Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(4): 8673–8687.
DOI: 10.3934/math.2025397
Received: 22 January 2025
Revised: 02 April 2025
Accepted: 03 April 2025
Published: 15 April 2025

Research article

Synchronization dynamics in fractional-order FitzHugh–Nagumo neural networks with time-delayed coupling

Canhong Long[1], **Zuozhi Liu**[1,2,*] **and Can Ma**[1]

[1] School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang, 550025, China

[2] School of Mathematics, Northwest University, Xi'an, 710069, China

* **Correspondence:** Email: liuzuo.zhi@163.com.

Abstract: Studying the synchronization of neural networks is crucial for understanding brain function and diagnosing neurological disorders. However, most existing research focuses on integer-order systems and overlooks the effects of time-delay coupling. To address this, this paper was devoted to investigating the synchronization behaviors of time-delay coupled fractional-order FitzHugh–Nagumo networks. The sufficient conditions for the synchronization of two coupled neurons were derived using the Lyapunov stability criterion. Furthermore, the synchronization factor was utilized to elucidate the combined effects of coupling strength and time delay, as well as the influence of time delay on fractional-order dynamics. The analysis began with two coupled systems, and the results were then extended to networks with a larger number of nodes. Numerical examples were presented to illustrate the obtained results.

Keywords: fractional-order; neuronal models; FithzHugh–Nagumo neuronal network; time-delayed coupling; synchronization
Mathematics Subject Classification: 26A33, 34K37

1. Introduction

The human brain consists of a complex network of interconnected neurons [1, 2]. Cognition and brain function result from the interplay of neurons, synaptic connections, and network structures, but their individual contributions are not well understood [3]. Significantly, to understand the fundamental nature of neurons, many researchers have focused on developing suitable neural network models to reveal the underlying principles of their interactions [4–9]. Among these models, the FitzHugh–Nagumo model, a simplified version of the Hodgkin–Huxley model, is widely used to describe neuronal dynamics [10–12]. Synchronization, a key phenomenon in neuronal systems, plays a crucial role in brain dynamics and is closely associated with several cognitive abilities [13–15]. Therefore, it is essential to understand the underlying causes of such behavior and to describe them rigorously for advancing progress in neuroscience.

Many studies have explored the synchronization and synchronization transitions in neuronal networks. This synchronous activity is crucial for information transmission and is closely linked to several neurological disorders. For example, epilepsy is a brain disorder characterized by a hypersynchronous state, which results from the excessive synchronization of large populations of neurons [16, 17]. This heightened synchronization often occurs at the end of seizures. Abnormal neural synchronization is also associated with several brain disorders, including Alzheimer's disease, schizophrenia, autism, and Parkinson's disease [18–20]. Therefore, research on the complex dynamic behaviors of neural networks, composed of diverse neurons, is crucial for the diagnosis and prevention of brain diseases, particularly those involving abnormal firing activities.

Synchronous behavior is affected by various factors, with coupling as an important one. In recent decades, numerous scholars have started to investigate different types of coupling in neural networks, including self-coupling, homotypic coupling, heterotypic coupling, and so on [21, 22]. Consequently, various coupling schemes, including electrical synapses, chemical synapses, and memristive synapses, have been used to model information transmission between neurons, thereby enhancing the understanding of synchronization in neural network behavior [23, 24].

It is important to note that the finite speed of signal transmission in coupled FitzHugh-Nagumo neuronal models through gap junctions can introduce time delays, which are an inherent characteristic of biological processes [25, 26]. Time delay has been found to play a significant role in the synchronization of neuronal systems. It can enable two coupled FitzHugh-Nagumo neurons to switch between neural rest and synchronized peaks, as well as enhance synchronization in networks of neuronal oscillators [27, 28]. Additionally, it affects the synchronization dynamics of identical FitzHugh-Nagumo neurons, with the impact varying depending on the number of time delays [29].

Most research has traditionally focused on integer-order neuronal networks. However, these models often fall short in capturing the inherent complexities of natural processes. In contrast, fractional-order models, grounded in robust mathematical principles, have emerged as a promising alternative, offering a more nuanced representation of such dynamics. This solid mathematical framework has not only deepened our understanding but also paved the way for a range of innovative applications. For example, one study applied fractional calculus to model the innate immune response in Parkinson's disease systems, yielding a more accurate depiction of nonlinear dynamics under therapeutic intervention [30]. In a related effort, researchers developed a fractional hierarchical algorithm that refines parameter estimation in complex control systems [31]. Similarly, advances in fractional adaptive processing have led to improved power signal estimation [32]. Furthermore, an innovative approach combining fractional dynamics with intelligent adaptive Bayesian networks has provided deeper insights into the irregular rhythms of brain electrical activity in Parkinson's disease [33]. Together, these developments underscore the versatility and enhanced representational capacity of fractional-order models compared to their integer-order counterparts.

