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Research article

Multiple periodic solutions of parameterized systems coupling asymmetric components and linear components

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Abstract: We investigated the multiplicity of periodic solutions for parameterized systems coupling asymmetric components and linear components, where the nonlinearity is allowed to be sign-changing. Our approach was based on an extended version of the Poincaré-Birkhoff theorem, a rotation number approach, and a new existence result.

Keywords: periodic solutions; Poincaré-Birkhoff theorem; parameterized; sign-changing
Mathematics Subject Classification: 34B15, 34C25

1. Introduction

In this paper, we consider the multiplicity of periodic solutions of weakly coupled parameterized systems of the form

$$\begin{cases} x_1'' + f_1(t, x_1, \cdots, x_N) = sp_1(t), \\ x_2'' + f_2(t, x_1, \cdots, x_N) = sp_2(t), \\ \vdots \\ x_N'' + f_N(t, x_1, \cdots, x_N) = sp_N(t). \end{cases} \tag{1.1}$$

We assume that $f = (f_1, \cdots, f_N) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $p = (p_1, \cdots, p_N) : \mathbb{R} \to \mathbb{R}^N$ are continuous functions, 2π-periodic with respect to the time variable. We also assume that there exists a continuous function denoted as $H : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, which is differentiable with respect to the variable x. This function satisfies the crucial relation

$$f_i(t, x) = \frac{\partial}{\partial x_i} H(t, x), \tag{1.2}$$

for every index i. Moreover, s is a positive parameter. Similar results can be obtained for a negative s. In the following, we will denote by $x \in \mathbb{R}^N$, the vector $x = (x_1, \cdots, x_N)$.

The study of parameterized differential equations or systems can be traced back to the Ambrosetti-Prodi problem originated from the seminal work of Ambrosetti and Prodi [1]. More precisely, a second-order equation

$$x'' + g(x) = sw(t) \tag{1.3}$$

was investigated by Berger and Podolak [2], where $g : \mathbb{R} \to \mathbb{R}$ is assumed to be C^2 such that $g'' > 0$ and $g'(-\infty) < \lambda_1 < g'(+\infty) < \lambda_2$, and $w(t) = \sin(\frac{\pi}{T}t)$ is the eigenfunction corresponding to $\lambda_1 = (\pi/T)^2$ for the Dirichlet problem on the interval $[0, T]$. Since then, there have been many significant achievements. Mawhin and Nkashama [3] initiated the investigation of the Ambrosetti-Prodi problem with periodic boundary conditions. Ortega [4] discussed the Ambrosetti-Prodi periodic problem for a damped Duffing equation from the point of view of the stability of the solutions. For additional contributions, concerning the existence and multiplicity of periodic solutions for second-order equations, see [5–7] and the references therein. The multiplicity of periodic solutions for the classical equation $u'' + g(u) = s(1 + h(t))$ was investigated in [8], Fonda and Ghirardelli [9] weakened the assumptions on the differentiability of the nonlinearity and obtained the multiplicity of solutions. Moreover, Calamai and Sfecci [10] studied the periodic boundary value problem for weakly coupled parameter-dependent equations. For more contributions, we refer to [11, 12] and the references therein.

In particular, Fonda and Ghirardelli [9] extended the results in [5, 8, 11] and investigated the multiplicity of solutions for the following periodic problem:

$$\begin{cases} x'' + g(t, x) = sw(t), \\ x(0) = x(T), \ x'(0) = x'(T). \end{cases} \tag{1.4}$$

They assumed $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ to be a Carathéodory function, and $w : [0, T] \times \mathbb{R} \to \mathbb{R}$ to be integrable. Then their result was extended by Calamai and Sfecci [10] to a parameterized weakly coupled system

$$\begin{cases} x_i'' + g_i(t, x) = sw_i(t), \\ x_i(0) = x_i(T), \ x_i'(0) = x_i'(T), \end{cases} \quad i = 1, \cdots, N, \tag{1.5}$$

where $g_i : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $w_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, and T-periodic with respect to the first variable.

Incorporating ideas from previous research, particularly from [9, 10], a natural question arises: whether are there multiple periodic solutions for parameterized systems coupling asymmetric and linear components? This paper delves into the investigation of multiplicity of periodic solutions for the weakly coupled parameterized system (1.1). However, it is important to note that the main tool employed in [9, 10], namely the Poincaré-Birkhoff theorem, is not directly applicable to the specific problem at hand. What adds to the excitement is the concerted effort among researchers to introduce innovative methodologies, exemplified by an expanded version of the higher-dimensional Poincaré-Birkhoff theorem as outlined in [13]. This advancement holds the potential to offer fresh insights into the present context and further propel research into the multiplicity of periodic solutions.

The extended version of the higher dimensional Poincaré-Birkhoff theorem outlined in [13] is suitable for area-preserving maps. Therefore, we postulate the existence of $H : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ in (1.2) in order to guarantee the Hamiltonian framework for the system delineated in (1.1). Within this paper, for every index $i = 1, 2, \cdots, N$, standard notations $x_i^+ := \max\{x_i, 0\}$ and $x_i^- := \max\{-x_i, 0\}$ are employed. Additionally, $\rho(\cdot)$ stands as a representation for the rotation number of a linear equation or a piecewise linear equation. The exact definition of this term is expounded in Section 2. Furthermore, t_0 is used to represent the initial time. We also introduce the notation $\check{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \mathbb{R}^{N-1}$ for concise representation.

We assume the following conditions:

(H_0) $f : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$, $p : \mathbb{R} \to \mathbb{R}^N$ are continuous functions, 2π-periodic with respect to the time variable, locally Lipschitz-continuous with respect to the second variable, and s is a positive parameter.

Fixing an integer $m < N$, we assume the following conditions for every index $i = 1, 2, \cdots, m$.

(H_1^i) There are two functions $\nu_1^i,\ \nu_2^i \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ such that

$$\nu_1^i(t) \le \liminf_{x_i \to -\infty} \frac{f_i(t,x)}{x_i} \le \limsup_{x_i \to -\infty} \frac{f_i(t,x)}{x_i} \le \nu_2^i(t)$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\check{x}_i \in \mathbb{R}^{N-1}$.

(H_2^i) There is a function $q_i(t) \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ such that

$$\lim_{x_i \to +\infty} \frac{f_i(t,x)}{x_i} = q_i(t)$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\check{x}_i \in \mathbb{R}^{N-1}$.

(H_3^i) There is an integer $m_i \ge 0$ such that

$$m_i < \rho(q_i) < m_i + 1,$$

where $\rho(q_i)$ denotes the rotation number of the equation $x_i'' + q_i(t)x_i = 0$. Moreover, the only 2π-periodic solution of

$$x_i'' + q_i(t)x_i = p_i(t) \tag{1.6}$$

is strictly positive.

(H_4^i) There is an integer $n_i \ge 0$ such that

$$\rho(\nu_1^i) > n_i, \quad \rho(\nu_2^i) < n_i + 1,$$

where $\rho(\nu_1^i)$ and $\rho(\nu_2^i)$ denote the rotation numbers of equations

$$x_i'' + q_i(t)x_i^+ - \nu_1^i(t)x_i^- = 0, \tag{1.7}$$

and

$$x_i'' + q_i(t)x_i^+ - \nu_2^i(t)x_i^- = 0, \tag{1.8}$$

respectively.

And for every $i \in \{m+1, \cdots, N\}$, we assume the following condition:

(H_5^i) $f_i(t,x) = M_i(t)x_i + e_i(t,x)$, with $e_i : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ being bounded, $\mathbb{M}(t) = \operatorname{diag}(M_{m+1}(t), \cdots, M_N(t))$ being a symmetric matrix, continuous and 2π-periodic, satisfying the nonresonance condition

$$x_{m,N}(t) \equiv 0 \quad \text{is the only } 2\pi - \text{periodic solution of } x_{m,N}'' = \mathbb{M}(t)x_{m,N}, \tag{1.9}$$

where $x_{m,N} = (x_{m+1}, \cdots, x_N)$.

Then we have the following result.

Theorem 1.1. *Suppose* (H_0) *holds. Moreover,* (H_1^i)–(H_4^i) *hold for* $i \in \{1, 2, \cdots, m\}$*, and* H_5^i *holds for* $i \in \{m+1, \cdots, N\}$*. Then there is a* $s_0 > 0$ *such that, for every* $s \geq s_0$*, Eq* (1.1) *has at least*

$$(m+1) \prod_{i=1}^{m} |n_i - m_i| + 1,$$

distinct 2π*-periodic solutions.*

Remark 1.1. *The nonresonance conditions outlined in* (H_1^i)–(H_4^i) *represent generalizations of the classical nonresonance conditions outlined in* $(H1^i)$–$(H4^i)$ *[10], where* v_1^i*,* v_2^i*, and* $a_i(t)$ *are required to be positive. However,* $v_1^i(t)$*,* $v_2^i(t)$*, and* $q_i(t)$ *are allowed to be sign-changing in* (H_1^i) *and* (H_2^i)*, which implies that* $\operatorname{sgn}(x_i) f_i(t, x)$ *could be sign-changing. The following is an interesting example regarding this. For every* $i \in \{1, 2, \cdots, m\}$*, define three sign-changing functions*

$$q_i(t) = \begin{cases} (2m_i + 1)^2, & t \in [0, \pi], \\ -\lambda_i^2, & t \in [\pi, 2\pi], \end{cases} \qquad v_j^i(t) = \begin{cases} (2\alpha_j^i + 1)^2, & t \in [0, \pi], \\ -(\mu_j^i)^2, & t \in [\pi, 2\pi], \end{cases}$$

where $j = 1, 2$*,* m_i*,* $\alpha_2^i \in \mathbb{N}^+$*, and* $\alpha_1^i \in \mathbb{R}^+$*,* $\arctan |\lambda_i| \leq \pi/(2(2m_i + 1))$*, and*

$$1 - \frac{n_i}{2\alpha_1^i + 1} - \frac{n_i}{2m_i + 1} \geq 2 \max\{\arctan |\lambda_i|, \ \arctan |\mu_1^i|\}/\pi > 0.$$

Additionally, assume that there exists an integer $n_i > 0$ *satisfying*

$$\frac{\pi}{m_i} + \frac{\pi}{\alpha_j^i} < \frac{2\pi}{n_i}, \quad \frac{\pi}{m_i + 1} + \frac{\pi}{\alpha_2^i + 1} > \frac{2\pi}{n_i + 1}, \quad j = 1, 2. \tag{1.10}$$

Then it can be proved (see the details in Section 5)

$$m_i < \rho(q_i) < m_i + 1, \quad \rho(v_1^i) > n_i, \quad \rho(v_2^i) < n_i + 1. \tag{1.11}$$

Therefore, the nonresonant conditions described in the sense of rotation numbers are different from the previous nonresonant conditions.

