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Research article

A new Beta distribution with interdisciplinary data analysis

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Abstract: Several families of Beta distributions, such as Beta of the first kind, Beta of the second kind, and Beta of the third kind, have been proposed in the literature for modeling random phenomena. This study introduced a new member of the Beta family called the New Beta (NE-Beta) distribution using a logarithmic transformation approach. This new model is highly flexible and capable of analyzing both positive and negative data, making it suitable for a wide range of interdisciplinary applications. The NE-Beta distribution exhibits nearly symmetric, right-skewed, or left-skewed density functions and featured an increasing or decreasing hazard functions, which are crucial for accurately modeling practical scenarios across various fields. Some properties of the new distribution were derived, and the parameter estimation was obtained by utilizing various approaches. To demonstrate the efficacy of the NE-Beta distribution, it was applied to multiple datasets, including exchange rate returns (finance), biomedical data, engineering reliability data, and hydrological data. The results indicate that the proposed NE-Beta model outperforms its competitors across these diverse domains.

Keywords: Beta distribution; finance; logarithmic transformation; skewness; Rényi entropy; economic resources
Mathematics Subject Classification: 60E05, 62F10, 62H12

1. Introduction

Statistical distributions are vital components in the analysis of empirical data, enabling the application of various statistical techniques and forming the basis of much research in applied probability and statistics, see [1, 2]. The accuracy of empirical analysis often depends on how well the assumed distribution models key characteristics such as variance, mean, kurtosis, and skewness [3]. Among the many available statistical distributions, the Beta family, including the Beta of the first kind (Beta-I), Beta of the second kind (Beta-II), and Beta of the third kind (Beta-III), plays a pivotal role due to its flexibility and wide-ranging applicability in modeling positive random variables [4].

The Beta-I distribution is widely recognized for its ability to model data confined within the interval (0, 1), making it an indispensable tool in fields such as finance, medicine, education, economics, and ecology, where probabilities, proportions, and percentages are commonly analyzed. However, there is a growing need for statistical models that can handle data spanning the entire real line, including both negative and positive values. This need arises in various interdisciplinary fields where the phenomena being studied are not limited to positive values alone. The Beta-I distribution is characterized by two shape parameters, α and β, which determine its form and attributes, such as the presence of a mode within the interval and the degree of symmetry. The probability density function (PDF) of the Beta-I distribution is provided as:

$$f_Y(y;\ \alpha,\ \theta) = \frac{1}{B(\alpha,\ \theta)} y^{\alpha-1}(1-y)^{\theta-1}, \quad 0 < y < 1;\ \ \alpha, \theta > 0, \tag{1.1}$$

where α and θ are the shape parameters, and $B(\alpha,\ \theta) = \int_0^1 x^{\alpha-1}(1-x)^{\theta-1}$ is the normalizing constant.

The versatility and adaptability of the Beta-I distribution have led to its application across a wide range of fields. In meteorological studies, for instance, [5] utilized the Beta distribution to model Malaysian sunshine data over ten years. The study demonstrated that the Beta-I distribution could accurately fit monthly sunshine data, providing valuable insights into the relationship between mean and standard deviation of sunshine duration. In the economic analysis, the Beta-I distribution has proven to be an effective model for income distributions, for instance, [6] studied the Beta Type I distribution to the survey data, stabilizing estimation and offering insights into the relationship between theoretical distribution characteristics and income indicators. Environmental research also benefits from the Beta-I distribution's capabilities. As reported in [7], the Beta-I distribution was employed to estimate relative humidity and runoff coefficients and established its usefulness for environmental data through goodness-of-fit tests. In project management, [8] introduced the Bayesian Beta S-curve method, which utilizes the Beta-I distribution for probabilistic forecasting. This method provided accurate predictions for project duration and cost at completion, proving advantageous for risk management in ongoing projects. Furthermore, the Beta-I distribution has been effectively applied in post-earthquake damage modeling and flood loss assessment. Recent developments in statistical inference for the Beta-I distribution, such as the novel closed-form point estimators [9], give alternative approaches that enhance computational efficiency and parameter estimate accuracy. These estimators provide analytical solutions that avoid iterative numerical techniques, making them particularly suitable for large-scale applications. In addition to point estimation, interval estimation is essential for evaluating the uncertainty in parameter estimates. Confidence intervals for estimated parameters and other relevant variables are critical for practical decision-making, especially in applications requiring risk assessment or predictive modeling.

