AIMS Mathematics, 10(4): 8777–8793.
DOI: 10.3934/math.2025402
Received: 31 October 2024
Revised: 23 January 2025
Accepted: 02 April 2025
Published: 17 April 2025

AIMS Mathematics

https://www.aimspress.com/journal/Math

Research article

Large deviations for transient random walks with asymptotically zero drifts

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Abstract: We consider transient nearest-neighbor random walks $X := \{X_n\}_{n\geq 0}$ on the half-line, whose transition probabilities are state-dependent with certain asymptotic perturbations. This is a specific case of Lamperti's random walks. Let $M_t := \max\{X_i,\ 0 \leq i \leq t\}$ be the maximum process of X, and $T_n := \inf\{t \geq 0, X_t = n\}$ be its inverse process. Hong&Yang (2019) provided the law of large numbers for T_n. In this paper, we study the large deviations for M_t and T_n with speeds less than n. This indicates that the perturbations slow down the random walk.

Keywords: nearest-neighbor; transient; random walk; large deviations; asymptotic perturbations
Mathematics Subject Classification: 60J80

1. Introduction

Consider a nearest-neighbor random walk defined as follows. $X = \{X_n\}_{n\geq 0}$ is a Markov chain on $\mathbb{N} = \{0, 1, 2, \cdots\}$ with $X_0 = 0$ and for $n \geq 1$, the transition probabilities are

$$p_i := P(X_{n+1} = i + 1 \mid X_n = i) = 1 - P(X_{n+1} = i - 1 \mid X_n = i) = \begin{cases} 1, & \text{if } i = 0, \\ \frac{1}{2} + \frac{B}{i^\alpha}, & \text{if } i \geq 1, \end{cases} \tag{1.1}$$

where $\alpha, B > 0$. This random walk X describes the motion of particles that starts at zero, moves on the nonnegative integers, and goes away from 0 with a larger probability than in the direction of 0. Obviously, $\frac{B}{i^\alpha}$ goes to 0 as $i \to \infty$. This means that the state 0 has a repelling power that decreases as the particle moves away from 0 (see [4]). This is a special case of the so-called Lamperti's random walk [7, 9].

The transience and recurrence for X are well-known results in the literature (e.g., Chung [3]). For $i \geq 1$, denote

$$\rho_i = \frac{1 - p_i}{p_i}.$$

Theorem A ([3]) *The random walk X defined in (1.1) is transient if and only if*

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n}\rho_i < \infty.$$

However, this criterion does not explicitly reveal the transient/recurrent classification for sequences of the form $\frac{B}{i^\alpha}$. Lamperti ([7, 9]) proved a more general theorem regarding the recurrence and transience of real nonnegative processes. Here we explicitly characterize the types of $\frac{B}{i^\alpha}$ sequences and discuss their implications for the transience and recurrence of the random walk X. These results also follow straightforwardly from Theorem A.

Transience criterion. For sufficiently large i:

· When $0 < \alpha < 1$, the random walk X is transient if $B > 0$ and recurrent if $B < 0$.

· When $\alpha = 1$, the random walk X is transient if $B > \frac{1}{4}$ and recurrent if $B \leq \frac{1}{4}$.

Numerous studies have investigated the limiting behavior of X depending on the sequence $\frac{B}{i^\alpha}$. Lamperti [8] established the weak convergence of X to a Bessel process. The law of the iterated logarithm for X was provided in [1, 12]. In [13], Voit proved the law of large numbers for X, which we restate here in our specific framework.

Theorem B ([13]) *If* $E_i = \frac{B}{i^\alpha}$, *where* $0 < \alpha < 1$ *and* $B > 0$, *then*

$$\lim_{n\to\infty}\frac{X_n}{n^{1/(1+\alpha)}} = [2B(1+\alpha)]^{1/(1+\alpha)} \textit{ almost surely.}$$

There is a minor mistake in [13] about the limit value. Hong&Yang [6] gave the correct form and provided the limit of T_n, which is defined as follows.

For $n \geq 0$, denote

$$T_n = \inf\{t \geq 0, X_t = n\}. \tag{1.2}$$

Theorem C ([6]) *If* $0 < \alpha < 1$ *and* $B > 0$, *then*

$$\lim_{n\to\infty}\frac{T_n}{n^{1+\alpha}} = \frac{1}{2B(1+\alpha)} \textit{ almost surely.}$$

On this basis, we investigate the large deviations for the sequences of hitting times $\{T_n, n \geq 1\}$. Additionally, for $t \in \mathbb{N}$, let

$$M_t = \max\{X_i,\ 0 \leq i \leq t\}$$

be the maximum of the random walk X up to time t. Note that T_n defined in (1.2) is the inverse process of M_t. Observing the relationship between M_t and T_n, we study the limit theorems and large deviations for M_t.

The paper is organized as follows. Section 2 presents the main results. Section 3 provides auxiliary results required for the proofs. The proofs of the main theorems are contained in Section 4.

2. Main results

This section presents the main results of the paper. It is divided into three parts. The first part establishes the law of large numbers for M_t, and the second part addresses the large deviation principles for T_n. Finally, we derive the large deviation principles for M_t.

Theorem 2.1. (Law of Large Numbers for M_t) *When* $0 < \alpha < 1$ *and* $B > 0$*, we have*

$$\lim_{n\to\infty} \frac{M_n}{n^{1/(1+\alpha)}} = [2B(1+\alpha)]^{1/(1+\alpha)} \text{ almost surely.}$$

Po-Ning Chen [2] established a generalization of the Gärtner–Ellis Theorem for arbitrary random sequences. All subsequent definitions in this section are adapted from reference [2].

Definition 2.1. *Define*

$$\Lambda_n(\lambda) = \frac{1}{n^{1-\alpha}} \log E \exp(\lambda n^{1-\alpha} \frac{T_n}{n^{1+\alpha}})$$

and

$$\overline{\Lambda}(\lambda) := \limsup_{n\to\infty} \Lambda_n(\lambda), \quad \underline{\Lambda}(\lambda) := \liminf_{n\to\infty} \Lambda_n(\lambda).$$

The sup-large deviation rate function of $\{T_n\}_{n=0}^{\infty}$ is defined as

$$\overline{\Lambda}^*(x) = \sup_{\{\lambda\in\mathbb{R},\ \overline{\Lambda}(\lambda)>-\infty\}} \{\lambda x - \overline{\Lambda}(\lambda)\}, \tag{2.1}$$

and the inf-large deviation rate function is defined as

$$\underline{\Lambda}^*(x) = \sup_{\{\lambda\in\mathbb{R},\ \underline{\Lambda}(\lambda)>-\infty\}} \{\lambda x - \underline{\Lambda}(\lambda)\}. \tag{2.2}$$

Actually, we have $\overline{\Lambda}(\lambda) \geq \underline{\Lambda}(\lambda) > -\infty$ for every $\lambda \in \mathbb{R}$ (see Property 3.2). So the ranges of the supremum operations in (2.1) and (2.2) are always $\{\lambda \in \mathbb{R}\}$. Hence, $\overline{\Lambda}^*(x)$ and $\underline{\Lambda}^*(x)$ are always defined.

