1(\tau) \leq C\cdot T^{2-pH}\cdot\epsilon^{\gamma},
$$

where $\gamma=(pH-1)\left(-\frac{p}{\mu(pH-1)}+\frac{1}{\mu}+1\right)$.

For the second term I_2, consider Minkowski's inequality and Hölder's inequality, and we can obtain

$$\begin{aligned}
I_2 &= \sup_{\tau\in\bar{J}}\mathbb{E}\left|\int_0^\tau\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\left(f(\varsigma,\varrho,v_\epsilon)-\overline{f}(\varrho,\overline{v})\right)d\varrho\,d\varsigma\right|^p\\
&\le T^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\left|\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\left(f(\varsigma,\varrho,v_\epsilon)-\overline{f}(\varrho,\overline{v})\right)d\varrho\right|^p d\varsigma\\
&\le 2^{p-1}T^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\left|\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\left(f(\varsigma,\varrho,v_\epsilon)-f(\varsigma,\varrho,\overline{v})\right)d\varrho\right|^p d\varsigma\\
&+ 2^{p-1}T^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\left|\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\left(f(\varsigma,\varrho,\overline{v})-\overline{f}(\varrho,\overline{v})\right)d\varrho\right|^p d\varsigma\\
&=: I_{21}+I_{22}.
\end{aligned}$$

Hölder's inequality is applied, and we have

$$\begin{aligned}
I_{21} &\le 2^{p-1}L_1^p\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\left|\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\left|f(\varsigma,\varrho,v_\epsilon)-f(\varsigma,\varrho,\overline{v})\right|d\varrho\right|^p d\varsigma\\
&\le 2^{p-1}L_1^p\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\left(\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^{\frac{p}{p-1}}d\varrho\right)^{p-1}\int_{\mathbb{R}}|f(\varsigma,\varrho,v_\epsilon)-f(\varsigma,\varrho,\overline{v})|^p\,d\varrho\,d\varsigma\\
&\le 2^{p-1}T^{p-1}L_1^pC_{\mu,H}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau(\tau-\varsigma)^{-\frac{1}{\mu}}\int_{\mathbb{R}}|v_\epsilon-\overline{v}|^p\,d\varrho\,d\varsigma\\
&\le \frac{\mu}{\mu-1}2^{p-1}L_1^pC_{\mu,H}T^{p-\frac{1}{\mu}}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\int_{\mathbb{R}}|v_\epsilon-\overline{v}|^p\,d\varrho\,d\varsigma.
\end{aligned}$$

Next, (H_4) may be used to get

$$\begin{aligned}
I_{22} &\le 2^{p-1}T^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\left(\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^{\frac{p}{p-1}}d\varrho\right)^{p-1}\\
&\quad\times\int_{\mathbb{R}}\left|f(\varsigma,\varrho,\overline{v})-\overline{f}(\varrho,\overline{v})\right|^p d\varrho\,d\varsigma\\
&\le 2^{p-1}T^{p-1}C_{\mu,H}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau(\tau-\varsigma)^{-\frac{1}{\mu}}\int_{\mathbb{R}}\left|f(\varsigma,\varrho,\overline{v})-\overline{f}(\varrho,\overline{v})\right|^p d\varrho\,d\varsigma\\
&\le \frac{\mu}{\mu-1}2^{p-1}C_{\mu,H}T^{p-\frac{1}{\mu}}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\int_{\mathbb{R}}\left|f(\varsigma,\varrho,\overline{v})-\overline{f}(\varrho,\overline{v})\right|^p d\varrho\,d\varsigma\\
&\le \frac{\mu}{\mu-1}2^{p-1}C_{\mu,H}T^{p-\frac{1}{\mu}}\sup_{\tau\in\bar{J}}K_2(\tau)\,(1+\mathbb{E}|\overline{v}|^p)\\
&= \frac{\mu}{\mu-1}2^{p-1}C_{\mu,H}T^{p-\frac{1}{\mu}}P_1,
\end{aligned}$$

where $P_1=\sup_{\tau\in\bar{J}}K_2(\tau)\,(1+\mathbb{E}|\overline{v}|^p)$.