The rest of the paper is organized as follows: In Section 2, we introduce the definitions and properties of rotation numbers and 2π-rotation numbers. In Section 3, we prove a crucial existence result. In Section 4, we give some preliminary lemmas. Finally, in Section 5, we prove Theorem 1.1, and give the proof of (1.11).

2. Rotation numbers and 2π-rotation numbers for component equations

Within this section, we present the precise definitions of 2π-rotation numbers for first-order component systems, as well as the rotation numbers for first-order piecewise linear systems. We also elucidate the interconnection between the rotation numbers and the 2π-rotation numbers. This discourse bears resemblance to the discussions found in [14, 15].

Consider a system of the form

$$x_i' = -y_i, \quad y_i' = h_i(t, x), \quad i = 1, 2, \cdots, N. \tag{2.1}$$

Let $x = (x_1, x_2, \cdots, x_N)$, $y = (y_1, y_2, \cdots, y_N)$, and $z = (x, y) \in \mathbb{R}^{2N}$. If the component $z_i(t; z_0)$ of a solution $z(t; z_0)$ with $z(t_0; z_0) = z_0$ does not cross the origin, we can pass to the polar coordinates

$$x_i = r_i \cos\theta_i, \quad y_i = r_i \sin\theta_i.$$

Then we have

$$\begin{cases} \theta_i' = \sin^2\theta_i + \dfrac{h_i(t, x)}{r_i}\cos\theta_i, \\ r_i' = -r_i \sin\theta_i \cos\theta_i + h_i(t, x)\sin\theta_i, \end{cases} \quad i = 1, 2, \cdots, N. \tag{2.2}$$

If $z(t; z_0)$ exists within the interval $[t_0, t_0 + 2\pi]$, we can define the 2π-rotation number associated with the i-th component $z_i(t; z_0)$ as follows:

$$\mathrm{Rot}_{h_i}(z_0) = \frac{\theta_i(t_0 + 2\pi; z_0) - \theta_i(t_0; z_0)}{2\pi} = \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} \frac{x_i h_i(t, x) + y_i^2}{x_i^2 + y_i^2} dt,$$

where $\theta_i(t; z_0)$ is the argument function of $z_i(t; z_0)$. Accordingly, $\mathrm{Rot}_{h_i}(z_0)$ represents the total algebraic count of the counterclockwise rotations of the component $z_i(t; z_0)$ around the origin, in the i-th projection plane during the time interval $[t_0, t_0 + 2\pi]$.

When the component of system (2.1) is a piecewise linear system

$$x_i' = -y_i, \quad y_i' = a_i^+(t)x_i^+ - a_i^-(t)x_i^-, \tag{2.3}$$

where $a_i^{\pm}(t) \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$. The component argument function $\theta_i(t; z_0)$ satisfies

$$\theta_i' = a_i^+(t)((\cos\theta_i)^+)^2 + a_i^-(t)((\cos\theta_i)^-)^2 + \sin^2\theta_i. \tag{2.4}$$

Hence, the value of $\theta_i(t; z_0)$ is solely determined by the initial time t_0 and the initial argument value, denoted as $\theta_i(t_0; z_0)$, belonging to the unit circle $\mathbb{S}^1$, which can be represented as $\mathbb{R}/(2\pi\mathbb{Z})$. In this particular scenario, we can express the 2π-rotation number of $z_i(t; z_0)$ as $\mathrm{Rot}a_i(\omega_0)$, where $\omega_0 = z_{0_i}/|z_{0_i}|$.

Moreover,

$$a_i^+(t)((\cos\theta_i)^+)^2 + a_i^-(t)((\cos\theta_i)^-)^2 + \sin^2\theta_i$$

is 2π-periodic in t and 2π-periodic in θ_i. Equation (2.4) is therefore a differential equation on a torus in this case. Hence we can define the rotation number of (2.4) as

$$\rho(a_i) = \lim_{t\to\infty} \frac{\theta_i(t_0 + t; \theta_0) - \theta_{0_i}}{t}, \tag{2.5}$$

where θ_{0_i} is the i-th component of θ_0. By extension, we refer to $\rho(a_i)$ as the rotation number of the system (2.3).

Now, we present the relationships between the rotation number $\rho(a_i)$ and the 2π-rotation number $\mathrm{Rot}_{a_i}(\omega_0)$ of system (2.3).

Lemma 2.1. *For an arbitrary integer n_i, we have*

(i) $\rho(a_i) > n_i \Leftrightarrow \mathrm{Rot}_{a_i}(\omega_0) > n_i$, $\forall\, \omega_0 \in \mathbb{S}_i^1$;

(ii) $\rho(a_i) < n_i \Leftrightarrow \mathrm{Rot}_{a_i}(\omega_0) < n_i$, $\forall\, \omega_0 \in \mathbb{S}_i^1$;

(iii) *$\rho(a_i) = n_i$ if and only if there is at least one nontrivial 2π-periodic solution $\theta_i(t; t_0, \theta_0)$ of system* (2.3) *with $\theta_i(t_0 + 2\pi; t_0, \theta_0) - \theta_{0_i} = 2n_i\pi$.*

The proofs of Lemma 2.1 and the following ones are similar to propositions or lemmas in [14, 15] and [16, 17], so we omit them. In fact, the core of the proof of Lemma 2.1 lies in utilizing the definition of the rotation number, the periodic property of the homeomorphism, and the idea of iteration. By analyzing the relationship between the extremal behavior of the mapping and its asymptotic linear growth, it establishes the equivalence between the rotation number and the extremal condition. Next, we give the following comparison result about the 2π-rotation numbers between the i-th component system (2.1) and system (2.3).

Lemma 2.2. *(Comparison lemma) Let $h_i : [t_0, t_0 + 2\pi] \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function, and let $a_i^{\pm} \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$, then*

(i) If

$$\liminf_{x_i \to \pm\infty} \frac{h_i(t, x)}{x_i} \geq a_i^{\pm}(t) \tag{2.6}$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and a certain i. Then, for each $\varepsilon > 0$, there exists $R_\varepsilon^i > 0$ such that

$$\mathrm{Rot}_{h_i}(z_0) \geq \mathrm{Rot}_{a_i}(\omega_0) - \varepsilon, \quad \forall\, t \in [t_0, t_0 + 2\pi],\ \omega_0 = z_{0_i}/|z_{0_i}| \tag{2.7}$$

holds for the i-th component of every solution of system (2.1) satisfying $|z_i(t)| \geq R_\varepsilon^i,\ \forall\, t \in [t_0, t_0 + 2\pi]$.

(ii) If

$$\limsup_{x_i \to \pm\infty} \frac{h_i(t, x)}{x_i} \leq a_i^{\pm}(t) \tag{2.8}$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and a certain i. Then, for each $\varepsilon > 0$, there exists $R_\varepsilon^i > 0$ such that

$$\mathrm{Rot}_{h_i}(z_0) \leq \mathrm{Rot}_{a_i}(\omega_0) + \varepsilon, \quad \forall\, t \in [t_0, t_0 + 2\pi],\ \omega_0 = z_{0_i}/|z_{0_i}| \tag{2.9}$$

holds for i-th component of every solution of system (2.1) satisfying $|z_i(t)| \geq R_\varepsilon^i,\ \forall\, t \in [t_0, t_0 + 2\pi]$.

By the above two lemmas, we have the following lemma about the 2π-rotation number of the system (2.1) and the rotation number of system (2.3).

Lemma 2.3. *Suppose $h_i : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, 2π-periodic with respect to the first variable, and let $a_i^{\pm} \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$, then*

(i) If $\rho(a_i) > n_i$ and (2.6) holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and a certain i, then there exists $R_i > 0$ such that $\mathrm{Rot}_{h_i}(z_0) > n_i$ holds for the i-th component of every solution $z(t; z_0)$ of Eq (2.1) satisfying $|z_i(t)| \geq R_i,\ \forall\, t \in [t_0, t_0 + 2\pi]$.

(ii) If $\rho(a_i) < n_i$ and (2.8) holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and a certain i, then there exists $R_i > 0$ such that $\mathrm{Rot}_{h_i}(z_0) < n_i$ holds for the i-th component of every solution $z(t; z_0)$ of Eq (2.1) satisfying $|z_i(t)| \geq R_i,\ \forall\, t \in [t_0, t_0 + 2\pi]$.

3. A new existence result

In this section, we consider the existence of 2π-periodic solutions for the following weakly coupled system

$$\begin{cases} x_1'' + g_1(t, x_1, \cdots, x_N) = \omega_1(t), \\ x_2'' + g_2(t, x_1, \cdots, x_N) = \omega_2(t), \\ \vdots \\ x_N'' + g_N(t, x_1, \cdots, x_N) = \omega_N(t). \end{cases} \tag{3.1}$$

Here, we assume the following conditions.

(S_0) $g = (g_1, \cdots, g_N) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $\omega = (\omega_1, \cdots, \omega_N) : \mathbb{R} \to \mathbb{R}^N$ are continuous functions, 2π-periodic with respect to the first variable, g is locally Lipschitz-continuous with respect to the second variable.

