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Research article

Geometric inequalities and equality conditions for slant submersions in Kenmotsu space forms

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Abstract: This study explored specific inequalities related to the scalar and Ricci curvatures of slant submersions in Kenmotsu space forms. We derived important geometric bounds and systematically investigated the conditions under which these bounds converge to equality. These findings enlarge the current setup of curvature inequalities and offer new findings on the geometric properties of slant submersions of contact structures.

Keywords: Ricci curvature; submersion; slant submersion; Kenmotsu space forms
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1. Introduction

The intricate relationship between intrinsic and extrinsic geometric properties has long been a central theme in differential geometry. Among the most studied interactions is the link between the squared mean curvature and the Ricci curvature of submanifolds in real space forms $R^n(c)$. This foundational concept was first introduced by B.-Y. Chen in 1999 [11], who established a sharp inequality involving these quantities. Later, in 2005, Chen extended this framework to encompass more general settings, further deepening the theoretical landscape [12]. These pioneering results initiated a rich vein of research exploring curvature inequalities across various ambient geometries [24, 26].

Following Chen's foundational work, many researchers have explored curvature-related inequalities in diverse settings. For instance, investigations in statistical manifolds and space forms have led to numerous inequalities [7, 8, 23, 25, 29, 33, 35, 40]. In parallel, a body of work has emerged focusing on optimal inequalities for submanifolds in complex, quaternionic, and cosymplectic manifolds [4–6, 36–38]. These studies not only generalize Chen's inequalities but also uncover new geometric invariants with deep implications for the theory of submanifolds.

In the context of differential geometry, smooth submersions ψ between (semi)-Riemannian manifolds (M, ϱ) and (N, ϱ) serve as a powerful framework for analyzing geometric structures. These mappings project the total space onto a base manifold while preserving essential geometric features. A classical example is the *Riemannian submersion*, introduced by B. O'Neill [28] and further developed by A. Gray [15], which has become foundational in the field.

This notion has since been extended in various directions. Almost Hermitian submersions were introduced in [39], and quaternionic submersions were explored in [20]. The concept of slant submersions has been studied extensively in [16, 18], while other structural generalizations such as anti-invariant [17], conformal anti-invariant [3], and semi-invariant submersions [30] have contributed to the classification and geometric analysis of submersions. Recent investigations also incorporate Ricci curvature bounds and harmonicity conditions [2,14,19,32], reflecting the continued development and application of submersion theory in modern differential geometry.

Holomorphic submersions, introduced by Watson [39], exemplify the interplay between almost complex structures and submersion geometry. Watson established that if the total space of a holomorphic submersion is Kähler, the base space naturally inherits a compatible structure. Building on this, Şahin [34] introduced slant submersions, which generalize holomorphic submersions by introducing a constant angle condition between vertical and horizontal distributions.

Let M denote an almost Hermitian manifold equipped with a complex structure F of type $(1, 1)$. Submanifolds of Kähler manifolds are categorized based on the interaction between their tangent spaces and the complex structure of the ambient space. Holomorphic submanifolds arise when the tangent space T_xM satisfies $F(T_xM) \subseteq T_xM$, whereas totally real submanifolds are characterized by $F(T_xM) \subseteq T_xM^{\perp}$. Chen [10] introduced the notion of slant submanifolds to unify and extend these classifications.

A submanifold is termed "slant" if the angle $\theta(X)$ between the tangent space T_xM and FX for any $X \in T_xM$ remains constant and lies within $\left[0, \frac{\pi}{2}\right]$. Holomorphic submanifolds correspond to $\theta = 0$, while totally real submanifolds occur when $\theta = \frac{\pi}{2}$. Submanifolds with intermediate constant angles are classified as proper slant submanifolds.

Recent work has extensively explored Chen-Ricci inequalities for submersions in real space forms and complex space forms, with particular emphasis on anti-invariant, semi-invariant, and Lagrangian submersions. This research seeks to extend these inequalities to slant Riemannian submersions in Kenmotsu space forms. The structure of this paper is as follows: The initial section reviews foundational definitions and concepts. Subsequent sections investigate Ricci and scalar curvature inequalities, focusing on vertical $(\ker \psi_*)$ and horizontal $(\ker \psi_*)^{\perp}$ distributions. Finally, we present generalized Chen-Ricci inequalities tailored to slant Riemannian submersions.

2. Basics on Riemannian submersions and Kenmotsu manifolds

Here we give the theoretical background required for Kenmotsu manifolds and Riemannian submersions. Consider a surjective smooth map $\psi : (M, \varrho) \to (N, \varrho)$ between two Riemannian manifolds (M, ϱ) and (N, ϱ), where M and N have dimensions m and n, respectively. If a map has maximal rank at every point of M and maintains the length of all horizontal vector fields, it is referred to as a Riemannian submersion.

By the implicit function theorem, the preimage of any point $p \in N$, denoted as $\psi^{-1}(p)$, is a smooth submanifold of M of dimension $m-n$. A vector field on M is classified as vertical (or tangent to the fibers) if it lies entirely within these preimages. Conversely, a vector field is called horizontal (or orthogonal to the fibers) if it is orthogonal to the vertical distribution.

We designate the vertical and horizontal components of a vector field X on M by $\mathcal{V}X$ and $\mathcal{H}X$, respectively. A vector field X on M is considered basic if it is horizontal and ψ-related to a vector field X_* on N, i.e., $\psi * X = X_{*F(p)}$ for all $p \in M$. The vertical and horizontal distributions are written as $\ker\psi*$ and $(\ker\psi*)^\perp$, respectively.

The horizontal distribution $\mathcal{H}$ determines an integrable foliation of M into fibers, where each vertical subspace $\mathcal{V}_p$ coincides with the tangent space of $\psi^{-1}(\{\psi(p)\})$ at any $p \in M$.

As per convention, the manifold (M, g) is recognized as the total manifold, while (N, g) is recognized as the base manifold of the submersion F.

$\mathcal{T}$ and Λ, the famous O'Neill tensors that characterize the geometry of Riemannian submersions, are presented as:

$$\begin{cases} \mathcal{T}_U V &= \mathcal{V}\nabla_U(\mathcal{H}V) + \mathcal{H}\nabla_U(\mathcal{V}V), \\ \Lambda_U V &= \mathcal{V}\nabla_U(\mathcal{V}V) + \mathcal{H}\nabla_U(\mathcal{H}V), \end{cases} \tag{2.1}$$

for all U, V in the vector fields on M and ∇ is the Levi-Civita connection induced by g.