Our main motivation in this paper is to investigate the synchronization dynamics of fractional-order time-delay coupled FitzHugh-Nagumo systems, focusing on how they are influenced by the coupling time delay. The main contributions of this paper include the following three aspects:

- Unlike traditional neuronal coupling models, our work considers the interplay between delayed coupling effects and fractional-order properties. This approach not only enhances the biological plausibility of neuronal modeling but also enables a more accurate representation of the memory and genetic characteristics of biological neurons;

- The analysis of synchronization in both two coupled neurons and large-scale fully connected networks, providing insights into the combined effects of coupling strength and time delay;
- The exploration of the complex dynamics and synchronization mechanisms in neural networks, offering a deeper understanding of the role of time-delay in shaping network behavior.

The rest of this paper is organized as follows. Section 2 reviews key concepts related to fractional-order systems, the FitzHugh-Nagumo model, and synchronization. Section 3 derives the synchronization conditions for the FitzHugh-Nagumo network. In Section 4, we present the simulation results of the network dynamics. Finally, Section 5 concludes the paper.

2. Preliminaries

2.1. Definitions and lemmas

In this section, some fundamental definitions and lemmas are introduced.

Dfinition 1. *[34] The fractional intergral of non-integer order $q > 0$ for function $f(t)$ is defined by*

$$ {}_{0}^{c}I_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-q}} d\tau, $$

where $\Gamma(\cdot)$ is the gamma and $\Gamma(\tau) = \int_0^\infty \frac{t^{\tau-1}}{e^t} dt$.

Dfinition 2. *[34] The Caputo fractional derivative of order $\beta \in (0,1)$ of function $f \in C([0,T];X)$ is defined by*

$$ {}_{0}^{c}D_t^q f(t) := \frac{d}{dt}[I_t^{1-q}(f(t) - f(0))] = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(\tau)}{(t-\tau)^q} d\tau, $$

Lemma 1. *[35] Let $f(t) \in R^n$ be a nonnegative, continuously differentiable function that satisfies the following conditions:*

$$ \frac{1}{2} D^q V(t) \leq f(t) D^q f(t) $$

Lemma 2. *[36] If $V(t) \in R^1$ is a continuous and differentiable function, then for any time instant, we have*

$$ \begin{aligned} &D^q V(t) \leq -aV(t) + bV(t-\tau), \forall q \in (0,1), \\ &V(t) = \theta(t) \geq 0, t \in [-\tau, 0], \end{aligned} $$

for all $\theta(t) > 0, \tau > 0$, if $0 < b < a$, then $\lim_{t\to+\infty} V(t) = 0$.

2.2. FitzHugh–Nagumo model

The FitzHugh-Nagumo model, a simplified representation of the Hodgkin-Huxley model, is a widely recognized spiking neuron model. Its dynamics are described by the following system of equations:

$$\begin{cases} \dfrac{dv(t)}{dt} = c\left(v(t) - \dfrac{v^3(t)}{3} - w(t)\right) + I_{ext}, \\ \dfrac{dw(t)}{dt} = d\left(v(t) + a - bw(t)\right), \end{cases}$$

where $v(t)$ represents the membrane potential, and $w(t)$ denotes the sodium gating variable. The term I_{ext} corresponds to an externally applied stimulus to the neuron. The parameters a and b are constants associated with the equilibrium states of the potassium and sodium potentials, respectively. The parameter c acts as a time constant governing the sodium gating rate. Together, these parameters define the physiological state and dynamic behavior of the neuron.

Building upon the simplified FitzHugh-Nagumo model, this paper investigates synchronization in two distinct scenarios: two coupled FitzHugh-Nagumo systems, and a fully connected network of such systems. The analysis integrates fractional-order dynamics and time delays to better represent biological neural networks. Time delays account for finite transmission times in neural pathways, while fractional-order dynamics capture memory effects and long-term dependencies in neural activity. By examining synchronization behaviors under these modifications, the study provides insights into the coherent dynamics and adaptive responses observed in real neural systems.