(S_1^i) There exist two functions $\xi_i,\ \zeta_i \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ such that

$$\xi_i(t) \le \liminf_{|x_i| \to +\infty} \frac{g_i(t,x)}{x_i} \le \limsup_{|x_i| \to +\infty} \frac{g_i(t,x)}{x_i} \le \zeta_i(t)$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\check{x}_i \in \mathbb{R}^{N-1}$.

(S_2^i) There exists an integer $m_i \ge 0$ such that

$$\rho(\xi_i) > m_i, \quad \rho(\zeta_i) < m_i + 1,$$

where $\rho(\xi_i)$ and $\rho(\zeta_i)$ denotes the rotation numbers of the equations

$$x_i'' + \xi_i(t) x_i = 0,$$

and

$$x_i'' + \zeta_i(t) x_i = 0,$$

respectively.

(S_3^i) There exists a function $\tilde{\xi}_i \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ such that

$$\lim_{|x_i| \to +\infty} \frac{g_i(t,x)}{x_i} = \tilde{\xi}_i(t)$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\check{x}_i \in \mathbb{R}^{N-1}$, where $\tilde{\xi}_i : \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic and satisfies the nonresonance condition

$$x_i(t) \equiv 0 \quad \text{is the only } 2\pi - \text{periodic solution of } x_i'' = \tilde{\xi}_i(t) x_i. \tag{3.2}$$

Then, we have the following result.

Theorem 3.1. *Assume (S_0). Moreover, (S_1^i) and (S_2^i) hold or (S_3^i) holds, for every $i = 1, 2, \cdots, N$. Then system (3.1) has at least one 2π-periodic solution.*

Remark 3.1. *In the proof of our main result, there is a crucial change of variables using the existence of 2π-periodic solutions of systems similar to system (3.1). The proof of Theorem 3.1 is based on a higher dimensional fixed-point theorem for the coupling of twist conditions and Poincaré-Bohl-type conditions in [18].*

For the convenience of readers, we state the higher dimensional fixed-point theorem in [18] as a lemma in the following and then provide the proof of Theorem 3.1 in the end of this section.

Lemma 3.1. *(Theorem 2.1 in [18]) Let Γ_i^+, Γ_i^- be simple closed curves in the i-th phase plane surrounding the origin O_i of the i-th phase plane, and $\Gamma_i^- \subset \mathrm{I}(\Gamma_i^+)$ with $\mathrm{I}(\Gamma_i^\pm)$ the interior region bounded by $\Gamma_i^\pm$, $\mathcal{A}_i = \overline{\mathrm{I}(\Gamma_i^+)} \setminus \mathrm{I}(\Gamma_i^-)$, $i = 1, 2, \cdots, k$. Denote $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k$, $\mathcal{B}^\pm = \mathrm{I}(\Gamma_1^\pm) \times \cdots \times \mathrm{I}(\Gamma_k^\pm)$. Let Ω_j be a bounded open set surrounding the origin O_j, whose boundary is a simple closed curve,*

$j = k+1, \cdots, N$, $\overline{\Omega} = \overline{\Omega}_{k+1} \times \cdots \times \overline{\Omega}_N$, where $0 \le k \le N$, $k = 0$ means that i does not take value and $k = N$ means that j does not take value. Define a continuous function

$$F : \overline{\mathcal{B}^+} \times \overline{\Omega} \to \mathbb{R}^{2N},$$

$$z = (z_1, z_2, \cdots, z_N) \mapsto (F_1(z), F_2(z), \cdots, F_N(z)).$$

For $i = 1, 2, \cdots, k$, let $U_i(O_i)$ be a neighborhood of O_i in the i-th phase plane, L_i be a real orthogonal matrix with $\det(L_i) = 1$, *and*

$$J_i = \{z \in \overline{\mathcal{B}^+} \times \overline{\Omega} : \ z_i \in \mathcal{A}_i, \ F_i(z) \notin U_i(O_i), \ \langle L_i z_i, F_i(z) \rangle = 0\}.$$

Suppose that

(i) For any continuous curve γ, if $\Pi_i(\gamma)$ connects Γ_i^- and Γ_i^+, then $\gamma \cap J_i \neq \emptyset$, where $\Pi_i(\gamma)$ is the projection of γ on the i-th phase plane, $i = 1, 2, \cdots, k$;

(ii) F_j satisfies

$$F_j(z) \neq \mu_j z_j, \quad for \quad \mu_j > 1, \quad z \in \overline{\mathcal{B}^+} \times \partial_j \Omega,$$

where $j = k+1, \cdots, N$ and $\partial_j \Omega = \overline{\Omega}_{k+1} \times \cdots \times \overline{\Omega}_{j-1} \times \partial\Omega_j \times \overline{\Omega}_{j+1} \times \cdots \times \overline{\Omega}_N$ for $j = k+2, \cdots, N-1$, $\partial_{k+1}\Omega = \partial\Omega_{k+1} \times \overline{\Omega}_{k+2} \times \cdots \times \overline{\Omega}_N$, $\partial_N \Omega = \overline{\Omega}_{k+1} \times \cdots \times \overline{\Omega}_{N-1} \times \partial\Omega_N$.

Then F has at least one fixed point z_0 in $\overline{\mathcal{B}^+} \times \overline{\Omega}$.

Consider the equivalent system of (3.1)

$$x_i' = -y_i, \quad y_i' = g_i(t, x_i) - \omega_i(t), \quad i = 1, 2, \cdots, N. \tag{3.3}$$

Denote $z_i = (x_i, y_i)$, $i = 1, 2, \cdots, N$. $z = (z_1, z_2, \cdots, z_N)$. Let $z(t; z_0)$ be the solution of system (3.3) satisfying the initial condition $z(t_0; z_0) = z_0$. If the component $z_i(t; z_0) \neq 0$, by the polar coordinate transformation

$$x_i = r_i \cos\theta_i, \quad y_i = r_i \sin\theta_i,$$

system (3.3) is equivalent to

$$\begin{cases} \theta_i' = \sin^2\theta_i + \dfrac{g_i(t, x) - \omega_i(t)}{r_i} \cos\theta_i, \\ r_i' = -r_i \sin\theta_i \cos\theta_i + (g_i(t, x) - \omega_i(t)) \sin\theta_i. \end{cases} \tag{3.4}$$

Then, we give a lemma concerning the uniqueness and global existence of solutions of the Cauchy problem associated with system (3.1).

Lemma 3.2. *Assume (S_0). Moreover, (S_1^i) or (S_3^i) holds for every $i \in \{i, 2, \cdots, N\}$. Then the solutions of the Cauchy problem associated with system* (3.1) *exists uniquely and globally.*

Proof. First, by the Lipschitz continuity of g, we obtain the uniqueness of solutions of the Cauchy problem associated with system (3.1).

Secondly, we prove the global existence of solutions of the Cauchy problem associated with system (3.1) using Gronwall's inequality.

By (S_1^i), there exist $0 < \varepsilon \le 1$ and $D_i > 0$ such that

$$\xi_i(t) - \varepsilon \le \frac{g_i(t, x) - \omega_i(t)}{x_i} \le \zeta_i(t) + \varepsilon$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\check{x}_i \in \mathbb{R}^{N-1}$ when $|x_i| \geq D_i$. Then, we have

$$|r_i'| \leq r_i(2 + \max\{|\xi_i(t)|, |\zeta_i(t)|\}),$$

by the Gronwall's inequality, we obtain

$$r_{0_i} e^{-\int_{t_0}^{t_0+2\pi}(2+\max\{|\xi_i(t)|,|\zeta_i(t)|\})dt} \leq r_i(t) \leq r_{0_i} e^{\int_{t_0}^{t_0+2\pi}(2+\max\{|\xi_i(t)|,|\zeta_i(t)|\})dt},$$

where r_{0_i} denotes the i-th component of r_0, and (r_0, θ_0) is the form in the polar coordinates of the initial value $z_0 = (x_0, y_0)$. Hence, when r_{0_i} is large enough, $r_i(t)$ will be large enough, so solutions of the Cauchy problem associated with system (3.1) exist globally.

Similarly, by (S_3^i), we can prove the global existence of solutions of the Cauchy problem associated with system (3.1). □

Next, we prove Theorem 3.1 by the higher dimensional fixed-point theorem for the coupling of twist conditions and Poincaré-Bohl-type conditions outlined in [18].

Proof of Theorem 3.1. We take $k = 0$ while applying Lemma 3.1. Suppose $z(t; z_0)$ is a solution of system (3.3) satisfying the initial condition $z(t_0; z_0) = z_0$, and define the Poincaré map

$$\begin{aligned} \mathcal{P}: \qquad \overline{\Omega} &\to \mathbb{R}^{2N}, \\ z_0 &\mapsto z(t_0 + 2\pi; z_0), \end{aligned}$$

where $\Omega = \Omega_1 \times \cdots \Omega_N$ is the region in the phase space of (3.3) described in Lemma 3.1. Denote $\mathcal{P} = (\mathcal{P}_1, \cdots, \mathcal{P}_N)$. From Lemma 3.2, by conditions (S_0), (S_1^i), and (S_3^i), we obtain the uniqueness and global existence of solutions for the initial value problem associated with system (3.1). Therefore, $\mathcal{P}$ is well defined, and is a continuous map. In addition, system (3.1) has a 2π-periodic solution if and only if $\mathcal{P}$ has a fixed point.

In the following we will prove that $\mathcal{P}$ has a fixed point using Lemma 3.1.

From (S_1^i), by Lemma 2.3, we have

$$m_i < \text{Rot}_{g_i}(z_0) < m_i + 1. \tag{3.5}$$

So we can verify that condition (ii) in Lemma 3.1 holds. Actually, since (3.5) holds, the solution $z(t; z_0)$ starting from z_0 cannot rotates integer numbers of turns in the i-th projected phase plane. So there exists a constant μ_i such that

$$\mathcal{P}_i(z) \neq \mu_i z_i, \quad \text{for } \mu_i > 1 \text{ and } z \in \partial_i \Omega, \tag{3.6}$$

where $\partial_i \Omega$ is defined as that in Lemma 3.1.