The Beta-II distribution, also known as the Beta prime distribution, is another crucial member of the Beta family. It is a versatile statistical model particularly effective for modeling positive data, as it is defined for values ranging from 0 to infinity. The Beta-II distribution and its variants have proven highly effective in several fields, including finance, environmental science, economics, and engineering, offering a flexible framework for capturing the variability and uncertainty inherent in diverse data types. The PDF of the Beta-II distribution is provided via:

$$f_T(t;\ \alpha,\ \theta) = \frac{1}{B(\alpha,\ \theta)}\left(\frac{t^{\alpha-1}}{(1+t)^{\alpha+\theta}}\right), \quad 0 < t < \infty;\ \ \alpha, \theta > 0. \tag{1.2}$$

Numerous generalizations of the Beta-II distribution have emerged in the statistical literature, each offering unique advantages for specific applications. For instance, [10] highlighted the application of the Beta-II distribution in engineering and environmental data, introducing the odd Beta prime generalized (OBP-G) family of distributions. Their study derived the OBP-logistic distribution's mathematical properties, such as moments and entropy, demonstrating its superior performance in modeling symmetric and skewed data. Similarly, [11] demonstrated the utility of the generalized Beta-II in flood frequency analysis, underscoring its relevance in environmental applications, while [12] expanded the OBP-G with the OBP-Burr X distribution, confirming its usefulness in modeling skewed data as well as varying hazard rates, especially in COVID-19 mortality and petroleum rock samples.

Additionally, [13] applied the generalized Beta-II distribution to financial data, specifically daily returns on the stock market, highlighting its suitability for financial modeling. The OBP-inverted Kumaraswamy distribution introduced by [14] showcased its applicability in biomedical sciences and engineering, outperforming competitors in modeling COVID-19 mortality data. The OBP-Fréchet distribution, suggested by [15], can accommodate both right-skewed and left-skewed data, including groundwater contamination data.

The Beta-III distribution is another important member of the Beta family, known for its versatility in statistical modeling. Like the Beta-I model, the Beta-III distribution is specified for the interval [0, 1]. The PDF of the Beta-III distribution is given by:

$$f_Z(z;\ \alpha,\ \theta) = \frac{2^{\alpha} z^{\alpha-1}(1-z)^{\theta-1}}{B(\alpha,\ \theta)(1+z)^{\alpha+\theta}}, \quad 0 < z < 1;\ \ \alpha, \theta > 0. \tag{1.3}$$

However, the Beta-III distribution often arises through transformations of the Beta-I distribution, and it can be extended to multivariate and matrix-variate cases, enhancing its applicability in complex statistical analyses. For example, in a similar study by [16], the authors introduced the matrix-variate Beta-III distribution. This distribution was obtained through matrix transformation techniques and exhibits a range of significant properties, including the Laplace transform and marginal distributions. The study also explores the relationships between the matrix-variate Beta-III model and its counterparts, the matrix-variate Beta-I as well as Beta-II models. Through a bilinear transformation of a random matrix, the matrix-variate Beta-III distribution is generated. The study's findings underscore the versatility of this distribution in capturing the properties of random matrices within the bounded interval [0, 1].

According to [17], the Beta-III distribution is derived from the Beta-I distribution. This change assures that the Beta-III distribution, defined on a bounded interval, can be used as a substitute for the Beta-I distribution in practical situations. The study delved into various properties of the Beta-III distribution and examined its relationships with the Beta-I and Beta-II distributions. Additionally, a multivariate generalization of the Beta-III distribution was developed, along with an exploration of its properties, further extending its applicability.

Despite the wide-ranging applicability of these Beta distributions, there remains a critical gap in modeling data that spans the entire real line, including both negative and positive values. Conventional

Beta distributions are constrained to specific intervals, limiting their use in scenarios where data may take on any real value. This limitation is particularly evident in fields such as finance, where returns data can be both negative and positive, and in engineering, biomedical, and hydrological studies, where data often exhibit more complex patterns.

To address this gap, the present paper introduces a new Extension of the Beta (NE-Beta) distribution. The proposed distribution extends the Beta-I distribution to the entire real line through a logarithmic transformation technique, thereby retaining the flexibility and adaptability of the Beta-I distribution while significantly broadening its applicability. The NE-Beta distribution is particularly useful for modeling data in scenarios where traditional Beta distributions are inadequate. It provides a more comprehensive tool for researchers and practitioners across various disciplines. To the best of our ability, the current study represents the first generalization of the Beta-I using a logarithmic transformation approach that can model both negative and positive data ranges. As the log-normal distribution originates via the normal distribution, the NE-Beta distribution is derived from the Beta-I. Some recent generalizations of statistical models using this logarithmic transformation approach can be found in [18–20], and the references therein. Other generalized distributions developed using various approaches can be found in [21–30].

