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Research article

A viscosity-based iterative method for solving split generalized equilibrium and fixed point problems of strict pseudo-contractions

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Abstract: In this paper, we developed a viscosity-based extragradient iterative algorithm to approximate the solution of the split generalized equilibrium problem, the variational inequality problem, and the fixed point problem for a finite family of ϵ-strict pseudo-contractive and a nonexpansive mapping in Hilbert space. The main purpose was to establish strong convergence of the proposed algorithm under suitable conditions. We presented a comprehensive computational analysis to illustrate the effectiveness of our method and compared its performance with existing approaches. Our results extend and unify several well-known results in the literature, contributing significantly to the field.

Keywords: fixed point problem; split generalized equilibrium problem; ϵ- strict pseudo-contractive mapping; variational inequality problem; nonexpansive mapping; iterative methods; extragradient iterative methods
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1. Introduction

Let Y_1 and Y_2 represent real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Define Q_1 and Q_2 as nonempty, closed, convex subsets of Y_1 and Y_2, respectively. This paper addresses the task of finding a unified solution for the split generalized equilibrium problem, variational inequality problem, and fixed point problem, focusing on a finite family of ϵ-strict pseudo-contractive and nonexpansive mappings within real Hilbert spaces. These problems are commonly encountered in a wide range of mathematical models, where equilibrium and fixed-point conditions play a crucial role. Notable examples include applications in game theory, as seen in Nash's foundational work [24], image reconstruction [12, 17], network optimization in telecommunications, public infrastructure planning [27], and the analysis of Nash equilibria in strategic decision-making [25]. The variational inequality problem (VIP) involves finding an element $s^* \in Q_1$ such that

$$\langle As^*, v - s^* \rangle \geq 0, \quad \forall v \in Q_1, \tag{1.1}$$

where $A : Q_1 \to Y_1$ is a nonlinear mapping, as introduced by Hartman and Stampacchia [13].

In 1994, Blum and Oettli [3] introduced and studied the following equilibrium problem (EP): finding $s^* \in Q_1$ that satisfies

$$f_1(s^*, v) \geq 0, \quad \forall v \in Q_1, \tag{1.2}$$

where $f_1 : Q_1 \times Q_1 \to \mathbb{R}$ is a bifunction, with the solution set denoted by Sol(EP(1.2)). EP(1.2) has been widely studied and extended in multiple directions over the past two decades due to its importance. For details on existence and iterative solution approximations, see [9, 10, 29, 31] and the references therein. Censor et al. [7] introduced the split feasibility problem (SFP) for finite-dimensional Hilbert spaces, primarily for applications in phase retrieval and medical imaging, defined as:

$$\text{Find } \ s^* \in Q_1 \text{ such that } Bs^* \in Q_2,$$

where $B : Y_1 \to Y_2$ is a bounded linear operator.

We introduce the split generalized equilibrium problem (SGEP) as follows. Let $f_j, \phi_j : Q_j \times Q_j \to \mathbb{R}, \ j = 1, 2$, be non-linear bifunctions, with $B : Y_1 \to Y_2$ as a bounded linear operator. SGEP aims to find $s^* \in Q_1$ that satisfies

$$f_1(s^*, v) + \phi_1(v, s^*) - \phi_1(s^*, s^*) \geq 0, \quad \forall v \in Q_1, \tag{1.3}$$

such that

$$t^* = Bs^* \in Q_2 \ \text{ solves } \ f_2(t^*, u) + \phi_2(u, t^*) - \phi_2(t^*, t^*) \geq 0, \quad \forall u \in Q_2. \tag{1.4}$$

If $\phi_1, \phi_2 \equiv 0$, SGEP becomes split equilibrium problem (SEP) as:

$$f_1(s^*, v) \geq 0, \quad \forall v \in Q_1, \tag{1.5}$$

such that

$$t^* = Bs^* \in Q_2 \ \text{ solves } f_2(t^*, u) \geq 0, \quad \forall u \in Q_2. \tag{1.6}$$

We note that SGEP generalizes the multiple-set split feasibility problem and includes split variational inequalities as a special case, which further extends split zero problems and split feasibility problems for the existence and iterative approaches (see, e.g., [5, 8, 11, 15, 20]).

The fixed point problem (in short, FPP) for a map $T : Q_1 \to Q_1$ is to find $v \in Q_1$ such that $Tv = v$. The fixed point set of T is denoted by Fix(T) and $\text{Fix}(T) = \{v \in Q_1 : v = Tv\}$. Fixed point theory is a cornerstone of mathematics, with applications in solving equations, optimization, and modeling in pure and applied sciences. The study of fixed points in the moduli spaces of vector bundles over algebraic curves is fundamental in understanding the geometry of these spaces, see [1]. It is foundational to fields like topology (Brouwer's theorem), analysis (Banach's contraction principle), and mathematical physics (e.g., Hitchin integrable systems and mirror symmetry). Fixed points also play a vital role in economics and game theory, such as in Nash equilibria, see [2, 14, 24, 34].

Korpelevich [16] introduced the extragradient iterative method in Hilbert space Y_1 for solving VIP (1.1):

$$\left.\begin{array}{l} v_0 \in Q_1, \\ \varrho_n = P_{Q_1}(v_n - \alpha A v_n), \\ v_{n+1} = P_{Q_1}(v_n - \alpha A \varrho_n), \end{array}\right\} \tag{1.7}$$

where $\alpha > 0$, $A : Q_1 \to Y_1$ is a monotone and Lipschitz continuous mapping, and P_{Q_1} denotes the metric projection onto Q_1. Under certain conditions, this sequence converges to a solution of VIP (1.1).

In 2006, Nadezkhina and Takahashi [21] introduced a modified form of (1.7) as follows::

$$\left.\begin{array}{l} v_0 \in Q_1, \\ u_n = P_{Q_1}(v_n - r_n A v_n), \\ v_{n+1} = \beta_n v_n + (1 - \beta_n) T P_{Q_1}(v_n - r_n A u_n). \end{array}\right\} \tag{1.8}$$

By setting appropriate conditions on control sequences, they examined the weak convergence of the generated sequence toward a common solution of Fix(T) and VIP (1.1).

In the same year, Nadezhkina and Takahashi [22] proposed an alternative extragradient approach. This method combined the hybrid method [23] with the extragradient iterative approach [16] and was formulated as:

$$\left.\begin{array}{l} v_0 \in Q_1, \\ \varrho_n = P_{Q_1}(v_n - r_n A v_n), \\ \mathfrak{z}_n = \alpha_n v_n + (1 - \alpha_n) T P_{Q_1}(v_n - r_n A \varrho_n), \\ P_n = \{v \in Q_1 : \|\mathfrak{z}_n - v\|^2 \le \|v_n - z\|^2\}, \\ Q_n = \{v \in Q_1 : \langle v_n - z, x - v_n\rangle \ge 0\}, \\ v_{n+1} = P_{P_n \cap Q_n} v_0. \end{array}\right\} \tag{1.9}$$

Using specific control sequences, they demonstrated the strong convergence of this iterative sequence to a common solution of Fix(T) and VIP (1.1). For additional generalizations of the iterative method (1.9), refer to [6].

Notably, only a few strong convergence theorems exist for extragradient iterative methods, other than the hybrid extragradient approach. Therefore, our primary objective is to develop a novel extragradient method distinct from the hybrid type.

A mapping $S : Q_1 \to Y_1$ is defined as an ϵ-strict pseudo-contractive if there exists $\epsilon \in [0, 1)$ such that:

$$\|S v_1 - S v_2\|^2 \le \|v_1 - v_2\|^2 + \epsilon\|(I - S)v_1 - (I - S)v_2\|^2, \quad \forall v_1, v_2 \in Q_1. \tag{1.10}$$

When $\epsilon = 0$, the mapping S is termed nonexpansive, and when $\epsilon = 1$, it is termed pseudo-contractive. S is said to be strongly pseudo-contractive if $\exists\ \eta \in (0, 1)$ with $\langle S v_1 - S v_2, v_1 - v_2\rangle \le \eta\|v_1 - v_2\|^2$, $\forall v_1, v_2 \in Q_1$. Thus, the ϵ-strict pseudo-contractive class lies between the nonexpansive and pseudo-contractive mappings. Note that the class of strongly pseudo-contractive mappings is independent from ϵ-strict pseudo-contractive (for more, see [4]). It is obvious that for real Hilbert space Y_1, (1.10) is equivalent to

$$\langle S v_1 - S v_2, v_1 - v_2\rangle \le \|v_1 - v_2\|^2 - \frac{1 - \epsilon}{2}\|(I - S)v_1 - (I - S)v_2\|^2, \quad \forall v_1, v_2 \in Q_1. \tag{1.11}$$

Moreover, iterative approaches for strict pseudo-contractive are less advanced than those for nonexpansive mappings, despite Browder and Petryshyn's early work in 1967 [4]. This gap may be due to the additional term on the right-hand side of (1.10), which complicates the convergence analysis for algorithms that locate a fixed point of the strict pseudo-contractive S. However, strict pseudo-contractive offer stronger applications than nonexpansive mappings for solving inverse problems (see [30]). This motivates the development of iterative methods to find a common solution to SGEP((1.3) and (1.4)) and fixed-point problems for nonexpansive mappings, as well as for a finite family of ϵ-strict pseudo-contractive mappings. For further reading, refer to [19, 33] and the references therein.

Inspired by previous contributions (e.g., [11,15,22,33]), we propose a viscosity-based extragradient iterative approach for approximating solutions to split generalized equilibrium, variational inequality, and fixed point problems involving nonexpansive and ϵ-strict pseudo-contractive mappings in Hilbert space. We discuss strong convergence and highlight specific results derived from our theorems, along with numerical analysis to demonstrate the significance of our findings.

This paper is structured as follows: Section 2 covers foundational concepts, lemmas, and assumptions. In Section 3, we present main results, numerical analyses, and graphical illustrations. Section 4 provides an interpretation of our findings.

2. Preliminaries

In this section, we compile key concepts and results needed for the presentation of this work. We denote strong and weak convergence by $\to$ and $\rightharpoonup$, respectively.