2.3. Synchronization definition

In neural networks, synchronization represents a fundamental phenomenon where neurons exhibit coherent dynamics due to coupling interactions and time delays. This concept is particularly significant in the fractional-order time-delay FitzHugh–Nagumo network, where synchronization reflects the coordinated evolution of neuronal states. The following definitions formalize the criteria for synchronization within this framework:

Dfinition 3. *Two systems are considered synchronized if their states converge over time, such that the differences between corresponding state variables approach zero. Specifically, synchronization is achieved if*

$$\lim_{t\to\infty} \|v_2(t) - v_1(t)\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \|w_2(t) - w_1(t)\| = 0.$$

This indicates that the membrane potentials (v_1, v_2) and recovery variables (w_1, w_2) evolve in unison, reflecting coherent behavior driven by coupling strength, time delay, and system dynamics.

Dfinition 4. *In the fractional-order time-delay FitzHugh–Nagumo network, synchronization occurs when the state differences (θ) or phase differences ($\Delta\Phi$) among neurons approach zero, and the synchronization indices R and R_p approach 1, indicating coherent behavior across the network.*

3. Main results

3.1. Synchronization of two fractional-order time-delay coupled FitzHugh–Nagumo systems

The paper begins by considering the simplest network configuration, comprising two fractional-order time-delay coupled FitzHugh–Nagumo systems, denoted as $i = 1, 2$. The dynamics of this network are described by the following equations:

$$\begin{cases} \dfrac{d^\alpha v_1(t)}{dt^q} = c\left(v_1(t) - w_1(t) - \dfrac{1}{3}v_1^3(t)\right) + I_{ext} + K\left(v_1(t-\tau) - v_2(t)\right), \\ \dfrac{d^\alpha w_1(t)}{dt^q} = d\left(v_1(t) + a - bw_1(t)\right), \\ \dfrac{d^\alpha v_2(t)}{dt^q} = c\left(v_2(t) - w_2(t) - \dfrac{1}{3}v_2^3(t)\right) + I_{ext} + K\left(v_2(t-\tau) - v_1(t)\right), \\ \dfrac{d^\alpha w_2(t)}{dt^q} = d\left(v_2(t) + a - bw_2(t)\right). \end{cases} \tag{1}$$

the coupling strength $K > 0$ governs the interaction between neurons, represented by $K\left(v_j(t-\tau) - v_i(t)\right), i \neq j$. This term captures the influence of the delayed states of other neuron at $t-\tau$ and the current state of the i neuron at t on its membrane potential.

To analyze the error dynamics between the two coupled neurons, the paper defines the error variables as $e_1(t) = v_2(t) - v_1(t)$ and $e_2(t) = w_2(t) - w_1(t)$. Substituting these definitions into system 1, the error dynamics are derived as:

$$\begin{cases} \dfrac{d^\alpha e_1(t)}{dt^q} = ce_1(t)\left(1 - \dfrac{1}{3}(v_2^2(t) + v_2(t)v_1(t) + v_1^2(t))\right) + Ke_1(t-\tau) - ce_2(t), \\ \dfrac{d^\alpha e_2(t)}{dt^q} = de_1(t) - dbe_2(t). \end{cases} \tag{2}$$

Letting $e(t) = \begin{bmatrix} e_1(t) & e_2(t) \end{bmatrix}^T$, the error system 2 can be expressed in matrix form as:

$$\begin{cases} \dfrac{d^\alpha e(t)}{dt^q} = -Ae(t) + Be(t) + Ce(t-\tau), \\ \quad e(t) = \theta(t), \quad t \in [-\tau, 0), \end{cases} \tag{3}$$

where the matrices A, B, and C are defined as:

$$A = \begin{bmatrix} -c & 0 \\ 0 & db \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{3}(v_2^2(t) + v_2(t)v_1(t) + v_1^2(t)) & -c \\ d & 0 \end{bmatrix}, \quad C = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}. \tag{4}$$

Here, c, d, b, and K are positive constants.

We now propose the following assumption and derive the synchronization criteria for the coupled neurons based on an appropriate control scheme.