The table below compares the NE-Beta distribution to three typical Beta distributions: Beta-I, Beta-II, and Beta-III. Table 1 summarizes the key differences in support, shape, and typical applications. While classical Beta distributions have constraints in terms of support and form flexibility, the NE-Beta distribution is more widely applicable. Its capacity to represent both positive and negative data, as well as its adaptability to varied shapes, make it an effective tool for a wide range of statistical modeling tasks.

Table 1. The mean, variance, skewness, and kurtosis for the NE-Beta distribution for various parameter combinations.

Distribution	Support	Shape	Applications
Beta-I	$(0, 1)$	Symmetric or skewed	Modeling proportions, probabilities
Beta-II	$(0, \infty)$	Right-skewed	Reliability analysis, survival analysis
Beta-III	$(0, 1)$	Symmetric or skewed	Multivariate applications
NE-Beta	$(-\infty, \infty)$	Near-symmetric, left-skewed, or right-skewed	Finance, biomedicine, engineering, hydrology

The motivation for this research stems from the need to develop a versatile statistical distribution that not only retains the core applicability of the Beta-I distribution but also extends its range to encompass all real numbers. This extension is critical for modeling phenomena that involve both positive and negative values, which are common across various fields such as finance, engineering, biomedicine, and hydrology. The proposed NE-Beta distribution achieves this by utilizing a logarithmic transformation of the Beta-I distribution, offering unique properties that make it suited for modeling diverse types of data.

In this study, we demonstrate that the NE-Beta distribution not only offers an excellent fit to financial returns data, which can be negative or positive, but also proves highly effective in interdisciplinary applications. The model can represent nearly symmetric, left-skewed, or right-skewed data distributions and features hazard rates that exhibit increasing trends. This flexibility allows the NE-Beta distribution to adapt to various data shapes, making it suitable for a wide array of practical scenarios. The key contributions of this paper are as follows:

i. To introduce the NE-Beta distribution, which extends the Beta-I distribution to cover the entire real line using a logarithmic transformation technique, thereby broadening its applicability across different fields.
ii. The NE-Beta distribution is designed to accommodate nearly symmetric, left-skewed, or right-skewed data distributions and features increasing or decreasing hazard rates, making it suitable for modeling diverse data shapes encountered in finance, engineering, biomedicine, and hydrology.
iii. To explore the important statistical properties of the NE-Beta distribution, including its mean, variance, skewness, kurtosis, and various moments and entropy measures, providing a deep understanding of its behavior.
iv. To employ the maximum likelihood estimation method to estimate the parameters of the NE-Beta distribution and validate its performance through Monte Carlo simulations using multiple estimation approaches, including least squares estimation (LSE), Cramér-von Mises (CVM), weighted least squares estimation (WLSE), and maximum likelihood estimation (MLE).
v. To demonstrate the broad applicability of the NE-Beta distribution, we apply it to multiple datasets, including financial returns data, engineering reliability data, biomedical data, and hydrological data. These applications showcase the model's ability to outperform competing models across various disciplines, highlighting its versatility and effectiveness.

The remainder of this paper is organized as follows: Section 2 defines the NE-Beta distribution and presents plots of its hazard and density functions. Various statistical properties and parameter estimation methods are discussed in Sections 3 and 4. In Section 5, we conduct Monte Carlo simulations using the quantile function and apply the NE-Beta distribution to interdisciplinary datasets. Finally, in Section 6, we conclude that the NE-Beta model offers superior performance compared to its competitors across different domains.

2. Materials and methods

This section offers a novel statistical distribution that will serve as an alternative to several forms of Beta distributions as well as some other conventional distributions with different probability distribution limits. The NE-Beta distribution is derived using a logarithmic transformation $X = \log\left(\frac{Y}{1-Y}\right)$ of the classical Beta-I distribution, and this can be provided as:

$$f_X(x;\ \alpha,\ \theta) = \frac{1}{B(\alpha,\ \theta)}\left(\frac{e^{\alpha x}}{(1+e^x)^{\alpha+\theta}}\right), \quad -\infty < x < \infty;\ \ \alpha, \theta > 0. \tag{2.1}$$

A random variable X with the PDF in Eq (2.1) can be expressed as $X \sim NE-Beta(\alpha, \theta)$, where X follows the New Beta distribution with parameters α and θ. The cumulative distribution function (CDF) of the NE-Beta distribution is obtained from Eq (2.1) as:

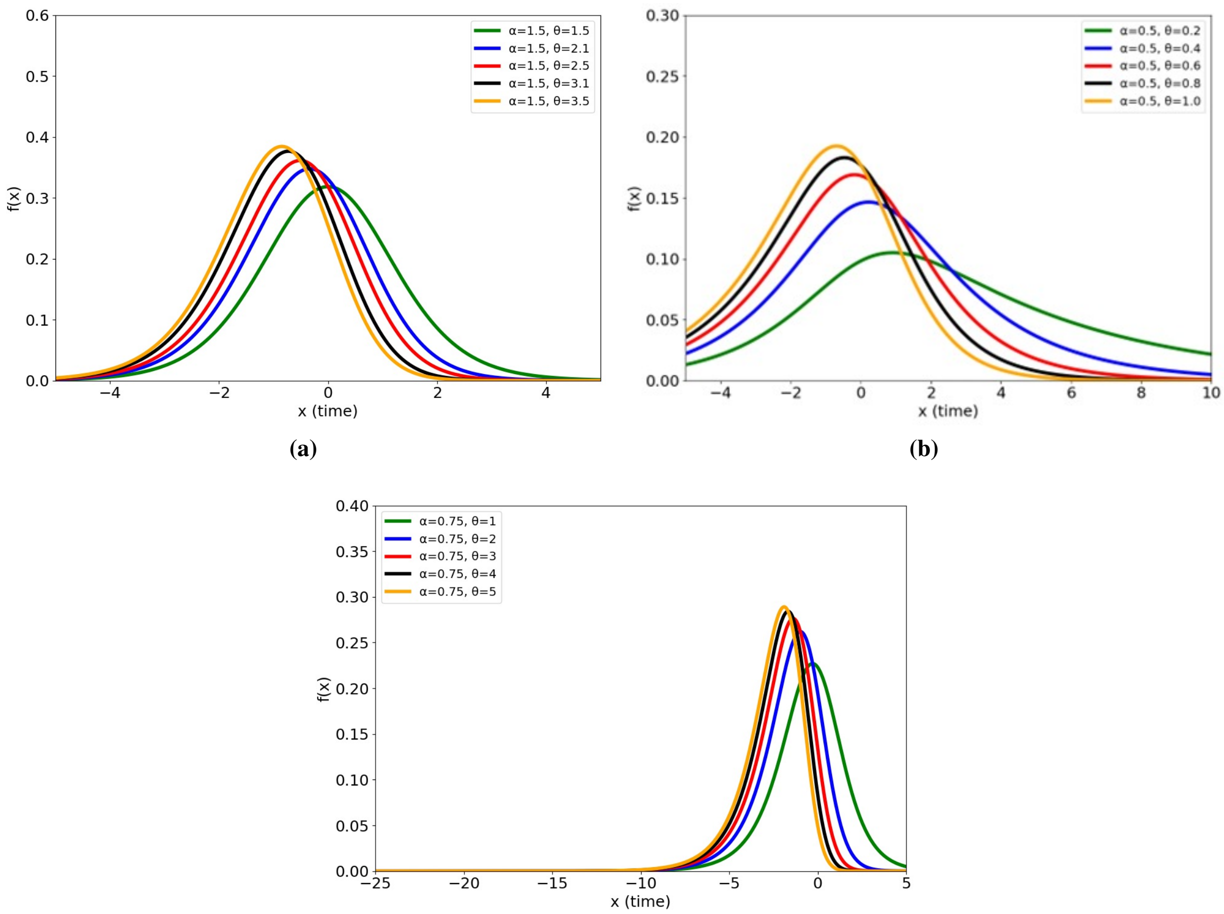
$$F_X(x;\ \alpha,\ \theta) = \frac{1}{B(\alpha,\ \theta)}\int_{-\infty}^{x}\frac{e^{\alpha t}}{(1+e^t)^{\alpha+\theta}}dt = \frac{1}{B(\alpha,\ \theta)}B_{e^x}(\alpha,\ \theta), \tag{2.2}$$

where $B_{e^x} = \int_0^{e^x} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\theta}} dt$ is the incomplete Beta function of the second kind. Nevertheless, the CDF presented in Eq (2.2) can be expressed as:

$$F_X(x;\ \alpha, \theta) = \mathrm{I}(e^x;\ \alpha,\ \theta), \tag{2.3}$$

which is represented as the regularized incomplete Beta function, where $\mathrm{I}(e^x;\ \alpha,\ \theta) = \frac{1}{B(\alpha,\theta)} B(e^x;\ \alpha,\ \theta)$.