For any $v_1 \in Y_1$ $\exists$ a unique nearest point to v_1 in Q_1 denoted by $P_{Q_1}v_1$ such that

$$\|v_1 - P_{Q_1}v_1\| \leq \|v_1 - v_2\|, \quad \forall v_2 \in Q_1.$$

The operator P_{Q_1} is called the metric projection of Y_1 onto Q_1. This projection is nonexpansive and satisfies

$$\langle v_1 - v_2, P_{Q_1}v_1 - P_{Q_1}v_2\rangle \geq \|P_{Q_1}v_1 - P_{Q_1}v_2\|^2, \quad \forall v_1, v_2 \in Y_1.$$

Additionally, $P_{Q_1}v_1$ is characterized by $P_{Q_1}v_1 \in Q_1$ and

$$\langle v_1 - P_{Q_1}v_1, v_2 - P_{Q_1}v_1\rangle \leq 0, \quad \forall v_2 \in Q_1.$$

This implies that

$$\|v_1 - v_2\|^2 \geq \|v_1 - P_{Q_1}v_1\|^2 + \|v_2 - P_{Q_1}v_1\|^2, \quad \forall v_1 \in Y_1,\ \forall v_2 \in Q_1.$$

In a real Hilbert space Y_1, it is known that

$$\|\beta v_1 + (1-\beta)v_2\|^2 = \beta\|v_1\|^2 + (1-\beta)\|v_2\|^2 - \beta(1-\beta)\|v_1 - v_2\|^2, \ \forall v_1, v_2 \in Y_1 \text{ and } \beta \in [0,1]; \tag{2.1}$$

and

$$\|v_1 + v_2\|^2 \leq \|v_1\|^2 + 2\langle v_2, v_1 + v_2\rangle, \ \forall v_1, v_2 \in Y_1. \tag{2.2}$$

Lemma 2.1. *[18] Let $\{b_n\}$ be a sequence of nonnegative real numbers with a subsequence $\{b_{n_i}\}$ such that $b_{n_i} < b_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then, there exists a non-decreasing sequence $\{m_j\} \subset \mathbb{N}$ such that $\lim_{j\to\infty} m_j = \infty$ and, for all sufficiently large $j \in \mathbb{N}$, the following hold:*

$$b_{m_j} \leq b_{m_{j+1}} \text{ and } b_j \leq b_{m_j}.$$

Moreover, m_j is the largest number n in the set $\{1, 2, 3, ..., j\}$ such that $b_n < b_{n+1}$.

Lemma 2.2. *[19] Assume that D is a strongly positive, self-adjoint, and bounded linear operator on a Hilbert space Y_1 with a positive coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|D\|^{-1}$. Then, it follows that $\|I - \rho D\| \leq 1 - \rho\overline{\gamma}$.*

Assumption 2.1. *Let $f_1, \phi_1 : Q_1 \times Q_1 \to \mathbb{R}$ be bimappings satisfying the following conditions:*

(1) $f_1(v_1, v_1) = 0, \quad \forall v_1 \in Q_1$;
(2) f_1 *is monotone, i.e.,*
$$f_1(v_1, v_2) + f_1(v_2, v_1) \leq 0, \quad \forall v_1, v_2 \in Q_1;$$
(3) *For each $v_2 \in Q_1$, $v_1 \to f_1(v_1, v_2)$ is weakly upper semicontinuous;*
(4) *For each $v_1 \in Q_1$, $v_2 \to f_1(v_1, v_2)$ is convex and lower semicontinuous;*
(5) *$\phi_1(.,.)$ is weakly continuous and $\phi_1(., v_2)$ is convex;*
(6) *ϕ_1 is skew-symmetric, i.e.,*

$$\phi_1(v_1, v_1) - \phi_1(v_1, v_2) + \phi_1(v_2, v_2) - \phi_1(v_2, v_1) \geq 0, \quad \forall v_1, v_2 \in Q_1.$$

Now, we define $F_r^{(f_1,\phi_1)} : Y_1 \to Q_1$ by

$$F_r^{(f_1,\phi_1)}(w) = \{v_1 \in Q_1 : f_1(v_1, v_2) + \phi_1(v_2, v_1) - \phi_1(v_1, v_1) + \frac{1}{r}\langle v_2 - v_1, v_1 - w\rangle \geq 0, \ \ \forall v_2 \in Q_1\}, \quad (2.3)$$

where r is a positive real number.

Lemma 2.3. *[28] Let f_1, ϕ_1 satisfy Assumption 2.1. Suppose that for each $w \in Y_1$ and for each $v_1 \in Q_1$, there exist a bounded subset $D_{v_1} \subseteq Q_1$ and $w_{v_1} \in Q_1$ such that for any $v_2 \in Q_1 \setminus D_{v_1}$,*

$$f_1(v_2, w_{v_1}) + \phi_1(w_{v_1}, v_2) - \phi_1(v_2, v_2) + \frac{1}{r}\langle w_{v_1} - v_2, v_2 - z\rangle < 0.$$

Let the mapping $F_r^{(f_1,\phi_1)}$ be defined by (2.3)*. Then, the following properties hold:*

(i) *$F_r^{(f_1,\phi_1)}(w)$ is nonempty for each $w \in Y_1$;*
(ii) *$F_r^{(f_1,\phi_1)}$ is single-valued;*
(iii) *$F_r^{(f_1,\phi_1)}$ is a firmly nonexpansive mapping, i.e., for all $w_1, w_2 \in Y_1$,*

$$\|F_r^{(f_1,\phi_1)}(w_1) - F_r^{(f_1,\phi_1)}(w_2)\|^2 \leq \langle F_r^{(f_1,\phi_1)}(w_1) - F_r^{(f_1,\phi_1)}(w_2), w_1 - w_2\rangle;$$

(iv) $\mathrm{Fix}(F_r^{(f_1,\phi_1)}) =$ *Sol(*GEP(1.3)*);*
(v) *Sol(*GEP(1.3)*) is closed and convex.*

Further, suppose $f_2, \phi_2 : Q_2 \times Q_2 \to \mathbb{R}$ satisfies Assumption 2.1. For $s > 0$ and $v_1 \in Y_2$, define the mapping $F_s^{(f_2,\phi_2)} : Y_2 \to Q_2$ as follows:

$$F_s^{(f_2,\phi_2)}(v_1) = \{v_2 \in Q_2 : f_2(v_1, v_3) + \phi_2(v_3, v_2) - \phi_2(v_2, v_2) + \frac{1}{s}\langle v_3 - v_2, v_2 - v_1\rangle \geq 0, \ \ \forall v_3 \in Q_2\}. \quad (2.4)$$

It follows that $F_s^{(f_2,\phi_2)}$ is nonempty, single-valued, and firmly nonexpansive, $\mathrm{Fix}(F_s^{(f_2,\phi_2)}) =$ Sol(GEP(1.4)), and Sol(GEP(1.4)) is closed and convex.

Lemma 2.4. *[28] Let f_1 and ϕ_1 satisfy Assumption 2.1 and let $F_r^{(f_1,\phi_1)}$ be defined by* (2.3). *Then, for $v_1, v_2 \in Y_1$ and $r_1, r_2 > 0$, we have*

$$\|F_{r_2}^{(f_1,\phi_1)}(v_2) - F_{r_1}^{(f_1,\phi_1)}(v_1)\| \leq \|v_2 - v_1\| + \frac{|r_2 - r_1|}{r_2}\|F_{r_2}^{(f_1,\phi_1)}(v_2) - v_2\|.$$

Lemma 2.5. *[35] Let $S : Q_1 \to Y_1$ be a ϵ_i-strictly pseudo-contractive mapping. Then,* $\mathrm{Fix}(S)$ *is closed convex and it yields that $P_{\mathrm{Fix}(S)}$ is well defined.*

Lemma 2.6. *[26] For any $u, v, w \in Y_1$, we have*

$$\|\sigma u + \gamma v + \mu w\|^2 = \sigma\|u\|^2 + \gamma\|v\|^2 + \mu\|w\|^2 - \sigma\gamma\|u - v\|^2 - \mu\gamma\|v - w\|^2 - \sigma\mu\|u - w\|^2,$$

where $\sigma,\ \gamma,\ \mu \in [0, 1]$ with $\sigma + \gamma + \mu = 1$.

Lemma 2.7. *[33] For each $i = 1, 2, 3, ..., \mathbb{N}$, where $\mathbb{N}$ is a natural number, consider $S_i : Q_1 \to Y_1$ to be a ϵ_i-strictly pseudo-contractive mapping for some $0 \leq \epsilon_i < 1$ with $\cap_{i=1}^{\mathbb{N}} \mathrm{Fix}(S_i) \neq \emptyset$. Let $\{\xi_i\}_{i=1}^{\mathbb{N}}$ be a positive sequence with $\sum_{i=1}^{\mathbb{N}} \xi_{i=1}^n = 1$. Then, $\sum_{i=1}^{\mathbb{N}} \xi_i S_i : Q_1 \to Y_1$ is ϵ-strictly pseudo-contractive with coefficient $\epsilon = \max_{1 \leq i < \mathbb{N}} \epsilon_i$ and $\mathrm{Fix}(\sum_{i=1}^{\mathbb{N}} \xi_i S_i) = \cap_{i=1}^{\mathbb{N}} \mathrm{Fix}(S_i)$.*

Lemma 2.8. *[32] Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ *and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < +\infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Results

Suppose $f_j, \phi_j : Q_j \times Q_j \to \mathbb{R}$ for $j = 1, 2$ are nonlinear bifunctions, and $B : Y_1 \to Y_2$ is a bounded linear operator. Let $A : Q_1 \to Y_1$ be a σ-inverse strongly monotone mapping and $h : Q_1 \to Q_1$ a δ-contraction mapping. Additionally, assume $T : Q_1 \to Y_1$ is a nonexpansive mapping, and for each $i = 1, 2, 3, ..., \mathbb{N}$, let $S_i : Q_1 \to Y_1$ be an ϵ_i-strictly pseudo-contractive mapping. Let $\{\xi_i^n\}_{i=1}^{\mathbb{N}}$ be a finite sequence of positive numbers satisfying $\sum_{i=1}^{\mathbb{N}} \xi_i^n = 1$. The algorithm we propose is as follows:

Algorithm 3.1.

Initialization: *Given* $v_1 \in Q_1$.

Iterative steps: *Iterate* v_{n+1} *using the following procedure:*
Step 1. *Compute:*

$$\begin{Bmatrix} \mathfrak{z}_n = F_{r_n}^{(f_1,\phi_1)}(v_n + \eta B^*((F_{r_n}^{(f_2,\phi_2)} - I)Bv_n), \\ \varrho_n = P_{Q_1}(\mathfrak{z}_n - \alpha_n A\mathfrak{z}_n), \end{Bmatrix}$$

and calculate the next iterate

$$v_{n+1} = \beta_n h(v_n) + (1-\beta_n)P_{Q_1}[\sigma_n v_n + \gamma_n T\varrho_n + \mu_n \sum_{i=1}^{\mathbb{N}} \xi_i^n S_i v_n], \quad n \geq 1,$$

where $F_{r_n}^{(f_1,\phi_1)}$ *is defined by* (2.3). *Set* $n := n+1$ *and move to Step 1.*

We consider control parameters in our main theorem as:

(i) $\beta_n, \sigma_n, \gamma_n, \mu_n \in (0,1)$ *and* $\sigma_n + \gamma_n + \mu_n = 1$,
(ii) $\lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \beta_n = \infty$,
(iii) $\sum_{n=1}^{\infty} \sum_{i=1}^{\mathbb{N}} |\xi_i^n - \xi_i^{n-1}| < +\infty$,
(iv) $0 \leq \epsilon_i \leq \sigma_n \leq c < 1, \ \lim_{n\to\infty} \sigma_n = c$,
(v) $\eta \in (0, \frac{1}{L})$, *L is the spectral radius of* B^*B *and* B^* *is the adjoint of B,*
(vi) $\alpha_n \in (0, 2\sigma)$.