It is straightforward to observe that both $\mathcal{T}_U$ and Λ_U are skew-symmetric operators on the tangent bundle of M, reversing the vertical and horizontal distributions. Furthermore, if V, W are vertical vector fields and X, Y are horizontal vector fields, the properties of $\mathcal{T}$ and Λ can be summarized as:

$$\begin{cases} \mathcal{T}_V W &= 0, \\ \Lambda_X Y &= -\Lambda_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \end{cases}$$

Moreover, from Eq (2.1), we obtain the following decompositions for the covariant derivative:

$$\begin{cases} \nabla_V W = \mathcal{T}_V W + \mathcal{V}\nabla_V W, \\ \nabla_V X = \mathcal{T}_V X + \mathcal{H}\nabla_V X, \\ \nabla_X V = \Lambda_X V + \mathcal{V}_X V, \\ \nabla_X Y = \mathcal{H}\nabla_X Y + \Lambda_X Y, \end{cases} \tag{2.2}$$

Lemma 2.1. *[28] Let $\psi : (M, \varrho) \to (N, \varrho)$ be a Riemannian submersion. Then, we have:*

$$\Lambda_{L_1} L_2 = -\Lambda_{L_2} L_1, L_1, L_2 \in \chi\left((\ker\psi*)^\perp\right); \tag{2.3}$$

$$\mathcal{T}_{G_1} G_2 = \mathcal{T}_{G_2} G_1, \quad G_1, G_2 \in \chi\left(\ker\psi*\right); \tag{2.4}$$

$$\varrho\left(\mathcal{T}_{G_1} L_2, L_3\right) = -\varrho\left(\mathcal{T}_{G_1} L_3, L_2\right), \quad G_1 \in \chi\left(\ker\psi*\right), \quad L_2, L_3 \in \chi\left(M\right); \tag{2.5}$$

$$\varrho_M\left(\Lambda_{L_1} L_2, L_3\right) = -\varrho\left(\Lambda_{L_1} L_3, L_2\right), \quad L_1 \in \chi\left((\ker\psi*)^\perp\right), L_2, L_3 \in \chi\left(M\right). \tag{2.6}$$

Let $R, R^N, R^{\ker\psi*}$, and $R^{(ker\psi*)^\perp}$ stand for the Riemannian curvature tensors of M, N, the vertical distribution $ker\psi*$, and the horizontal distribution $(ker\psi*)^\perp$, respectively.

Lemma 2.2. *[28] Let $\psi:(M,\varrho)\to(N,\varrho)$ be a Riemannian submersion. Then, we have:*

$$
\begin{aligned}
R(G_1,G_2,G_3,G_4) &= R^{ker\psi*}(G_1,G_2,G_3,G_4)+\varrho(\mathcal{T}_{G_1}G_4,\mathcal{T}_{G_2}G_3) \\
&\quad - \varrho(\mathcal{T}_{G_2}G_4,\mathcal{T}_{G_1}G_3), \qquad (2.7)\\
R(L_1,L_2,L_3,L_4) &= R^{(ker\psi*)^{\perp}}(L_1,L_2,L_3,L_4)-2\varrho(\Lambda_{L_1}L_2,\Lambda_{L_3}L_4) \\
&\quad + \varrho(\Lambda_{L_2}L_3,\Lambda_{L_1}L_4)-\varrho(\Lambda_{L_1}L_3,\Lambda_{L_2}L_4), \qquad (2.8)\\
R(L_1,G_1,L_2,G_2) &= \varrho((\nabla\mathcal{T})(G_1,G_2),L_2)+\varrho((\nabla\Lambda)(L_1,L_2),G_2) \\
&\quad - \varrho(\mathcal{T}_{G_1}L_1,\mathcal{T}_{G_2}L_2)+\varrho(\Lambda_{L_2}G_2,\Lambda_{L_1}G_1), \qquad (2.9)
\end{aligned}
$$

$$\forall\, G_1,G_2,G_3,G_4\in\chi(k\,\mathrm{er}\,\psi*),\, L_1,L_2,L_3,L_4\in\chi((\ker\psi*)^{\perp}).$$

In this context, the mean curvature H of each fiber in the ψ Riemannian submersion is specified as:

$$H=\frac{1}{r}\mathcal{N},\quad \mathcal{N}=\Xi_{p=1}^{r}\mathcal{T}_{\varepsilon_p}\varepsilon_p, \qquad (2.10)$$

where $\{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_r\}$ is an orthonormal basis of the vertical distribution $ker\psi*$.

A $(2m+1)$-dimensional smooth manifold M is referred to as a Kenmotsu manifold if it admits a 1-form μ that satisfies the following criteria, a structure vector field ξ, and an endomorphism ϕ of its tangent bundle TM:

$$
\begin{cases}
\phi^2=-I+\mu\otimes\xi,\quad \mu(\xi)=1,\quad \mu\circ\phi=0,\\
g(\phi X,\phi Y)=g(X,Y)-\mu(X)\mu(Y),\quad \mu(X)=g(X,\xi),\\
(\overline{\nabla}_X\phi)Y=g(\phi X,Y)\xi-\mu(Y)\phi X,\\
\overline{\nabla}_X\xi=X-\mu(X)\xi,
\end{cases}
$$

for any X,Y tangent to M.

A Kenmotsu manifold M with constant ϕ- sectional curvature c is called a Kenmotsu space form and is denoted by $M(c)$. The curvature R for a Kenmotsu space form $M(c)$ is given by [22]

$$
\begin{aligned}
R(G_1,G_2,G_3,G_4) &= \frac{(c-3)}{4}\Big\{\varrho(G_2,G_3)\varrho(G_1,G_4)-\varrho(G_1,G_3)\varrho(G_2,G_4)\Big\}\\
&+ \frac{(c+1)}{4}\Big\{\mu(G_1)\mu(G_3)\varrho(G_2,G_4)+\mu(G_2)\mu(G_4)\varrho(G_1,G_3)\\
&- \mu(G_2)\mu(G_3)\varrho(G_1,G_4)-\mu(G_1)\mu(G_4)\varrho(G_2,G_3)\\
&+ \varrho(\psi G_1,G_4)\varrho(\psi G_2,G_3)-\varrho(\psi G_1,G_3)\varrho(\psi G_2,G_4)\\
&+ 2\varrho(G_1,\psi G_2)\varrho(\psi G_3,G_4)\Big\}. \qquad (2.11)
\end{aligned}
$$

A Riemannian submersion ψ from an almost contact manifold (M,ϱ) on a Riemannian manifold (N,ϱ) qualifies as a proper slant submersion if and only if there is a constant λ satisfying

$$\phi^2X=-\lambda X \qquad (2.12)$$

for $X\in\Gamma(ker\psi_*)$ and $\lambda=-\cos^2\theta$. Hence, we have

$$\varrho(\phi X,\phi Y)=\cos^2\theta(\varrho(X,Y)-\mu(X)\mu(Y)), \qquad (2.13)$$

$$\varrho(\omega X,\omega Y)=\sin^2\theta(\varrho(X,Y)-\mu(X)\mu(Y)). \qquad (2.14)$$

3. Basic Chen inequalities

Given that ψ is a slant submersion, we may use (2.11) and (2.7) to obtain [31]:

Lemma 3.1. *Let us represent $\psi : M(c) \to N$ as a slant submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ).*
Then for $G_1, G_2, G_3, G_4 \in \chi\,(ker\psi)$ we obtain

$$\begin{aligned}
R^{ker\psi*}(G_1, G_2, G_3, G_4) &= \frac{(c-3)}{4}\{\varrho(G_2, G_3)\varrho(G_1, G_4) - \varrho(G_1, G_3)\varrho(G_2, G_4)\} \\
&+ \frac{(c+1)}{4}\{\mu(G_1)\mu(G_3)\varrho(G_2, G_4) + \mu(G_2)\mu(G_4)\varrho(G_1, G_3) \\
&- \mu(G_2)\mu(G_3)\varrho(G_1, G_4) - \mu(G_1)\mu(G_4)\varrho(G_2, G_3) \\
&+ \varrho(\psi G_1, G_4)\varrho(\psi G_2, G_3) - \varrho(\psi G_1, G_3)\varrho(\psi G_2, G_4) \\
&+ 2\varrho(G_1, \psi G_2)\varrho(\psi G_3, G_4)\} + \varrho(h(G_1, G_4), h(G_2, G_3)) \\
&- \varrho(h(G_1, G_3), h(G_2, G_4)) - \varrho(\mathcal{T}_{G_1}G_4, \mathcal{T}_{G_2}G_3) + \varrho(\mathcal{T}_{G_2}G_4, \mathcal{T}_{G_1}G_3) \\
&+ \mu(G_1)\mu(G_4)\varrho(\psi G_2, G_3) - \mu(G_1)\mu(G_3)\varrho(\psi G_2, G_4) \\
&- \mu(G_2)\mu(G_4)\varrho(\psi G_1, G_3) + \mu(G_2)\mu(G_3)\varrho(\psi G_1, G_4) \qquad (3.1)
\end{aligned}$$