Figure 1 presents the different density patterns of the NE-Beta distribution, including (a) nearly symmetric, (b) right-skewed, and (c) left-skewed shapes with different parameter settings. As demonstrated in these figures, the distribution's shape is greatly influenced by its parameters, α and θ. When α is considerably larger than θ, the distribution approaches near-symmetry, making it highly suitable for modeling data that is not strongly skewed in either direction. On the other hand, as θ increases relative to α, the distribution becomes progressively (b) right-skewed, or (c) left-skewed, enabling it to effectively model data with a pronounced long tail to the right or left. This adaptability is particularly valuable in domains such as finance, engineering, biomedicine, and hydrology, where data often exhibits such characteristics.



(a) **(b)** **(c)**

Figure 1. The PDF plots for the NE-Beta distribution.

Moreover, the NE-Beta distribution offers several advantages over conventional Beta distributions. By extending the domain to the entire real line, it surpasses the limitations of Beta distributions, which are confined to the interval (0, 1). This enhanced flexibility allows the NE-Beta distribution to model a wider range of data, including those with negative or unbounded values. This flexibility enhances its utility for diverse datasets across various domains.

2.1. Related distributions

The NE-Beta distribution is related to other statistical distributions, including Beta types I and II. Some of the relationships are as follows:

i. If $X \sim NE - Beta\,(\alpha,\, \theta)$, then $e^X \sim Beta - II\,(\alpha,\, \theta)$,
ii. If $X \sim NE - Beta\,(\alpha,\, \theta)$, then $\frac{e^X}{1+e^X} \sim Beta - I\,(\alpha,\, \theta)$.

3. Properties of the NE-Beta distribution

This section discusses the quantile function, survival function, hazard function, cumulative hazard function, mixture representations, moment generating function, moments, and Rényi as well as Tsallis entropies of the NE-Beta distribution.

3.1. Quantile function

The quantile function provides a direct link between the distribution and its application in generating random samples. This is particularly useful in simulation studies and Monte Carlo experiments, where the quantile function is employed to efficiently generate data points from the new distribution. The quantile function of the NE-Beta distribution with the CDF defined in Eq (2.3) can be determined using the technique applied in [10] as:

$$e^{x_q} = \mathrm{I}^{-1}\,(u;\ \alpha,\, \theta)\,. \tag{3.1}$$

Taking the logarithm of both sides of Eq (3.1) we may obtain the quantile function of the NE-Beta distribution

$$x_q = \log\left\{\mathrm{I}^{-1}\,(u;\ \alpha,\, \theta)\right\}, \tag{3.2}$$

where u is a real number whose values range from 0 to 1.

3.2. Survival, hazard, and cumulative hazard functions

The survival function is crucial in reliability analysis and survival studies, as it represents the probability of an event occurring beyond a specific time. Its flexible form in the new distribution enables the modeling of complex lifetime data, including cases with heavy tails or varying hazard rates. Thus, the survival function of the NE-Beta distribution can be determined according to Eq (2.2), whereby

$$S\,(x;\ \alpha,\, \theta) = 1 - \left(\frac{B_{e^x}\,(\alpha,\, \theta)}{B\,(\alpha,\, \theta)}\right), \quad -\infty < x < \infty;\ \ \alpha, \theta > 0. \tag{3.3}$$

The hazard function highlights the instantaneous risk of an event occurring at a specific time, given that it has not occurred earlier. The flexibility of the hazard function derived from the new distribution makes it suitable for modeling real-world phenomena with increasing, decreasing, or bathtub-shaped hazard rates, which are commonly encountered in reliability and biomedical applications. Then, the hazard function of the NE-Beta distribution is derived from Eqs (2.1) and (3.3):

$$h\,(x;\ \alpha,\, \theta) = \frac{e^{\alpha x}\,(1 + e^x)^{-(\alpha+\theta)}}{B\,(\alpha,\, \theta)\left\{1 - \frac{B_{e^x}(\alpha,\, \theta)}{B(\alpha,\, \theta)}\right\}}, \quad -\infty < x < \infty;\ \ \alpha, \theta > 0. \tag{3.4}$$