These parameters play a crucial role in the convergence and behavior of our algorithm, providing flexibility and adaptability across iterations.

Theorem 3.1. *Let* Q_1 *and* Q_2 *be non-empty closed convex subsets of Hilbert spaces* Y_1 *and* Y_2, *respectively. Let* $f_j, \phi_j : Q_j \times Q_j \to \mathbb{R}$, *where* $j = 1, 2$, *be non-linear bifunctions that satisfy Assumption* 2.1, *and* $B : Y_1 \to Y_2$ *be a bounded linear operator. Let* $A : Q_1 \to Y_1$ *and* $h : Q_1 \to Q_1$ *be a* σ*-inverse strongly monotone mapping and* δ*-contraction mapping, respectively. Further, assume that* $T : Q_1 \to Y_1$ *is a nonexpansive mapping and for each* $i = 1, 2, 3, ..., \mathbb{N}$, $S_i : Q_1 \to Y_1$ *is a* ϵ_i*-strict pseudo-contractive mapping. Let* $\{\xi_i^n\}_{i=1}^{\mathbb{N}}$ *be a finite sequence of positive numbers with* $\sum_{i=1}^{\mathbb{N}} \xi_i^n = 1$. *Assume* $\Omega := \cap_{i=1}^{\mathbb{N}} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(T) \cap \mathrm{Sol}(\mathrm{SGEP}(1.3-1.4)) \cap \mathrm{Sol}(\mathrm{VIP}(1.1)) \neq \emptyset$. *Then, the sequence* $\{v_n\}$ *generated by Algorithm* 3.1 *converges strongly to* $\bar{v} \in \Omega$, *where* $\bar{v} = P_\Omega h(\bar{v})$.

For convenience, we split the proof of our main Theorem 3.1 into some lemmas as follows:

Lemma 3.1. *The sequences* $\{v_n\}$, $\{\mathfrak{z}_n\}$, *and* $\{\varrho_n\}$ *generated by iterative Algorithm* 3.1 *are bounded.*

Proof. We claim that $\{v_n\}$ is bounded. We set $s_n = \sigma_n v_n + \gamma_n T\varrho_n + \mu_n \sum_{i=1}^{\mathbb{N}} \xi_i^n S_i v_n$. Let $s^* \in \Omega$. Using the concept of non-expansivity of $I - \alpha_n A$ and T , we compute

$$\|\varrho_n - s^*\| = \|P_{Q_1}(\mathfrak{z}_n - \alpha_n A\mathfrak{z}_n) - s^*\|$$

$$\begin{aligned} &\le \|(I-\alpha_n A)\mathfrak{z}_n-(I-\alpha_n A)s^*\| \\ &\le \|\mathfrak{z}_n-s^*\|, \end{aligned} \tag{3.1}$$

and thus

$$\|T\varrho_n-s^*\|\le\|\varrho_n-s^*\|.$$

We calculate

$$\begin{aligned} \|\mathfrak{z}_n-s^*\|^2 &= \|F_{r_n}^{(f_1,\phi_1)}(v_n+\eta B^*(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n)-s^*\|^2 \\ &\le \|v_n+\eta B^*(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n)-s^*\|^2 \\ &\le \|v_n-s^*\|^2+\eta^2\|B^*(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\|^2 \\ &\quad +2\eta\langle v_n-s^*,B^*(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|\mathfrak{z}_n-s^*\|^2 &\le \|v_n-s^*\|^2+\eta^2\langle(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n,\ BB^*(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle \\ &\quad +2\eta\langle v_n-s^*,B^*(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle. \end{aligned} \tag{3.2}$$

Thus,

$$\begin{aligned} &\eta^2\langle(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n,BB^*(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle \\ &\qquad\le L\eta^2\langle(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n,(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle \\ &\qquad= L\eta^2\|(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\|^2. \end{aligned} \tag{3.3}$$

Assume that $\Pi:=2\eta\langle v_n-s^*,B^*(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle$, and we have

$$\begin{aligned} \Pi &= 2\eta\langle v_n-s^*,B^*(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle \\ &= 2\eta\langle B(v_n-s^*),(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle \\ &= 2\eta\langle B(v_n-s^*)+(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n \\ &\quad -(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n,(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle \\ &= 2\eta\Big\{\langle F_{r_n}^{(f_2,\phi_2)}Bv_n-Bs^*,(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\rangle \\ &\quad -\|(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\|^2\Big\} \\ &\le 2\eta\Big\{\frac{1}{2}\|(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\|^2-\|(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\|^2\Big\} \\ &\le -\eta\|(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\|^2. \end{aligned} \tag{3.4}$$

By (3.2)–(3.4), we get

$$\|\mathfrak{z}_n-s^*\|^2\le\|v_n-s^*\|^2+\eta(L\eta-1)\|(F_{r_n}^{(f_2,\phi_2)}-I)Bv_n\|^2. \tag{3.5}$$

As $\eta\in(0,\frac{1}{L})$, we have

$$\|\mathfrak{z}_n-s^*\|\le\|v_n-s^*\|.$$

We compute

$$\|v_{n+1}-s^*\| = \|\beta_n h(v_n)+(1-\beta_n)P_{Q_1}s_n-s^*\|$$

$$
\begin{aligned}
&\leq \beta_n\|h(v_n) - s^*\| + (1-\beta_n)\|P_{Q_1}s_n - s^*\| \\
&\leq \beta_n\|h(v_n) - s^*\| + (1-\beta_n)\|s_n - s^*\|. \qquad (3.6)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\|h(v_n) - s^*\| &\leq \|h(v_n) - h(s^*)\| + \|h(s^*) - s^*\| \\
&\leq \delta\|v_n - s^*\| + \|h(s^*) - s^*\|. \qquad (3.7)
\end{aligned}
$$

Setting $G_n = \sum_{i=1}^{\mathbb{N}} \xi_i^n S_i$ and using Lemma 2.7, we observe that the mapping $G_n : Q_1 \to Y_1$ is ϵ-strictly pseudo-contractive with $\epsilon = \max_{1\leq i<\mathbb{N}} \epsilon_i$ and $\mathrm{Fix}(G_n) = \cap_{i=1}^{\mathbb{N}}\mathrm{Fix}(S_i)$. Thus, by Lemma 2.6, we estimate

$$
\begin{aligned}
\|s_n - s^*\|^2 &= \|\sigma_n v_n + \gamma_n T\varrho_n + \mu_n G_n v_n - s^*\|^2 \\
&= \|\sigma_n(v_n - s^*) + \gamma_n(T\varrho_n - s^*) + \mu_n(G_n v_n - s^*)\|^2 \\
&= \sigma_n\|v_n - s^*\|^2 + \gamma_n\|T\varrho_n - s^*\|^2 + \mu_n\|G_n v_n - s^*\|^2 \\
&\quad - \sigma_n\gamma_n\|(v_n - T\varrho_n\|^2 - \gamma_n\mu_n\|T\varrho_n - G_n v_n\|^2 - \sigma_n\mu_n\|v_n - G_n v_n\|^2 \\
&\leq \sigma_n\|v_n - s^*\|^2 + \gamma_n\|\varrho_n - s^*\|^2 + \mu_n(\|v_n - s^*\|^2 + \epsilon\|v_n - G_n v_n\|^2) \\
&\quad - \sigma_n\gamma_n\|(v_n - T\varrho_n\|^2 - \gamma_n\mu_n\|T\varrho_n - G_n v_n\|^2 - \sigma_n\mu_n\|v_n - G_n v_n\|^2 \qquad (3.8) \\
&= (\sigma_n + \gamma_n + \mu_n)\|v_n - s^*\|^2 - \mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2 \\
&\quad - \sigma_n\gamma_n\|v_n - T\varrho_n\|^2 - \gamma_n\mu_n\|T\varrho_n - G_n v_n\|^2 \\
&= \|v_n - s^*\|^2 - \mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2 \\
&\quad - \sigma_n\gamma_n\|v_n - T\varrho_n\|^2 - \gamma_n\mu_n\|T\varrho_n - G_n v_n\|^2 \qquad (3.9)
\end{aligned}
$$

that implies

$$
\|s_n - s^*\| \leq \|v_n - s^*\|. \qquad (3.10)
$$

Thus, by (3.6), (3.7), and (3.10), we have

$$
\begin{aligned}
\|v_{n+1} - s^*\| &\leq \beta_n[\delta\|v_n - s^*\| + \|h(s^*) - s^*\|] + (1-\beta_n)\|v_n - s^*\| \\
&\leq [1 - \beta_n(1-\delta)]\|s_n - s^*\| + \beta_n\|h(s^*) - s^*\|.
\end{aligned}
$$

By induction, we get

$$
\|v_{n+1} - s^*\| \leq \max\{\|v_0 - s^*\|, \frac{1}{1-\delta}\|h(s^*) - s^*\|\}, \quad \forall n \geq 1,
$$

which shows that $\{v_n\}$ is bounded and hence, $\{\mathfrak{z}_n\}$ and $\{\varrho_n\}$ are also bounded. □

Lemma 3.2. *For each $n \geq 1$, prove that $\lim_{n\to\infty}\|v_{n+1} - v_n\| = 0$, $\lim_{n\to\infty}\|(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\| = 0$, $\lim_{n\to\infty}\|\mathfrak{z}_n - v_n\| = 0$, and $\lim_{n\to\infty}\|\varrho_n - \mathfrak{z}_n\| = 0$. Also, show that the sequence $\{v_n\}$ strongly converges to s^*, where $s^* = P_\Omega h(s^*)$.*

Proof. As $s^* \in \Omega$, therefore we compute

$$
\begin{aligned}
\|v_{n+1} - s^*\|^2 &= \|\beta_n(h(v_n) - s^*) + (1-\beta_n)(P_{Q_1}s_n - s^*)\|^2 \\
&\leq (1-\beta_n)\|P_{Q_1}s_n - s^*\|^2 + 2\langle\beta_n(h(v_n) - s^*), v_{n+1} - s^*\rangle
\end{aligned}
$$

$$\leq \quad (1-\beta_n)\|s_n - s^*\|^2 + 2\beta_n\langle h(v_n) - s^*, v_{n+1} - s^*\rangle. \tag{3.11}$$