and for $G_1 = G_3$, $G_2 = G_4$, we get

$$\begin{aligned}
K^{ker\psi*}(G_1, G_2) &= \frac{c-3}{4}\left\{\varrho^2(G_1, G_2) - \|G_1\|^2\|G_2\|^2\right\} \\
&+ \frac{c+1}{4}\left\{2\mu(G_1)^2 - \mu(G_1)^2\varrho(G_1, G_2) + 3\varrho^2(G_1, \phi G_2)\right\} \\
&+ \varrho(\mathcal{T}_{G_2}G_2, \mathcal{T}_{G_1}G_1) + \mu(G_1)\mu(G_2)\varrho(\psi G_1, G_2) \\
&+ \mu(G_2)\mu(G_3)\varrho(G_1, \psi G_2) - \left\|\mathcal{T}_{G_1}G_2\right\|^2, \qquad (3.2)
\end{aligned}$$

where $K^{kerp\psi*}$ is a bi-sectional curvature of $\ker\psi*$.

Using the previously obtained result, we show the following.

Theorem 3.2. *Let us represent $\psi : M(c) \to N$ as a slant submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ). Then we have*

$$\begin{aligned}
Ric^{ker\psi*}(G_1) &\geq \frac{c-3}{4}\left\{(r-1)\varrho(G_1, G_1)\right\} - 2r\varrho(\mathcal{T}_{G_1}G_1, \mathcal{H}) \\
&+ \frac{c+1}{4}\left\{2\mu(G_1)^2 - r\mu(G_1)^2 + 3\cos^2\theta(\varrho(G_1, G_1) - \mu(G_1)^2)\right\} \qquad (3.3)
\end{aligned}$$

and the condition for achieving equality in the inequality is if and only if every fiber is totally geodesic.

Proof. Let $\psi : M \to N$ be an slant submersion. For every node $k \in M$, let $\{\varepsilon_1, \ldots, \varepsilon_r, \varepsilon_{r+1} = \xi, \varepsilon_{r+2}, \ldots, \varepsilon_{2r+1}\}$ be an orthonormal basis of T_kM such that $ker\psi* = \text{span}\{\varepsilon_1, \ldots, \varepsilon_r\}$, $(ker\psi*)^\perp =$

span$\{\varepsilon_{r+2} = e_1, \ldots, \varepsilon_{2r+1} = e_u\}$. Now, if we take $G_4 = G_1$ and $G_2 = G_3 = \varepsilon_i, i = 1, 2, \ldots, r$, in (3.1) and using (2.13), then we arrive at

$$\begin{aligned} Ric^{ker\psi_*}(G_1) &= \frac{(c-3)}{4}\{\varrho(G_i, G_i)\varrho(G_1, G_1) - \varrho(G_1, G_i)\varrho(G_i, G_1)\} \\ &+ \frac{(c+1)}{4}\{\mu(G_1)\mu(G_i)\varrho(G_i, G_1) + \mu(G_i)\mu(G_1)\varrho(G_1, G_i) \\ &- \mu(G_i)\mu(G_i)\varrho(G_1, G_1) - \mu(G_1)\mu(G_1)\varrho(G_i, G_i) \\ &+ \varrho(\psi G_1, G_1)\varrho(\psi G_i, G_i) - \varrho(\psi G_1, G_i)\varrho(\psi G_i, G_1) \\ &+ 2\varrho(G_1, \psi G_i)\varrho(\psi G_i, G_1)\} + \varrho(h(G_1, G_1), h(G_i, G_i)) \\ &- \varrho(h(G_1, G_i), h(G_i, G_i)) - \varrho(\mathcal{T}_{u_1}G_1, \mathcal{T}_{u_i}G_i) \\ &+ \varrho(\mathcal{T}_{G_i}G_1, \mathcal{T}_{G_1}G_i) + \mu(G_1)^2\varrho(\psi G_i, G_i) \\ &- \mu(G_i)\mu(G_1)\varrho(\psi G_i, G_1) - \mu(G_1)\mu(G_i)\varrho(\psi G_1, G_i) \\ &+ \mu(G_i)\mu(G_i)\varrho(\psi G_1, G_1), \\ &= \frac{c-3}{4}\Big\{(r-1)\varrho(G_1, G_1)\Big\} - 2r\varrho(\mathcal{T}_{G_1}G_1, \mathcal{H}) + \Xi_{i=1}^{2r}\varrho(\mathcal{T}_{\varepsilon_i}G_1, \mathcal{T}_{G_1}\varepsilon_i) \\ &+ \frac{c+1}{4}\Big\{2\mu(G_1)^2 - r\mu(G_1)^2 + 3\cos^2\theta(\varrho(G_1, G_1)\mu(G_1)^2)\Big\}. \end{aligned} \tag{3.4}$$

□

The following outcome can be declared as a consequence.

Corollary 3.3. *Let us represent $\psi : M(c) \to N$ as an anti-invariant ($\theta = \frac{\pi}{2}$) Riemannian submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ). Then we have*

$$Ric^{ker\psi_*}(G_1) \geq \frac{c-3}{4}\Big\{(r-1)\varrho(G_1, G_1)\Big\} + \frac{c+1}{4}\Big\{2\mu(G_1)^2 - r\mu(G_1)^2\Big\} - 2r\varrho(\mathcal{T}_{G_1}G_1, \mathcal{H}) \tag{3.5}$$

and the condition for achieving equality in the inequality is if and only if every fiber is totally geodesic.

Theorem 3.4. *Let us represent $\psi : M(c) \to N$ as a slant submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ). Then we have*

$$2scal^{ker\psi_*} \geq \frac{c-3}{4}\Big\{r(r-1)\Big\} + \frac{c+1}{4}\Big\{3r\cos^2\theta\Big\} - 4r^2\|\mathcal{H}\|^2 \tag{3.6}$$

and the condition for achieving equality in the inequality is if and only if every fiber is totally geodesic.

Proof. Taking $G_1 = \varepsilon_j, j = 1, \ldots, r$, in (3.4) and using (2.10), then we obtain the scalar curvature of the fiber as

$$2scal^{ker\psi_*} = \frac{c-3}{4}\Big\{r(r-1)\Big\} + \frac{c+1}{4}\Big\{3r\cos^2\theta\Big\} - 4r^2\|\mathcal{H}\|^2 + \Xi_{i,j=1}^{2r}\varrho\left(\mathcal{T}_{\varepsilon_i}\varepsilon_j, \mathcal{T}_{\varepsilon_i}\varepsilon_j\right). \tag{3.7}$$

This infers the desired result. □

From (2.11) and (2.8), for a slant submersion of Kenmotsu space form, the curvature tensor of $(ker\psi*)^{\perp}$ satisfies:

Lemma 3.5. *Let us represent $\psi : M(c) \to N$ as a slant submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ). Then, for $L_1, L_2, L_3, L_4 \in \chi\left((\ker\psi*)^{\perp}\right)$ we have*

$$
\begin{aligned}
R^{(ker\psi*)^{\perp}}(L_1, L_2, L_3, L_4) &= \frac{(c-3)}{4}\Big\{\varrho(L_2, L_3)\varrho(L_1, L_4) - \varrho(L_1, L_3)\varrho(L_2, L_4)\Big\} \\
&+ \frac{(c+1)}{4}\Big\{\mu(L_1)\mu(L_3)\varrho(L_2, L_4) + \mu(L_2)\mu(L_4)\varrho(L_1, L_3) \\
&- \mu(L_2)\mu(L_3)\varrho(L_1, L_4) - \mu(L_1)\mu(L_4)\varrho(L_2, L_3) \\
&+ \varrho(CL_1, L_4)\varrho(CL_2, L_3) - \varrho(CL_1, L_3)\varrho(CL_2, L_4) \\
&+ 2\varrho(L_1, CL_2)\varrho(CL_3, L_4)\Big\} + 2\varrho\left(\Lambda_{L_1}L_2, \Lambda_{L_3}L_4\right) \\
&- \varrho\left(\Lambda_{L_2}L_3, \Lambda_{L_1}L_4\right) + \varrho\left(\Lambda_{L_1}L_3, \Lambda_{L_2}L_4\right), \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
K^{(ker\psi*)^{\perp}}(L_1, L_2) &= \frac{c-3}{4}\left\{\varrho^2(L_1, L_2) - \|L_1\|^2\|L_2\|^2\right\} \\
&- \frac{3(c+1)}{4}\varrho^2(CL_1, L_2) + 3\left\|\Lambda_{L_1}L_2\right\|^2 \qquad (3.9)
\end{aligned}
$$

here $K^{(ker\psi*)^{\perp}}$ is a bi-sectional curvature of $(ker\psi*)^{\perp}$.

Theorem 3.6. *Let us represent $\psi : M(c) \to N$ as a slant submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ). Then*

$$
\begin{aligned}
Ric^{(ker\psi*)^{\perp}}(L_1) &\leq \frac{c-3}{4}\Big\{(u-1)\varrho(L_1, L_1)\Big\} \\
&+ \frac{c+1}{4}\Big\{2\mu(L_1)^2 - n\mu(L_1)^2 + 3\varrho^2(CL_1, L_1)\Big\} \qquad (3.10)
\end{aligned}
$$

and the condition for achieving equality in the inequality is if and only if horizontal distribution is integrable.

Proof. If we take $L_1 = L_4$ and $L_2 = L_3 = csc\theta\omega e_j,\ \ j = 1, \ldots, u$:

$$
\begin{aligned}
Ric^{(ker\psi*)^{\perp}}(L_1) &= \frac{(c-3)}{4}\Big\{\varrho(\csc\theta\omega e_j, \csc\theta\omega e_j)\varrho(L_1, L_1) \\
&- \varrho(L_1, \csc\theta\omega e_j)\varrho(\csc\theta\omega e_j, L_1)\Big\} \\
&+ \frac{(c+1)}{4}\Big\{\mu(L_1)\mu(\csc\theta\omega e_j)\varrho(\csc\theta\omega e_j, L_1) \\
&+ \mu(\csc\theta\omega e_j)\mu(L_1)\varrho(L_1, \csc\theta\omega e_j) \\
&- \mu(\csc\theta\omega e_j)\mu(\csc\theta\omega e_j)\varrho(L_1, L_1) \\
&- \mu(L_1)\mu(L_1)\varrho(\csc\theta\omega e_j, \csc\theta\omega e_j)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \varrho(CL_1, L_1)\varrho(C \csc\theta\omega e_j, \csc\theta\omega e_j) \\
&- \quad \varrho(CL_1, \csc\theta\omega e_j)\varrho(C \csc\theta\omega e_j, L_1) \\
&+ \quad 2\varrho(L_1, C \csc\theta\omega e_j)\varrho(C \csc\theta\omega e_j, L_1)\Big\} \\
&+ \quad 2\varrho(\Lambda_{L_1} csc\theta\omega e_j, \Lambda_c sc\theta\omega e_j L_1) \\
&- \quad \varrho(\Lambda_{csc\theta\omega} e_j csc\theta\omega e_j, \Lambda_{L_1} L_1) \\
&+ \quad \varrho(\Lambda_{L_1} csc\theta\omega e_j, \Lambda_{csc\theta\omega e_j} L_1).
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
Ric^{(ker\psi*)^{\perp}}(L_1) \;&=\; \frac{c-3}{4}\Big\{(u-1)\varrho(L_1, L_1)\Big\} + \frac{c+1}{4}\Big\{2\mu(L_1)^2 - u\mu(L_1)^2 \\
&+\; 3\varrho^2(CL_1, L_1)\Big\} + 3\Xi_{j=1}^{u}\varrho\left(\Lambda_{L_1} \csc\theta\omega e_j, \Lambda_{\csc\theta\omega e_j} L_1\right).
\end{aligned}
\tag{3.12}
$$

From here, the result follows. □

Theorem 3.7. *Let us represent $\psi : M(c) \to N$ as a slant submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ). Then we have*

$$
2scal^{(ker\psi*)^{\perp}} \leq \frac{c-3}{4}\Big\{u(u-1)\Big\} + \frac{c+1}{4}\Big\{3\|C\|^2\Big\}
\tag{3.13}
$$

and equality status of (3.13) satisfies if and only if $(ker\psi_*)^{\perp}$ horizontal distribution is integrable.

Proof. Taking $L_1 = \csc\theta\omega e_i, i = 1, 2, \ldots, u$, in (3.12), then we obtain:

$$
\begin{aligned}
2\ \mathrm{scal}^{\ (ker\psi*)^{\perp}} \;&=\; \frac{c-3}{4}\Big\{u(u-1)\Big\} + \frac{c+1}{4}\Big\{3\|C\|^2\Big\} \\
&+\; 3\Xi_{j=1}^{u}\varrho\left(\Lambda_{\csc\theta\omega e_i} \csc\theta\omega e_j, \Lambda_{\csc\theta\omega e_j} \csc\theta\omega e_i\right).
\end{aligned}
\tag{3.14}
$$

Then, we write

$$
2scal^{(ker\psi*)^{\perp}} \leq \frac{c-3}{4}\Big\{u(u-1)\Big\} + \frac{c+1}{4}\Big\{3\|C\|^2\Big\},
\tag{3.15}
$$

which is the required result. □

4. Chen-Ricci inequalities

The investigation of Chen-Ricci inequalities has become a promising research area in differential geometry that provides new information on the relationship between the intrinsic curvature and extrinsic curvature of submanifolds. This section seeks to discuss the basic feature of the Chen-Ricci inequalities.

We represent

$$
\mathcal{T}_{ij}^{t} = \varrho(\mathcal{T}_{\varepsilon_1}\varepsilon_j, \csc\theta\omega e_t),
\tag{4.1}
$$

where $i, j = 1, \ldots, r$ and $t = 1, \ldots, u$.
Consequently, we denote

$$\Lambda_{ij}^t = \varrho(\Lambda_{\csc\theta\omega e_i} e_j, \varepsilon_t), \tag{4.2}$$

$1 \leq i, j \leq u$, *and* $1 \leq t \leq r$.
We also use

$$\delta(\mathcal{N}) = \Xi_{i=1}^u \Xi_{k=1}^r \varrho\left((\nabla\mathcal{T})_{\varepsilon_k}\varepsilon_k, \csc\theta\omega e_i\right). \tag{4.3}$$

First we prove the following result.