For the term I_3, keep Minkowski's inequality in mind. Assuming Hölder's inequality is applied, we have

$$I_3=\sup_{\tau\in\bar{J}}\mathbb{E}\left|\int_0^\tau\int_{\mathbb{R}} G_\mu(\tau-\varsigma,\zeta-\varrho)\,(g(\varsigma,\varrho,v_\epsilon)-\overline{g}(\varrho,\overline{v}))\,\psi(\varsigma,\varrho)\,d\varrho\,d\varsigma\right|^p$$

$$\le T^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\left|\int_{\mathbb{R}}G_\mu(\tau-\varsigma,\zeta-\varrho)\,|g(\varsigma,\varrho,v_\epsilon)-\overline{g}(\varrho,\overline{v})|\,\psi(\varsigma,\varrho)\,d\varrho\right|^p d\varsigma$$
$$\le 2^{p-1}T^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\left|\int_{\mathbb{R}}G_\mu(\tau-\varsigma,\zeta-\varrho)\,|g(\varsigma,\varrho,v_\epsilon)-g(\varsigma,\varrho,\overline{v})|\,\psi(\varsigma,\varrho)\,d\varrho\right|^p d\varsigma$$
$$+\,2^{p-1}T^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\left|\int_{\mathbb{R}}G_\mu(\tau-\varsigma,\zeta-\varrho)\,|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|\,\psi(\varsigma,\varrho)\,d\varrho\right|^p d\varsigma$$
$$=: I_{31}+I_{32}.$$

Using Hölder's inequality again and considering (H_3), we get

$$I_{31}\le 2^{p-1}L_1^p\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\left|\int_{\mathbb{R}}G_\mu(\tau-\varsigma,\zeta-\varrho)\,|g(\varsigma,\varrho,v_\epsilon)-g(\varsigma,\varrho,\overline{v})|\,\psi(\varsigma,\varrho)\,d\varrho\right|^p d\varsigma$$
$$\le 2^{p-1}L_1^p\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\left(\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^{\frac{p-3}{p}}\,d\varrho\right)^{\frac{p}{p-3}}$$
$$\times\int_{\mathbb{R}}|g(\varsigma,\varrho,v_\epsilon)-g(\varsigma,\varrho,\overline{v})|^p\,d\varrho\left(\int_{\mathbb{R}}|\psi(\varsigma,\varrho)|^{\frac{p}{2}}\,d\varrho\right)^2 d\varsigma$$
$$\le 2^{p-1}T^{p-1}L_1^pC_{\mu,H}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}(\tau-\varsigma)^{\frac{3}{\mu(p-3)}}\int_{\mathbb{R}}|v_\epsilon-\overline{v}|^p\,d\varrho\left(\int_{\mathbb{R}}|\psi(\varsigma,\varrho)|^{\frac{p}{2}}\,d\varrho\right)^2 d\varsigma$$
$$\le\frac{\mu(p-3)}{\mu(p-3)+3}2^{p-1}L_1^pC_{\mu,H}T^{\frac{3}{\mu(p-3)}+p}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\int_{\mathbb{R}}|v_\epsilon-\overline{v}|^p\,d\varrho\left(\int_{\mathbb{R}}|\psi(\varsigma,\varrho)|^{\frac{p}{2}}\,d\varrho\right)^2 d\varsigma$$
$$\le\frac{\mu(p-3)}{\mu(p-3)+3}2^{p-1}C_\psi L_1^pC_{\mu,H}T^{\frac{3}{\mu(p-3)}+p}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\int_{\mathbb{R}}|v_\epsilon-\overline{v}|^p\,d\varrho\,d\varsigma,$$