Definition 2.2. *Define the sup-Gärter–Ellis set as*

$$\overline{\mathscr{G}} \triangleq \bigcup_{\{\lambda\in\mathbb{R},\ \overline{\Lambda}(\lambda)>-\infty\}} \overline{\mathscr{G}}(\lambda)$$

where

$$\overline{\mathscr{G}}(\lambda) \triangleq \{x \in \mathbb{R} : \limsup_{t\downarrow 0} \frac{\overline{\Lambda}(\lambda+t) - \overline{\Lambda}(\lambda)}{t} \leq x \leq \liminf_{t\downarrow 0} \frac{\overline{\Lambda}(\lambda) - \overline{\Lambda}(\lambda-t)}{t}\}.$$

Define the inf-Gärter–Ellis set as

$$\underline{\mathscr{G}} \triangleq \bigcup_{\{\lambda\in\mathbb{R},\ \underline{\Lambda}(\lambda)>-\infty\}} \underline{\mathscr{G}}(\lambda)$$

where

$$\underline{\mathscr{G}}(\lambda) \triangleq \{x \in \mathbb{R} : \limsup_{t\downarrow 0} \frac{\underline{\Lambda}(\lambda+t) - \underline{\Lambda}(\lambda)}{t} \leq x \leq \liminf_{t\downarrow 0} \frac{\underline{\Lambda}(\lambda) - \underline{\Lambda}(\lambda-t)}{t}\}.$$

Let us briefly remark on the sup-Gärter–Ellis set defined above. The definitions of $\overline{\mathscr{G}}$ and $\underline{\mathscr{G}}$ are special cases of Definitions 3.4 and 3.5 in [2], which only require $h(x) = x$. By Property 3.2, we can see that $\overline{\Lambda}'(\lambda)$ and $\underline{\Lambda}'(\lambda)$ exist for $|\lambda| \leq 2B^2$. So it can be derived that $\{x = \overline{\Lambda}'(\lambda) : |\lambda| \leq 2B^2\} \subset \overline{\mathscr{G}}$ and $\{x = \underline{\Lambda}'(\lambda) : |\lambda| \leq 2B^2\} \subset \underline{\mathscr{G}}$.

Theorem 2.2. (Large Deviation Principles for T_n)
(1) For any closed set $F \subset \mathbb{R}$, we have

$$\limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in F\right\} \leq -\inf_{x\in F} \overline{\Lambda}^*(x) \tag{2.3}$$

and

$$\liminf_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in F\right\} \leq -\inf_{x\in F} \underline{\Lambda}^*(x). \tag{2.4}$$

(2) For any $(a, b) \subset \overline{\mathscr{G}}$, we have

$$\limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in (a, b)\right\} \geq -\inf_{x\in(a,b)} \overline{\Lambda}^*(x).$$

(3) For any $(a, b) \subset \underline{\mathscr{G}}$, we have

$$\liminf_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in (a, b)\right\} \geq -\inf_{x\in(a,b)} \underline{\Lambda}^*(x).$$

Theorem 2.3. (Large Deviation Principles for M_t) *Define*

$$\overline{I}(x) = x^{1-\alpha}\overline{\Lambda}^*(\frac{1}{x^{1+\alpha}}),$$
$$\underline{I}(x) = x^{1-\alpha}\underline{\Lambda}^*(\frac{1}{x^{1+\alpha}}).$$

Then,
(1) for any closed set $F \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\left(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \in F\right) \leq -\inf_{x\in F} \overline{I}(x),$$

and

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\left(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \in F\right) \leq -\inf_{x\in F} \underline{I}(x),$$

(2) for any open set G,

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\left(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \in G\right) \geq -\inf_{x\in G\cap\{v, \frac{1}{v^{1+\alpha}} \in \underline{\mathscr{G}}^o \cap \overline{\mathscr{G}}^o\}} \underline{I}(x),$$

where $\underline{\mathscr{G}}^o$ and $\overline{\mathscr{G}}^o$ represent the interior of $\underline{\mathscr{G}}$ and $\overline{\mathscr{G}}$ respectively.

3. Preliminaries

Property 3.1. $\overline{\Lambda}^*(x)$ *and* $\underline{\Lambda}^*(x)$ *are the sup- and inf-large deviation rate functions of* $\{T_n\}_{n=0}^{\infty}$ *respectively. Denote* $\overline{x} = \frac{1}{2B(1+\alpha)}$, *then*
(1) $\overline{\Lambda}^*(x)$ and $\underline{\Lambda}^*(x)$ are always defined;
(2) $\overline{\Lambda}^*(x)$ and $\underline{\Lambda}^*(x)$ are both convex rate function;
(3) $\overline{\Lambda}^*(x)$ is continuous over $\{x \in \mathbb{R} : \overline{\Lambda}^*(x) < \infty\}$. Likewise, $\underline{\Lambda}^*(x)$ is continuous over $\{x \in \mathbb{R} : \underline{\Lambda}^*(x) < \infty\}$;
(4) $\overline{\Lambda}^*(\overline{x}) = \underline{\Lambda}^*(\overline{x}) = 0$.

Before proving Property 3.1, we provide some preparatory works.

Lemma A ([11], Lemma 17) *Let $p = (p_i)_{i\in\mathbb{Z}}$ and $q = (q_i)_{i\in\mathbb{Z}}$ such that*

$$p_i \le q_i,\ \forall i \in \mathbb{Z}.$$

Then we can construct the random walks $\{X_n, n \in \mathbb{N}\}$ and $\{Y_n, n \in \mathbb{N}\}$, respectively associated with p and q, and starting from any common point $d \in \mathbb{Z}$, such that

$$\forall n \in \mathbb{N},\ X_n \le Y_n \text{ almost surely.}$$

In [14], Ke Zhou provided explicit expressions for the generating functions of hitting times of the skip-free Markov chain on $\mathbb{Z}^+$. The chain's upward jumps are restricted to unit size; moreover, it starts at state 0 and is absorbed by state d. The result relevant to our case is stated as follows.

Assume that the random walk defined in 1.1 is absorbed by state d and that $p_i \equiv p \in (0, 1)$ for $1 \le i \le d - 1$. We denote this random walk as X^*. Let P be the transition probability matrix of X^*, and let $\tau_{d-1,d}$ denote the hitting time of state d when starting from state $d - 1$.

Lemma B ([14]) *Denote $f_{d-1}(s)$ as the generating function of $\tau_{d-1,d}$; then we have*

$$f_{d-1}(s) = ps\frac{\det[A_{d-1}(s)]}{\det[A_d(s)]}$$

where matrix $A_i(s)$ is the first i rows and first i lines of $I_{d+1} - P$ for $i = d - 1, d$. I_{d+1} is $(d + 1) \times (d + 1)$ unit matrix.

Actually, **Lemma B** can be derived from the proof of **Theorem 1.1** in [14]. Specifically, we have

$$f_{d-1}(s) = \frac{\varphi_d(s)}{\varphi_{d-1}(s)},$$

where $\varphi_d(s)$ is a symbol defined in [14], representing the generating function of the hitting time from state 0 to state d. We omit further details here.