Theorem 4.1. *Let us represent $\psi : M(c) \to N$ as a slant submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ). Then we have*

$$Ric^{ker\psi*}(\varepsilon_1) \geq \frac{c-3}{4}\Big\{2(r-1)\Big\} - \frac{3(c+1)}{4}\cos^2\theta - r^2\|\mathcal{H}\|^2 \tag{4.4}$$

and the condition for achieving equality in the inequality is if and only if

$$\mathcal{T}_{11}^t = \mathcal{T}_{22}^t + \cdots + \mathcal{T}_{rr}^t,$$
$$\mathcal{T}_{1j}^t = 0, \quad j = 2, \ldots, r.$$

Proof. Referring to (3.7), we can write:

$$\begin{aligned} 2scal^{ker\psi*} &= \frac{c-3}{4}\{r(r-1)\} + \frac{c+1}{4}\{3r\cos^2\theta\} - 4r^2\|\mathcal{H}\|^2 + \Xi_{i,j=1}^r g\left(\mathcal{T}_{\varepsilon_i}\varepsilon_j, \mathcal{T}_{\varepsilon_i}\varepsilon_j\right) \\ &= \frac{c-3}{4}\{r(r-1)\} + \frac{c+1}{4}\{3r\cos^2\theta\} - 4r^2\|\mathcal{H}\|^2 + \Xi_{i,j=1}^r \Xi_{t=1}^u (\mathcal{T}_{ij})^2. \end{aligned} \tag{4.5}$$

The equation that follows between the tensor fields $\mathcal{T}$ can be derived from the Binomial Theorem.

$$\begin{aligned} \Xi_{t=1}^u \Xi_{i,j=1}^r \left(\mathcal{T}_{ij}^t\right)^2 &= \frac{1}{2}4r^2\|H\|^2 + \frac{1}{2}\Xi_{t=1}^u \left(\mathcal{T}_{11}^t - \mathcal{T}_{22}^t - \ldots - \mathcal{T}_{rr}^t\right)^2 \\ &+ 2\Xi_{t=1}^u \Xi_{j=2}^r \left(\mathcal{T}_{1j}^t\right)^2 - 2\Xi_{t=1}^u \Xi_{2\leq i<j\leq r}\left(\mathcal{T}_{ii}^t \mathcal{T}_{ij}^t - \left(\mathcal{T}_{ij}^t\right)^2\right). \end{aligned} \tag{4.6}$$

Using (4.6) in (4.5), we arrive at

$$\begin{aligned} 2scal^{ker\psi*} &= \frac{c-3}{4}\{r(r-1)\} + \frac{c+1}{4}\{3r\cos^2\theta\} - 4r^2\|\mathcal{H}\|^2 \\ &+ \frac{1}{2}4r^2\|H\|^2 + \frac{1}{2}\Xi_{t=1}^u \left(\mathcal{T}_{11}^t - \mathcal{T}_{22}^t - \ldots - \mathcal{T}_{rr}^t\right)^2 \\ &+ 2\Xi_{t=1}^u \Xi_{j=2}^{2r} \left(\mathcal{T}_{1j}^t\right)^2 - 2\Xi_{t=1}^u \Xi_{2\leq i<j\leq r}\left(\mathcal{T}_{ii}^t \mathcal{T}_{ij}^t - \left(\mathcal{T}_{ij}^t\right)^2\right). \end{aligned} \tag{4.7}$$

Thus, it follows that

$$2scal^{ker\psi*} \geq \frac{c-3}{4}\{r(r-1)\} + \frac{c+1}{4}\{3r\cos^2\theta\} - 2r^2\|\mathcal{H}\|^2 \tag{4.8}$$

$$- \quad 2\Xi_{t=1}^{u}\Xi_{2\leq i<j\leq r}\left(\mathcal{T}_{ii}^{t}\mathcal{T}_{ij}^{t}-\left(\mathcal{T}_{ij}^{t}\right)^{2}\right).$$

On the other hand taking $G_1 = G_4 = \varepsilon_i$ *and* $G_2 = G_3 = \varepsilon_j$, in (2.7), we have

$$2\Xi_{2\leq i<j\leq r}R\left(\varepsilon_i,\varepsilon_j,\varepsilon_j,\varepsilon_i\right) = 2\Xi_{2\leq i<j\leq r}R^{ker\psi*}\left(\varepsilon_i,\varepsilon_j,\varepsilon_j,\varepsilon_i\right) + 2\Xi_{t=1}^{u}\Xi_{2\leq i<j\leq r}\left(\mathcal{T}_{ii}^{t}\mathcal{T}_{jj}^{t}-\left(\mathcal{T}_{ij}^{t}\right)^{2}\right). \tag{4.9}$$

Using (4.9) in (4.8), we obtain

$$\begin{aligned} 2scal^{\ker\psi*} \;&\geq\; \frac{c-3}{4}\{r(r-1)\} + \frac{c+1}{4}\{3r\cos^2\theta\} - 2r^2\|\mathcal{H}\|^2 \\ &+\; 2\Xi_{2\leq i<j\leq r}R^{ker\psi*}\left(\varepsilon_i,\varepsilon_j,\varepsilon_j,\varepsilon_i\right) - 2\Xi_{2\leq i<j\leq r}R\left(\varepsilon_i,\varepsilon_j,\varepsilon_j,\varepsilon_i\right). \end{aligned} \tag{4.10}$$

Furthermore, we have

$$2scal^{ker\psi*} = 2\Xi_{2\leq i<j\leq r}R^{ker\psi*}\left(\varepsilon_i,\varepsilon_j,\varepsilon_j,\varepsilon_i\right) + 2\Xi_{j=1}^{r}R^{ker\psi*}\left(\varepsilon_1,\varepsilon_j,\varepsilon_j,\varepsilon_1\right). \tag{4.11}$$

Using the above equation in (4.10),

$$\begin{aligned} Ric^{ker\psi*}(\varepsilon_1) \;&\geq\; \frac{c-3}{4}\{r(r-1)\} \\ &+\; \frac{c+1}{4}\{3r\cos^2\theta\} - r^2\|\mathcal{H}\|^2 - \Xi_{2\leq i<j\leq r}R^{\ker\psi*}\left(\varepsilon_i,\varepsilon_j,\varepsilon_j,\varepsilon_i\right). \end{aligned} \tag{4.12}$$

Since M is a Kenmotsu space form, from (2.11), after some simplification, we infer

$$Ric^{ker\psi*}(\varepsilon_1) \;\geq\; \frac{c-3}{4}\{2(r-1)\} - \frac{c+1}{4}3\cos^2\theta - r^2\|\mathcal{H}\|^2, \tag{4.13}$$

which completes the proof. □

Theorem 4.2. *Let us represent $\psi : M(c) \to N$ as a slant submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ). Then we have*

$$2Ric^{(ker\psi*)^{\perp}}(\csc\theta\omega e_1) \leq \frac{c-3}{4}\Big\{2(u-1)\Big\} + \frac{c+1}{4}\Big\{-3\|C\|^2\Big\} \tag{4.14}$$

and the condition for achieving equality in the inequality is if and only if $A_{1j} = 2,\dots,u$.