Also, we estimate

$$\begin{aligned}\langle h(v_n) - s^*, v_{n+1} - s^*\rangle &= \langle h(v_n) - s^*, v_n - s^*\rangle + \langle h(v_n) - s^*, v_{n+1} - v_n\rangle \\ &\leq \|h(v_n) - h(s^*)\|\|v_n - s^*\| + \frac{M}{2}\|v_{n+1} - v_n\| + \langle h(v_n) - s^*, v_n - s^*\rangle \\ &\leq \delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\| + \langle h(v_n) - s^*, v_n - s^*\rangle,\end{aligned} \tag{3.12}$$

where $M = \sup_n \|h(v_n) - s^*\|$. Using (3.9), (3.11), and (3.12), we have

$$\begin{aligned}\|v_{n+1} - s^*\|^2 &\leq (1 - \beta_n(1-2\delta))\|v_n - s^*\|^2 + \beta_n M\|v_{n+1} - v_n\| + 2\beta_n\langle h(v_n) - s^*, v_n - s^*\rangle \\ &\quad - \mu_n(\sigma_n - \epsilon)(1-\beta_n)\|v_n - G_n v_n\|^2 - (1-\beta_n)\sigma_n\gamma_n\|v_n - T\varrho_n\|^2 \\ &\quad - (1-\beta_n)\gamma_n\mu_n\|T\varrho_n - G_n v_n\|^2 \end{aligned} \tag{3.13}$$

$$\begin{aligned}\|v_{n+1} - s^*\|^2 &\leq (1 - \beta_n(1-2\delta))\|v_n - s^*\|^2 + \beta_n M\|v_{n+1} - v_n\| \\ &\quad + 2\beta_n\langle h(v_n) - s^*, v_n - s^*\rangle.\end{aligned} \tag{3.14}$$

Set $q_n = \|v_n - s^*\|^2$. Consider the two cases on $\{q_n\}$ as:
Case 1. For every $n \geq m_0$ where $m_0 \in \mathbb{N}$, consider the sequence $\{q_n\}$ as decreasing, therefore it must be convergent. Applying the conditions in (3.13), we get

$$\lim_{n\to\infty}\|v_n - T\varrho_n\| = 0, \quad \lim_{n\to\infty}\|T\varrho_n - G_n v_n\| = 0, \quad \text{and} \quad \lim_{n\to\infty}\|v_n - G_n v_n\| = 0. \tag{3.15}$$

Notice that $\{v_n\}$ is bounded, therefore $\exists$ a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ with $v_{n_j} \rightharpoonup p \in Q_1$ and satisfies

$$\limsup_{n\to\infty}\langle h(s^*) - s^*, v_n - s^*\rangle = \lim_{j\to\infty}\langle h(s^*) - s^*, v_{n_j} - s^*\rangle. \tag{3.16}$$

Define $H_n = \kappa_n v + (1-\kappa_n)G_n v, \ \ \forall v \in Q_1$ and $\kappa_n \in [\delta, 1)$. Applying Lemma 2.5, $H_n : Q_1 \to Y_1$ is nonexpansive and we have

$$\begin{aligned}\|v_n - H_n v_n\| &= \|v_n - (\kappa_n v_n + (1-\kappa_n)G_n v_n)\| \\ &= \|(\kappa_n + (1-(\kappa_n)v_n) - (\kappa_n v_n + (1-\kappa_n)G_n v_n)\| \\ &= (1-\kappa_n)\|v_n - G_n v_n\|.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\|v_n - H_n v_n\| = 0. \tag{3.17}$$

Applying the given conditions, we may consider that $\xi_i^n \to \xi_i$ as $n \to \infty, \ \ \forall i$. By Lemma 2.7, the map $G : Q_1 \to Y_1$ with $Gv = (\sum_{i=1}^{\mathbb{N}} \xi_i S_i)v, \ \forall v \in Q_1$, is ϵ-strict pseudo-contractive and $\text{Fix}(G) = \cap_{i=1}^{\mathbb{N}}\text{Fix}(S_i)$. Applying Lemma 2.7, given the conditions and boundedness of v_n, we get

$$\begin{aligned}\|v_n - Gv_n\| &\leq \|v_n - G_n v_n\| + \|G_n v_n - Gv_n\| \\ &\leq \|v_n - G_n v_n\| + \sum_{i=1}^{\mathbb{N}}|\xi_i^n - \xi_i|\|S_i v_n\|,\end{aligned}$$

and thus

$$\lim_{n\to\infty} \|v_n - Gv_n\| = 0. \tag{3.18}$$

As

$$\|G_n v_n - Gv_n\| \leq \|G_n v_n - v_n\| + \|v_n - Gv_n\|,$$

this yields by (3.14) and (3.18) that

$$\lim_{n\to\infty} \|G_n v_n - Gv_n\| = 0. \tag{3.19}$$

Again, we notice that the map $H = tv + (1-t)Gv,\ \ \forall v \in Q_1$ and $t \in [\delta, 1)$, and $\mathrm{Fix}H = \mathrm{Fix}G$. Thus, we obtain

$$\begin{aligned} \|v_n - Hv_n\| &\leq \|v_n - H_n v_n\| + \|H_n v_n - Hv_n\| \\ &\leq \|v_n - H_n v_n\| + \|\kappa_n v_n + (1-\kappa_n)H_n v_n - tv - (1-t)Gv\| \\ &\leq \|v_n - H_n v_n\| + |\kappa_n - t|\|v_n - Gv_n\| + (1-\kappa_n)\|H_n v_n - Hv_n\|. \end{aligned}$$

Applying (3.17)–(3.19), we have

$$\lim_{n\to\infty} \|v_n - Hv_n\| = 0.$$

As $v_n \in Q_1$, therefore

$$\|v_{n+1} - v_n\| \leq \beta_n \|hv_n - v_n\| + (1-\beta_n)[\sigma_n \|T\varrho_n - v_n\| + \mu_n \|v_n - G_n v_n\|].$$

Using the given conditions and (3.14), we get

$$\lim_{n\to\infty} \|v_{n+1} - v_n\| = 0. \tag{3.20}$$

Applying (3.11) and (3.12), we estimate

$$\|v_{n+1} - s^*\|^2 \leq (1-\beta_n)\|s_n - s^*\|^2 + 2\beta_n[\delta \|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|]. \tag{3.21}$$

Using (3.1), (3.5), and (3.8), we compute

$$\begin{aligned} \|s_n - s^*\|^2 &\leq (1-\gamma_n)\|v_n - s^*\|^2 + \gamma_n \|\mathfrak{z}_n - s^*\|^2 &(3.22)\\ &\leq \|v_n - s^*\|^2 + \eta(L\eta - 1)\gamma_n \|(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\|^2. &(3.23) \end{aligned}$$

By (3.21) and (3.23), we get

$$\begin{aligned} \|v_{n+1} - s^*\|^2 &\leq (1-\beta_n)\|v_n - s^*\|^2 + \eta(L\eta - 1)(1-\beta_n)\gamma_n \|(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\|^2 \\ &\quad + 2\delta\beta_n \|v_n - s^*\|^2 + M\beta_n \|v_{n+1} - v_n\| \\ \eta(1 - L\eta)(1-\beta_n)\gamma_n \|(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\|^2 &\leq \|v_n - s^*\|^2 - \|v_{n+1} - s^*\|^2 \\ &\quad + 2\delta\beta_n \|v_n - s^*\|^2 + M\beta_n \|v_{n+1} - v_n\| \\ &\leq (\|v_n - s^*\| + \|v_{n+1} - s^*\|)\|v_n - v_{n+1}\| \\ &\quad + 2\delta\beta_n \|v_n - s^*\|^2 + M\beta_n \|v_{n+1} - v_n\|. \end{aligned}$$

Applying the given condition and (3.20), we get

$$\lim_{n\to\infty} \|(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\| = 0. \tag{3.24}$$

Next, we compute

$$\begin{aligned}
\|\mathfrak{z}_n - s^*\|^2 &= \|F_{r_n}^{(f_1,\phi_1)}(v_n + \eta B^*(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n) - s^*\|^2 \\
&\leq \|F_{r_n}^{(f_1,\phi_1)}(v_n + \eta B^*(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n) - F_{r_n}^{(f_1,\phi_1)}s^*\|^2 \\
&\leq \langle \mathfrak{z}_n - s^*, v_n + \eta B^*(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n - s^*\rangle \\
&= \frac{1}{2}\Big\{\|\mathfrak{z}_n - s^*\|^2 + \|v_n + \eta B^*(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n - s^*\|^2 \\
&\quad - \|(\mathfrak{z}_n - s^*) - [v_n + \eta B^*(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n - s^*]\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|\mathfrak{z}_n - s^*\|^2 + \|v_n - s^*\|^2 \\
&\quad - \|\mathfrak{z}_n - v_n - \eta B^*(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|\mathfrak{z}_n - s^*\|^2 + \|v_n - s^*\|^2 \\
&\quad - [\|\mathfrak{z}_n - v_n\|^2 + \eta^2\|B^*(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\|^2 \\
&\quad - 2\eta\langle \mathfrak{z}_n - v_n, B^*(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\rangle]\Big\}.
\end{aligned}$$

Thus,

$$\|\mathfrak{z}_n - s^*\|^2 \leq \|v_n - s^*\|^2 - \|\mathfrak{z}_n - v_n\|^2 + 2\eta\|B(z_n - v_n)\|\|(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\|. \tag{3.25}$$

Using (3.22) and (3.25) in (3.21), we get

$$\begin{aligned}
\|v_{n+1} - s^*\|^2 &\leq (1-\beta_n)\|s_n - s^*\|^2 + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|] \\
&\leq (1-\beta_n)(1-\gamma_n)\|v_n - s^*\|^2 + (1-\beta_n)\gamma_n\|v_n - s^*\|^2 \\
&\quad - \gamma_n(1-\beta_n)\|\mathfrak{z}_n - v_n\|^2 \\
&\quad + 2\eta(1-\beta_n)\gamma_n\|B(z_n - v_n)\|\|(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\| \\
&\quad + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|] \\
\implies \quad \gamma_n(1-\beta_n)\|\mathfrak{z}_n - v_n\|^2 &\leq \|v_n - s^*\|^2 - \|v_{n+1} - s^*\|^2 \\
&\quad + 2\eta(1-\beta_n)\gamma_n\|B(z_n - v_n)\|\|(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\| \\
&\quad + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|] \\
&\leq (\|v_n - s^*\| + \|v_{n+1} - s^*\|)\|v_n - v_{n+1}\| \\
&\quad + 2\eta(1-\beta_n)\gamma_n\|B(z_n - v_n)\|\|(F_{r_n}^{(f_2,\phi_2)} - I)Bv_n\| \\
&\quad + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|].
\end{aligned} \tag{3.26}$$