Lemma 3.1. *Let $q = 1 - p$; we have*

$$\frac{\det[A_{d-1}(s)]}{\det[A_d(s)]} = \frac{(\eta^{d-3} - \beta^{d-3})g_3(s) - (\eta^{d-3}\beta - \eta\beta^{d-3})g_2(s)}{(\eta^{d-2} - \beta^{d-2})g_3(s) - (\eta^{d-2}\beta - \eta\beta^{d-2})g_2(s)} \tag{3.1}$$

where η, β are the roots of equation $x^2 - x + pqs^2 = 0$, and $g_2(s) = 1 - s^2q,\ g_3(s) = 1 - s^2q - pqs^2$.

Proof. We will prove this result using mathematical induction. We can easily calculate that $\det[A_2(s)] = 1 - s^2q = g_2(s)$ and $\det[A_3(s)] = 1 - s^2q - pqs^2 = g_3(s)$. Clearly, (3.1) holds for $d = 3$. Assume (3.1) is also hodes for $d = k$. To calculate $\det[A_{k+1}(s)]$, we expand along the last row and obtain

$$\det[A_{k+1}(s)] = \det[A_k(s)] - pqs^2 \det[A_{k-1}(s)].$$

By the quadratic formula, we have

$$\beta = \frac{1 - \sqrt{1 - 4pqs^2}}{2},\ \ \eta = \frac{1 + \sqrt{1 - 4pqs^2}}{2},$$

satisfying $\eta\beta = pqs^2$ and $\eta + \beta = 1$. Therefore, for $d = k + 1$, we have

$$\begin{aligned}\frac{\det[A_k(s)]}{\det[A_{k+1}(s)]} &= \frac{\det[A_k(s)]}{\det[A_k(s)] - \eta\beta\det[A_{k-1}(s)]} = \frac{1}{1 - \eta\beta\frac{\det[A_{k-1}(s)]}{\det[A_k(s)]}} \\ &= \frac{1}{1 - \eta\beta\frac{(\eta^{k-3}-\beta^{k-3})g_3(s)-(\eta^{k-3}\beta-\eta\beta^{k-3})g_2(s)}{(\eta^{k-2}-\beta^{k-2})g_3(s)-(\eta^{k-2}\beta-\eta\beta^{k-2})g_2(s)}} \\ &= \frac{(\eta^{k-2}-\beta^{k-2})g_3(s)-(\eta^{k-2}\beta-\eta\beta^{k-2})g_2(s)}{(\eta^{k-2}-\beta^{k-2}-\eta^{k-2}\beta+\eta\beta^{k-2})g_3(s)-(\eta^{k-2}\beta-\eta\beta^{k-2}-\eta^{k-2}\beta^2+\eta^2\beta^{k-2})g_2(s)} \\ &= \frac{(\eta^{k-2}-\beta^{k-2})g_3(s)-(\eta^{k-2}\beta-\eta\beta^{k-2})g_2(s)}{(\eta^{k-1}-\beta^{k-1})g_3(s)-(\eta^{k-1}\beta-\eta\beta^{k-1})g_2(s)},\end{aligned}$$

that is to say, (3.1) is true for $d = k + 1$. In conclusion, (3.1) is true for all d. This completes the inductive step. □

The assumptions $0 < \alpha < 1, B > 0$ will be used throughout the paper. Recall the definition of T_n in (1.2); let

$$\tau_i = T_i - T_{i-1}, \text{ for } i \geq 1. \tag{3.2}$$

Lemma 3.2. *For $\lambda \leq 2B^2$, we have*

$$Ee^{\frac{\lambda}{n^{2\alpha}}\tau_i} \to 1, \quad as\ n \to \infty$$

uniform about $1 \leq i \leq n$.

Proof. By **Lemma A**, let $p^{(1)} = (p_1, p_2, \cdots, p_{i-1})$ and $p^{(2)} = (p_i, \cdots, p_i)$ be $(i-1)$-dimensional vectors. Obviously, for every $1 \leq j \leq i-1$, $p_j > p_i$. We construct the random walks $\{X_n^{(1)}\}_{n\in\mathbb{N}}$ and $\{X_n^{(2)}\}_{n\in\mathbb{N}}$, associated with $p^{(1)}$ and $p^{(2)}$ respectively , starting from $i-1$, reflected at 0, and absorbed at i, such that

$$\forall n \in \mathbb{N},\ X_n^{(1)} \geq X_n^{(2)} \text{ almost surely.}$$

For $j = 1, 2$, denote $\tau_i^{(j)} = \inf\{n \in \mathbb{N},\ X_n^{(j)} = i\}$ as the passage time of $X^{(j)}$ starting from $i-1$ and ending at i. Then

$$\tau_i^{(1)} \leq \tau_i^{(2)} \text{ almost surely.}$$

implying $Es^{\tau_i^{(1)}} \leq Es^{\tau_i^{(2)}}$. By **Lemma B** and Lemma 3.1, the generating function of $\tau_i^{(2)}$ is:

$$\begin{aligned}f_{i-1}(s) = Es^{\tau_i^{(2)}} &= sp_i\frac{\det[A_{i-1}(s)]}{\det[A_i(s)]} \\ &= sp_i\frac{(\eta^{i-3}-\beta^{i-3})g_3(s)-(\eta^{i-3}\beta-\eta\beta^{i-3})g_2(s)}{(\eta^{i-2}-\beta^{i-2})g_3(s)-(\eta^{i-2}\beta-\eta\beta^{i-2})g_2(s)} \\ &= sp_i\frac{1+\frac{\eta g_2(s)-g_3(s)}{g_3(s)-\beta g_2(s)}\gamma^{i-3}}{\eta+\beta\frac{\eta g_2(s)-g_3(s)}{g_3(s)-\beta g_2(s)}\gamma^{i-3}},\end{aligned} \tag{3.3}$$

where η, β are the roots of equation $x^2 - x + p_iq_is^2 = 0$, and

$$g_3(s) = 1 - s^2q_i - p_iq_is^2,$$

$$g_2(s) = 1 - s^2 q_i,$$
$$\gamma = \frac{\beta}{\eta}.$$

By the definitions of X_n in (1.1) and τ_i in (3.2), the random variables $\tau_i^{(1)}$ and τ_i share the same distribution. Consequently, their generating functions $Es^{\tau_i} = Es^{\tau_i^{(1)}} \le Es^{\tau_i^{(2)}}$. Letting $s = e^{\frac{\lambda}{n^{2\alpha}}}$ and considering sufficiently large n large enough, we employ the representation $\tau_i^{(2)}$ from (3.3) to derive that

$$Ee^{\frac{\lambda}{n^{2\alpha}}\tau_i^{(2)}} \le Ee^{\frac{\lambda}{n^{2\alpha}}\tau_n^{(2)}} = sp_n \frac{1 + \frac{\eta g_2(s) - g_3(s)}{g_3(s) - \beta g_2(s)}\gamma^{n-3}}{\eta + \beta \frac{\eta g_2(s) - g_3(s)}{g_3(s) - \beta g_2(s)}\gamma^{n-3}}, \quad \text{for } \forall\ 1 \le i \le n.$$