The cumulative hazard function (CHF) offers insights into the aggregate risk of an event over time. This property is particularly beneficial in reliability engineering and failure rate analysis, where the cumulative effect of risks must be quantified to design more robust systems. Thus, the CHF is derived from Eq (3.3) as:

$$H(x;\ \alpha,\ \theta) = -\log(S(x;\ \alpha,\ \theta)) = -\log\left(1-\left(\frac{B_{e^x}(\alpha,\ \theta)}{B(\alpha,\ \theta)}\right)\right),\quad -\infty < x < \infty;\ \ \alpha,\theta > 0. \tag{3.5}$$

Figure 2(a) demonstrates that the curve of the NE-Beta distribution goes higher and remains flat at where the y-axis is equal to 1. This indicates that the NE-Beta distribution is a legitimate statistical distribution. In Figure 2(b), the cumulative hazard plots of the NE-Beta distribution display an increasing shape, indicating that the risk of occurrence has risen over time. Figure 3 shows the structure of the hazard functions for the NE-Beta distribution, displaying (a) increasing and (b) decreasing shapes across various parameter values. This characteristic is essential for modeling failure rates in interdisciplinary datasets, where data can exhibit diverse patterns and complexities, particularly in fields such as finance, engineering, biomedicine, and hydrology.

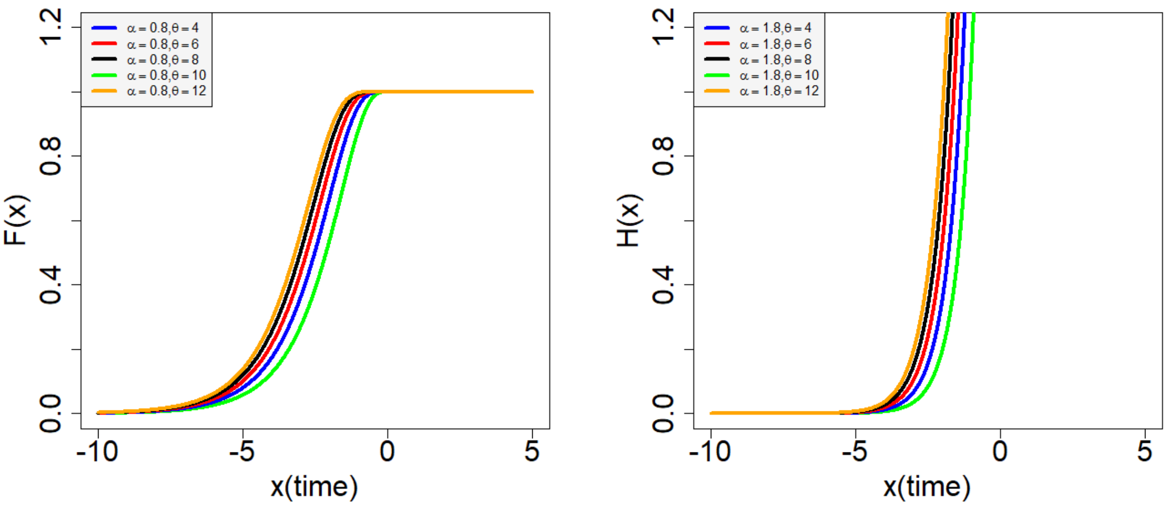


Figure 2. The CDF and CHF plots for the NE-Beta distribution.

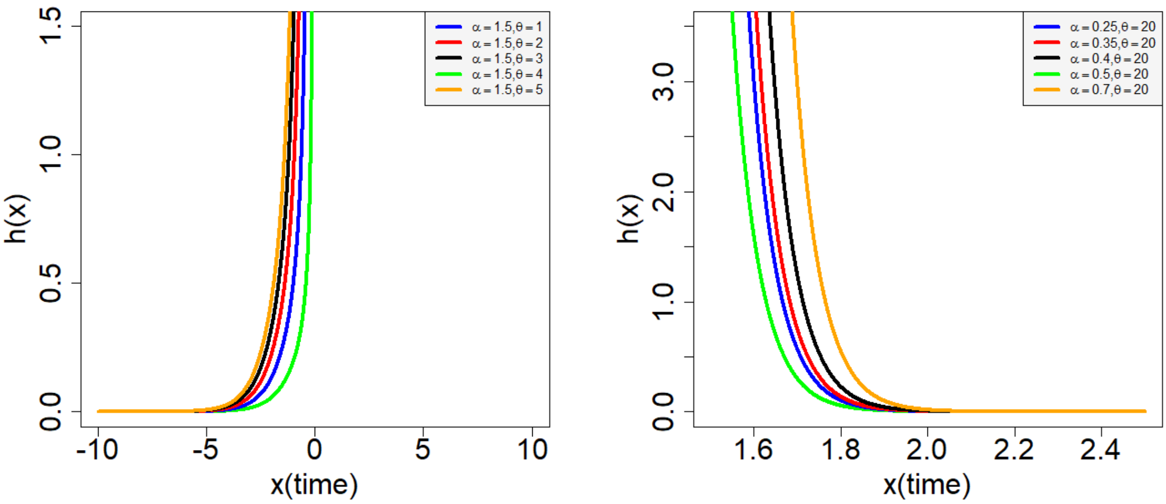


Figure 3. The hazard function plots for the NE-Beta distribution.