By (S_3^i), we can conclude that if $x_i(t)$ is a nontrivial solution of $x_i'' = \tilde{\xi}_i(t) x_i$, then $x_i(t)$ can not perform integer turns around the the i-th origin O_i, and we can also prove (3.6).

Hence by Lemma 3.1 we conclude that $\mathcal{P}$ has at least one fixed point, which implies the existence of 2π-periodic solution of system (3.1). □

4. Preliminary lemmas

In this section, we prepare some preliminary lemmas for the proofs of the main result. It is worth noting that our discussion is based on the sign-varying conditions. We will always suppose that $s \geq 1$, denoting by $\|\cdot\|_p$ the usual norm in $L^p([t_0, t_0 + 2\pi])$.

Lemma 4.1. *There exist three positive constants ε_0, c_0, and C_0 such that, for every index $i \in \{1, 2, \cdots, N\}$, if β and γ are 2π-periodic functions, and $\beta,\ \gamma \in L^1([t_0, t_0 + 2\pi], \mathbb{R})$ satisfy*

$$\|\beta\|_1 \leq \varepsilon_0, \quad \|\gamma - a_i\|_1 \leq \varepsilon_0,$$

then the linear equation

$$u'' + \gamma(t)u = p_i(t) + \beta(t)$$

has a unique 2π-periodic solution u, and $c_0 \leq u(t) \leq C_0$, for every $t \in [t_0, t_0 + 2\pi]$.

The proof of Lemma 4.1 is similar to that of Lemma 1 in [9], and we can take ε_0 such that

$$0 < \varepsilon_0 < \min\{\rho(q_i) - m_i, m_i + 1 - \rho(q_i)\} \tag{4.1}$$

holds throughout the proof.

Lemma 4.2. *Assume (H_1^i), (H_2^i), and (H_5^i). Let $\varepsilon_0 > 0$ satisfy $\varepsilon_0 < \min\{\rho(\nu_1^i) - n_i,\ n_i + 1 - \rho(\nu_2^i)\}$ besides satisfying* (4.1)*, then for each function f_i, we can write it as*

$$f_i(t, x) = \tilde{a}_i(t, x)x_i^+ - \tilde{b}_i(t, x)x_i^- + r_i(t, x),$$

where $\tilde{a}_i, \tilde{b}_i, r_i : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are continuous functions such that, for almost every $t \in [t_0, t_0 + 2\pi]$, every $x \in \mathbb{R}^N$ and every $i = 1, 2, \cdots, m$,

$$q_i(t) - \varepsilon_0 \leq \tilde{a}_i(t, x) \leq q_i(t) + \varepsilon_0, \tag{4.2}$$

$$\nu_1^i(t) - \varepsilon_0 \leq \tilde{b}_i(t, x) \leq \nu_2^i(t) + \varepsilon_0, \tag{4.3}$$

and for every $i = m + 1, \cdots, N$,

$$M_i(t) - \varepsilon_0 \leq \tilde{a}_i(t, x) = \tilde{b}_i(t, x) \leq M_i(t) + \varepsilon_0, \tag{4.4}$$

and $r_i(t, x)$ is bounded: there is a 2π-periodic function $\tilde{r}_i(t)$ with $\tilde{r}_i \in L^1(t_0, t_0 + 2\pi)$, such that, for almost every $t \in [t_0, t_0 + 2\pi]$ and every $x \in \mathbb{R}^N$,

$$|r_i(t, x)| \leq \tilde{r}_i(t). \tag{4.5}$$

Proof. We will prove this lemma in two cases.

First, for $i \in \{1, 2, \cdots, m\}$, by (H_2^i), there exists a constant $D_i^+ > 0$ such that, when $x_i \geq D_i^+$,

$$q_i(t) - \varepsilon_0 \leq \frac{f_i(t, x)}{x_i} \leq q_i(t) + \varepsilon_0$$

holds for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\check{x}_i \in \mathbb{R}^{N-1}$. So, we define

$$\tilde{a}_i(t,x)=\begin{cases}\dfrac{f_i(t,x)}{x_i}, & x_i>D_i^+,\\ \dfrac{f(t,D_i^+,\check{x}_i)}{D_i^+}, & x_i\le D_i^+.\end{cases}$$

Similarly, by (H_1^i), there exists a constant $D_i^-<0$ such that, when $x_i\le D_i^-$,

$$\nu_1^i(t)-\varepsilon_0\le \frac{f_i(t,x)}{x_i}\le \nu_2^i(t)+\varepsilon_0$$

holds for a.e. $t\in[t_0,t_0+2\pi]$ and $\check{x}_i\in\mathbb{R}^{N-1}$. So, we define

$$\tilde{b}_i(t,x)=\begin{cases}\dfrac{f_i(t,x)}{x_i}, & x_i<D_i^-,\\ \dfrac{f(t,D_i^-,\check{x}_i)}{D_i^-}, & x_i\ge D_i^-.\end{cases}$$

Then, take

$$r_i(t,x)=f_i(t,x)-\tilde{a}_i(t,x)x_i^+ +\tilde{b}_i(t,x)x_i^-.$$

By the definitions of $\tilde{a}_i(t,x)$ and $\tilde{b}_i(t,x)$, we have $r_i(t,x)=0,\ x\notin[D_i^-,D_i^+]$, so the proof can be easily completed.

Second, for $i\in\{m+1,\cdots,N\}$, similar to above, by (H_5^i), there exists a constant $\tilde{D}_i>0$ such that, when $|x_i|\ge\tilde{D}_i$,

$$M_i(t)-\varepsilon_0\le\frac{f_i(t,x)}{x_i}\le M_i(t)+\varepsilon_0$$

holds for a.e. $t\in[t_0,t_0+2\pi]$ and $\check{x}_i\in\mathbb{R}^{N-1}$. So, we define

$$\tilde{a}_i(t,x)=\tilde{b}_i(t,x)=\begin{cases}\dfrac{f_i(t,x)}{x_i}, & |x_i|>\tilde{D}_i,\\ \dfrac{f(t,\tilde{D}_i,\check{x}_i)}{D_i^-}, & |x_i|\le\tilde{D}_i.\end{cases}$$

Then, take

$$r_i(t,x)=f_i(t,x)-\tilde{a}_i(t,x)x_i^+ +\tilde{b}_i(t,x)x_i^-.$$

By the definitions of $\tilde{a}_i(t,x)$ and $\tilde{b}_i(t,x)$, we have $r_i(t,x)=0,\ x\notin[-\tilde{D}_i,\tilde{D}_i]$, so the proof can also be completed. □

Now, we introduce a change of variable. In (1.1), we set

$$u(t)=\frac{1}{s}x(t),$$

with $u=(u_1,u_2,\cdots,u_N)$. Then Eq (1.1) is equivalent to the following system

$$u_i''+\frac{f_i(t,su)}{s}=p_i(t),\quad i=1,2,\cdots,N. \tag{4.6}$$

Lemma 4.3. *Assume* (H_0), (H_1^i)–(H_3^i) *and* (H_5^i). *Then there exists a* $s_1 \geq 1$ *such that for every* $s \geq s_1$, *system* (4.6) *has a* 2π*-periodic solution* $u(s,\cdot)$ *with the i-th component* $u_i(s,t)$ *satisfying*

$$c_0 \leq u_i(s,t) \leq C_0, \tag{4.7}$$

for every $t \in [t_0, t_0 + 2\pi]$ *and* $i = 1, 2, \cdots, N$, *where* c_0 *and* C_0 *are two positive constants given in Lemma 4.1.*

Proof. By Lemma 4.2, we can rewrite (4.6) as follows:

$$u_i'' + \tilde{a}_i(t,su)u_i^+ - \tilde{b}_i(t,su)u_i^- = p_i(t) - \frac{r_i(t,su)}{s}, \quad i = 1, 2, \cdots, N. \tag{4.8}$$

Next we will prove that (4.8) has a 2π-periodic solution with every component positive. If such a solution exists, it satisfies

$$u_i'' + \tilde{a}_i(t,su)u_i = p_i(t) - \frac{r_i(t,su)}{s}, \quad i = 1, 2, \cdots, N. \tag{4.9}$$

The converse is also true. Now we set $G_i(t,su) = \tilde{a}_i(t,su)u_i - p_i(t) + \dfrac{r_i(t,su)}{s}$. Next, we will discuss in two cases.

Case 1. For $i = 1, 2, \cdots, m$, by (4.2) and (4.5) we have

$$\lim_{u_i \to +\infty} \frac{G_i(t,su)}{u_i} = \lim_{x_i \to +\infty} \frac{f_i(t,x)}{x_i} = q_i(t). \tag{4.10}$$

We define a function

$$\tilde{G}_i(t,su) = \begin{cases} G_i(t,su), & u_i \geq 0, \\ 0, & u_i < 0, \end{cases}$$

then system (4.9) is changed into

$$u_i'' + \tilde{G}_i(t,su) = 0, \quad i = 1, 2, \cdots, m, \tag{4.11}$$

where $\tilde{G}_i(t,su)$ satisfies

$$\lim_{u_i \to +\infty} \frac{\tilde{G}_i(t,su)}{u_i} = q_i(t), \quad \lim_{u_i \to -\infty} \frac{\tilde{G}_i(t,su)}{u_i} = 0.$$

Then the solutions of the initial value problem associated with system (4.11) exist uniquely and globally. Indeed, on the one hand, by (H_0), there is a positive constant L_i such that

$$|\tilde{G}_i(t,su) - \tilde{G}_i(t,sv)| \leq |\frac{f_i(t,su)}{s} - \frac{f_i(t,sv)}{s}| \leq L_i|u_i - v_i|.$$

Thus every solution of the initial value problem associated with Eq (4.11) exists uniquely. On the other hand, by (4.2) and (4.5), we have

$$\begin{aligned} |\tilde{G}_i(t,su)| &\leq |G_i(t,su)| \leq |\tilde{a}_i(t,su)||u_i| + |p_i(t)| + \frac{1}{s}|r_i(t,su)| \\ &\leq (|q_i(t)| + \varepsilon_0)|u_i| + |p_i(t)| + \frac{1}{s}\tilde{r}_i(t) \leq C|u_i|_\infty + P_r, \end{aligned}$$

where $C = \max_{i=1,2,\cdots,N}\{|q_i(t)| + \varepsilon_0,\ t \in [t_0, t_0 + 2\pi]\}$, $P_r = \max_{i=1,2,\cdots,N}\{|p_i(t)| + \frac{1}{s}\tilde{r}_i(t),\ t \in [t_0, t_0 + 2\pi]\}$. Thus the solutions of the Cauchy problem associated with system (4.11) exist globally.