Assumption 1. *There exist positive constants M_1 and M_2 such that:*

$$|v_1(t)| \leq M_1, \quad |v_2(t)| \leq M_1. \tag{5}$$

Theorem 1. *When $0 < q < 1$, assuming that Assumption 1 holds, and the condition $0 < K < L$ is satisfied, the coupled system described by 1 achieves asymptotic synchronization, where $L = \min\{c + cM_1^2, db\}$.*

Proof: Define the Lyapunov candidate function as:

$$V(t) = \frac{1}{2}(e_1^2(t) + e_2^2(t)). \tag{6}$$

Taking the fractional-order derivative of $V(t)$ along the trajectories of $e(t)$ using Lemma 1, we have:

$$D^q V(t) \leq e_1(t) D^q e_1(t) + e_2(t) D^q e_2(t). \tag{7}$$

Substituting 2 and 3 into the derivative, we obtain:

$$\begin{aligned} D^q V(t) &\leq e_1(t)\Big[ce_1\left(1 - \frac{1}{3}(v_2^2(t) + v_2(t)v_1(t) + v_1^2(t))\right) + Ke_1(t-\tau) - ce_2(t)\Big] \\ &\quad + e_2(t)[de_1(t) - dbe_2(t)] \\ &= \left(c + \frac{1}{3}c(v_2^2(t) + v_2(t)v_1(t) + v_1^2(t))\right)e_1^2(t) + Ke_1(t-\tau)e_1(t) \\ &\quad + (d-c)e_1(t)e_2(t) - dbe_2^2(t). \end{aligned} \tag{8}$$

Using Assumption 1 and the definition 1 of L, we simplify:

$$D^q V(t) \leq -(c + cM_1^2)e_1^2(t) - dbe_2^2(t) + Ke_1^2(t-\tau). \tag{9}$$

Thus:

$$D^q V(t) \leq -LV(t) + KV(t-\tau). \tag{10}$$

According to Theorem 1, when $0 < q < 1$ and $0 < K < L$, the coupled FitzHugh–Nagumo system achieves asymptotic synchronization. This completes the proof.

3.2. Synchronization of fractional-order time-delay coupled FitzHugh–Nagumo networks

Consider the fractional-order time-delay FitzHugh–Nagumo network, which is modeled as a fully connected network. The dynamic equations for the network are expressed as:

$$\begin{cases} \dfrac{d^\alpha v_i(t)}{dt^q} = c\left(v_i(t) - w_i(t) - \dfrac{1}{3}v_i^3(t)\right) + I_{ext} + \displaystyle\sum_{j\neq i} K\left(v_j(t-\tau) - v_i(t)\right), \\ \dfrac{d^\alpha w_i(t)}{dt^q} = d\left(v_i(t) + a - bw_i(t)\right), \end{cases} \tag{11}$$

where $i = 1, 2, \ldots, N$ represents the indices of the neurons in the network, and N denotes the total number of neurons. The coupling strength between neurons is represented by $K > 0$, and the interaction among neurons is described by the term $\sum_{j\neq i} K\left(v_j(t-\tau) - v_i(t)\right)$. This term indicates that the membrane potential of the i-th neuron is influenced by the delayed states of all other neurons in the network at time $t - \tau$, as well as its own current state at time t.

This formulation reflects an all-to-all coupling structure, where every neuron is directly connected to all others. The inclusion of the delay term τ introduces a temporal effect, highlighting the fact that neural interactions are not instantaneous but instead occur after a finite delay due to signal transmission times or synaptic processes. The delayed coupling models realistic physiological scenarios, such as the propagation of action potentials across neural pathways or communication in distributed neural networks.

Neuronal synchronization within the framework of the all-to-all topology in the fractional-order time-delay coupled FitzHugh–Nagumo network is assessed using metrics for complete synchronization and phase synchronization. The measure of complete synchronization, $\theta(t)$, is mathematically expressed as:

$$\theta(t) = \sqrt{e_x^2(t) + e_y^2(t)} = \sqrt{(v_1(t) - v_2(t))^2 + (w_1(t) - w_2(t))^2}. \tag{12}$$

To characterize the phase relationship between neurons, the phase difference and instantaneous phase are defined as:

$$\begin{cases} \Delta\Phi(t) = \sqrt{\Phi_v^2(t) + \Phi_y^2(t)} = \sqrt{(\Phi_v(t) - \Phi_v'(t))^2 + (\Phi_w(t) - \Phi_w'(t))^2}, \\ \Phi_s(t) = \arctan \dfrac{\hat{s}(t)}{s(t)}, \quad \hat{s}(t) = -\dfrac{1}{\pi} \displaystyle\int_{-\infty}^{+\infty} \dfrac{s(\tau)}{t - \tau}\, d\tau, \end{cases} \tag{13}$$

where $s(\tau)$ represents the membrane potential time series of the network at time τ, and $\hat{s}(t)$ denotes the Hilbert transform of $s(t)$.