which is the same as I_{21}. Therefore, according to I_{22}, one can show that

$$I_{32}\le 2^{p-1}T^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\left(\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^{\frac{p-3}{p}}\,d\varrho\right)^{\frac{p}{p-3}}\int_{\mathbb{R}}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p\,d\varrho\left(\int_{\mathbb{R}}|\psi(\varsigma,\varrho)|^{\frac{p}{2}}\,d\varrho\right)^2 d\varsigma$$
$$\le 2^{p-1}T^{p-1}C_{\mu,H}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}(\tau-\varsigma)^{\frac{3}{\mu(p-3)}}\int_{\mathbb{R}}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p\,d\varrho\left(\int_{\mathbb{R}}|\psi(\varsigma,\varrho)|^{\frac{p}{2}}\,d\varrho\right)^2 d\varsigma$$
$$\le\frac{\mu(p-3)}{\mu(p-3)+3}2^{p-1}C_{\mu,H}T^{\frac{3}{\mu(p-3)}+p}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\int_{\mathbb{R}}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p\,d\varrho\left(\int_{\mathbb{R}}|\psi(\varsigma,\varrho)|^{\frac{p}{2}}\,d\varrho\right)^2 d\varsigma$$
$$\le\frac{\mu(p-3)}{\mu(p-3)+3}2^{p-1}C_\psi C_{\mu,H}T^{\frac{3}{\mu(p-3)}+p}\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^{\tau}\int_{\mathbb{R}}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p\,d\varrho\,d\varsigma$$
$$\le\frac{\mu(p-3)}{\mu(p-3)+3}2^{p-1}C_\psi C_{\mu,H}T^{\frac{3}{\mu(p-3)}+p}\sup_{\tau\in\bar{J}}K_3(\tau)\,(1+\mathbb{E}|\overline{v}|^p)$$
$$=\frac{\mu(p-3)}{\mu(p-3)+3}2^{p-1}C_\psi C_{\mu,H}T^{\frac{3}{\mu(p-3)}+p}P_2,$$

where $P_2=\sup_{\tau\in\bar{J}}K_3(\tau)\,(1+\mathbb{E}|\overline{v}|^p)$.

For the last term, note Minkowski's inequality:

$$I_4=\sup_{\tau\in\bar{J}}\mathbb{E}\left|\int_0^{\tau^+}\int_{\mathbb{R}}\int_{E_2}G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)\,(g(\varsigma,\varrho,v_\epsilon)-\overline{g}(\varrho,\overline{v}))\,M(\,dz,\,d\varrho,\,d\varsigma)\right|^p$$

$$
\begin{aligned}
&\leq 2^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\left|\int_0^\tau\int_{\mathbb{R}}\int_{E_2}G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)\left(g(\varsigma,\varrho,v_\epsilon)-g(\varsigma,\varrho,\overline{v})\right)M(\,dz,\,d\varrho,\,d\varsigma)\right|^p\\
&\quad+2^{p-1}\sup_{\tau\in\bar{J}}\mathbb{E}\left|\int_0^\tau\int_{\mathbb{R}}\int_{E_2}G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)\left(g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})\right)M(\,dz,\,d\varrho,\,d\varsigma)\right|^p\\
&=: I_{41}+I_{42}.
\end{aligned}
$$