3.3. *Expansion for the PDF*

The PDF expansion decomposes the new distribution into simpler components, allowing for better statistical inference, as presented in [27]. The PDF for NE-Beta can be represented as follows:

Consider the generalized binomial expansion for $\phi > 0$ as follows:

$$(1+w)^{-\phi} = \sum_{k=0}^{\infty} \frac{(-\iota)^k \Gamma(\phi+k)}{k!\Gamma(\phi)} w^k, \qquad w > 0. \tag{3.6}$$

Utilizing Eq (3.6) into the denominator in Eq (2.1) yields:

$$\begin{aligned} f_X(x;\ \alpha,\ \theta) &= \frac{1}{B(\alpha,\ \theta)} e^{\alpha x} \sum_{k=0}^{\infty} \frac{(-\iota)^k \Gamma(\alpha+\theta+k)}{k!\Gamma(\alpha+\theta)} (e^x)^k \\ &= \frac{1}{B(\alpha,\ \theta)} \sum_{k=0}^{\infty} \frac{(-\iota)^k \Gamma(\alpha+\theta+k)}{k!\Gamma(\alpha+\theta)} e^{x(k+\alpha)}, \end{aligned} \tag{3.7}$$

which is the PDF of the NE-Beta distribution.

3.4. *Moment generating function*

The moment generating function (MGF) is instrumental in deriving moments, cumulants, and other statistical measures of the distribution. Its existence for the new distribution highlights its potential use in risk management, financial modeling, and decision-making processes that rely on higher-order moments. The MGF for the NE-Beta distribution can be constructed according to Eq (2.1) as follows:

$$M_X(t) = \frac{1}{B(\alpha,\ \theta)} \int_{-\infty}^{\infty} \frac{e^{tx} \times e^{\alpha x}}{(1+e^x)^{\alpha+\theta}} dx$$

$$= \frac{1}{B(\alpha,\theta)} \int_{-\infty}^{\infty} \frac{e^{x(t+\alpha)}}{(1+e^{x})^{\alpha+\theta}} dx. \tag{3.8}$$

Following some computations, Eq (3.8) is presented as:

$$M_X(t) = \frac{1}{B(\alpha,\theta)} B(t+\alpha, \theta-t), \quad \theta > t, \tag{3.9}$$

which indicates that the MGF of the NE-Beta distribution can only exist when $\theta > t$.

3.5. Moments

Moments provide critical insights into the central tendency, dispersion, skewness, and kurtosis of the distribution. These measures are valuable in understanding the shape and variability of the data, which is essential in fields like finance, environmental science, and quality control. The r^{th} moments of X about the origin of the NE-Beta distribution can be determined using the MGF obtained in Eq (3.9). This can be achieved by differentiating Eq (3.9) r times for t and then setting $t = 0$, as explained in [31]. The steps to obtain moments are as follows:

$$\begin{aligned}
E(X^r) &= \left|\frac{d^r}{dt^r}\{M_X(t)\}\right|_{t=0} \\
&= \frac{1}{B(\alpha,\theta)}\left|\frac{d^r}{dt^r}\{B(t+\alpha,\theta-t)\}\right|_{t=0} \\
&= \frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)\Gamma(\theta)}\left|\frac{d^r}{dt^r}\{B(t+\alpha,\theta-t)\}\right|_{t=0} \\
&= \frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)\Gamma(\theta)}\left|\frac{d^r}{dt^r}\left\{\frac{\Gamma(t+\alpha)\Gamma(\theta-t)}{\Gamma(t+\alpha+\theta-t)}\right\}\right|_{t=0} \\
&= \frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)\Gamma(\theta)}\left|\frac{d^r}{dt^r}\left\{\frac{\Gamma(t+\alpha)\Gamma(\theta-t)}{\Gamma(\alpha+\theta)}\right\}\right|_{t=0} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\theta)}\left|\frac{d^r}{dt^r}\{\Gamma(t+\alpha)\Gamma(\theta-t)\}\right|_{t=0}.
\end{aligned} \tag{3.10}$$