To quantify the degree of synchronization in the network, a synchronization index is utilized. This index ranges from 0 to 1, with a value close to 1 indicating strong synchronization among the neurons, while a value near 0 signifies weak or no synchronization. The complete synchronization factor R and the phase synchronization factor R_p are defined as follows:

$$\begin{cases} R = \dfrac{\langle F^2 \rangle - \langle F \rangle^2}{\frac{1}{N} \sum_{n=1}^{N} \langle v_n^2(t) \rangle - \langle v_n(t) \rangle^2}, \quad R_p = \langle F_p \rangle, \\ F = \dfrac{1}{N} \displaystyle\sum_{n=1}^{N} v_n(t), \quad F_p = \left| \dfrac{1}{N} \displaystyle\sum_{n=1}^{N} e^{j\Phi_n(t)} \right|. \end{cases} \tag{14}$$

In these equations, n denotes the node index, N represents the total number of neurons, and $\langle * \rangle$ indicates the time-averaged mean value. The synchronization factors R and R_p quantify the degree of neuronal synchronization and evolve over time based on the network dynamics. Specifically, R depends on the temporal variations of membrane potentials $v_n(t)$, while R_p is determined by the instantaneous phases $\Phi_n(t)$.

4. Numerical simulation

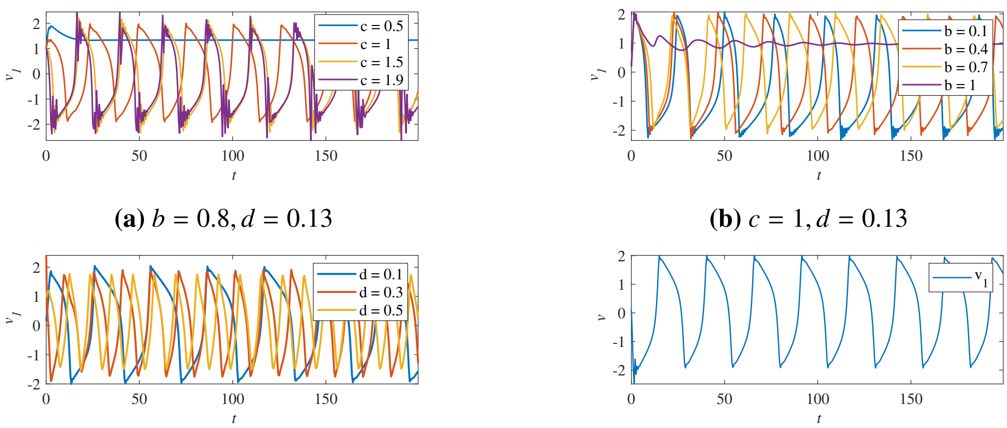
This section explores the synchronization behaviors of the fractional-order delayed FitzHugh-Nagumo systems, first focusing on synchronization between two coupled neurons and then extending the analysis to a fully connected network. The study of synchronization between two neurons offers insights into the conditions necessary for neural synchronization, while the investigation of synchronization dynamics in the network examines the collective behavior of a larger system with multiple interconnected nodes.

The numerical simulations in this study were conducted using MATLAB (R2018a). The differential equations were solved using a custom function, ddeFractionalSolver, which implements a numerical scheme for solving fractional-order delay differential equations based on the Grünwald-Letnikov method. This method is commonly employed for approximating fractional derivatives in a time-stepping procedure. The Hilbert transform used for phase synchronization analysis was computed using the built-in Hilbert function of the MATLAB Signal Processing Toolbox. All visualizations were generated with the built-in plotting functions of the MATLAB.

4.1. Synchronization between two neurons

In the simulation, the dynamics of two fractional-order time-delay coupled FitzHugh-Nagumo systems were considered, with parameters carefully selected to ensure excitability and spike-like behavior, which are essential for modeling neuronal dynamics. Specifically, $a = 0.7$ and $b = 0.8$ were chosen to maintain the system in an excitable regime, while $c = 1$ was set to normalize the system dynamics, and $d = 0.13$ was selected to ensure appropriate recovery behavior. Additionally, the fractional order was set to $\alpha = 0.81$, and the time delay $\tau = 2$ was introduced to account for memory effects and delayed interactions in neuronal systems.

As shown in Figure 1, to further examine the robustness of the model, we analyzed the impact of varying c, d, and b on the membrane potential v. The results indicate that excessively large or small values of these parameters significantly alter the spike-like behavior. Larger values tend to induce non-physiological oscillations, whereas smaller values suppress spike generation, leading to deviations from realistic neuronal activity. These findings suggest that the chosen parameter values are crucial for maintaining biologically plausible spike dynamics.