Applying the given conditions, (3.20) and (3.24) in (3.26), we get

$$\lim_{n\to\infty} \|\mathfrak{z}_n - v_n\| = 0. \tag{3.27}$$

Further, we estimate

$$
\begin{aligned}
\|\varrho_n - s^*\|^2 &= \|P_{Q_1}(\mathfrak{z}_n - \alpha_n A\mathfrak{z}_n) - P_{Q_1}(s^* - \alpha_n A s^*)\|^2 \\
&\le \langle \varrho_n - s^*, (\mathfrak{z}_n - \alpha_n A\mathfrak{z}_n) - (s^* - \alpha_n A s^*)\rangle \\
&\le \frac{1}{2}\{\|\varrho_n - s^*\|^2 + \|(\mathfrak{z}_n - \alpha_n A\mathfrak{z}_n) \\
&\quad - (s^* - \alpha_n A s^*)\|^2 - \|(\varrho_n - \mathfrak{z}_n) + \alpha_n(A\mathfrak{z}_n - As^*)\|^2\} \\
&\le \frac{1}{2}\{\|\varrho_n - s^*\|^2 + \|\mathfrak{z}_n - s^*\|^2 - \|(\varrho_n - \mathfrak{z}_n) + \alpha_n(A\mathfrak{z}_n - As^*)\|^2\} \\
&\le \|\mathfrak{z}_n - s^*\|^2 - \|\varrho_n - \mathfrak{z}_n\|^2 - \alpha_n^2\|A\mathfrak{z}_n - As^*\|^2 \\
&\quad + 2\alpha_n\langle \varrho_n - \mathfrak{z}_n, A\mathfrak{z}_n - As^*\rangle \\
&\le \|\mathfrak{z}_n - s^*\|^2 - \|\varrho_n - \mathfrak{z}_n\|^2 + 2\alpha_n\|\varrho_n - \mathfrak{z}_n\|\|A\mathfrak{z}_n - As^*\| \\
&\le \|v_n - s^*\|^2 - \|\varrho_n - \mathfrak{z}_n\|^2 + 2\alpha_n\|\varrho_n - \mathfrak{z}_n\|\|A\mathfrak{z}_n - As^*\|. \qquad (3.28)
\end{aligned}
$$

From (3.8), we obtain

$$
\begin{aligned}
\|s_n - s^*\|^2 &\le \sigma_n\|v_n - s^*\|^2 + \gamma_n\|\varrho_n - s^*\|^2 + \mu_n\|v_n - s^*\|^2 + \mu_n\epsilon\|v_n - G_n v_n\|^2 \\
&\le (1-\gamma_n)\|v_n - s^*\|^2 + \gamma_n\|\varrho_n - s^*\|^2 - \mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2. \qquad (3.29)
\end{aligned}
$$

From (3.21) and (3.29), we estimate

$$
\begin{aligned}
\|v_{n+1} - s^*\|^2 &\le (1-\beta_n)[(1-\gamma_n)\|v_n - s^*\|^2 + \gamma_n\|\varrho_n - s^*\|^2 - \mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2] \\
&\quad + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|] \\
&\le (1-\beta_n)(1-\gamma_n)\|v_n - s^*\|^2 + (1-\beta_n)\gamma_n[P_{Q_1}(\mathfrak{z}_n - \alpha_n A\mathfrak{z}_n) - P_{Q_1}(s^* - \alpha_n A s^*)] \\
&\quad - (1-\beta_n)\mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2 + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|] \\
&\le (1-\beta_n)(1-\gamma_n)\|v_n - s^*\|^2 + (1-\beta_n)\gamma_n[\|\mathfrak{z}_n - s^*\|^2 + \alpha_n(\alpha_n - 2\sigma)\|A\mathfrak{z}_n - As^*\|^2] \\
&\quad - (1-\beta_n)\mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2 + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|] \\
&\le (1-\beta_n)(1-\gamma_n)\|v_n - s^*\|^2 + (1-\beta_n)\gamma_n[\|v_n - s^*\|^2 + \alpha_n(\alpha_n - 2\sigma)\|A\mathfrak{z}_n - As^*\|^2] \\
&\quad - (1-\beta_n)\mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2 + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|] \\
&\le \|v_n - s^*\|^2 + (1-\beta_n)\gamma_n\alpha_n(\alpha_n - 2\sigma)\|A\mathfrak{z}_n - As^*\|^2 \\
&\quad - (1-\beta_n)\mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2 + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|],
\end{aligned}
$$

which implies

$$
\begin{aligned}
(1-\beta_n)\gamma_n\alpha_n(2\sigma - \alpha_n)\|A\mathfrak{z}_n - As^*\|^2 &\le \|v_n - s^*\|^2 - \|v_{n+1} - s^*\|^2 - (1-\beta_n)\mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2 \\
&\quad + 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|] \\
&\le (\|v_n - s^*\| + \|v_{n+1} - s^*\|)\|v_n - v_{n+1}\| \\
&\quad - (1-\beta_n)\mu_n(\sigma_n - \epsilon)\|v_n - G_n v_n\|^2
\end{aligned}
$$

$$+ 2\beta_n[\delta\|v_n - s^*\|^2 + \frac{M}{2}\|v_{n+1} - v_n\|]. \tag{3.30}$$

Applying the given conditions, (3.15) and (3.20) in (3.30), we obtain

$$\lim_{n\to\infty} \|A\mathfrak{z}_n - As^*\| = 0. \tag{3.31}$$

Using (3.11), (3.12), (3.28), and (3.29), we compute

$$\begin{aligned}
\|v_{n+1} - s^*\|^2 &\leq (1-\beta_n)(1-\gamma_n)\|v_n - s^*\|^2 + (1-\beta_n)\gamma_n[\|v_n - s^*\|^2 - \|\varrho_n - \mathfrak{z}_n\|^2 \\
&\quad + 2\alpha_n\|\varrho_n - \mathfrak{z}_n\|\|A\mathfrak{z}_n - As^*\|] + (1-\beta_n)\mu_n\epsilon\|v_n - G_nv_n\|^2 \\
&\quad + 2\beta_n\delta\|v_n - s^*\|^2 + \beta_nM\|v_{n+1} - v_n\| + 2\beta_n\langle h(v_n) - s^*, v_n - s^*\rangle \\
\implies \quad (1-\beta_n)\gamma_n\|\varrho_n - \mathfrak{z}_n\|^2 &\leq \|v_n - s^*\|^2 - \|v_{n+1} - s^*\|^2 \\
&\quad + 2(1-\beta_n)\gamma_n\alpha_n\|\varrho_n - \mathfrak{z}_n\|\|A\mathfrak{z}_n - As^*\| + (1-\beta_n)\mu_n\epsilon\|v_n - G_nv_n\|^2 \\
&\quad + 2\beta_n\delta\|v_n - s^*\|^2 + \beta_nM\|v_{n+1} - v_n\| + 2\beta_n\langle h(v_n) - s^*, v_n - s^*\rangle \\
&\leq (\|v_n - s^*\| + \|v_{n+1} - s^*\|)\|v_n - v_{n+1}\| \\
&\quad + 2(1-\beta_n)\gamma_n\alpha_n\|\varrho_n - \mathfrak{z}_n\|\|A\mathfrak{z}_n - As^*\| + (1-\beta_n)\mu_n\epsilon\|v_n - G_nv_n\|^2 \\
&\quad + 2\beta_n\delta\|v_n - s^*\|^2 + \beta_nM\|v_{n+1} - v_n\| \\
&\quad + 2\beta_n\langle h(v_n) - s^*, v_n - s^*\rangle.
\end{aligned}$$

Applying the given conditions, (3.14), (3.20), and (3.31), we get

$$\lim_{n\to\infty} \|\varrho_n - \mathfrak{z}_n\| = 0.$$

Now, we show that $s^* \in \mathrm{Fix}(H) = \mathrm{Fix}(G) = \mathrm{Fix}(G_n) = \cap_{i=1}^{\mathbb{N}}\mathrm{Fix}(S_i)$. Let $s^* \notin \mathrm{Fix}(H)$. As $v_{n_j} \rightharpoonup s^*$ and $s^* \neq Hs^*$, then by the Opial condition, we obtain

$$\begin{aligned}
\liminf_{j\to\infty} \|v_{n_j} - s^*\| &< \liminf_{j\to\infty} \|v_{n_j} - Hs^*\| \\
&\leq \liminf_{j\to\infty}[\|v_{n_j} - Hv_{n_j}\| + \|Hv_{n_j} - Hs^*\|] \\
&\leq \liminf_{j\to\infty} \|v_{n_j} - s^*\|,
\end{aligned} \tag{3.32}$$

which contradicts to our supposition. Hence, $s^* \in \mathrm{Fix}(H) = \mathrm{Fix}(G) = \mathrm{Fix}(G_n) = \cap_{i=1}^{\mathbb{N}}\mathrm{Fix}(S_i)$. By (3.14), we observe that $\{v_n\}$ and $\{\varrho_n\}$ have the same asymptotic behavior, and therefore $\exists$ a subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ with $\varrho_{n_j} \rightharpoonup s^*$. Again from (3.14) and the opial condition we have that $s^* \in \mathrm{Fix}(T)$. Next, we prove that $s^* \in \mathrm{Sol}(\mathrm{SGEP}(1.3-1.4))$. Set $\tau_n := v_n + \eta B^*((F_{r_n}^{(f_2,\phi_2)} - I)Bv_n$. Then, $\mathfrak{z}_n = F_{r_n}^{(f_1,\phi_1)}\tau_n$. For any $v \in Q_1$, we get

$$\begin{aligned}
f_1(\mathfrak{z}_n, v) + \phi_1(v, \mathfrak{z}_n) - \phi_1(\mathfrak{z}_n, \mathfrak{z}_n) + \frac{1}{r_n}\langle v - \mathfrak{z}_n, \mathfrak{z}_n - \tau_n\rangle &\geq 0 \\
\phi_1(v, \mathfrak{z}_n) - \phi_1(\mathfrak{z}_n, \mathfrak{z}_n) + \frac{1}{r_n}\langle v - \mathfrak{z}_n, \mathfrak{z}_n - \tau_n\rangle &\geq f_1(v, \mathfrak{z}_n) \\
\implies \quad \phi_1(v, \mathfrak{z}_{n_j}) - \phi_1(\mathfrak{z}_{n_j}, \mathfrak{z}_{n_j}) + \langle v - \mathfrak{z}_{n_j}, \frac{\mathfrak{z}_{n_j} - \tau_{n_j}}{r_{n_j}}\rangle &\geq f_1(v, \mathfrak{z}_{n_j}).
\end{aligned} \tag{3.33}$$