Proof. As seen in (3.15), we obtain

$$2scal^{(ker\psi*)^{\perp}} \;=\; \frac{c-3}{4}\{u(u-1)\} + \frac{c+1}{4}\{3\|C\|^2\} - 3\Xi_{i,j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{ij}^{t})^2. \tag{4.15}$$

$$\begin{aligned} &=\; \frac{c-3}{4}\{u(u-1)\} + \frac{c+1}{4}\{3\|C\|^2\} \\ &-\; 6\Xi_{j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{1j}^{t})^2 - 6\Xi_{2\leq i,j\leq u}\Xi_{t=1}^{r}(\Lambda_{ij}^{t})^2. \end{aligned} \tag{4.16}$$

If we put $L_1 = L_4 = \csc\theta\omega e_i$ and $L_2 = L_3 = \csc\theta\omega e_j$ in Eq (2.8),

$$R(\csc\theta\omega e_i,\csc\theta\omega e_j,\csc\theta\omega e_j,\csc\theta\omega e_i) \;=\; R^{ker\psi*}(\csc\theta\omega e_i,\csc\theta\omega e_j,\csc\theta\omega e_j,\csc\theta\omega e_i)$$

$$
\begin{aligned}
&- && 2\varrho(\Lambda_{\csc\theta\omega e_i}\csc\theta\omega e_j, \Lambda_{\csc\theta\omega e_j}\csc\theta\omega e_i)\\
&- && 2\varrho(\Lambda_{\csc\theta\omega e_j}\csc\theta\omega e_j, \Lambda_{\csc\theta\omega e_i}\csc\theta\omega e_i)\\
&- && \varrho(\Lambda_{\csc\theta\omega e_i}\csc\theta\omega e_j, \Lambda_{\csc\theta\omega e_j}\csc\theta\omega e_i)\\
&= && R^{(ker\psi*)^{\perp}}(\csc\theta\omega e_i, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_i)\\
&+ && 6\Xi_{2\leq i,j\leq u}\Xi_{t=1}^{r}(\Lambda_{ij}^{t})^2. \qquad (4.17)
\end{aligned}
$$

From (4.16) and (4.17), we have

$$
\begin{aligned}
2scal^{(ker\psi*)^{\perp}} &= \frac{c-3}{4}\{u(u-1)\} + \frac{c+1}{4}\{3\|C\|^2\} - 6\Xi_{j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{1j}^{t})^2\\
&+ 2\Xi_{2\leq i,j\leq u}R^{(ker\psi*)^{\perp}}(\csc\theta\omega e_i, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_i)\\
&- 2\Xi_{2\leq i,j\leq u}R(\csc\theta\omega e_i, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_i). \qquad (4.18)
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
2\text{ scal }^{(ker\psi*)^{\perp}} &= 2\Xi_{2\leq i,j\leq n}^{u}R^{(ker\psi*)^{\perp}}(\csc\theta\omega e_i, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_i) \qquad (4.19)\\
&+ \Xi_{j=1}^{u}R^{(ker\psi*)^{\perp}}(\csc\theta\omega e_1, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_1).
\end{aligned}
$$

Now,

$$
\begin{aligned}
&2\Xi_{2\leq i,j\leq u}R(\csc\theta\omega e_i, \csc\theta\omega e_j, \csc\theta\omega e_j, \csc\theta\omega e_i)\\
&\qquad = \frac{(c-3)}{4}\{(u-2)(u-1)\} + \frac{(c+1)}{4}\{3\|C\csc\theta\omega e_1\|^2\}. \qquad (4.20)
\end{aligned}
$$

By leveraging (4.18), (4.19), and (4.20), it can be shown that

$$
2Ric^{(ker\psi*)^{\perp}}(\csc\theta\omega e_1) = \frac{c-3}{4}\{2(u-1)\} + \frac{c+1}{4}\{3\|C\csc\theta\omega e_1\|^2\} - 6\Xi_{j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{1j}^{t})^2. \qquad (4.21)
$$

Therefore, we have

$$
2Ric^{(ker\psi*)^{\perp}}(\csc\theta\omega e_1) \leq \frac{c-3}{4}\{2(u-1)\} + \frac{c+1}{4}\{3\|C\csc\theta\omega e_1\|^2\}. \qquad (4.22)
$$

□

We now proceed to compute the Chen-Ricci inequality on $ker\psi*$ and its orthogonal complement $(ker\psi*)^{\perp}$.

Theorem 4.3. *Let us represent $\psi : M(c) \to N$ as a slant submersion originating from the Kenmotsu space form $(M(c), \varrho)$ and targeting the Riemannian manifold (N, ϱ). Then we have*

$$
\begin{aligned}
Ric^{ker\psi*}(\varepsilon_1) &\geq \frac{c-3}{4}\{r(r+u-1) - 4u + 1\}\\
&+ \frac{c+1}{4}\{-3cos^2\theta + 3\|C\|^2 + 6\|B\|^2\}\\
&- Ric^{(ker\psi*)^{\perp}}(\csc\theta\omega e_1) - 2r^2\|\mathcal{H}\|^2 + 2\delta(\mathcal{N})
\end{aligned}
$$

$$- \quad 6\Xi_{j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{1j}^{t})^2 - 2\|\mathcal{T}^{ker\psi*}\|^2 + \|\Lambda^{ker\psi*}\|^2 \tag{4.23}$$

and the condition for achieving equality in the inequality is if and only if

$$\mathcal{T}_{11}^{t} = \mathcal{T}_{22}^{t} + \cdots + \mathcal{T}_{rr}^{t},$$
$$\mathcal{T}_{1j}^{t} = 0, \quad j = 2, \ldots, r.$$

Proof. Mathematically, the scalar curvature *scal* of M is

$$\begin{aligned} 2scal &= \Xi_{i=1}^{u}Ric(\csc\theta\omega e_t, \csc\theta\omega e_t) + \Xi_{k=1}^{r}Ric(\varepsilon_k, \csc\theta\omega e_t) \qquad (4.24)\\ &= \Xi_{j,k=1}^{r}R(\varepsilon_j, \varepsilon_k, \varepsilon_k, \varepsilon_j) + \Xi_{i=1}^{u}\Xi_{k=1}^{r}R(\csc\theta\omega e_i, \varepsilon_k, \varepsilon_k \csc\theta\omega e_i)\\ &+ \Xi_{i,t=1}^{u}R(\csc\theta\omega e_i, \csc\theta\omega e_t, \csc\theta\omega e_t, \csc\theta\omega e_i)\\ &+ \Xi_{t=1}^{u}\Xi_{j=1}^{r}R(\varepsilon_j \csc\theta\omega e_t, \csc\theta\omega e_t, \varepsilon_j)\\ &= \frac{c-1}{4}r(r-1) + \frac{c+1}{4}3r\cos^2\theta + \frac{c-1}{r}(u-1) + \frac{c+1}{4}3\|B\|^2\\ &+ \frac{c-1}{4}u(u-1) + \frac{c+1}{4}3\|C\|^2 + \frac{c-1}{4}u(r-1) + \frac{c+1}{4}3\|B\|^2\\ &= \frac{c-1}{4}\{(u+r)^2 - (u+1)(r+1)\}\\ &+ \frac{c+1}{4}\{3r\cos^2\theta + 6\|B\|^2 + 3\|C\|^2\}. \qquad (4.25) \end{aligned}$$