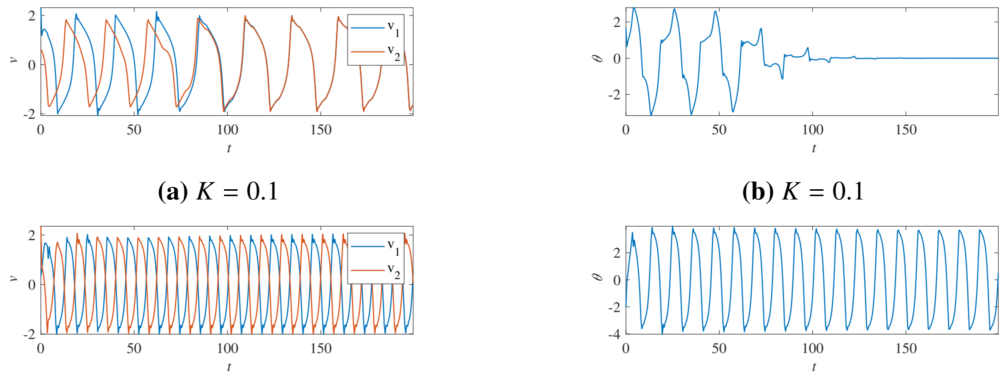


(a) $b = 0.8, d = 0.13$ **(b)** $c = 1, d = 0.13$

(c) $c = 1, b = 0.8$ **(d)** $c = 1, b = 0.8, d = 0.13$

Figure 1. Effect of parameter variations (c, d, b) on membrane potential v.

The numerical results, shown in Figure 2, validate the synchronization condition $0 < K < db$ derived previously. In Figure 2(a) and (b), where the coupling strength K satisfies the synchronization condition, the two neurons achieve complete synchronization. In contrast, Figure 2(c) and (d) shows that when the synchronization condition is not met, the system fails to synchronize, demonstrating that the coupling strength must fall within the specified range for synchronization to occur.



(a) $K = 0.1$ **(b)** $K = 0.1$

(c) $K = 0.5$ **(d)** $K = 0.5$

Figure 2. Dynamics of two fractional-order time-delay coupled FitzHugh-Nagumo systems with different coupling strengths.

4.2. Synchronization dynamics in the network

First, the section consider the dynamics of the fractional-order delayed FitzHugh-Nagumo fully connected network with 10 nodes. Figure 3(a) illustrates the network structure. Figure 3(b) shows the evolution of the complete synchronization factor R and the phase synchronization factor R_p. Within a certain range of coupling strength K, an increase in K enhances synchronization behaviors. In Figure 3(c), the dynamics of the activators for all nodes are presented, with subplots corresponding to $K = 0$, $K = 0.05$, $K = 0.10$, and $K = 0.15$. When $K = 0$, the network exhibits a relatively low level of synchronization. As the coupling strength K increases within a certain range, the synchronization level also improves. The parameters are set as follows: $c = 1$, $d = 0.15$, $a = 0.7$, $b = 0.8$, $I = 1$, $\tau = 2$, and $\alpha = 0.91$. The values of the state variables for the nodes are color-coded. The results demonstrate that for the selected parameters, synchronization is achieved for the activator values. However, it is important to note that excessively large coupling strengths lead to desynchronization of the neuronal network, indicating that there exists an optimal range of coupling strength K that ensures synchronization without destabilizing the system.

Next, the section examines the dynamics of the fractional-order delayed FitzHugh-Nagumo fully connected network consisting of 50 nodes. Figure 4(a) illustrates the network structure. Figure 4(b) shows the evolution of the complete synchronization factor R and the phase synchronization factor R_p. Within a certain range, an increase in the time delay τ weakens the synchronization behavior. In Figure 4(c), the dynamics of the activators for all nodes are presented, with subplots corresponding to $\tau = 0$, $\tau = 5$, $\tau = 10$, and $\tau = 15$. Despite the increase in delay time, synchronization is still achieved for all cases. However, as the delay time increases, the dynamical behavior becomes more complex, reflecting the influence of time delay on the stability and synchronization of the system. The parameters are set as follows: $c = 1$, $d = 0.15$, $a = 0.7$, $b = 0.8$, $I = 1$, $K = 0.02$, and $\alpha = 0.91$. The state variable values of the nodes are color-coded. From the results, it is evident that for the selected parameters, synchronization is achieved for the activator values. However, it is also observed that as the delay increases, the dynamics of the system become increasingly intricate, indicating the role of time delay in modulating the synchronization characteristics and overall behavior of the network.