Assume $\omega_\varsigma := (1-\varsigma)s^* + \varsigma v,\ \ \forall \varsigma \in (0,1]$. As $v, s^* \in Q_1$, therefore $\omega_\varsigma \in Q_1$. Hence, by (3.33)

$$\begin{aligned} 0 &\le f_1(\omega_\varsigma, \mathfrak{z}_{n_j}) - \phi_1(\omega_\varsigma, \mathfrak{z}_{n_j}) + \phi_1(\mathfrak{z}_{n_j}, \mathfrak{z}_{n_j}) \\ &\quad - \langle \omega_\varsigma - \mathfrak{z}_{n_j}, \frac{\mathfrak{z}_{n_j} - v_{n_j}}{r_{n_j}} + \eta B^* \frac{((F^{(f_2,\phi_2)}_{r_{n_j}} - I)Bv_{n_j}}{r_{n_j}} \rangle. \end{aligned}$$

Using the given conditions (3.24) and (3.27), we get

$$\phi_1(\omega_\varsigma, s^*) - \phi_1(s^*, s^*) \le f_1(\omega_\varsigma, s^*).$$

Thus,

$$\begin{aligned} 0 &= f_1(\omega_\varsigma, \omega_\varsigma) \\ &= \varsigma f_1(\omega_\varsigma, v) + (1-\varsigma) f_1(\omega_\varsigma, s^*) \\ &\ge \varsigma f_1(\omega_\varsigma, v) + (1-\varsigma)[\phi_1(\omega_\varsigma, s^*) - \phi_1(s^*, s^*)] \\ &\ge \varsigma f_1(\omega_\varsigma, v) + (1-\varsigma)\varsigma[\phi_1(v, s^*) - \phi_1(s^*, s^*)] \\ &\ge f_1(\omega_\varsigma, v) + (1-\varsigma)[\phi_1(v, s^*) - \phi_1(s^*, s^*)]. \end{aligned}$$

Assuming $\varsigma \to 0$, we obtain

$$f_1(s^*, v) + \phi_1(v, s^*) - \phi_1(s^*, s^*) \ge 0,\ \ \forall v \in Q_1.$$

This implies that $s^* \in$ Sol(GEP(1.3)). Further, we prove that $Bs^* \in$ Sol(GEP(1.4)). As $\|z_n - v_n\| \to 0$, $z_n \rightharpoonup s^*$ as $n \to \infty$ and $\{v_n\}$ is bounded and, $\exists$ a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ with $v_{n_j} \rightharpoonup s^*$ and $Bv_{n_j} \rightharpoonup Bs^*$ because B is a bounded linear operator.

Set $q_{n_j} = Bv_{n_j} - F^{(f_2,\phi_2)}_{r_n} Bv_{n_j}$. Using (3.24), we get $\lim_{j\to\infty} q_{n_j} = 0$ and $Bv_{n_j} - q_{n_j} = F^{(f_2,\phi_2)}_{r_n} Bv_{n_j}$. Applying Lemma 2.3, we get

$$\begin{aligned} f_2(Bv_{n_j} - q_{n_j}, v) &+ \phi_1(v, \mathfrak{z}_{n_j}) - \phi_1(\mathfrak{z}_{n_j}, \mathfrak{z}_{n_j}) \\ &+ \frac{1}{r_{n_j}} \langle v - (Bv_{n_j} - q_{n_j}), (Bv_{n_j} - q_{n_j}) - Bv_{n_j} \rangle \ge 0,\ \ \forall v \in Q_1. \end{aligned} \tag{3.34}$$

Taking the limit superior in (3.34) as $j \to \infty$, using the concept of upper semicontinuity in the first argument of f_2, and applying the given conditions, we get

$$f_2(Bs^*, v) + \phi_1(v, s^*) - \phi_1(s^*, s^*) \ge 0,\ \ \forall v \in Q_1,$$

which implies $Bs^* \in$ Sol(GEP(1.3)). Thus, $s^* \in$ Sol(SGEP(1.3 − 1.4)).

Next, we show that $s^* \in$ Sol(VIP(1.1)). As $\lim_{n\to\infty} \|\mathfrak{z}_n - \varrho_n\| = 0$, $\exists$ $\{\mathfrak{z}_{n_j}\}$ and $\{\varrho_{n_j}\}$ subsequences of $\{\mathfrak{z}_n\}$ and $\{\varrho_n\}$ with $\mathfrak{z}_{n_j} \rightharpoonup s^*$ and $\varrho_{n_j} \rightharpoonup s^*$.

Let

$$\Delta(s) = \begin{cases} A(s) + N_{Q_1}(s^*), & \text{if } s^* \in Q_1, \\ \emptyset, & \text{if } s^* \notin Q_1, \end{cases}$$

where $N_{Q_1}(s^*) := \{t \in Y_1 : \langle s^* - v, t\rangle \ge 0,\ \forall t \in Q_1\}$ is the normal cone to Q_1 at $s^* \in Y_1$. Hence, Δ is maximal monotone and $0 \in \Delta s^* \Leftrightarrow s^* \in$ Sol(VIP(1.1)). Let $(s^*, w) \in$ graph(Δ). Then, $w \in$

$\Delta s^* = As^* + N_{Q_1}(s^*)$ and hence $w - As^* \in N_{Q_1}(s^*)$. Thus, $\langle s^* - t, w - As^*\rangle \geq 0, \quad \forall t \in Q_1$. Since, $\varrho_n = P_{Q_1}(\mathfrak{z}_n - \alpha_n A\mathfrak{z}_n)$ and $s^* \in Q_1$, therefore

$$\begin{aligned} \langle (\mathfrak{z}_n - \alpha_n A\mathfrak{z}_n) - \varrho_n, \varrho_n - s^*\rangle &\geq 0 \\ \implies \quad \langle t - \varrho_n, \frac{\varrho_n - \mathfrak{z}_n}{\alpha_n} + A\mathfrak{z}_n\rangle &\geq 0, \ \forall p \in Q_1. \end{aligned}$$

As $\langle p - t, w - Ap\rangle \geq 0$, for all $p \in Q_1$ and $\varrho_{n_j} \in Q_1$, monotonicity of A, we obtain

$$\begin{aligned} \langle p - \varrho_{n_j}, w\rangle &\geq \langle p - \varrho_{n_j}, As^*\rangle \\ &\geq \langle p - \varrho_{n_j}, Ap\rangle - \langle p - \varrho_{n_j}, \frac{\varrho_{n_j} - \mathfrak{z}_{n_j}}{\alpha_{n_j}} + A\mathfrak{z}_{n_j}\rangle \\ &= \langle p - \varrho_{n_j}, Ap - A\mathfrak{z}_{n_j}\rangle + \langle p - \varrho_{n_j}, A\varrho_{n_j} - A\mathfrak{z}_{n_j}\rangle \\ &\quad - \langle p - \varrho_{n_j}, \frac{\varrho_{n_j} - \mathfrak{z}_{n_j}}{\alpha_{n_j}}\rangle \\ &\geq \langle p - \varrho_{n_j}, A\varrho_{n_j} - A\mathfrak{z}_{n_j}\rangle - \langle p - \varrho_{n_j}, \frac{\varrho_{n_j} - \mathfrak{z}_{n_j}}{\alpha_{n_j}}\rangle. \end{aligned}$$

Taking $j \to \infty$ and by the continuity of A, we get $\langle p - s^*, w\rangle \geq 0$. As Δ is maximal monotone, $s^* \in \Delta^{-1}(0)$ and hence $s^* \in \mathrm{Sol(VIP(1.1))}$. Hence, $s^* \in \Omega$.

As $s^* = P_\Omega h(s^*)$, therefore by (3.16)

$$\limsup_{n\to\infty}\langle h(s^*) - s^*, v_n - s^*\rangle = \lim_{j\to\infty}\langle h(s^*) - s^*, v_{n_j} - s^*\rangle \leq 0. \tag{3.35}$$

Applying the given conditions, (3.13), (3.20), (3.35), and Lemma 2.8, we obtain $q_n \to 0$ as $n \to \infty$. Hence, $\{v_n\}$ strongly converges to $s^* = P_\Omega h(s^*)$.

Case 2. Consider $\{q_{t_j}\}$ to be a subsequence of $\{q_t\}$ with $q_{t_j} < q_{t_{j+1}}, \quad \forall j \geq 0$. Then followed by Lemma 2.1, construct a nondecreasing sequence $\{m_t\} \subset \mathbb{N}$ with $m_t \to \infty$, as $t \to \infty$ and $\max\{q_{m_t}, q_t\} \leq q_{m_{t+1}}, \ \forall t$. As $r_t \in [c, d] \subset (0, \sigma^{-1}), \ t \geq 0, \sigma_t, \gamma_t, \mu_t \in (0, 1)$ with the given condition and (3.13), and we get

$$\lim_{t\to\infty}\|v_{m_t} - Ty_{m_t}\| = 0, \quad \lim_{t\to\infty}\|Ty_{m_t} - G_{m_t}v_{m_t}\| = 0, \quad \text{and} \quad \lim_{t\to\infty}\|v_{m_t} - G_{m_t}v_{m_t}\| = 0.$$

By applying the same steps as in Case 1, we get

$$\limsup_{t\to\infty}\langle h(s^*) - s^*, v_{m_t} - s^*\rangle \leq 0.$$

As $\{v_t\}$ is bounded and $\lim_{t\to\infty}\beta_t = 0$, we obtain from (3.15), (3.17), and (3.20) that

$$\lim_{t\to\infty}\|v_{m_{t+1}} - v_{m_t}\| = 0.$$

As $q_{m_t} \leq q_{m_{t+1}}, \ \forall t$, we obtain from (3.14) that

$$(1 - 2\delta)q_{m_{t+1}} \leq M\|v_{m_{t+1}} - v_{m_t}\| + 2\langle h(s^*) - s^*, v_{m_t} - s^*\rangle.$$

Taking $t \to \infty$, we get $q_{m_{t+1}} \to 0$. As $q_{m_t} \leq q_{m_{t+1}}, \ \forall t$, therefore $q_t \to 0$ as $t \to \infty$. Thus, $v_t \to 0$ as $t \to \infty$. Hence, we have proved that the sequence $\{v_n\}$ strongly converges to $s^* = P_\Omega h(s^*)$. □

Following this approach, we present several remarks that stem from the conclusions of Theorem 3.1. These remarks provide a concise overview of the theoretical results and pave the way for broader exploration and application of the proposed iterative scheme across various mathematical and computational settings.