Hence, to finish the proof of the lemma, we just need to verify that $Ee^{\frac{\lambda}{n^{2\alpha}}\tau_n^{(2)}} \to 1$, as $n \to \infty$. To ensure the existence of η, β, we require

$$s^2 = e^{\frac{2\lambda}{n^{2\alpha}}} \le \frac{1}{4p_n q_n},$$

where $p_n = 1 - q_n = \frac{1}{2} + \frac{B}{n^\alpha}$. In other words,

$$\lambda \le \frac{1}{2} \log \frac{1}{[(1 - \frac{4B^2}{n^{2\alpha}})^{\frac{n^{2\alpha}}{4B^2}}]^{4B^2}} \downarrow 2B^2.$$

Now,

$$\gamma = \frac{\beta}{\eta} = \frac{1 - \sqrt{1 - 4p_n q_n e^{\frac{2\lambda}{n^{2\alpha}}}}}{1 + \sqrt{1 - 4p_n q_n e^{\frac{2\lambda}{n^{2\alpha}}}}} \sim 1 - \frac{2\sqrt{4B^2 - 2\lambda}}{n^\alpha}, \ as\ n \to \infty,$$

so $\lim_{n\to\infty} \gamma^{n-3} = 0$.
Additionally, note that *as* $n \to \infty$,

$$\cdot s = e^{\frac{\lambda}{n^{2\alpha}}} \to 1,$$
$$\cdot p_n = \frac{1}{2} + \frac{B}{n^\alpha} \to \frac{1}{2},$$
$$\cdot q_n = 1 - p_n \to \frac{1}{2}.$$

From this, we can conclude that

$$\eta = \frac{1 + \sqrt{1 - 4p_n q_n e^{\frac{2\lambda}{n^{2\alpha}}}}}{2} \to \frac{1}{2},\ \beta = \frac{1 - \sqrt{1 - 4p_n q_n e^{\frac{2\lambda}{n^{2\alpha}}}}}{2} \to \frac{1}{2},\ as\ n \to \infty.$$

Similarly, we have

$$\frac{\eta g_2(s) - g_3(s)}{g_3(s) - \beta g_2(s)} = \frac{\eta(1 - s^2 q_n) - (1 - s^2 q_n - p_n q_n s^2)}{(1 - s^2 q_n - p_n q_n s^2) - \beta(1 - s^2 q_n)} \to constant,\ as\ n \to \infty.$$

Hence, for $\lambda \le 2B^2$,

$$Ee^{\frac{\lambda}{n^{2\alpha}}\tau_n^{(2)}} \to 1,\ as\ n \to \infty.$$

□

Lemma 3.3. *When* $|\lambda| \leq 2B^2$, *the series*

$$\sum_{j=1}^{\infty} \frac{\lambda^j}{j!} \frac{(2j-3)!!}{(4B^2)^{\frac{2j-1}{2}}}$$

are convergent.

Proof. Indeed,

$$\frac{(2(j+1)-3)!!}{(j+1)!} / \frac{(2j-3)!!}{j!} \to 2, \ as \ j \to \infty.$$

Therefore, when $|\frac{\lambda}{4B^2}| < \frac{1}{2}$, i.e., $|\lambda| < 2B^2$, the series

$$\sum_{j=1}^{\infty} \frac{\lambda^j}{j!} \frac{(2j-3)!!}{(4B^2)^{\frac{2j-1}{2}}}$$

converges.

When $\lambda = 2B^2$,

$$j[1 - (\frac{(2B^2)^{j+1}}{(j+1)!} \frac{(2(j+1)-3)!!}{(4B^2)^{\frac{2(j+1)-1}{2}}}) / (\frac{(2B^2)^j}{j!} \frac{(2j-3)!!}{(4B^2)^{\frac{2j-1}{2}}})] = j(\frac{3}{2j+2}) \to \frac{3}{2} \geq r > 1, \ as \ j \to \infty.$$

By the label discriminant, we can conclude that the series $\sum_{j=1}^{\infty} \frac{(2B^2)^j}{j!} \frac{(2j-3)!!}{(4B^2)^{\frac{2j-1}{2}}}$ is convergent.

Similarly, when $\lambda = -2B^2$, the series $\sum_{j=1}^{\infty} \frac{(-2B^2)^j}{j!} \frac{(2j-3)!!}{(4B^2)^{\frac{2j-1}{2}}}$ is also convergent. □

Lemma 3.4. *Suppose* $0 < \alpha < 1$*; then we have*

$$-\infty < \sum_{j=1}^{\infty} \frac{\lambda^j}{j!} L(j) = \underline{\Lambda}(\lambda) \leq \overline{\Lambda}(\lambda) = \sum_{j=1}^{\infty} \frac{\lambda^j}{j!} U(j) < \infty$$

for $|\lambda| \leq 2B^2$, *where*

$$L(j) := \liminf_{n\to\infty} \frac{1}{n^{1+(2j-1)\alpha}} \sum_{i=1}^{n} E(\tau_i)^j,$$

$$U(j) := \limsup_{n\to\infty} \frac{1}{n^{1+(2j-1)\alpha}} \sum_{i=1}^{n} E(\tau_i)^j.$$

Proof. Recalling that the random variables $\tau_i, \ i \geq 1$, defined in (3.2), are independent, we observe that for any fixed $\lambda \leq 2B^2$, the following holds:

$$\Lambda_n(\lambda) = \frac{1}{n^{1-\alpha}} \log E \exp(\lambda n^{1-\alpha} \frac{T_n}{n^{1+\alpha}}) = \frac{1}{n^{1-\alpha}} \log E \exp(\frac{\lambda}{n^{2\alpha}} \sum_{i=1}^{n} \tau_i)$$

$$= \frac{1}{n^{1-\alpha}} \sum_{i=1}^{n} \log(1 + Ee^{\frac{\lambda}{n^{2\alpha}}\tau_i} - 1)$$

$$
\begin{aligned}
&\sim \frac{1}{n^{1-\alpha}}\sum_{i=1}^{n}(Ee^{\frac{\lambda}{n^{2\alpha}}\tau_i}-1)\\
&=\frac{1}{n^{1-\alpha}}\sum_{i=1}^{n}(\sum_{j=1}^{\infty}\frac{E(\frac{\lambda}{n^{2\alpha}}\tau_i)^j}{j!})\\
&=\frac{1}{n^{1-\alpha}}\sum_{j=1}^{\infty}\frac{1}{j!}(\frac{\lambda}{n^{2\alpha}})^j\sum_{i=1}^{n}E(\tau_i)^j\\
&=\sum_{j=1}^{\infty}\frac{\lambda^j}{j!}\frac{1}{n^{1+(2j-1)\alpha}}\sum_{i=1}^{n}E(\tau_i)^j.
\end{aligned}
$$