Secondly, we verify the nonresonance conditions. By (4.10) and the definition of $\tilde{G}_i(t, su)$, we have

$$\lim_{u_i\to+\infty} \frac{\tilde{G}_i(t, su)}{u_i} = q_i(t), \quad \lim_{u_i\to-\infty} \frac{\tilde{G}_i(t, su)}{u_i} = 0.$$

Consider the limit piecewise linear equations in the i-th projection plane

$$u_i'' + q_i(t)u_i^+ - 0u_i^- = 0,$$

that is

$$u_i'' + q_i(t)u_i = 0,$$

coupled with (H_3^i), we have that (S_1^i) and (S_2^i) hold in Theorem 3.1.

Case 2. For $i = m + 1, \cdots, N$, by (4.4) and (4.5) we have

$$\lim_{|x_i|\to+\infty} \frac{\tilde{G}_i(t, su)}{u_i} = M_i(t), \tag{4.12}$$

thus by (H_5^i), similar to that in Case 1, we can prove that any solution of the Cauchy problem associated with system (4.11) exists uniquely and globally. Then we consider the validity of the nonresonance condition for $i = m + 1, \cdots, N$. By (4.12), the limit linear equation in the i-th projection plane is

$$u_i'' + M_i(t)u_i = 0, \tag{4.13}$$

take $\tilde{\xi}_i(t) = M_i(t)$, and combing (H_5^i), we have that (S_3^i) holds.

Therefore, by Theorem 3.1, we have the existence of a 2π-periodic solution of system (4.11), which means that (4.9) has a 2π-periodic solution.

Next we will prove that every component of such a solution $u(s, t)$ is positive for s large enough. Notice that every component of $u(s, t)$ solves the following equation:

$$u_i'' + \tilde{a}_i(t, su(s, t))u_i = p_i(t) - \frac{r_i(t, su(s, t))}{s}. \tag{4.14}$$

Now set $s_1 = \dfrac{1}{\varepsilon_0}\|\tilde{r}_i\|_1$, for every $s \geq s_1$, by (4.2) and (4.5), we have

$$\|\tilde{a}_i(\cdot, su(s, \cdot)) - q_i(\cdot)\|_1 \leq \varepsilon_0, \quad \|\frac{r_i(\cdot, su(s, \cdot))}{s}\|_1 \leq \varepsilon_0.$$

Thus by Lemma 4.1, for $s \geq s_1$, Eq (4.14) has a unique 2π-periodic solution, which must coincide with $u_i(s, t)$ and satisfy $c_0 \leq u_i(s, t) \leq C_0$. □

Next, by another change of variables

$$v(t) = u(t) - u(s, t),$$

with $v = (v_1, v_2, \cdots, v_N)$. Equation (4.6) is changed into

$$v_i'' + \frac{f_i(t, s(v + u(s, t))) - f_i(t, su(s, t))}{s} = 0, \quad i = 1, 2, \cdots, N, \tag{4.15}$$

then, we can see that $v = 0$ is a solution of (4.15).

Lemma 4.4. *For every $i \in \{1, 2, \cdots, m\}$,*

$$\lim_{s\to+\infty} \frac{f_i(t, s(v + u(s,t))) - f_i(t, su(s,t))}{s} = q_i(t)v_i$$

holds uniformly for every $t \in [t_0, t_0 + 2\pi]$ and $v \in \mathbb{R}^N$ with $v_i \in [-\frac{1}{2}c_0, \frac{1}{2}c_0]$.

Proof. For every $i \in \{1, 2, \cdots, m\}$, by (4.7) and (H_2^i), we can deduce that

$$\lim_{s\to+\infty} \frac{f_i(t, s(v + u(s,t)))}{s(v_i + u_i(s,t))} = q_i(t),$$

and

$$\lim_{s\to+\infty} \frac{f_i(t, su(s,t))}{su_i(s,t)} = q_i(t),$$

hold uniformly for every $t \in [t_0, t_0 + 2\pi]$ and $v \in \mathbb{R}^N$ with $v_i \in [-\frac{1}{2}c_0, \frac{1}{2}c_0]$. Therefore, for every $\varepsilon > 0$ and s large enough, we can deduce that

$$\begin{aligned}
&|\tilde{f}_i(s,t,v) - q_i(t)v_i|\\
=&\left|\frac{f_i(t, s(v + u(s,t))) - f_i(t, su(s,t))}{s} - q_i(t)v_i\right|\\
\leq&\left|\frac{f_i(t, s(v + u(s,t))) - q_i(t)s(v_i + u_i(s,t))}{s(v_i + u_i(s,t))}\right| \cdot |v_i + u_i(s,t)| + \left|\frac{f_i(t, su(s,t)) - q_i(t)su_i(s,t)}{su_i(s,t)}\right| \cdot |u_i(s,t)|\\
\leq&(c_0/2 + 2C_0)\varepsilon,
\end{aligned}$$

holds for every $t \in [t_0, t_0 + 2\pi]$ and $v \in \mathbb{R}^N$ with $v_i \in [-\frac{1}{2}c_0, \frac{1}{2}c_0]$. □

5. Proof of Theorem 1.1

Set

$$\tilde{f}_i(s,t,v) = \frac{f_i(t, s(v + u(s,t))) - f_i(t, su(s,t))}{s},$$

then system (4.15) is changed into

$$v_i'' + \tilde{f}_i(s,t,v) = 0, \quad i = 1, 2, \cdots, N. \tag{5.1}$$

Specifically, the actual form of system (5.1) is

$$\begin{cases}
v_1'' + \tilde{f}_1(s,t,v) = 0,\\
\cdots\\
v_m'' + \tilde{f}_m(s,t,v) = 0,\\
v_{m+1}'' + M_i(t)v_{m+1}(s,t,v) + \tilde{e}_{m+1}(s,t,v) = 0,\\
\cdots\\
v_N'' + M_i(t)v_N(s,t,v) + \tilde{e}_N(s,t,v) = 0,
\end{cases}$$

where $\tilde{e}_i(s,t,v) = \dfrac{e_i(t, s(v+u(s,t))) - e_i(t, su(s,t))}{s}$, $i = m+1, \cdots, N$. For the convenience of description, we still use (5.1) to represent the above system.

Consider the equivalent system

$$v_i' = -w_i, \quad w_i' = \tilde{f}_i(s,t,v), i = 1, 2, \cdots, N \tag{5.2}$$

associated with system (5.1), where $\tilde{f}_i(s,t,v)$ satisfies the following conditions, $i = 1, 2, \cdots, m$.

$(H_0)'$ $\tilde{f} : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, 2π-periodic in the first variable, and locally Lipschitz-continuous in the second variable, $\tilde{f}(s,t,0) = 0$.

$(H_1^i)'$ For the function $v_1^i(t), v_2^i \in L^1([t_0, t_0+2\pi], \mathbb{R})$ in (H_1^i), we have

$$v_1^i(t) \le \liminf_{v_i \to -\infty} \frac{\tilde{f}_i(s,t,v)}{v_i} \le \limsup_{v_i \to -\infty} \frac{\tilde{f}_i(s,t,v)}{v_i} \le v_2^i(t) \tag{5.3}$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\check{x}_i \in \mathbb{R}^{N-1}$.

$(H_2^i)'$ For the function $q_i(t) \in L^1([t_0, t_0+2\pi], \mathbb{R})$ in (H_2^i), we have

$$\lim_{v_i \to +\infty} \frac{\tilde{f}_i(s,t,v)}{v_i} = q_i(t)$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\check{x}_i \in \mathbb{R}^{N-1}$.

And for every $i \in \{m+1, \cdots, N\}$, we assume the following condition:

$(H_5^i)'$ $\tilde{f}_i(s,t,v) = M_i(t)v_i + \tilde{e}_i(s,t,v)$, with $e_i : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ being bounded, $\mathbb{M}(t) = \mathrm{diag}(M_{m+1}(t), \cdots, M_N(t))$ being a symmetric matrix, continuous and 2π-periodic with respect to the time variable, satisfying the nonresonance condition

$$v_{m,N}(t) \equiv 0 \quad \text{is the only } 2\pi-\text{periodic solution of } v_{m,N}'' = \mathbb{M}(t)v_{m,N}, \tag{5.4}$$

where $v_{m,N} = (v_{m+1}, \cdots, v_N)$.

Now, we will outline the deduction process for the aforementioned conditions based on (H_0), (H_1^i), (H_2^i) and (H_5^i).

Firstly, by assumption (H_0), we can conclude that $\tilde{f}_i : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function, 2π-periodic with respect to the first variable, and $\tilde{f}(s,t,0) = 0$. Moreover, using the Lipschitz-continuity of f with respect to the second variable as presented in (H_0), we can deduce that, for arbitrary $t \in \mathbb{R}$ and for $v_I, v_{II} \in U(v(s,t_0))$, where $U(v(s,t_0))$ represents an arbitrary neighborhood of $v(s,t_0)$, there exists a positive constant $\tilde{L}_i$ such that

$$\left|\tilde{f}_i(s,t,v_I) - \tilde{f}_i(s,t,v_{II})\right| = \frac{1}{s}\left|f_i(t, s(v_I + u(s,t))) - f_i(t, s(v_{II} + u(s,t)))\right| \le \tilde{L}_i |v_I^i - v_{II}^i|,$$

where v_I^i and v_{II}^i are the i-th component of v_I^i, v_{II}^i, respectively.