Using Lemma 2.2 and the B-D-G inequality, we can present

$$
\begin{aligned}
I_{41}&\leq 2^{p-1}C_{p,T}\sup_{\tau\in\bar{J}}\Bigg(\int_0^\tau\int_{\mathbb{R}}\int_{E_2}(\mathbb{E}|G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)\\
&\qquad\times[g(\varsigma,\varrho,v_\epsilon)-g(\varsigma,\varrho,\overline{v})]|^p)^{\frac{2}{p}}\varphi_2(\,dz)\,d\varrho\,d\varsigma\Bigg)^{\frac{p}{2}}\\
&\leq 2^{p-1}C_{p,T}L_1^p\sup_{\tau\in\bar{J}}\left(\int_0^\tau\int_{\mathbb{R}}\int_{E_2}|G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)|^2\times(\mathbb{E}|v_\epsilon-\overline{v}|^p)^{\frac{2}{p}}\varphi_2(\,dz)\,d\varrho\,d\varsigma\right)^{\frac{p}{2}}\\
&\leq 2^{p-1}C_{p,T}L_1^p\sup_{\tau\in\bar{J}}\left(\int_0^\tau\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^2\int_{E_2}|\omega(\varsigma,\varrho,z)|^2\varphi_2(\,dz)(\mathbb{E}|v_\epsilon-\overline{v}|^p)^{\frac{2}{p}}\,d\varrho\,d\varsigma\right)^{\frac{p}{2}}\\
&\leq 2^{p-1}C_{p,T}L_1^pC\varphi_2(E_2)\sup_{\tau\in\bar{J}}\left(\int_0^\tau\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^2\times(\mathbb{E}|v_\epsilon-\overline{v}|^p)^{\frac{2}{p}}\,d\varrho\,d\varsigma\right)^{\frac{p}{2}}\\
&\leq 2^{p-1}T^{\frac{p-2}{2}}C_{p,T}L_1^pC\varphi_2(E_2)\sup_{\tau\in\bar{J}}\int_0^\tau\left(\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^{\frac{2p}{p-2}}\,d\varrho\right)^{\frac{p-2}{2}}\int_{\mathbb{R}}\mathbb{E}|v_\epsilon-\overline{v}|^p\,d\varrho\,d\varsigma\\
&\leq 2^{p-1}T^{\frac{p-2}{2}}C_{p,T}C_{\mu,H}L_1^pC\varphi_2(E_2)\sup_{\tau\in\bar{J}}\int_0^\tau(\tau-\varsigma)^{-\frac{p+2}{2\mu}}\int_{\mathbb{R}}\mathbb{E}|v_\epsilon-\overline{v}|^p\,d\varrho\,d\varsigma\\
&\leq\frac{2\mu}{2\mu-p-2}2^{p-1}T^{\frac{(\mu-1)p-2}{2\mu}}C_{p,T}C_{\mu,H}L_1^pC\varphi_2(E_2)\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\int_{\mathbb{R}}|v_\epsilon-\overline{v}|^p\,d\varrho\,d\varsigma.
\end{aligned}
$$

From condition (H_4), employing the same method as in I_{41}, we can get

$$
\begin{aligned}
I_{42}&\leq 2^{p-1}C_{p,T}\sup_{\tau\in\bar{J}}\left(\int_0^\tau\int_{\mathbb{R}}\int_{E_2}(\mathbb{E}|G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)[g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})]|^p)^{\frac{2}{p}}\varphi_2(\,dz)\,d\varrho\,d\varsigma\right)^{\frac{p}{2}}\\
&\leq 2^{p-1}C_{p,T}\sup_{\tau\in\bar{J}}\left(\int_0^\tau\int_{\mathbb{R}}\int_{E_2}|G_\mu(\tau-\varsigma,\zeta-\varrho)\omega(\varsigma,\varrho,z)|^2(\mathbb{E}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p)^{\frac{2}{p}}\varphi_2(\,dz)\,d\varrho\,d\varsigma\right)^{\frac{p}{2}}\\
&\leq 2^{p-1}C_{p,T}\sup_{\tau\in\bar{J}}\left(\int_0^\tau\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^2\int_{E_2}|\omega(\varsigma,\varrho,z)|^2\varphi_2(\,dz)(\mathbb{E}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p)^{\frac{2}{p}}\,d\varrho\,d\varsigma\right)^{\frac{p}{2}}\\
&\leq 2^{p-1}C_{p,T}C\varphi_2(E_2)\sup_{\tau\in\bar{J}}\left(\int_0^\tau\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^2\ (\mathbb{E}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p)^{\frac{2}{p}}\,d\varrho\,d\varsigma\right)^{\frac{p}{2}}\\
&\leq 2^{p-1}T^{\frac{p-2}{2}}C_{p,T}C\varphi_2(E_2)\sup_{\tau\in\bar{J}}\int_0^\tau\left(\int_{\mathbb{R}}|G_\mu(\tau-\varsigma,\zeta-\varrho)|^{\frac{2p}{p-2}}\,d\varrho\right)^{\frac{p-2}{2}}\int_{\mathbb{R}}\mathbb{E}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p\,d\varrho\,d\varsigma\\
&\leq 2^{p-1}T^{\frac{p-2}{2}}C_{p,T}C_{\mu,H}C\varphi_2(E_2)\sup_{\tau\in\bar{J}}\int_0^\tau(\tau-\varsigma)^{-\frac{p+2}{2\mu}}\int_{\mathbb{R}}\mathbb{E}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p\,d\varrho\,d\varsigma
\end{aligned}
$$