Hence,

$$
\sum_{j=1}^{\infty}\frac{\lambda^j}{j!}L(j)=\liminf_{n\to\infty}\Lambda_n(\lambda)=\underline{\Lambda}(\lambda)\le\overline{\Lambda}(\lambda)=\limsup_{n\to\infty}\Lambda_n(\lambda)=\sum_{j=1}^{\infty}\frac{\lambda^j}{j!}U(j),
$$

where,

$$
\begin{aligned}
L(j)&:=\liminf_{n\to\infty}\frac{1}{n^{1+(2j-1)\alpha}}\sum_{i=1}^{n}E(\tau_i)^j,\\
U(j)&:=\limsup_{n\to\infty}\frac{1}{n^{1+(2j-1)\alpha}}\sum_{i=1}^{n}E(\tau_i)^j.
\end{aligned}
$$

Furthermore, by (3.3), we obtain

$$
E(\frac{\tau_n^{(2)}}{n^{2\alpha}})^j=f_{n-1}^{(j)}(0)\sim\frac{(2j-3)!!}{(4B^2)^{\frac{2j-1}{2}}}\frac{1}{n^{\alpha}},\ as\ n\to\infty.
$$

Consequently, for all $1\le i\le n$, it follows that

$$
\frac{1}{n^{1+(2j-1)\alpha}}\sum_{i=1}^{n}E(\tau_i)^j\le\frac{nE(\tau_n^{(2)})^j}{n^{1+(2j-1)\alpha}}=\frac{E(\tau_n^{(2)})^j}{n^{(2j-1)\alpha}}\to\frac{(2j-3)!!}{(4B^2)^{\frac{2j-1}{2}}},\ as\ n\to\infty.
$$

Hence, $L(j)\le U(j)\le\frac{(2j-3)!!}{(4B^2)^{\frac{2j-1}{2}}}$.

By Lemma 3.3, we have $-\infty<\underline{\Lambda}(\lambda)\le\overline{\Lambda}(\lambda)<\infty$ for $|\lambda|\le 2B^2$. □

Proposition 3.1. *For $j=1,2$, $L(j)=U(j)$.*

Proof. For $j=1$, we have

$$
\lim_{n\to\infty}\frac{\sum_{i=1}^{n}E\tau_i}{n^{1+\alpha}}=\lim_{n\to\infty}\frac{ET_n}{n^{1+\alpha}}=\frac{1}{2B(1+\alpha)},
$$

so, $L(1) = U(1) = \frac{1}{2B(1+\alpha)}$.
For $j = 2$, we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} E(\tau_i)^2}{n^{1+3\alpha}} = \lim_{n\to\infty} \frac{VarT_n + \sum_{i=1}^{n}(E\tau_i)^2}{n^{1+3\alpha}} = \frac{1}{4B^3(1+3\alpha)}. \tag{3.4}$$

By Proposition 2.1 [6],

$$\lim_{n\to\infty} \frac{E\tau_n}{n^\alpha} = \frac{1}{2B}.$$

Hence,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(E\tau_i)^2}{n^{1+2\alpha}} = \frac{1}{4B^2(1+2\alpha)}. \tag{3.5}$$

According to Theorem1.2 [6],

$$\lim_{n\to\infty} \frac{Var(T_n)}{n^{1+3\alpha}} = \frac{1}{4B^3(1+3\alpha)}. \tag{3.6}$$

The combination of (3.4, 3.5, and 3.6) yields:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} E(\tau_i)^2}{n^{1+3\alpha}} = \frac{1}{4B^3(1+3\alpha)}.$$

Hence, $L(2) = U(2) = \frac{1}{4B^3(1+3\alpha)}$. □

Property 3.2. $\overline{\Lambda}'(\lambda)$ *and* $\underline{\Lambda}'(\lambda)$ *exist for* $|\lambda| \le 2B^2$.

Proof. By Lemma 3.4, $\overline{\Lambda}(\lambda)$ and $\underline{\Lambda}(\lambda)$ can be expressed as power series when $|\lambda| \le 2B^2$. According to the properties of power series, it follows that $\overline{\Lambda}'(\lambda)$ and $\underline{\Lambda}'(\lambda)$ exist for $|\lambda| \le 2B^2$. □

Proposition 3.2. *For every* $\lambda \in \mathbb{R}$, $\overline{\Lambda}(\lambda) \ge \underline{\Lambda}(\lambda) > -\infty$.

Proof. Since for every $n \ge 0$, T_n is non-negative, we have $\underline{\Lambda}(\lambda) > -\infty$ for every $\lambda \in \mathbb{R}^+$. Let $\mathscr{D}_{\underline{\Lambda}} \triangleq \{\lambda : \underline{\Lambda}(\lambda) > -\infty\}$, and let $\mathscr{D}^o_{\underline{\Lambda}}$ be the interior of $\mathscr{D}_{\underline{\Lambda}}$. By Lemma 3.4, we know that $0 \in \mathscr{D}^o_{\underline{\Lambda}}$, so there exists $\lambda_0 < 0$ such that $\underline{\Lambda}(\lambda_0) > -\infty$. Hence, for every $\lambda < \lambda_0$, by Jensen's inequality we have the following:

$$E\exp(\lambda n^{1-\alpha}\frac{T_n}{n^{1+\alpha}}) = E[\exp(\lambda_0 n^{1-\alpha}\frac{T_n}{n^{1+\alpha}})]^{\frac{\lambda}{\lambda_0}} \ge [E\exp(\lambda_0 n^{1-\alpha}\frac{T_n}{n^{1+\alpha}})]^{\frac{\lambda}{\lambda_0}}.$$

So,

$$\begin{aligned}\underline{\Lambda}(\lambda) &= \liminf_{n\to\infty} \frac{1}{n^{1-\alpha}} \log E\exp(\lambda n^{1-\alpha}\frac{T_n}{n^{1+\alpha}}) \\ &\ge \frac{\lambda}{\lambda_0} \liminf_{n\to\infty} \frac{1}{n^{1-\alpha}} \log E\exp(\lambda_0 n^{1-\alpha}\frac{T_n}{n^{1+\alpha}}) = \frac{\lambda}{\lambda_0}\underline{\Lambda}(\lambda_0) > -\infty.\end{aligned}$$

Hence, we have $\overline{\Lambda}(\lambda) \ge \underline{\Lambda}(\lambda) > -\infty$ for every $\lambda \in \mathbb{R}$. □

The proof of Property 3.1

By Proposition 3.2, it follows that $\overline{\Lambda}^*(x)$ and $\underline{\Lambda}^*(x)$ are always defined. Properties (2) and (3) follow from the results in [2]. Specifically, $\overline{x} = \frac{1}{2B(1+\alpha)}$ is the limiting value of $\frac{T_n}{n^{1+\alpha}}$ as stated in Theorem C. The idea of the proof of (4) comes from Lemma2.2.5 in [5]. For all $\lambda \in \mathbb{R}$, by Jensen's inequality,

$$\begin{aligned}\overline{\Lambda}(\lambda) &= \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log E \exp(\lambda n^{1-\alpha} \frac{T_n}{n^{1+\alpha}}) \\ &\geq \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} E \log[\exp(\lambda n^{1-\alpha} \frac{T_n}{n^{1+\alpha}})] \\ &= \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} E(\lambda n^{1-\alpha} \frac{T_n}{n^{1+\alpha}}) = \lambda \limsup_{n\to\infty} \frac{ET_n}{n^{1+\alpha}} = \lambda\overline{x},\end{aligned}$$