Remark 3.1. *Let $T = I$, where I is the identity mapping and $\epsilon_i = 0$, that is, S_i is a finite family of nonexpansive mappings in Theorem 3.1. Then,* $\Omega := \mathrm{Fix}(S_i) \cap \mathrm{Sol}(\mathrm{SGEP}(1.3-1.4)) \cap \mathrm{Sol}(\mathrm{VIP}(1.1)) \neq \emptyset$.

Remark 3.2. *Let $B = I$, where I is the identity mapping, $Y_1 = Y_2$, $Q_1 = Q_2$, $f_1 = f_2$, and $\phi_1 = \phi_2$ in Theorem 3.1. Then,* $\Omega := \cap_{i=1}^{\mathbb{N}} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(T) \cap \mathrm{Sol}(\mathrm{GEP}(1.3)) \cap \mathrm{Sol}(\mathrm{VIP}(1.1)) \neq \emptyset$.

4. Numerical example

We now provide examples to illustrate the main theorem.

Example 4.1. *Let $Y_1 = Y_2 = \mathbb{R}$ and $Q_1 = Q_2 = [0, +\infty)$. Define the mappings: $f_1(v_1, v_2) = v_1(v_2 - v_1)$, $\forall v_1, v_2 \in Q_1$; $f_2(t_1, t_2) = t_1(t_2 - t_1)$, $\forall t_1, t_2 \in Q_2$, and $\phi_1(v_1, v_2) = \phi_2(v_1, v_2) = v_1 v_2$, $\forall v_1, v_2 \in Q_1$. It is straightforward to verify that the functions f_1, f_2, ϕ_1, and ϕ_2 satisfy the conditions of Assumption* 2.1. *Now, consider the additional mappings: $h(v) = \frac{v}{5}$, $Av = 3v$, $v \in Q_1$; $B(s) = \frac{1}{2}s$, $s \in Y_1$; $T(v) = \frac{v}{4}$, $v \in Q_1$, and $S_i(v) = -(1+i)v$, $v \in Q_1$, $i = 1, 2, 3$. These mappings can also be easily checked to satisfy the requirements of Theorem 3.1. The execution of the algorithms involves specific parameter settings. Let $r_n = 1$, $\alpha_n = \{\frac{1}{5}\}$, $\eta = \frac{1}{6}$, $\beta_n = \{\frac{1}{10n}\}$, $\sigma_n = 0.7 + \frac{0.1}{n^2}$, $\gamma_n = 0.2 - \frac{0.2}{n^2}$, $\mu_n = 0.1 + \frac{0.1}{n^2}$, and $\{\xi_i^n\} = \{\frac{1}{3}\}$. Under these configurations, the sequence produced by Algorithm* 3.1 *converges to* $q = \{0\} \in \Omega$.

The computations and graphical visualizations for this algorithm were carried out using MATLAB R2015a on a standard HP laptop featuring an Intel Core i7 processor and 8 GB of RAM. The stopping criterion is set as $\|v_{n+1} - v_n\| < 10^{-10}$. Various initial points v_1 are tested, and the results are summarized in Tables 1 and 2, where we also compare our findings with those in [16, 21]. Additionally, the convergence behavior is illustrated in Figures 1 and 2. Upon analyzing the figures and the table, on taking distinct initial points, we observe that our proposed algorithm tends to complete tasks more quickly, typically measured in seconds, compared to other methods. However, it is challenging to identify a clear trend from these results.

Table 1. Comparison of our main results for initial point $v_1 = 0.9$.

No. of iterations	Main Theorem cpu time (in seconds)	Korpelevich [16] cpu time (in seconds)	Nadezkhina et al. [21] cpu time (in seconds)
1	0.180000	0.684000	0.705600
2	0.062481	0.519840	0.544723
3	0.023582	0.395078	0.418347
4	0.009167	0.300260	0.320454
5	0.003614	0.228197	0.245083
6	0.001436	0.173430	0.187244
7	0.000574	0.131807	0.142947
8	0.000230	0.100173	0.109069
9	0.000092	0.076132	0.083183
10	0.000037	0.057860	0.063419
11	0.000015	0.043974	0.048337
12	0.000006	0.033420	0.036833
13	0.000002	0.025399	0.028061
14	0.000001	0.019303	0.021374
15	0.000000	0.014671	0.016279

Table 2. Comparison of our main results for initial point $v_1 = 2.1$.

No. of iterations	Main Theorem cpu time (in seconds)	Korpelevich [16] cpu time (in seconds)	Nadezkhina et al. [21] cpu time (in seconds)
1	0.420000	1.500000	1.560000
2	0.145790	0.900000	0.990000
3	0.055024	0.684000	0.760320
4	0.021390	0.519840	0.582405
5	0.008433	0.395078	0.445423
6	0.003351	0.300260	0.340304
7	0.001338	0.228197	0.259797
8	0.000536	0.173430	0.198225
9	0.000215	0.131807	0.151180
10	0.000087	0.100173	0.115260
11	0.000035	0.076132	0.087849
12	0.000014	0.057860	0.066941
13	0.000006	0.043974	0.050999
14	0.000002	0.033420	0.038846
15	0.000001	0.025399	0.029585
16	0.000000	0.019303	0.022529

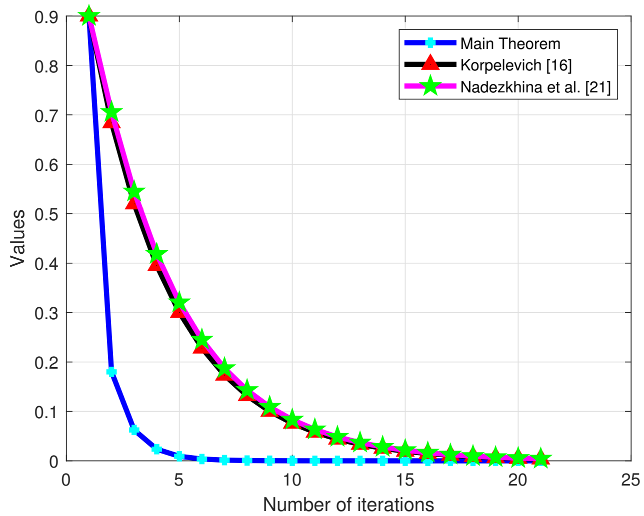

Figure 1. Convergence of $\{v_n\}$ at $v_1 = 0.9$.

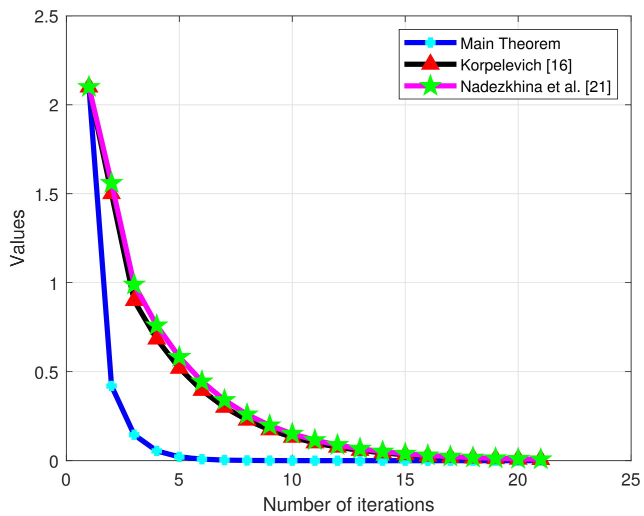

Figure 2. Convergence of $\{v_n\}$ at $v_1 = 2.1$.

Example 4.2. *Let $Y_1 = Y_2 = l_2$ be real Hilbert spaces, where l_2 consists of square-summable infinite sequences of real numbers. Define $Q_1 = Q_2 = \{w \in l_2 : \|w\| \leq 3\}$. The mappings are defined as follows: $f_1(u, v) = (4v + 5u)(v - u)$, $f_2(u, v) = (2v + 3u)(v - u)$, where $\forall u = \{u_1, u_2, ..., u_n, ...\}, v = \{v_1, v_2, ..., v_n, ...\}$. The norm and inner product on l_2 are defined by: $\|u\| = (\sum\limits_{j=1}^{\infty} |u_j|^2)^{\frac{1}{2}}$, $\langle u, v\rangle = \sum\limits_{j=1}^{\infty} u_j v_j$. Additional mappings are given as: $\phi_1(u, v) = (5v - 4u)u$, $\phi_2(u, v) = (3v - 2u)u$. It is straightforward to verify that the functions f_1, f_2, ϕ_1, and ϕ_2 satisfy the conditions of Assumption 2.1. Now, consider the additional mappings: $h(u) = \frac{1}{2}u$, $Au = 10u$, $u \in Q_1$; $B(s) = \frac{1}{5}s$, $s \in Y_1$; $T(u) = \frac{1}{100}u$, $u \in Q_1$, and $S_i(v) = \frac{1}{2(i+1)}u$, $u \in Q_1$, $i = 1, 2, 3$. These mappings can also be verified to satisfy the requirements of Theorem 3.1. The execution of the algorithms involves specific parameter settings. Let $r_n = 1$,*

$\alpha_n = \{\frac{1}{13}\}$, $\eta = \frac{1}{7}$, $\beta_n = \{\frac{1}{10n}\}$, $\sigma_n = 0.7 + \frac{0.1}{n^2}$, $\gamma_n = 0.2 - \frac{0.2}{n^2}$, $\mu_n = 0.1 + \frac{0.1}{n^2}$, and $\{\xi_i^n\} = \{\frac{1}{3}\}$. Under these configurations, the sequence produced by Algorithm 3.1 *converges to $q = \{0\} \in \Omega$.*

The computations and graphical visualizations for this algorithm were carried out using MATLAB R2015a on a standard HP laptop featuring an Intel Core i7 processor and 8 GB of RAM. The stopping criterion is set to $\|v_{n+1} - v_n\| < 10^{-10}$. Several initial points v_1 are tested, and the convergence behavior is illustrated in Figures 3 and 4.