$$\leq \frac{2\mu}{2\mu-p-2}2^{p-1}T^{\frac{(\mu-1)p-2}{2\mu}}C_{p,T}C_{\mu,H}C\varphi_2(E_2)\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\int_{\mathbb{R}}|g(\varsigma,\varrho,\overline{v})-\overline{g}(\varrho,\overline{v})|^p\,d\varrho\,d\varsigma$$
$$\leq \frac{2\mu}{2\mu-p-2}2^{p-1}T^{\frac{(\mu-1)p-2}{2\mu}}C_{p,T}C_{\mu,H}C\varphi_2(E_2)\sup_{\tau\in\bar{J}}K_3(\tau)\,(1+\mathbb{E}|\overline{v}|^p)$$
$$\leq \frac{2\mu}{2\mu-p-2}2^{p-1}T^{\frac{(\mu-1)p-2}{2\mu}}C_{p,T}C_{\mu,H}C\varphi_2(E_2)P_3,$$

where $P_3 = \sup_{\tau\in\bar{J}} K_3(\tau)\,(1+\mathbb{E}|\overline{v}|^p)$.

Now, substituting the above analysis, we have

$$\begin{aligned}
&\sup_{\tau\in\bar{J}}\mathbb{E}|v_\epsilon(\tau,\zeta)-\overline{v}(\tau,\zeta)|^p\\
&\leq \mathscr{Q}_1\epsilon^{\gamma+\frac{p}{2}}+\mathscr{Q}_2\epsilon^p+\mathscr{Q}_3\epsilon^p\sup_{\tau\in\bar{J}}\mathbb{E}\int_0^\tau\int_{\mathbb{R}}|v_\epsilon-\overline{v}|^p\,d\varrho\,ds\\
&\leq \mathscr{Q}_1\epsilon^{\gamma+\frac{p}{2}}+\mathscr{Q}_2\epsilon^p+\mathscr{Q}_3\epsilon^p\sup_{\tau\in\bar{J}}\mathbb{E}\|v_\epsilon-\overline{v}\|^p,
\end{aligned}$$

where we denote that

$$\begin{aligned}
\mathscr{Q}_1 &= 4^{p-1}C_pC_{\mu,H}T^{2-pH}K_1(t),\\
\mathscr{Q}_2 &= 8^{p-1}\Bigg[\frac{\mu}{\mu-1}C_{\mu,H}T^{p-\frac{1}{\mu}}P_1+\frac{\mu(p-3)}{\mu(p-3)+3}C_\psi C_{\mu,H}T^{\frac{3}{\mu(p-3)}+p}P_2\\
&\quad+\frac{2\mu}{2\mu-p-2}T^{\frac{(\mu-1)p-2}{2\mu}}C_{p,T}C_{\mu,H}C\varphi_2(E_2)P_3\Bigg],\\
\mathscr{Q}_3 &= 8^{p-1}\Bigg[\frac{\mu}{\mu-1}C_{\mu,H}T^{p-\frac{1}{\mu}}+\frac{\mu(p-3)}{\mu(p-3)+3}C_\psi C_{\mu,H}T^{\frac{3}{\mu(p-3)}+p}\\
&\quad+\frac{2\mu}{2\mu-p-2}T^{\frac{(\mu-1)p-2}{2\mu}}C_{p,T}C_{\mu,H}C\varphi_2(E_2)\Bigg].
\end{aligned}$$