Since $\overline{\Lambda}(0) = 0$, we obtain $\overline{\Lambda}^*(\overline{x}) = \sup_{\{\lambda\in\mathbb{R}\}}\{\lambda\overline{x} - \overline{\Lambda}(\lambda)\} = 0$. Similarly, we also have $\underline{\Lambda}^*(\overline{x}) = 0$. □

Proposition 3.3. *For all $x \geq \overline{x}$*

$$\overline{\Lambda}^*(x) = \sup_{\{\lambda\geq 0\}}\{\lambda x - \overline{\Lambda}(\lambda)\}, \quad \underline{\Lambda}^*(x) = \sup_{\{\lambda\geq 0\}}\{\lambda x - \underline{\Lambda}(\lambda)\}$$

is a non-decreasing function. Similarly, for all $x \leq \overline{x}$,

$$\overline{\Lambda}^*(x) = \sup_{\{\lambda\leq 0\}}\{\lambda x - \overline{\Lambda}(\lambda)\}, \quad \underline{\Lambda}^*(x) = \sup_{\{\lambda\leq 0\}}\{\lambda x - \underline{\Lambda}(\lambda)\}$$

is a non-increasing function.

Proof. For every $x \geq \overline{x}$ and every $\lambda < 0$,

$$\lambda x - \overline{\Lambda}(\lambda) \leq \lambda\overline{x} - \overline{\Lambda}(\lambda) \leq \overline{\Lambda}^*(\overline{x}) = 0,$$

so $\overline{\Lambda}^*(x) = \sup_{\{\lambda\geq 0\}}\{\lambda x - \overline{\Lambda}(\lambda)\}$. This also implies the monotonicity of $\overline{\Lambda}^*(x)$ on $(\overline{x}, \infty)$, since for every $\lambda \geq 0$, the function $\lambda x - \overline{\Lambda}(\lambda)$ is nondecreasing as a function of x.

For every $x \leq \overline{x}$ and every $\lambda > 0$,

$$\lambda x - \overline{\Lambda}(\lambda) \leq \lambda\overline{x} - \overline{\Lambda}(\lambda) \leq \overline{\Lambda}^*(\overline{x}) = 0,$$

so $\overline{\Lambda}^*(x) = \sup_{\{\lambda\leq 0\}}\{\lambda x - \overline{\Lambda}(\lambda)\}$. This also implies the monotonicity of $\overline{\Lambda}^*(x)$ on $(-\infty, \overline{x})$, since for every $\lambda \leq 0$, the function $\lambda x - \overline{\Lambda}(\lambda)$ is nonincreasing as a function of x.

Similarly, we can also obtain analogous properties for $\underline{\Lambda}^*(x)$. □

4. Proof of main results

Proof of Theorem 2.1

Let k_n be the unique (random) integers such that

$$T_{k_n} \leq n \leq T_{k_n+1}, \tag{4.1}$$

Since T_{k_n} represents the time when the random walk first reaches position k_n, and $T_{k_n} \leq n$, the maximum position M_n must satisfy $M_n \geq k_n$. Similarly, because $n \leq T_{k_n+1}$ and the definition of T_{k_n+1}, it follows that $M_n \leq k_n + 1$. Therefore, we conclude $k_n \leq M_n \leq k_n + 1$. Hence,

$$\frac{k_n}{n^{\frac{1}{1+\alpha}}} \leq \frac{M_n}{n^{\frac{1}{1+\alpha}}} \leq \frac{k_n+1}{n^{\frac{1}{1+\alpha}}}.$$

Note that **Theorem C** states that $\lim_{n\to\infty} \frac{T_{k_n}}{(k_n)^{1+\alpha}} = \lim_{n\to\infty} \frac{T_{k_n+1}}{(k_n)^{1+\alpha}} = \frac{1}{2B(1+\alpha)}$. As a consequence, dividing both sides of inequality (4.1) by $(k_n)^{1+\alpha}$ yields $\lim_{n\to\infty} \frac{n}{(k_n)^{1+\alpha}} = \frac{1}{2B(1+\alpha)}$. Thus,

$$[2B(1+\alpha)]^{1/(1+\alpha)} \geq \limsup_{n\to\infty} \frac{M_n}{n^{\frac{1}{1+\alpha}}} \geq \liminf_{n\to\infty} \frac{M_n}{n^{\frac{1}{1+\alpha}}} \geq [2B(1+\alpha)]^{1/(1+\alpha)}.$$

The proof is completed.

Proof of Theorem 2.2

The idea originates from the proof of Cramér's Theorem in [5] and Theorem 3 in [10]. Let F be a non-empty closed set. Note that (2.3) holds trivially if

$$\inf_{x\in F} \overline{\Lambda}^*(x) = 0.$$

Assume instead that

$$\inf_{x\in F} \overline{\Lambda}^*(x) > 0.$$

Since $\overline{\Lambda}^*(\overline{x}) = 0$ (see Property 3.1), $\overline{x}$ must lie in the open set F^c. Let (x_-, x_+) denote the union of all open intervals $(a, b) \subset F^c$ containing $\overline{x}$. Observe that $x_- < x_+$ and at least one of x_- or x_+ must be finite (since F is non-empty).

(1) If x_- is finite and $x_+ = +\infty$, then $x_- \in F \subset (-\infty, x_-)$. Consequently,

$$\overline{\Lambda}^*(x_-) \geq \inf_{x\in F} \overline{\Lambda}^*(x).$$

For every $\lambda \leq 0$,

$$\begin{aligned} P(\frac{T_n}{n^{1+\alpha}} \in F) &\leq P(\frac{T_n}{n^{1+\alpha}} \in (-\infty, x_-]) = P(\frac{T_n}{n^{1+\alpha}} - x_- \leq 0) = E[I_{\frac{T_n}{n^{1+\alpha}} - x_- \leq 0}] \\ &\leq E[\exp\{n^{1-\alpha}\lambda(\frac{T_n}{n^{1+\alpha}} - x_-)\}] = \exp\{-n^{1-\alpha}\lambda x_-\} E\exp\{n^{1-\alpha}\lambda\frac{T_n}{n^{1+\alpha}}\}. \end{aligned}$$

Observe that the random variable $\exp\{n^{1-\alpha}\lambda(\frac{T_n}{n^{1+\alpha}} - x_-)\} \geq 1$ on the set $\{\frac{T_n}{n^{1+\alpha}} - x_- \leq 0\}, for\ \lambda \leq 0$, we used Chebyshev's inequality in the last inequality above. Next, by taking the logarithm on both sides of the above inequality and considering the upper limit with proper scaling, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in F\right\} &\leq \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P(\frac{T_n}{n^{1+\alpha}} \in (-\infty, x_-]) \\ &\leq \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log\{\exp\{-n^{1-\alpha}\lambda x_-\} E\exp\{n^{1-\alpha}\lambda\frac{T_n}{n^{1+\alpha}}\}\} \end{aligned}$$

$$= -\lambda x_- + \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log E \exp\{n^{1-\alpha}\lambda \frac{T_n}{n^{1+\alpha}}\}$$
$$= -(\lambda x_- - \overline{\Lambda}(\lambda)). \tag{4.2}$$

The above inequality holds for all $\lambda \le 0$; consequently,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in F\right\} &\le \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P(\frac{T_n}{n^{1+\alpha}} \in (-\infty, x_-]) \\ &\le \inf_{\lambda\le 0}\{-(\lambda x_- - \overline{\Lambda}(\lambda))\} = -\sup_{\lambda\le 0}\{\lambda x_- - \overline{\Lambda}(\lambda)\} = -\overline{\Lambda}^*(x_-) \\ &\le -\inf_{x\in F} \overline{\Lambda}^*(x),\end{aligned} \tag{4.3}$$

where the last equality follows from Proposition 3.3. The final inequality holds trivially since $x_- \in F$.