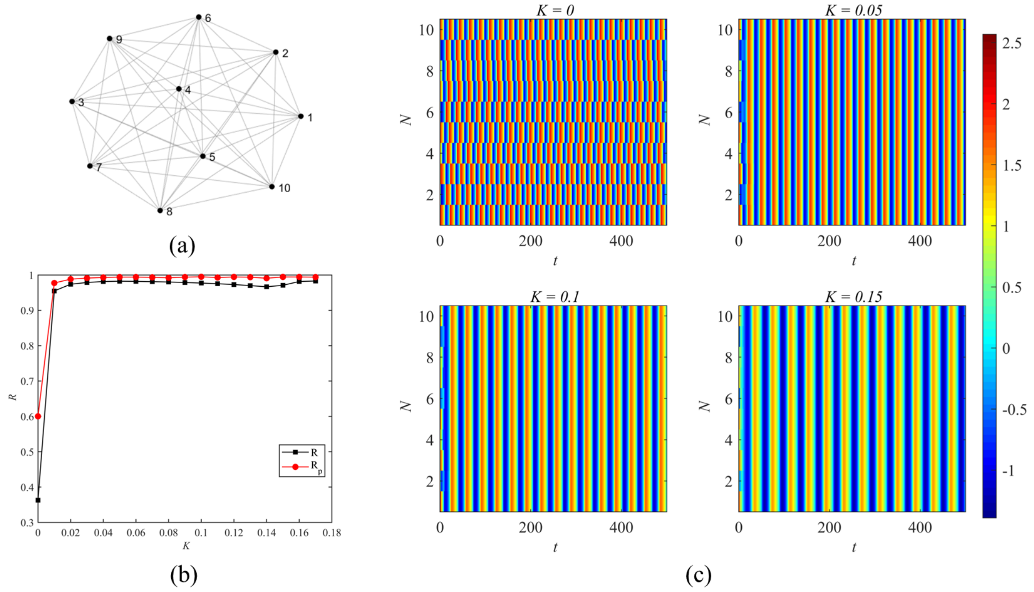


Figure 3. Dynamics of the fractional-order delayed FitzHugh-Nagumo fully connected network with 10 nodes. (a) Illustration of the network structure. (b) Evolution of the complete synchronization factor R and the phase synchronization factor R_p. (c) Dynamics of the activators for all nodes, with subplots displaying values of $K = 0$, $K = 0.05$, $K = 0.10$, and $K = 0.15$.

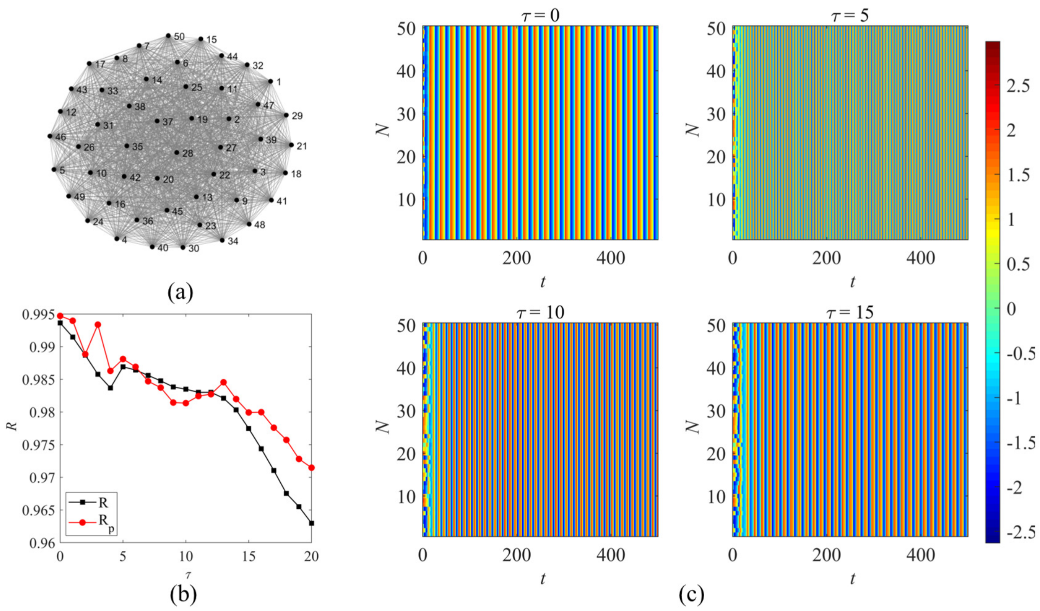
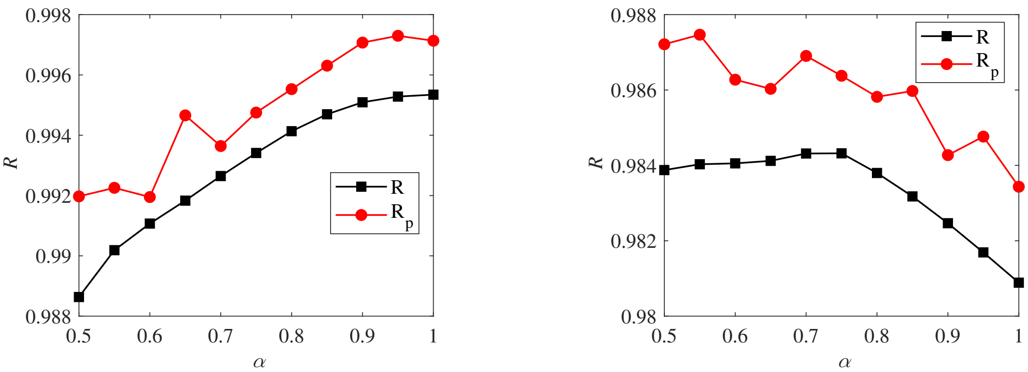