With the help of (2.7), (2.8), and (2.9), we conclude

$$\begin{aligned} 2scal &= 2scal^{ker\psi*} + 2\,\mathrm{scal}\,l^{(ker\psi*)^{\perp}} + 4r^2\|\mathcal{H}\|^2 + \Xi_{j,k=1}^{r}\varrho\left(\mathcal{T}_{\varepsilon_k}\varepsilon_j, \mathcal{T}_{\varepsilon_k}\varepsilon_j\right) \qquad (4.26)\\ &+ 3\Xi_{i,t=1}^{u}\varrho\left(\Lambda_{\csc\theta\omega e_i}\csc\theta\omega e_t, \Lambda_{\csc\theta\omega e_i}\csc\theta\omega e_t\right)\\ &- \Xi_{i=1}^{u}\Xi_{k=1}^{r}\varrho\left((\nabla\mathcal{T}_{\varepsilon_k}\varepsilon_k, \csc\theta\omega e_i\right) - \Xi_{t=1}^{u}\Xi_{j=1}^{r}\varrho\left((\nabla\mathcal{T})_{\varepsilon_j}\varepsilon_j, \csc\theta\omega e_t\right)\\ &+ \Xi_{i=1}^{u}\Xi_{k=1}^{r}\left\{\begin{array}{l}\varrho\left(\mathcal{T}_{\varepsilon_k}\csc\theta\omega e_i, \mathcal{T}_{\varepsilon_k}\csc\theta\omega e_i\right)\\ -\varrho\left(\Lambda_{\csc}\theta\omega e_i\varepsilon_k, \Lambda_{\csc}\theta\omega e_i\varepsilon_k\right)\end{array}\right\}\\ &+ \Xi_{t=1}^{u}\Xi_{j=1}^{r}\left\{\begin{array}{l}\varrho\left(\mathcal{T}_{\varepsilon_j}\csc\theta\omega e_t, \mathcal{T}_{\varepsilon_j}\csc\theta\omega e_t\right)\\ -\varrho\left(\Lambda_{\csc}\theta\omega e_t\varepsilon_j, \Lambda_{\csc}\theta\omega e_t\varepsilon_j\right)\end{array}\right\}. \end{aligned}$$

Utilizing (4.1) and (4.6), we derive

$$\begin{aligned} 2scal &= 2scal^{ker\psi*} + 2\,\mathrm{scal}\,l^{(ker\psi*)^{\perp}} + 2r^2\|\mathcal{H}\|^2\\ &+ \frac{1}{2}\Xi_{t=1}^{u}\left(\mathcal{T}_{11}^{t} - \mathcal{T}_{22}^{t} - \ldots - \mathcal{T}_{rr}^{t}\right)^2 - 2\delta(\mathcal{N})\\ &- 2\Xi_{t=1}^{u}\Xi_{2\le i<j\le r}\left(\mathcal{T}_{ii}^{t}\mathcal{T}_{ij}^{t} - \left(\mathcal{T}_{ij}^{t}\right)^2\right) + 2\Xi_{t=1}^{u}\Xi_{j=2}^{r}\left(\mathcal{T}_{1j}^{t}\right)^2\\ &- 6\Xi_{j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{1j}^{t})^2 - 6\Xi_{2\le i,j\le u}\Xi_{t=1}^{r}(\Lambda_{ij}^{t})^2\\ &+ \Xi_{i=1}^{u}\Xi_{k=1}^{r}\left\{\begin{array}{l}\varrho\left(\mathcal{T}_{\varepsilon_k}\csc\theta\omega e_i, \mathcal{T}_{\varepsilon_k}\csc\theta\omega e_i\right)\\ -\varrho\left(\Lambda_{\csc}\theta\omega e_i\varepsilon_k, \Lambda_{\csc}\theta\omega e_i\varepsilon_k\right)\end{array}\right\}\\ &+ \Xi_{t=1}^{u}\Xi_{j=1}^{r}\left\{\begin{array}{l}\varrho\left(\mathcal{T}_{\varepsilon_j}\csc\theta\omega e_t, \mathcal{T}_{\varepsilon_j}\csc\theta\omega e_t\right)\\ -\varrho\left(\Lambda_{\csc}\theta\omega e_t\varepsilon_j, \Lambda_{\csc}\theta\omega e_t\varepsilon_j\right)\end{array}\right\}. \end{aligned}$$

$$
\begin{aligned}
2scal &= \Xi_{2\le i,j\le u} R^{ker\psi*}(\varepsilon_i,\varepsilon_j,\varepsilon_j,\varepsilon_i) + \Xi_{j=1}^{r} R^{ker\psi*}(\varepsilon_1,\varepsilon_j,\varepsilon_j,\varepsilon_1) \\
&+ \Xi_{2\le i,j\le u} R^{(ker\psi*)^\perp}(\csc\theta\omega e_s,\csc\theta\omega e_t,\csc\theta\omega e_t,\csc\theta\omega e_s) \qquad (4.27)\\
&+ \Xi_{t=1}^{u} R^{(ker\psi*)^\perp}(\csc\theta\omega e_1,\csc\theta\omega e_t,\csc\theta\omega e_t,\csc\theta\omega e_1) - 2\delta(\mathcal{N}) \\
&- 2\Xi_{t=1}^{u}\Xi_{2\le i<j\le r}\left(\mathcal{T}_{ii}^{t}\mathcal{T}_{ij}^{t} - \left(\mathcal{T}_{ij}^{t}\right)^2\right) + 2\Xi_{t=1}^{u}\Xi_{j=2}^{r}\left(\mathcal{T}_{1j}^{t}\right)^2 \\
&- 6\Xi_{j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{1j}^{t})^2 - 6\Xi_{2\le i,j\le u}\Xi_{t=1}^{r}(\Lambda_{ij}^{t})^2 \\
&+ \Xi_{i=1}^{u}\Xi_{k=1}^{r}\left\{\begin{array}{l} \varrho\left(\mathcal{T}_{\varepsilon_k}\csc\theta\omega e_i,\mathcal{T}_{\varepsilon_k}\csc\theta\omega e_i\right) \\ -\varrho\left(\Lambda_{\csc}\theta\omega e_i\varepsilon_k,\Lambda_{\csc}\theta\omega e_i\varepsilon_k\right)\end{array}\right\} \\
&+ \Xi_{t=1}^{u}\Xi_{j=1}^{r}\left\{\begin{array}{l} \varrho\left(\mathcal{T}_{\varepsilon_j}\csc\theta\omega e_t,\mathcal{T}_{\varepsilon_j}\csc\theta\omega e_t\right) \\ -\varrho\left(\Lambda_{\csc}\theta\omega e_t\varepsilon_j,\Lambda_{\csc}\theta\omega e_t\varepsilon_j\right)\end{array}\right\} \\
&+ 2r^2\|\mathcal{H}\|^2 + \frac{1}{2}\Xi_{t=1}^{u}\left(\mathcal{T}_{11}^{t} - \mathcal{T}_{22}^{t} - \ldots - \mathcal{T}_{rr}^{t}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
2scal &= \Xi_{2\le i,j\le u} R^{ker\psi*}(\varepsilon_i,\varepsilon_j,\varepsilon_j,\varepsilon_i) + Ric^{ker\psi*}(\varepsilon_1) + Ric^{(ker\psi*)^\perp}(\csc\theta\omega e_1) \\
&+ \Xi_{2\le i,j\le u} R^{(ker\psi*)^\perp}(\csc\theta\omega e_s,\csc\theta\omega e_t,\csc\theta\omega e_t,\csc\theta\omega e_s) \qquad (4.28)\\
&+ 2r^2\|\mathcal{H}\|^2 + \frac{1}{2}\Xi_{t=1}^{u}\left(\mathcal{T}_{11}^{t} - \mathcal{T}_{22}^{t} - \ldots - \mathcal{T}_{rr}^{t}\right)^2 - 2\delta(\mathcal{N}) \\
&- 2\Xi_{t=1}^{u}\Xi_{2\le i<j\le r}\left(\mathcal{T}_{ii}^{t}\mathcal{T}_{ij}^{t} - \left(\mathcal{T}_{ij}^{t}\right)^2\right) + 2\Xi_{t=1}^{u}\Xi_{j=2}^{r}\left(\mathcal{T}_{1j}^{t}\right)^2 \\
&- 6\Xi_{j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{1j}^{t})^2 - 6\Xi_{2\le i,j\le u}\Xi_{t=1}^{r}(\Lambda_{ij}^{t})^2 \\
&+ \Xi_{i=1}^{u}\Xi_{k=1}^{r}\left\{\begin{array}{l} \varrho\left(\mathcal{T}_{\varepsilon_k}\csc\theta\omega e_i,\mathcal{T}_{\varepsilon_k}\csc\theta\omega e_i\right) \\ -\varrho\left(\Lambda_{\csc}\theta\omega e_i\varepsilon_k,\Lambda_{\csc}\theta\omega e_i\varepsilon_k\right)\end{array}\right\} \\
&+ \Xi_{t=1}^{u}\Xi_{j=1}^{r}\left\{\begin{array}{l} \varrho\left(\mathcal{T}_{\varepsilon_j}\csc\theta\omega e_t,\mathcal{T}_{\varepsilon_j}\csc\theta\omega e_t\right) \\ -\varrho\left(\Lambda_{\csc}\theta\omega e_t\varepsilon_j,\Lambda_{\csc}\theta\omega e_t\varepsilon_j\right)\end{array}\right\}.
\end{aligned}
$$