By a similar argument, if x_+ is finite and $x_- = -\infty$, then

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in F\right\} &\le \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P(\frac{T_n}{n^{1+\alpha}} \in [x_+, \infty)) \\ &\le -\overline{\Lambda}^*(x_+) \le -\inf_{x\in F} \overline{\Lambda}^*(x)\end{aligned} \tag{4.4}$$

(2) If x_-, x_+ are all finite, then $x_-, x_+ \in F$, and $x_- < \overline{x} < x_+$.

$$\begin{aligned}\frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in F\right\} &\le \frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in (-\infty, x_-] \cup [x_+, \infty)\right\} \\ &\le \frac{1}{n^{1-\alpha}} \log \max\{2P(\frac{T_n}{n^{1+\alpha}} \le x_-), 2P(\frac{T_n}{n^{1+\alpha}} \ge x_+)\} \\ &\le \frac{\log 2}{n^{1-\alpha}} + \max\{\frac{1}{n^{1-\alpha}} \log P(\frac{T_n}{n^{1+\alpha}} \le x_-), \frac{1}{n^{1-\alpha}} \log P(\frac{T_n}{n^{1+\alpha}} \ge x_+)\},\end{aligned}$$

Combining (4.3) with (4.4), we obtain

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P\left\{\frac{T_n}{n^{1+\alpha}} \in F\right\} \\ &\le \max\{\limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P(\frac{T_n}{n^{1+\alpha}} \le x_-), \limsup_{n\to\infty} \frac{1}{n^{1-\alpha}} \log P(\frac{T_n}{n^{1+\alpha}} \ge x_+)\} \\ &\le -\inf_{x\in F} \overline{\Lambda}^*(x).\end{aligned} \tag{4.5}$$

In summary, we have completed the proof of (2.3). The proof of (2.4) follows by taking the limit inferior in (4.2), (4.3), (4.4) and (4.5).

The large deviations lower bound of T_n (statements (2) and (3) in Theorem 2.2) follow directly from Theorem3.5-3.6 in [2] with $h(x) = x$; therefore, we omit the details here.

Proof of Theorem 2.3
Let $v_0 = (2B(1+\alpha))^{\frac{1}{1+\alpha}}$, which is the limit value of $M_n/n^{\frac{1}{1+\alpha}}$ in Theorem 2.1. For every $v > v_0$, i.e. $\frac{1}{v^{1+\alpha}} < \frac{1}{v_0^{1+\alpha}} = \frac{1}{2B(1+\alpha)} = \overline{x}$, note that

$$P\left(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \ge v\right) = P\left(M_n \ge n^{\frac{1}{1+\alpha}} v\right) \le P\left(T_{\lfloor n^{\frac{1}{1+\alpha}} v\rfloor} \le n\right) = P\left(\frac{T_{\lfloor n^{\frac{1}{1+\alpha}} v\rfloor}}{(\lfloor n^{\frac{1}{1+\alpha}} v\rfloor)^{1+\alpha}} \le \frac{n}{(\lfloor n^{\frac{1}{1+\alpha}} v\rfloor)^{1+\alpha}}\right).$$

The event $M_n \geq n^{\frac{1}{1+\alpha}}v$ implies that the random walk has reached $n^{\frac{1}{1+\alpha}}v$ before time n, and thus

$$\{M_n \geq n^{\frac{1}{1+\alpha}}v\} \subset \{T_{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor} \leq n\},$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x, and $\lceil x\rceil$ denotes the smallest integer greater than or equal to x. Taking logarithms on both sides of the inequality and taking the upper limit with proper scaling yields

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{(\lfloor n^{\frac{1}{1+\alpha}}v\rfloor)^{1-\alpha}} \log P\left(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \geq v\right) \\
&\qquad\qquad \leq \limsup_{n\to\infty} \frac{1}{(\lfloor n^{\frac{1}{1+\alpha}}v\rfloor)^{1-\alpha}} \log P\left(\frac{T_{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor}}{(\lfloor n^{\frac{1}{1+\alpha}}v\rfloor)^{1+\alpha}} \leq \frac{n}{(\lfloor n^{\frac{1}{1+\alpha}}v\rfloor)^{1+\alpha}}\right) \\
&\qquad\qquad \leq -\overline{\Lambda}^*(\frac{1}{v^{1+\alpha}}),
\end{aligned}$$

where the last inequality follows from (4.3), noting that $\lim_{n\to\infty} \frac{n}{(\lfloor n^{\frac{1}{1+\alpha}}v\rfloor)^{1+\alpha}} = \frac{1}{v^{1+\alpha}}$ and the continuity of $-\overline{\Lambda}^*(x)$ (see Property 3.1). Multiplying both sides by $v^{1-\alpha}$ gives

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\left(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \geq v\right) \leq -v^{1-\alpha}\overline{\Lambda}^*(\frac{1}{v^{1+\alpha}}) = -\overline{I}(v).$$

Next, we derive an upper bound for $P(M_n/n^{\frac{1}{1+\alpha}} \leq v)$, where $v < v_0$ (i.e., $\frac{1}{v^{1+\alpha}} > \frac{1}{v_0^{1+\alpha}} = \overline{x}$). Observe that

$$P(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \leq v) \leq P(M_n \leq n^{\frac{1}{1+\alpha}}v) \leq P(T_{\lceil n^{\frac{1}{1+\alpha}}v\rceil} \geq n),$$

Thus, (4.4) implies

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\lceil n^{\frac{1}{1+\alpha}}v\rceil^{1-\alpha}} \log P(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \leq v) \\
&\leq \limsup_{n\to\infty} \frac{1}{\lceil n^{\frac{1}{1+\alpha}}v\rceil^{1-\alpha}} \log\left[P(\frac{T_{\lceil n^{\frac{1}{1+\alpha}}v\rceil}}{\lceil n^{\frac{1}{1+\alpha}}v\rceil^{1+\alpha}} \geq \frac{n}{\lceil n^{\frac{1}{1+\alpha}}v\rceil^{1+\alpha}})\right] \\
&\leq -\overline{\Lambda}^*(\frac{1}{v^{1+\alpha}})
\end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \leq v) \leq -v^{1-\alpha}\overline{\Lambda}^*(\frac{1}{v^{1+\alpha}}) = -\overline{I}(v). \tag{4.6}$$

The remainder of the proof follows similarly to the large deviations upper bound for T_n (Theorem 2.2). We have completed the proof of part (1) in Theorem 2.3.