Figure 4. Dynamics of the fractional-order delayed FitzHugh-Nagumo fully connected network with 50 nodes. (a) Illustration of the network structure. (b) Evolution of the complete synchronization factor R and the phase synchronization factor R_p. (c) Dynamics of the activators for all nodes, with subplots displaying values of $\tau = 0$, $\tau = 5$, $\tau = 10$, and $\tau = 15$.

Finally, considering a fully connected fractional-order delayed FitzHugh-Nagumo network with 10 nodes under a fixed $K = 0.1$, we investigate the effect of different delay times τ (with values of 2 and 6) and varying α on the synchronization level. As shown in Figure 5, the results indicate that when $\tau = 2$, the synchronization level increases with increasing α. However, when $\tau = 6$, the synchronization level decreases as α increases. Therefore, the effect of the delay time τ on synchronization depends on the variation of α, and its influence mechanism differs for different values of τ.



(a) $\tau = 2$ **(b)** $\tau = 6$

Figure 5. Synchronization level variation with α in a fractional-order FitzHugh-Nagumo network under different delay times.

5. Conclusions

This paper investigates the dynamics and synchronization behaviors of the fractional-order delayed FitzHugh-Nagumo system, offering insights into both pairwise neuron interactions and large-scale fully connected networks. Theoretical analysis establishes the synchronization condition $0 < K < db$, demonstrating that two coupled neurons achieve synchronization within this range. In larger networks, excessive coupling strength may lead to desynchronization, suggesting the presence of an optimal coupling range K that maintains synchronization without compromising stability. Furthermore, as the time delay τ increases, the system exhibits increasingly complex dynamical behaviors, underscoring the critical role of delay in modulating network synchronization and overall system dynamics. Numerical experiments further reveal that the impact of different time delays ($\tau = 2, 6$) on synchronization is dependent on the fractional-order parameter α, with distinct synchronization mechanisms emerging for different τ values. It is hoped that the findings of this study will contribute to the further understanding of neural network dynamics.

Future research could explore synchronization in biologically plausible networks, such as small-world or scale-free structures, to better understand neural coordination. Investigating the effects of noise and external perturbations may provide insights into the robustness of synchronization, while further theoretical analysis of stability conditions could enhance the understanding of fractional-order delayed neural dynamics.

Author contributions

Canhong Long conducted the entire research and wrote all sections of the proposed article. Zuozhi Liu supervised and guided Canhong Long throughout the research process. Can Ma reviewed the manuscript for clarity, and coherence. All authors approved the final version of the manuscript for submission.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

This work was supported by the Science and Technology Foundation of Guizhou Province (No.QKHJC-ZK[2022] YB024), the Graduate Research Fund Project of Guizhou Province (NO.2024YJSKYJJ258) and the Innovative Exploration and Academic Emerging Foundation of Guizhou University of Finance and Economics (No.2022XSXMB09).

Conflict of interest

The authors have no conflict of interest that might be perceived to influence the results in this paper.

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