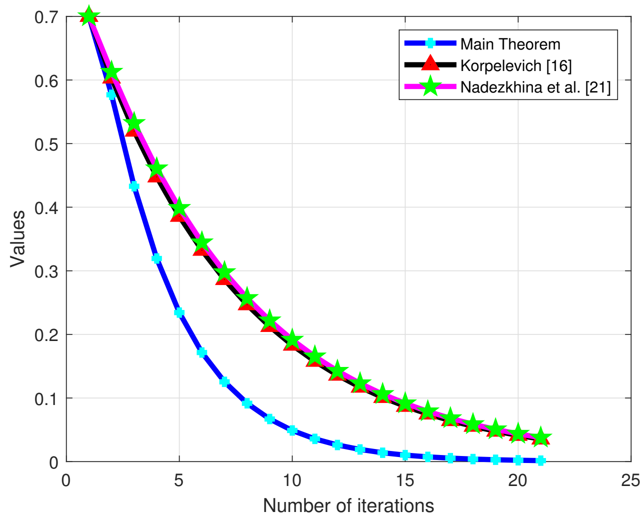

Figure 3. Convergence of $\{v_n\}$ at initial point $v_1 = \{0.7, 0.7, ..., 0.7, ...\}$.

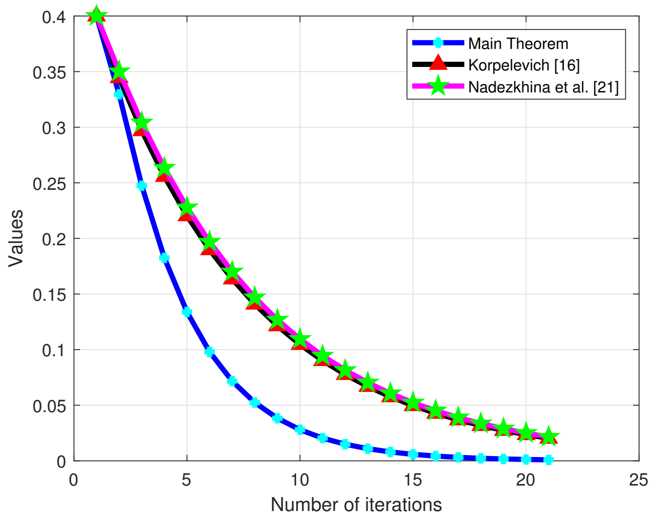

Figure 4. Convergence of $\{v_n\}$ at initial point $v_1 = \{0.4, 0.4, ..., 0.4, ...\}$.

Application in optimization problems: We explore the application of our algorithms to optimization problems. Let $M_1 : Q_1 \to \mathbb{R}$ and $M_2 : Q_2 \to \mathbb{R}$ be two functions. Define $f_1(u_1, v_1) = M_1(v_1) -$

$M_1(u_1)$, $\forall u_1, v_1 \in Q1$, and $f_2(u_2, v_2) = M_2(v_2) - M_2(u_2)$, $\forall u_2, v_2 \in Q2$. The objective is to determine $u \in Q1$ such that

$$F_1(u) \leq F_1(u^*), \quad \forall u^* \in Q1 \tag{4.1}$$

and ensure that

$$v = Bu \in Q_2 \text{ solves } F_2(v) \leq F_2(v^*), \quad \forall v^* \in Q_2. \tag{4.2}$$

Denote the solution set of these optimization problems (4.1) and (4.2) by Γ and assume that $\Gamma \neq \emptyset$. It is straightforward to verify that Assumption 2.1, $1-4$, hold. Consequently, we have $\Gamma = \Omega$.

5. Conclusions

In this paper, we proposed a viscosity-based extragradient iterative algorithm for solving the split generalized equilibrium problem, the variational inequality problem, and the fixed point problem for a finite family of ϵ-strict pseudo-contractive and a nonexpansive mapping in Hilbert space. The strong convergence of the algorithm was established under appropriate assumptions. To demonstrate the practical applicability of the proposed algorithm, we presented results in the form of two comprehensive tables and four illustrative figures. These include comparisons with existing methods and a detailed analysis of convergence behavior, highlighting the effectiveness and efficiency of our approach.

This study extends and unifies various well-known results in the literature, offering a versatile tool for tackling a range of problems in optimization and computational mathematics.

However, the algorithm has certain limitations. Its convergence heavily depends on precise parameter tuning, which may pose challenges in practical applications. Additionally, the framework is currently restricted to Hilbert spaces, limiting its generalization to Banach spaces or other settings. Despite these limitations, the results presented in this paper extend and unify numerous previously established outcomes in this particular research domain.

Author contributions

Mohammad Farid: Writing − Original Draft, Software; Saud Fahad Aldosary: Review and Editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. S. Antón, Fixed points of automorphisms of the vector bundle moduli space over a compact Riemann surface, *Mediterr. J. Math.*, **21** (2024), 20. https://doi.org/10.1007/s00009-023-02559-z
2. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181
3. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63** (1994), 123–145.
4. F. E. Browder, W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, *J. Math. Anal. Appl.*, **20** (1967), 197–228. https://doi.org/10.1016/0022-247X(67)90085-6
5. C. Byrne, Y. Censor, A. Gibali, S. Reich, Weak and strong convergence of algorithms for the split common null point problem, 2011.
6. L. C. Ceng, N. Hadjisavvas, N. C. Wong, Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems, *J. Glob. Optim.*, **46** (2010), 635–646. https://doi.org/10.1007/s10898-009-9454-7
7. Y. Censor, T. Elfving, A multiprojection algorithms using Bragman prtojection in a product space, *Numer. Algor.*, **8** (1994), 221–239. https://doi.org/10.1007/BF02142692
8. Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity modulated radiation therapy, *Phys. Med. Biol.*, **51** (2006), 2353. https://doi.org/10.1088/0031-9155/51/10/001
9. P. L. Combettes, S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, **6** (2005), 117–136.
10. M. Farid, The subgradient extragradient method for solving mixed equilibrium problems and fixed point problems in Hilbert spaces, *J. Appl. Numer. Optim.*, **1** (2019), 335–345. https://doi.org/10.23952/jano.1.2019.3.10
11. M. Farid, K. R. Kazmi, A new mapping for finding a common solution of split generalized equilibrium problem, variational inequality problem and fixed point problem, *Korean J. Math.*, **27** (2019), 295–325. https://doi.org/10.11568/kjm.2019.27.2.297
12. M. Farid, W. Cholamjiak, R. Ali, K. R. Kazmi, A new shrinking projection algorithm for a generalized mixed variational-like inequality problem and asymptotically quasi-ϕ-nonexpansive mapping in a Banach space, *RACSAM*, **115** (2021), 114. https://doi.org/10.1007/s13398-021-01049-9
13. P. Hartman, G. Stampacchia, On some non-linear elliptic differential-functional equation, *Acta Math.*, **115** (1966), 271–310. https://doi.org/10.1007/BF02392210
14. N. Hitchin, Stable bundles and integrable systems, *Duke Math. J.*, **54** (1987), 91–114. https://doi.org/10.1215/S0012-7094-87-05408-1

15. K. R. Kazmi, S. H. Rizvi, M. Farid, A viscosity Cesàro mean approximation method for split generalized vector equilibrium problem and fixed point problem, *J. Egypt. Math. Soc.*, **23** (2015), 362–370. https://doi.org/10.1016/j.joems.2014.05.001

16. G. M. Korpelevich, The extragradient method for finding saddle points and other problems, *Matecon*, **12** (1976), 747–756.

17. D. Kundur, D. Hatzinakos, Blind image deconvolution, *IEEE Signal Proc. Mag.*, **13** (1996), 43–64. https://doi.org/10.1109/79.489268

18. P. E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Anal.*, **16** (2008), 899–912. https://doi.org/10.1007/s11228-008-0102-z

19. G. Marino, H. K. Xu, Weak and strong convergence theorems for k-strict pseudo-contractive in Hilbert spaces, *J. Math. Anal. Appl.*, **329** (2007), 336–346. https://doi.org/10.1016/j.jmaa.2006.06.055

20. A. Moudafi, The split common fixed point problem for demicontractive mappings, *Inverse Probl.*, **26** (2010), 5. https://doi.org/10.1088/0266-5611/26/5/055007

21. N. Nadezhkina, W. Takahashi, Strong convergence theorem by a hybrid method for nonexpansive mapping and Lipschitz continuous monotone mappings, *SIAM J. Optim.*, **16** (2006), 1230–1241. https://doi.org/10.1137/050624315

22. N. Nadezhkina, W. Takahashi, Weak convergence theorem by a extragradient method for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.*, **128** (2006), 191–201. https://doi.org/10.1007/s10957-005-7564-z

23. K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, **279** (2003), 372–379. https://doi.org/10.1016/S0022-247X(02)00458-4

24. J. Nash, Equilibrium points in n-person games, *Proc. Natl. Acad. Sci. USA*, **36** (1950), 48–49. https://doi.org/10.1073/pnas.36.1.48

25. J. Nash, Non-Cooperative Games, *Ann. Math.*, **54** (1951), 286–295. https://doi.org/10.2307/1969529

26. M. O. Osilike, D. I. Igbokewe, Weak and strong convergence theorems for fixed points of pseudo-contractive and solutions of monotone type operator equations, *Comput. Math. Appl.*, **40** (2000), 559–567. https://doi.org/10.1016/S0898-1221(00)00179-6

27. M. Patriksson, *The traffic assignment problem: Models and methods*, Courier Dover Publications, 2015.

28. B. D. Rouhani, K. R. Kazmi, M. Farid, Common solutions to some systems of variational inequalities and fixed point problems, *Fixed Point Theor.*, **18** (2017), 167–190. https://doi.org/10.24193/fpt-ro.2017.1.14

29. S. Reich, S. Sabach, Three strong convergence theorems regarding iterative methods for solving equilibrium problems in reflexive Banach spaces, *Contemp. Math.*, **568** (2012), 225–240.

30. O. Scherzer, Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems, *J. Math. Anal. Appl.*, **194** (1991), 911–933. https://doi.org/10.1006/jmaa.1995.1335

31. S. Takahashi, W. Takahashi, Viscosity approximation method for equilibrium problems and fixed point problems in Hilbert space, *J. Math. Anal. Appl.*, **331** (2007), 506–515. https://doi.org/10.1016/j.jmaa.2006.08.036

32. H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.*, **66** (2002), 240–256. https://doi.org/10.1112/S0024610702003332

33. G. L. Acedo, H. K. Xu, Iterative methods for strict pseudo-contractive in Hilbert spaces, *Nonlinear Anal. Theor.*, **67** (2007), 2258–2271. https://doi.org/10.1016/j.na.2006.08.036

34. M. Younis, L. Chen, D. Singh, *Recent developments in fixed point theory: Theoretical foundations and real-world applications*, Singapore: Springer, 2024. https://doi.org/10.1007/978-981-99-9546-2

35. H. Y. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractive in Hilbert space, *Nonlinear Anal. Theor.*, **69** (2008), 456–462. https://doi.org/10.1016/j.na.2007.05.032