Secondly, by (H_1^i), we can conclude that

$$\liminf_{v_i \to -\infty} \frac{\tilde{f}_i(s,t,v)}{v_i} = \liminf_{v_i \to -\infty} \frac{f_i(t, s(v+u(s,t))) - f_i(t, su(s,t))}{sv_i} = \liminf_{v_i \to -\infty} \frac{f_i(t, s(v+u(s,t)))}{sv_i}$$
$$= \liminf_{v_i \to -\infty} \frac{f_i(t, s(v+u(s,t)))}{s(v_i + u_i(s,t))} \cdot \frac{s(v_i + u_i(s,t))}{sv_i} = \liminf_{v_i \to -\infty} \frac{f(t, s(v+u(s,t)))}{s(v_i + u_i(s,t))} \ge v_1^i(t),$$

and

$$\limsup_{v_i\to-\infty}\frac{\tilde{f}_i(s,t,v)}{v_i}=\limsup_{v_i\to-\infty}\frac{f_i(t,s(v+u(s,t)))-f_i(t,su(s,t))}{sv_i}\le v_2^i(t).$$

Similarly, by (H_2^i), we have

$$\begin{aligned}\lim_{v_i\to+\infty}\frac{\tilde{f}_i(s,t,v)}{v_i}&=\lim_{v_i\to+\infty}\frac{f_i(t,s(v+u(s,t)))-f_i(t,su(s,t))}{sv_i}=\lim_{v_i\to+\infty}\frac{f_i(t,s(v+u(s,t)))}{sv_i}\\&=\lim_{v_i\to+\infty}\frac{f_i(t,s(v+u(s,t)))}{s(v_i+u_i(s,t))}\cdot\frac{s(v_i+u_i(s,t))}{sv_i}=\lim_{v_i\to+\infty}\frac{f(t,s(v+u(s,t)))}{s(v_i+u_i(s,t))}=q_i(t).\end{aligned}$$

Finally, by the boundedness of e_i outlined in (H_5^i), we can conclude that $\tilde{e}_i$ is bounded. Therefore, $(H_5^i)'$ holds.

Denote by $\tilde{z}(s,t):=(v(s,t),w(s,t))$ a solution of Eq (5.2) with initial value $\tilde{z}_0:=\tilde{z}(s,t_0)=(v(s,t_0),w(s,t_0))$, $\tilde{z}=(\tilde{z}_1,\cdots,\tilde{z}_N)$, $\tilde{z}_0=(\tilde{z}_{0_1},\cdots,\tilde{z}_{0_N})$. If the component $\tilde{z}_i=(\tilde{v}_i,\tilde{w}_i)$ does not cross the origin, we can pass to the standard polar coordinates

$$v_i=r_i\cos\theta_i,\quad w_i=r_i\sin\theta_i,$$

we have

$$\begin{cases}\theta_i'=\sin^2\theta_i+\dfrac{\tilde{f}_i(s,t,v)}{r_i}\cos\theta_i,\\ r_i'=-r_i\sin\theta_i\cos\theta_i+\tilde{f}_i(s,t,v)\sin\theta_i,\end{cases}\qquad i=1,2,\cdots,N. \tag{5.5}$$

Denote by $(\tilde{\theta}(s,t),\tilde{r}(s,t)):=(\theta(s,t;r_0,\theta_0),r(s,t;r_0,\theta_0))$ a solution of system (5.5) with $(\tilde{\theta}(s,t_0),\tilde{r}(s,t_0))=(\theta_0,r_0)$. $(\tilde{\theta},\tilde{r})=((\tilde{\theta}_1,\tilde{r}_1),\cdots,(\tilde{\theta}_N,\tilde{r}_N))$, $(\theta_0,r_0)=((\theta_{0_1},r_{0_1}),\cdots,(\theta_{0_N},r_{0_N}))$.

Next, we give some general properties of system (5.1), containing global existence and rotational property.

Lemma 5.1. *Assume $(H_0)'$–$(H_2^i)'$ and $(H_5^i)'$, then solutions of system* (5.1) *exist globally.*

Proof. Using Lemma 4.2, we can write function $\tilde{f}_i(s,t,v)$ as

$$\tilde{f}_i(s,t,v)=\tilde{a}_i(s,t,v)v_i^+-\tilde{b}_i(s,t,v)v_i^-+r_i(s,t,v),$$

where

$$\tilde{a}_i(s,t,v)=\tilde{a}_i(t,s(v+u(s,t))),\quad \tilde{b}_i(s,t,v)=\tilde{b}_i(t,s(v+u(s,t))).$$

Therefore, for almost every $t\in[t_0,t_0+2\pi]$ and every $v\in\mathbb{R}^N$, it follows that

$$q_i(t)-\varepsilon_0\le\tilde{a}_i(s,t,v)\le q_i(t)+\varepsilon_0,$$

$$v_1^i(t)-\varepsilon_0\le\tilde{b}_i(s,t,v)\le v_2^i(t)+\varepsilon_0.$$

Furthermore, by Lemma 4.3, we have

$$0\le(v_i+u_i(s,t))^+-v_i^+\le u_i(s,t)\le C_0,\quad -C_0\le-u_i(s,t)\le(v_i+u_i(s,t))^--v_i^-,$$

then we can conclude that $r_i(s,t,v)$ is bounded independently of $s\ge1$, namely,

$$|r_i(s,t,v)| \le \tilde{R}(t), \quad \text{for } a.e.\ t \in [t_0, t_0 + 2\pi],$$

with $\tilde{R}(t) = (2|q_i(t)| + |v_2^i(t)| + 3\varepsilon_0) + 2\tilde{r}_i(t)$. Especially, by $(H_1^i)'$ and $(H_2^i)'$, for $s \ge 1$, it follows that

$$|\tilde{f}_i(s,t,v)| \le \tilde{C}_i|v_i| + \tilde{R}_i(t), \quad \text{for } a.e.\ t \in [t_0, t_0 + 2\pi] \text{ and } \check{v}_i \in \mathbb{R}^{N-1},$$

where $\tilde{C}_i = \max_{t\in[t_0,t_0+2\pi]}\{|q_i(t)|, |v_2^i(t)|\} + \varepsilon_0$. Therefore, $\tilde{f}_i(s,t,v)$ grows at most linearly in v_i, and so the solution to the Cauchy problem associated with system (5.2) exists globally. □

In order to find the inner boundary of a suitable annulus in the Poincaré-Birkhoff theorem, we need the following lemma, which comes from Lemma 2.5 in [10].

Lemma 5.2. *There are three positive constants δ, r_i, and s_2, with $\delta < r_i < \frac{1}{2}c_0$ and $s_2 \ge s_1$, such that, for every $s \ge s_2$, if $r_i(t_0) = r_i$, then we have*

$$\delta < r_i(t) < \frac{1}{2}c_0,$$

for every $t \in [t_0, t_0 + 2\pi]$ and $i \in \{1, 2, \cdots, m\}$.

Proof. We first prove that $r_i(t) < \dfrac{1}{2}c_0$ for every $t \in [t_0, t_0 + 2\pi]$. We assume by contradiction that there exists a $\bar{t} \in [t_0, t_0 + 2\pi]$ such that

$$r_i(t) < \frac{1}{2}c_0 \quad \text{for every } t \in [t_0, \bar{t}), \quad \text{and} \quad r_i(\bar{t}) = \frac{1}{2}c_0. \tag{5.6}$$

Set

$$\tilde{r}_i = \frac{1}{8}c_0 e^{-2(1+\|q\|_\infty)\pi}, \quad \delta = \frac{1}{4}\tilde{r}_i e^{-2(1+\|q\|_\infty)\pi} \quad \text{and} \quad \varepsilon = \frac{\tilde{r}_i}{2\pi}.$$

It is clear that $0 < \delta < \tilde{r}_i < \dfrac{1}{2}c_0$. By Lemma 4.4, there exists a $s_2 \ge s_1$ such that, for every $s \ge s_2$, almost every $t \in [t_0, \bar{t}]$ and every $v_i \in [-\dfrac{1}{2}c_0, \dfrac{1}{2}c_0]$,

$$|\tilde{f}_i(s,t,v) - q_i(t)v_i| \le \varepsilon.$$

Then by (5.5), we have

$$|r_i'(t)| = |r_i \sin\theta_i \cos\theta_i - \tilde{f}_i(s,t,v)\sin\theta_i| \le r_i(t) + |\tilde{f}_i(s,t,v)| \le (1 + \|q_i\|_\infty)r_i(t) + \varepsilon.$$

By Gronwall's inequality, we have

$$r_i(t) \le (\tilde{r}_i + \varepsilon\bar{t})e^{(1+\|q_i\|_\infty)t},$$

thus for the solution of (5.1), we have

$$r_i(\bar{t}) \le (\tilde{r}_i + 2\varepsilon\pi)e^{2(1+\|q_i\|_\infty)\pi} = \frac{1}{4}c_0,$$

which contradicts (5.6). By a similar discussion, we can prove that $r_i(t) > \delta$ for every $t \in [t_0, t_0+2\pi]$. □

In order to complete the proof of Theorem 1.1, we apply a consequence of an extended version of the higher dimensional Poincaré-Birkhoff theorem for Hamiltonian systems (see Corollary 5.2 in [13]), which is a direct consequence of Theorem 1.1 in [13].

Proof of Theorem 1.1. We will divide the proof into three steps.
Step 1. Define a set

$$\Omega_i := \{\tilde{z}_i \in \mathbb{R}^2 : \delta < |\tilde{z}_i| < \frac{1}{2}c_0\},$$

and let

$$\Gamma_i^- := \{\tilde{z}_i(s,t) : |\tilde{z}_i(s,t)| = r_i\},$$

for $i = 1, 2, \cdots, m$.