Thus, the calculations above lead to

$$\sup_{\tau\in\bar{J}}\mathbb{E}|v_\epsilon(\tau,\zeta)-\overline{v}(\tau,\zeta)|^p\leq\frac{\mathscr{Q}_1\epsilon^{\gamma+\frac{p}{2}}+\mathscr{Q}_2\epsilon^p}{1-\mathscr{Q}_3\epsilon^p},$$

and then, as $\epsilon\to 0$, we have

$$\lim_{\epsilon\to 0}\sup_{\tau\in\bar{J}}\mathbb{E}|v_\epsilon(\tau,\zeta)-\overline{v}(\tau,\zeta)|^p=0.$$

This completes the proof. □

5. Example and analysis

Recalling $\dot{L}$, $\dot{M}$, $\dot{N}$ defined in Lemma 2.1 and Eq (3.2), we consider the equation below:

$$\begin{cases}
\dfrac{\partial v_\epsilon(\tau,\zeta)}{\partial\tau}=\triangle_\mu v_\epsilon(\tau,\zeta)+\epsilon a_1v_\epsilon(\tau,\zeta)\cos^2(\tau)+\sqrt{\epsilon}\displaystyle\int_{U_0}2z^5\sin^2(\tau)v_\epsilon(\tau,\zeta)\left(\frac{1}{1+\zeta^2}\right)^p\dot{M}(d\varrho,\zeta,\tau)\\
\qquad+\displaystyle\int_{E_2\setminus U_0}2z^4\sin^2(\tau)v_\epsilon(\tau,\zeta)\left(\frac{1}{1+\zeta^2}\right)^p\dot{N}(d\varrho,\zeta,\tau)+\sqrt{\epsilon}a_2\sin^2(\tau)v_\epsilon(\tau,\zeta)\dot{B}^H,\\
v_\epsilon(0,\zeta)=v_0(\zeta),\qquad\tau\in[0,\pi],\zeta\in\mathbb{R}.
\end{cases}\tag{5.1}$$

In the above, $a_{1,2}$ are constants, $\omega_1(\tau,\zeta,z) = \left(\frac{1}{1+\zeta^2}\right)^p z$, $\omega_2(\tau,\zeta,z) = \left(\frac{1}{1+\zeta^2}\right)^p$, for $p \in \left(\frac{2(\mu+1)}{\mu-1}, +\infty\right)$, and $\mu \in (1,2)$. Then

$$\begin{aligned}\|\psi(\tau,\varsigma)\|_p^p &= \left(\int_{\mathbb{R}} \left|\int_{E_2\setminus U_0} \omega_2(\tau,\varrho,z)\varphi_2(\,dz)\right|^p dx\right)^{\frac{1}{p}} \\ &= C_p \pi^{\frac{1}{p}} \varphi_2(E_2\setminus U_0) \\ &< \infty,\end{aligned}$$