Since M is a Kenmotsu space form,

$$
\begin{aligned}
2scal &= Ric^{ker\psi*}(\varepsilon_1) + Ric^{(ker\psi*)^\perp}(\csc\theta\omega e_1) + 2r^2\|\mathcal{H}\|^2 \\
&+ \frac{1}{2}\Xi_{t=1}^{u}\left(\mathcal{T}_{11}^{t} - \mathcal{T}_{22}^{t} - \ldots - \mathcal{T}_{rr}^{t}\right)^2 - 2\delta(\mathcal{N}) + 2\Xi_{t=1}^{u}\Xi_{j=2}^{r}\left(\mathcal{T}_{1j}^{t}\right)^2 \\
&- 6\Xi_{j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{1j}^{t})^2 + 2\|\mathcal{T}^{ker\psi*}\|^2 - \|\Lambda^{ker\psi*}\|^2 \\
&+ \frac{c-3}{4}\{(u-1)(u-2)\} + \frac{c+1}{4}\{3(r-1)cos^2\theta\}. \qquad (4.29)
\end{aligned}
$$

As a consequence of (4.25) and (4.29), we deduce

$$
\begin{aligned}
&\frac{c-3}{4}\{r(r+u-1) - 4u + 1\} + \frac{c+1}{4}\{-3cos^2\theta + 3\|C\|^2 + 6\|B\|^2\} \\
&\quad \le Ric^{ker\psi*}(\varepsilon_1) + Ric^{(ker\psi*)^\perp}(\csc\theta\omega e_1) + 2r^2\|\mathcal{H}\|^2 - 2\delta(\mathcal{N}) \\
&\quad + 6\Xi_{j=1}^{u}\Xi_{t=1}^{r}(\Lambda_{1j}^{t})^2 + 2\|\mathcal{T}^{ker\psi*}\|^2 - \|\Lambda^{ker\psi*}\|^2. \qquad (4.30)
\end{aligned}
$$

□

Example 4.4. Let $\mathbb{R}^7$ be a Kenmotsu manifold with coordinates $(u_1, u_2, u_3, v_1, v_2, v_3, t)$, equipped with the standard Kenmotsu structure (ϕ, ξ, η, g) defined by:

$$\xi = \frac{\partial}{\partial t}, \quad \eta = dt,$$

and the $(1, 1)$-tensor field ϕ acts as:

$$\phi\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial}{\partial v_i}, \quad \phi\left(\frac{\partial}{\partial v_i}\right) = -\frac{\partial}{\partial u_i}, \quad \text{for } i = 1, 2, 3,$$
$$\phi\left(\frac{\partial}{\partial t}\right) = 0.$$

Let $(\mathbb{R}^4, g_{\mathbb{R}^4})$ be a Riemannian manifold endowed with the metric

$$g_{\mathbb{R}^4} = \sum_{i=1}^{4} dy_i^2.$$

Define a smooth map $F : \mathbb{R}^7 \to \mathbb{R}^4$ by:

$$F(u_1, u_2, u_3, v_1, v_2, v_3, t) = \left(u_1, u_2, u_3, \frac{v_1 + v_2}{\sqrt{2}} e^t\right).$$

Then, by direct calculations, the Jacobian matrix of F is:

$$F_* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} e^t & \frac{1}{\sqrt{2}} e^t & 0 & \frac{v_1+v_2}{\sqrt{2}} e^t \end{bmatrix}.$$

The rank of F_* is equal to 4. Thus, the map F is a submersion. After some computations, we obtain:

$$\ker F_* = \operatorname{span}\left\{ \bar{X}_1 = \frac{\partial}{\partial v_3}, \quad \bar{X}_2 = \xi = \frac{\partial}{\partial t}, \quad \bar{X}_3 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}\right) \right\},$$

and

$$(\ker F_*)^{\perp} = \operatorname{span}\left\{ \bar{Z}_1 = \frac{\partial}{\partial u_1}, \quad \bar{Z}_2 = \frac{\partial}{\partial u_2}, \quad \bar{Z}_3 = \frac{\partial}{\partial u_3}, \quad \bar{Z}_4 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}\right) \right\}.$$

A straightforward computation shows that the angle θ between $\phi(\bar{X}_3)$ and the horizontal distribution is constant and equal to $\frac{\pi}{2}$.

Hence, the map F is a slant Riemannian submersion from the Kenmotsu manifold $\mathbb{R}^7$ to $\mathbb{R}^4$.

5. Conclusions and future directions

In this paper, we derived precise inequalities for the scalar and Ricci curvatures associated with slant submersions from Kenmotsu space forms. These results generalize classical curvature inequalities and deepen our understanding of the interplay between submersion geometry and the contact metric structure. Furthermore, the characterization of the equality cases provides insights into the geometric behavior of the vertical and horizontal distributions, offering a potential framework for analyzing related submanifolds in contact geometry.

Beyond pure mathematics, the geometric structures explored here, particularly those involving curvature conditions and soliton solutions, have growing relevance in physical applications. In general relativity, geometric flows and solitons such as Ricci and Yamabe solitons play a role in describing self-similar solutions to Einstein's field equations. Additionally, in nonlinear optics, soliton theory contributes to the study of pulse propagation in optical fibers, where geometric structures are used to model wave phenomena with curvature-type constraints.

Future research could build upon the present results by extending the analysis to broader classes of submersions, including those defined on Sasakian, trans-Sasakian, or (κ, μ)-contact manifolds. Another promising direction is the investigation of curvature inequalities in the context of geometric flows and soliton equations, such as f-Ricci solitons, which naturally arise in both theoretical physics and geometric analysis. Applications to Lorentzian geometry and warped product spacetimes also offer valuable pathways, especially in modeling gravitational and cosmological systems.

For further exploration, readers may refer to recent studies such as Nagaraja and Premalatha [27], Kılıç and Meriç [21], and Agrawal [1], which examine solitonic and curvature-related structures in contact manifolds and their applications in theoretical physics. In addition, Blair's monograph [9] provides a broader geometric context, and Chow et al. [13] discuss soliton solutions relevant to general relativity. These works offer new methods and perspectives that complement the framework established in this paper.

Author contributions

M.A.: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Visualization, Writing–original draft, Supervision, Project administration, Funding acquisition, Writing–review & editing; I.A-D.: Methodology, Validation, Formal analysis, Investigation, Writing–original draft, Writing–review & editing; M.I.: Validation, Formal analysis, Investigation, Writing–original draft, Writing–review & editing; M.A.K.: Validation, Formal analysis, Investigation, Writing–original draft, Supervision, Writing–review & editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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