Now, consider a solution $\tilde{z}(s,t)$ of (5.2) with $\tilde{z}_{0_i} \in \Gamma_i^-$. By Lemma 5.2, there is a positive constant s_2 with $s_2 \geq s_1$, such that $\tilde{z}_i(s,t) \in \Omega_i$ when $s \geq s_2$, that is

$$\delta < |\tilde{z}_i(s,t)| < \frac{1}{2}c_0, \quad t \in [t_0, t_0 + 2\pi].$$

Therefore, it follows that the component $v_i(s,t)$ satisfies $0 < |v_i(s,t)| \leq \frac{1}{2}c_0$. Consequently, by Lemma 4.4, it follows that

$$\lim_{s\to+\infty} \tilde{f}_i(s,t,v) = q_i(t)v_i$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\tilde{z}_i = (v_i, w_i) \in \Omega_i$. So we have

$$\lim_{s\to+\infty} \frac{\tilde{f}_i(s,t,v)}{v_i} = q_i(t) \tag{5.7}$$

holds uniformly for a.e. $t \in [t_0, t_0 + 2\pi]$ and $\tilde{z}_i = (v_i, w_i) \in \Omega$. Furthermore, we observe that

$$s \to +\infty \iff s(v_i + u_i(s,t)) \to +\infty,$$

where $v_i + u_i(s,t) \in [\frac{1}{2}c_0, \frac{1}{2}c_0 + C_0]$. Then by (H_3^i) and Lemma 2.3, we have

$$m_i < \mathrm{Rot}_{\tilde{f}_i}(\tilde{z}_0) < m_i + 1, \quad \text{for} \quad \tilde{z}_{0_i} \in \Gamma_i^-. \tag{5.8}$$

Step 2. By $(H_1^i)'$ and $(H_2^i)'$, we have

$$\nu_1^i(t) \leq \liminf_{v_i\to-\infty} \frac{\tilde{f}_i(s,t,v)}{v_i} \leq \limsup_{v_i\to-\infty} \frac{\tilde{f}_i(s,t,v)}{v_i} \leq \nu_2^i(t), \qquad \lim_{v_i\to+\infty} \frac{\tilde{f}_i(s,t,v)}{v_i} = q_i(t).$$

Therefore, by (H_4^i) and Lemma 2.3, for $s \geq s_2$, there exists a $R_i > 0$ such that if the component $\tilde{z}_i(s,t)$ of a solution of system (5.2) satisfying $|\tilde{z}_i(s,t)| \geq R_i$, $t \in [t_0, t_0 + 2\pi]$, it follows that

$$n_i < \mathrm{Rot}_{\tilde{f}_s}(\tilde{z}_0) < n_i + 1. \tag{5.9}$$

Let

$$\Gamma_i^+ := \{z_i : |z_i| = R_i\},$$

consider the solution of system (5.2) starting with $\tilde{z}_{0_i} \in \Gamma_i^+$. From (4.1), we have

$$n_i < \mathrm{Rot}_{\tilde{f}_i}(\tilde{z}_0) < n_i + 1, \quad \text{for} \quad \tilde{z}_{0_i} \in \Gamma_i^+. \tag{5.10}$$

Step 3. Consider the annulus $\mathcal{A} = \overline{\mathrm{int}(\Gamma_i^+)} \setminus \mathrm{int}(\Gamma_i^-)$. For every $i \in \{1, 2, \cdots, m\}$, take $l_i = m_i + 1, m_i + 2, \cdots, n_i$ if $n_i > m_i$ (take $l_i = n_i + 1, \cdots, m_i$ if $m_i > n_i$), then for each l_i, by (5.8), (5.10), and Corollary 5.2 [13], system (5.2) has at least $m + 1$ distinct 2π-periodic solutions such that

$$\mathrm{Rot}_{\tilde{f}_i}(\tilde{z}_0) = l_i. \tag{5.11}$$

So that system (5.2) has at least $(m+1)\prod_{i=1}^m |n_i - m_i|$ distinct 2π-periodic solutions.

And recalling the zero solution of system (4.15) that corresponds to the solution of $u(s,t)$ of (5.1). Therefore, we get $(m+1)\prod_{i=1}^m |n_i - m_i| + 1$ distinct 2π-periodic solutions of system (5.1). Therefore, system (1.1) has at least $(m+1)\prod_{i=1}^m |n_i - m_i| + 1$ distinct 2π-periodic solutions. □

Remark 5.1. *In order to illustrate the application of the main result, we present an example. Consider the coupled system*

$$\begin{cases} x_1'' + f_1(t, x_1) + e_1(t, x_1, x_2) = sp_1(t), \\ x_2'' + M_2(t)x_2 + e_2(t, x_1, x_2) = sp_2(t), \end{cases} \tag{5.12}$$

where e_i, $i = 1, 2$ are bounded functions, $M_2(t)$ is defined as that in (H_5^i), and $f_1(t, x_1)$ is defined as

$$f_1(t, x_1) = \begin{cases} v_1^1(t)x_1, & x_1 \le 0, \\ q_1(t)x_1, & x_1 > 0, \end{cases}$$

where $v_1^1(t)$ and $q_1(t)$ are the functions defined in Remark 1.1. This means that we let $m = 1$ and $N = 2$ in Theorem 1.1. Then we can deduce that

$$v_1^1(t) \le \liminf_{x_1 \to -\infty} \frac{f_1(t, x_1) + e_1(t, x_1, x_2)}{x_1} \le \limsup_{x_1 \to -\infty} \frac{f_1(t, x_1) + e_1(t, x_1, x_2)}{x_1} \le v_2^1(t), \tag{5.13}$$

and

$$\lim_{x_1 \to +\infty} \frac{f_1(t, x_1) + e_1(t, x_1, x_2)}{x_1} = q_1(t) \tag{5.14}$$

hold uniformly for a.e. $t \in [0, 2\pi]$. Therefore, (H_1^1) and (H_2^1) hold. By Remark 1.1, we can verify the validity of (H_3^1) and (H_4^1) holding. Then for the system (5.12), there is a $s_0 \ge 0$ such that, for every $s \ge s_0$, it has at least $2|n_1 - m_1| + 1$ distinct 2π-periodic solutions by Theorem 1.1.

In order to facilitate the proof of (1.11), we prepare the following lemma. This lemma can be seen as a slight generalization of Proposition 1 in [14].

Lemma 5.3. *For the piecewise linear system (2.3), the following two statements hold:*

(i) If $a_i^+(t) \ge \eta^2$, $a_i^-(t) \ge \tau^2$, $t \in [t_0, t_0 + 2\pi]$. Then

$$\rho(a_i) \ge \frac{2\eta\tau}{\eta + \tau}.$$

(ii) If $a_i^+(t) \ge \eta^2$, $a_i^-(t) \ge \tau^2$, $t \in [t_0, t_0 + \chi]$; and $a_i^+(t)$, $a_i^-(t)$ take other values for $t \in (t_0 + \chi, t_0 + 2\pi]$. Then

$$\rho(a_i) \ge [\frac{\eta\tau\chi}{(\eta + \tau)\pi}].$$

Proof of (1.11). (i) We show that $\rho(q_i) > m_i$. By Lemma 4.4 in [16], it holds $\rho(q_i) \geq m_i$. If $\rho(q_i) = m_i$, from Lemma 2.1, there is at least one nontrivial 2π-periodic solution $\theta_i(t)$ of the torus differential equation

$$\theta_i' = q_i(t)\cos^2\theta_i + \sin^2\theta_i, \tag{5.15}$$

with $\theta_i(2\pi) - \theta_i(0) = 2m_i\pi$. Using a simple computation, it holds

$$\theta_i\Big(\frac{2m_i\pi}{2m_i+1}\Big) - \theta_i(0) = 2m_i\pi.$$

Then $\theta_i(2\pi) - \theta_i\Big(\frac{2m_i\pi}{2m_i+1}\Big) = 0$. Using Lemma 4.5 in [16], we have $\theta_i(2\pi) - \theta_i(\pi) > -2\arctan|\lambda_i|$, then it follows that

$$\theta_i(\pi) - \theta_i\Big(\frac{2m_i\pi}{2m_i+1}\Big) < 2\arctan|\lambda_i|.$$

Since

$$\theta_i' = \sin^2\theta_i + (2m_i+1)^2\cos^2\theta_i > 1, \quad \text{for } t \in \Big(\frac{2m_i\pi}{2m_i+1}, \pi\Big),$$

then

$$\theta_i(\pi) - \theta_i\Big(\frac{2m_i\pi}{2m_i+1}\Big) > \pi/(2m_i+1).$$

Thus $\arctan|\lambda_i| > \pi/(2(2m_i+1))$. This contradicts to the definitions of λ_i in Remark 1.1. Therefore, $\rho(q_i) > m_i$.