and

$$\begin{aligned}\left\|\int_{E_2} |\omega(\tau,\varrho,z)|^2 \varphi_2(\,dz)\right\|_{\frac{p}{2}}^{\frac{p}{2}} &= \left\|\int_{E_2} [\omega_1(\tau,\varrho,z)I_{U_0}(z) + \omega_2(\tau,\varrho,z)I_{E_2\setminus U_0}(z)]^2 \varphi_2(\,dz)\right\|_{\frac{p}{2}}^{\frac{p}{2}} \\ &\le \left(\int_{\mathbb{R}} \left(\int_{U_0} \omega_1^2(\tau,\zeta,\varrho)\phi_2(dz) + \int_{E_2\setminus U_0} \omega_2^2(\tau,\zeta,\varrho)\phi_2(dz)\right)^{\frac{p}{2}} dx\right)^{\frac{2}{p}} \\ &\le C_p \pi^{\frac{2}{p}} \left(\varphi_2(E_2\setminus U_0) + \int_{U_0} z^2 \varphi_2(\,dz)\right) \\ &< \infty.\end{aligned}$$

So (H_1)–(H_3) are satisfied, and we can conclude that Theorem 3.1 holds.

Next, take $T = \pi$ and

$$\begin{aligned}\overline{\sigma}(\varrho) &= \frac{1}{\pi}\int_0^\pi \sigma(\varsigma,\varrho)\,d\varsigma = \frac{a_2}{2}\overline{v}, \\ \overline{f}(\varrho,\overline{v}) &= \frac{1}{\pi}\int_0^\pi f(\varsigma,\varrho,\overline{v})\,d\varsigma = \frac{a_1}{2}\overline{v}, \\ \overline{g}(\varrho,\overline{v}) &= \frac{1}{\pi}\int_0^\pi g(\varsigma,\varrho,\overline{v})\,d\varsigma = z^4\overline{v}.\end{aligned}$$

All of the assumptions (H_1) to (H_4) listed in Theorem 4.1 can be easily verified. As a result, Eq (5.1)'s averaged equation could be written as follows:

$$\begin{cases} \dfrac{\partial\overline{v}(\tau,\zeta)}{\partial\tau} = \triangle_\mu \overline{v}(\tau,\zeta) + \dfrac{a_1}{2}\epsilon\overline{v}(\tau,\zeta) + \sqrt{\epsilon}\displaystyle\int_{U_0} z^5\overline{v}(\tau,\zeta)\left(\frac{1}{1+\zeta}\right)^p \dot{M}(\,d\varrho,\zeta,\tau) \\ \qquad + \displaystyle\int_{E_2\setminus U_0} z^4\overline{v}(\tau,\zeta)\left(\frac{1}{1+\zeta}\right)^p \dot{N}(\,d\varrho,\zeta,\tau) + \sqrt{\epsilon}\frac{a_2}{2}\overline{v}(\tau,\zeta)\dot{B}^H, \\ \overline{v}(0,\zeta) = \overline{v}_0(\zeta), \qquad \tau\in[0,\pi], \zeta\in\mathbb{R}. \end{cases} \tag{5.2}$$

6. Conclusions

Our research focuses on the averaging principle of SFPDEs driven by Lévy space-time white noise and fBm. First, using the fixed point theorem, the existence and uniqueness of the mild solution are proven. Through rigorous mathematical derivation, we obtain convergence estimates for v_ϵ

converging to $\overline{v}$. We have found convergence to the limit process as the scale parameter ϵ approaches zero. The averaging principle proposed in this study eliminates temporal dependence by averaging the time component τ in nonlinear functions, thereby significantly reducing computational complexity and simplifying the application of such equations in practical modeling scenarios.

Focusing on practical applications, the long-range dependence of fBm confers unique modeling and prediction advantages across various fields, such as image processing, control systems, and, for instance, long-memory processes in signal processing, high-frequency financial volatility modeling, etc. In particular, the original equation can be approximated using the simplified equation obtained by averaging, while retaining its dynamic characteristics as much as possible. This significantly reduces the computational difficulty in solving more similar practical problems of the above kind. Moreover, we explicitly highlight the computational challenges as a direction for future research.

Author contributions

Conceptualization, Y. W. and H. G.; formal analysis, R. A.; writing–original draft preparation, Y. W. and H. G.; writing–review and editing, R.A.. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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