Next, we consider the large deviation lower bound for M_n. For $v < v_0$ (i.e., $\frac{1}{v^{1+\alpha}} > \overline{x}$) satisfying $\frac{1}{v^{1+\alpha}} \in \overline{\mathscr{G}}^o$, there exists a neighborhood $(\frac{1}{v^{1+\alpha}} - \delta, \frac{1}{v^{1+\alpha}} + \delta) \subset \overline{\mathscr{G}}^o$. For any $0 < \epsilon < \delta/2$, we have

$$P(\frac{M_n}{n^{\frac{1}{1+\alpha}}} < v) = P(M_n < n^{\frac{1}{1+\alpha}}v) \geq P(T_{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor} > n)$$

$$\geq P\Big(\frac{T_{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor}}{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor^{1+\alpha}} > \frac{n}{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor^{1+\alpha}}\Big)$$
$$\geq P\Big(\frac{T_{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor}}{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor^{1+\alpha}} > \frac{1}{v^{1+\alpha}} + \epsilon\Big)$$

for sufficiently large n. By Theorem 2.2, we obtain

$$\limsup_{n\to\infty} \frac{1}{(\lfloor n^{\frac{1}{1+\alpha}}v\rfloor)^{1-\alpha}} \log P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} < v\Big)$$
$$\geq \limsup_{n\to\infty} \frac{1}{(\lfloor n^{\frac{1}{1+\alpha}}v\rfloor)^{1-\alpha}} \log P\Big(\frac{T_{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor}}{\lfloor n^{\frac{1}{1+\alpha}}v\rfloor^{1+\alpha}} > \frac{1}{v^{1+\alpha}} + \epsilon\Big)$$
$$\geq -\inf_{x\in(\frac{1}{v^{1+\alpha}}+\epsilon, \frac{1}{v^{1+\alpha}}+2\epsilon)} \overline{\Lambda}^*(x)$$
$$\geq -\overline{\Lambda}^*\Big(\frac{1}{v^{1+\alpha}} + \frac{3}{2}\epsilon\Big)$$

Since $\frac{1}{v^{1+\alpha}} \in \overline{\mathscr{G}}^o$, $\overline{\Lambda}^*(\frac{1}{v^{1+\alpha}}) < \infty$, and $\overline{\Lambda}^*(x)$ is continuous at $\frac{1}{v^{1+\alpha}}$ (see (3) in Property 3.1), letting $\epsilon \to 0$ yields

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} < v\Big) \geq -v^{1-\alpha}\overline{\Lambda}^*\Big(\frac{1}{v^{1+\alpha}}\Big) = -\overline{I}(v).$$

Combining this with (4.6), we conclude

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} < v\Big) = \limsup_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \leq v\Big) = -\overline{I}(v). \tag{4.7}$$

Similarly, for $v < v_0$, satisfying $\frac{1}{v^{1+\alpha}} \in \underline{\mathscr{G}}^o$, we have

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} < v\Big) = \liminf_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \leq v\Big) = -\underline{I}(v) \tag{4.8}$$

Hence, for any v with $\frac{1}{v^{1+\alpha}} \in \overline{\mathscr{G}}^o \cap \underline{\mathscr{G}}^o$, there exists a neighborhood $(\frac{1}{v^{1+\alpha}} - \delta, \frac{1}{v^{1+\alpha}} + \delta) \subset \overline{\mathscr{G}}^o \cap \underline{\mathscr{G}}^o$. Assume $v < v_0$ (the case $v > v_0$ is analogous). Choose $\delta_1 < \delta$ and $\delta_2 < \delta$, combining (4.7) with (4.8), we obtain

$$-\underline{I}(v+\delta_2) = \liminf_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} < v+\delta_2\Big) > \limsup_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \leq v-\delta_1\Big) = -\overline{I}(v-\delta_1). \tag{4.9}$$

Hence, for any $\epsilon' > 0$ and sufficiently large n,

$$P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} < v+\delta_2\Big) \geq \exp\{-n^{\frac{1-\alpha}{1+\alpha}}(\underline{I}(v+\delta_2) + \epsilon')\}$$
$$P\Big(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \leq v-\delta_1\Big) \leq \exp\{-n^{\frac{1-\alpha}{1+\alpha}}(\overline{I}(v-\delta_1) - \epsilon')\}$$

Thus,

$$P\Big(v+\delta_2 > \frac{M_n}{n^{\frac{1}{1+\alpha}}} > v-\delta_1\Big)$$

$$= P(\frac{M_n}{n^{\frac{1}{1+\alpha}}} < v + \delta_2) - P(\frac{M_n}{n^{\frac{1}{1+\alpha}}} \le v - \delta_1)$$
$$\ge \exp\{-n^{\frac{1-\alpha}{1+\alpha}}(\underline{I}(v + \delta_2) + \epsilon')\} - \exp\{-n^{\frac{1-\alpha}{1+\alpha}}(\overline{I}(v - \delta_1) - \epsilon')\}$$
$$= \exp\{-n^{\frac{1-\alpha}{1+\alpha}}(\underline{I}(v + \delta_2) + \epsilon')\}\{1 - \exp\{-n^{\frac{1-\alpha}{1+\alpha}}(\overline{I}(v - \delta_1) - \underline{I}(v + \delta_2) - 2\epsilon')\}\}$$

Since $\overline{I}(v - \delta_1) > \underline{I}(v + \delta_2)$ (by (4.9)); choosing ϵ', δ_1, and δ_2 such that $\overline{I}(v - \delta_1) - \underline{I}(v + \delta_2) - 2\epsilon' > 0$ yields

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P(v + \delta_2 > \frac{M_n}{n^{\frac{1}{1+\alpha}}} > v - \delta_1) \ge -\underline{I}(v + \delta_2) - \epsilon'.$$

By the continuity $\underline{I}(v)$, for any $\epsilon > 0$, we can choose δ_2 sufficiently small such that $\underline{I}(v + \delta_2) < \underline{I}(v) + \epsilon$. Consequently,

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P(v + \delta_2 > \frac{M_n}{n^{\frac{1}{1+\alpha}}} > v - \delta_1) \ge -\underline{I}(v) - \epsilon - \epsilon'.$$

Since ϵ and ϵ' are arbitrary, we conclude

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{1-\alpha}{1+\alpha}}} \log P(v + \delta_2 > \frac{M_n}{n^{\frac{1}{1+\alpha}}} > v - \delta_1) \ge -\underline{I}(v).$$

5. Conclusions

In recent years, random walks with asymptotic perturbations in transition probabilities have received widespread attention from scholars. These perturbations bring many new phenomena to random walks. For transient near-critical random walks, Voit [13] established the law of large numbers for the random walks, showing that the escape velocity of such processes is significantly slower than that of simple random walks. Building on this foundation, we derive large deviation principles for such random walks and demonstrate that their velocity order is substantially sublinear in n. This result further indicates that asymptotic perturbations reduce the wandering speed of random walks.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author thanks the anonymous referees for their meticulous reading of the manuscript. This work was partly supported by the National Natural Science Foundation of China (Grant Nos. 12001558, 71971228, 72371261).

Conflict of interest

The author declares no conflicts of interest in this paper.

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