(ii) We prove $\rho(q_i) < m_i + 1$. Let $\bar{\theta}_i(t)$ be a solution of (5.15) with $\bar{\theta}_i(0) = 0$. Since $q_i(t) = (2m_i+1)^2$ for $t \in [0, \pi]$, it follows that $\bar{\theta}_i(\pi) = (2m_i+1)\pi$. Furthermore, we can observe that $y_i' = -\mu_i^2 x$ for $t \in [\pi, 2\pi]$, and it implies that nonzero solutions of (5.15) can never perform counterclockwise rotations at x_i-axis when $t \in [\pi, 2\pi]$. Therefore, we have

$$\bar{\theta}_i(t) < (2m_i+1)\pi, \quad \text{for } t \in [\pi, 2\pi].$$

So it follows that $\bar{\theta}_i(2\pi) < (2m_i+1)\pi$. Furthermore, by the uniqueness of the solution for Cauchy problem associated with Eq (5.15), we have

$$\bar{\theta}_i(2k_i\pi) < \bar{\theta}_i(2(k_i-1)\pi) + (2m_i+1)\pi < (2m_i+1)k_i\pi, \quad \text{for } k_i \in \mathbb{N}.$$

Therefore, by the definition of rotation number in Section 2, we have

$$\rho(q_i) = \lim_{k_i\to+\infty} \frac{\bar{\theta}_i(2k_i\pi) - \bar{\theta}_i(0)}{2k_i\pi} \leq m_i + \frac{1}{2} < m_i + 1.$$

(iii) We prove $\rho(\nu_1^i) > n_i$. By (ii) of Lemma 5.3 and (1.10), we have $\rho(\nu_1^i) \geq n_i$. If $\rho(\nu_1^i) = n_i$, from Lemma 2.1, we can conclude that there is at least one nontrivial 2π-periodic solution $\theta_i(t)$ of the torus differential equation

$$\theta_i' = q_i(t)((\cos\theta_i)^+)^2 + \nu(t)((\cos\theta_i)^-)^2 + \sin^2\theta_i,$$

with $\theta_i(2\pi) - \theta_i(0) = 2n_i\pi$. Using a simple computation, it holds

$$\theta_i\Big(\frac{n_i\pi}{2m_i+1} + \frac{n_i\pi}{2\alpha_1^i+1}\Big) - \theta_i(0) = 2n_i\pi.$$

Then $\theta_i(2\pi)-\theta_i\Big(\frac{n_i\pi}{2m_i+1}+\frac{n_i\pi}{2\alpha_1^i+1}\Big)=0$. Similar to Lemma 4.5 in [16], it follows that

$$-\max\{2\arctan|\lambda_i|,\ 2\arctan|\mu_1^i|\}<\theta_i(2\pi)-\theta_i(\pi)<\pi-\arctan|\lambda_i|-\arctan|\mu_1^i|,$$

which implies

$$\theta_i(\pi)-\theta_i\Big(\frac{n_i\pi}{2m_i+1}+\frac{n_i\pi}{2\alpha_1^i+1}\Big)<2\max\{\arctan|\lambda_i|,\ \arctan|\mu_1^i|\}.$$

Since

$$\theta_i'=\sin^2\theta_i+(2m_i+1)^2\cos^2\theta_i>1,\quad \text{for}\quad t\in\Big(\frac{n_i\pi}{2m_i+1}+\frac{n_i\pi}{2\alpha_1^i+1},\pi\Big),$$

then

$$\theta_i(\pi)-\theta_i\Big(\frac{n_i\pi}{2m_i+1}+\frac{n_i\pi}{2\alpha_1^i+1}\Big)>\pi-\frac{n_i\pi}{2m_i+1}-\frac{n_i\pi}{2\alpha_1^i+1}.$$

Thus we have $2\max\{\arctan|\lambda_i|,\ \arctan|\mu_i|\}>\pi-n_i\pi/(2m_i+1)-n_i\pi/(2\alpha_1^i+1)$. This contradicts to the definitions of λ_i and μ_i in Remark 1.1. Therefore, $\rho(\nu_1^i)>n_i$.

(iv) We prove $\rho(\nu_2^i)<n_i+1$. Consider the first-order linear system

$$x_i'=-y_i,\quad y'=q_i(t)x_i^+-\nu_2^i(t)x_i^-, \tag{5.16}$$

associated with Eq (1.8). Denote by $\bar\theta_i(t)$ a nontrivial 2π-periodic solution of the torus differential equation

$$\theta_i'=q_i(t)((\cos\theta_i)^+)^2+\nu_2^i(t)((\cos\theta_i)^-)^2+\sin^2\theta_i$$

associated to system (5.16), with $\bar\theta_i(0)=\pi/2$. Let t_1 denote the instant at which the solution component $z_i(t;0,\pi/2)$ first intersects the negative y-axis in a counterclockwise manner, and t_2 denote the instant at which the solution component $z_i(t;0,\pi/2)$ first intersects the positive y-axis in a counterclockwise manner. Then, by two simple calculations, we have

$$t_1=\int_0^\pi\frac{d\theta_i}{(2\alpha_2^i+1)^2\cos^2\theta_i+\sin^2\theta_i}=\frac{\pi}{2\alpha_2^i+1},$$

and

$$t_2-t_1=\int_0^\pi\frac{d\theta_i}{(2m_i+1)^2\cos^2\theta_i+\sin^2\theta_i}=\frac{\pi}{2m_i+1}.$$

Therefore, we can calculate the time Δt_{n_i} used by the solution component to perform n_i turns around the origin, meanwhile, by (1.10), we can conclude that

$$\Delta t_{n_i}=\frac{n_i\pi}{2\alpha_2^i+1}+\frac{n_i\pi}{2m_i+1}<\pi.$$

For the time Δt_{n_i+1} used by the solution component to perform n_i+1 turns around the origin, by (1.10), it follows that

$$\Delta t_{n_i+1}=\frac{(n_i+1)\pi}{2\alpha_2^i+1}+\frac{(n_i+1)\pi}{2m_i+1}>\pi.$$

On the other hand, we can observe that $y_i'=-\lambda_i^2x_i$ or $y_i'=-\mu_i^2x_i$ for $t\in[\pi,2\pi]$, and it implies that nonzero solutions of (5.16) can never perform counterclockwise rotations at x_i-axis when $t\in[\pi,2\pi]$.

Therefore, we can deduce that the solution component has completed fewer than n_i+1 full turns around the origin, namely

$$\mathrm{Rot}_{v_2^i}(\omega_0) < n_i + 1, \quad \omega_0 \in \mathbb{S}^1.$$

Using Lemma 2.1 again, it follows that $\rho(v_2^i) < n_i + 1$. □

6. Conclusions

In this paper, we studied the multiplicity of periodic solutions for parameterized systems coupling asymmetric components and linear components, and obtained the multiplicity of periodic solutions. It is formulated in an original way, relying on sufficiently general assumptions.

Variable transformation is a key step in this paper, which requires the existence of periodic solutions of system (3.1), so we present a new existence theorem in Section 3. Moreover, we address the challenges arising from the sign-changing nature of the nonlinearity.

Author contributions

Haihua Lu and Tianjue Shi: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. The final version of the manuscript was read and approved by all authors.

Use of Generative-AI tools declaration

We declare that we have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

We hereby declare that there is no conflict of interest.

References

1. A. Ambrosetti, G. Prodi, On the inversion of some differentiable mappings with singularities between Banach spaces, *Ann. Mat. Pur. Appl.*, **93** (1972), 231–246. https://doi.org/10.1007/BF02412022
2. M. Berger, E. Podolak, J. Moser, On the solutions of a nonlinear Dirichlet problem, *Indiana Univ. Math. J.*, **24** (1975), 837–846.

3. C. Fabry, J. Mawhin, M. Nkashama, A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations, *Bull. Lond. Math. Soc.*, **18** (1986), 173–180. https://doi.org/10.1112/blms/18.2.173
4. R. Ortega, Stability of a periodic problem of Ambrosetti-Prodi type, *Differ. Integral Equ.*, **3** (1990), 275–284. https://doi.org/10.57262/die/1371586143
5. M. A. Del Pino, R. F. Manásevich, A. Murua, On the number of 2π-periodic solutions for $u'' + g(u) = s(1 + h(t))$ using the Poincaré-Birkhoff theorem, *J. Differ. Equations*, **95** (1992), 240–258. https://doi.org/10.1016/0022-0396(92)90031-H
6. C. Rebelo, F. Zanolin, Multiplicity results for periodic solutions of second order odes with asymmetric nonlinearities, *Trans. Amer. Math. Soc.*, **348** (1996), 2349–2389. https://doi.org/10.1090/S0002-9947-96-01580-2
7. C. Rebelo, F. Zanolin, Multiple periodic solutions for a second order equation with one-sided superlinear growth, *Dyn. Contin. Discrete Impuls. Syst. A*, **2** (1996), 1–27.
8. C. Zanini, F. Zanolin, A multiplicity result of periodic solutions for parameter dependent asymmetric non-autonomous equations, *Dyn. Contin. Discrete Impuls. Syst. A*, **12** (2005), 1–12.
9. A. Fonda, L. Ghirardelli, Multiple periodic solutions of scalar second order differential equations, *Nonlinear Anal.-Theor.*, **72** (2010), 4005–4015. https://doi.org/10.1016/j.na.2010.01.032
10. A. Calamai, A. Sfecci, Multiplicity of periodic solutions for systems of weakly coupled parametrized second order differential equations, *Nonlinear Differ. Equ. Appl.*, **24** (2017), 4. https://doi.org/10.1007/s00030-016-0427-5
11. A. Lazer, P. McKenna, Large scale oscillatory behaviour in loaded asymmetric systems, *Ann. I. H. Poincaré C*, **4** (1987), 243–274. https://doi.org/10.1016/S0294-1449(16)30368-7
12. E. Sovrano, Nonlinear differential equations having non-sign-definite weights, Ph. D Thesis, University of Udine, 2018.
13. A. Fonda, P. Gidoni, Coupling linearity and twist: an extension of the Poincaré-Birkhoff theorem for Hamiltonian systems, *Nonlinear Differ. Equ. Appl.*, **27** (2020), 55. https://doi.org/10.1007/s00030-020-00653-9
14. D. Qian, P. Torres, P. Wang, Periodic solutions of second order equations via rotation numbers, *J. Differ. Equations*, **266** (2019), 4746–4768. https://doi.org/10.1016/j.jde.2018.10.010
15. S. Gan, M. Zhang, Resonance pockets of Hill's equations with two-step potentials, *SIAM J. Math. Anal.*, **32** (2000), 651–664. https://doi.org/10.1137/S0036141099356842
16. C. Liu, D. Qian, P. Torres, Non-resonance and double resonance for a planar system via rotation numbers, *Results Math.*, **76** (2021), 91. https://doi.org/10.1007/s00025-021-01401-w
17. C. Liu, Non-resonance with one-sided superlinear growth for indefinite planar systems via rotation numbers, *AIMS Mathematics*, **7** (2022), 14163–14186. https://doi.org/10.3934/math.2022781
18. C. Liu, D. Qian, A new fixed point theorem and periodic solutions of nonconservative weakly coupled systems, *Nonlinear Anal.*, **192** (2020), 111668. https://doi.org/10.1016/j.na.2019.111668

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