The first moments can be obtained by differentiating Eq (3.10) for t as:

$$\begin{aligned}
E(X) &= \frac{1}{\Gamma(\alpha)\Gamma(\theta)}\left\{\Gamma(\theta-t)\frac{d}{dt}\{\Gamma(t+\alpha)\} + \Gamma(t+\alpha)\frac{d}{dt}\{\Gamma(\theta-t)\}\right\} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\theta)}\{\Gamma(\theta-t)\psi(t+\alpha)\Gamma(t+\alpha) + \Gamma(t+\alpha)\{-\psi(\theta-t)\Gamma(\theta-t)\}\} \\
&= \frac{\Gamma(t+\alpha)\Gamma(\theta-t)}{\Gamma(\alpha)\Gamma(\theta)}\{\psi(t+\alpha) - \psi(\theta-t)\},
\end{aligned} \tag{3.11}$$

where $\psi(t+c) = \frac{d}{dt}\ln\Gamma(t+c) = \frac{\Gamma'(t+c)}{\Gamma(t+c)}$, which implies $\Gamma'(t+c) = \Gamma(t+c)\psi(t+c)$.

Putting $t = 0$ into Eq (3.11) will give the first moments for the NE-Beta distribution obtained as:

$$E(X) = \psi(\alpha) - \psi(\theta). \tag{3.12}$$

The second moments can be derived by differentiating Eq (3.11) for t as:

$$E\left(X^2\right) = \frac{1}{\Gamma(\alpha)\Gamma(\theta)}\frac{d}{dt}\left\{\Gamma(t+\alpha)\Gamma(\theta-t)\left\{\psi(t+\alpha)-\psi(\theta-t)\right\}\right\}$$
$$= \frac{\Gamma(t+\alpha)\Gamma(\theta-t)}{\Gamma(\alpha)\Gamma(\theta)}\left\{\psi(1,\, t+\alpha)+\psi(1,\, \theta-t)-2\psi(t+\alpha)\psi(\theta-t)+\psi^2(t+\alpha)+\psi^2(\theta-t)\right\}. \tag{3.13}$$

Substituting $t = 0$ into Eq (3.13), we obtain the second moments for the NE-Beta distribution as:

$$E\left(X^2\right) = \psi(1,\, \alpha)+\psi(1,\, \theta)-2\psi(\alpha)\psi(\theta)+\psi^2(\alpha)+\psi^2(\theta). \tag{3.14}$$

Then, the third and fourth moments are, respectively presented as:

$$E\left(X^3\right) = \psi(2,\, \alpha)-\psi(2,\, \theta)-3\psi(\theta)\psi(1,\, \alpha)+3\psi(\alpha)\psi(1,\, \alpha)-3\psi(\theta)\psi(1,\, \theta)+3\psi(\alpha)\psi(1,\, \theta)$$
$$-\psi(\theta)\psi^2(\alpha)+\psi(\alpha)\psi^2(\theta)+\psi^3(\alpha)-\psi^3(\theta), \tag{3.15}$$

and

$$E\left(X^4\right) = \psi(3,\, \alpha)+\psi(3,\, \theta)-4\psi(\theta)\psi(2,\, \alpha)+4\psi(\alpha)\psi(2,\, \alpha)+4\psi(\theta)\psi(2,\, \theta)-4\psi(\alpha)\psi(2,\, \theta)+$$
$$6\psi(1,\, \alpha)\psi(1,\, \theta)-8\psi(\alpha)\psi(\theta)\psi(1,\, \alpha)+6\psi^2(\alpha)\psi(1,\, \alpha)+3\psi^2(1,\, \alpha)+3\psi^2(\theta)\psi(1,\, \alpha)+$$
$$3\psi^2(1,\, \theta)+3\psi^2(\theta)\psi(1,\, \theta)-4\psi^2(\theta)\psi(1,\, \theta)-6\psi(\alpha)\psi(\theta)\psi(1,\, \theta)+4\psi^2(\alpha)\psi(1,\, \theta)+$$
$$2\psi^2(\alpha)\psi^2(\theta)-2\psi(\alpha)\psi(\theta)\psi(1,\, \theta)-\psi^2(\alpha)\psi(\theta)-2\psi(\alpha)\psi^3(\theta)-\psi^3(\alpha)\psi(\theta)$$
$$\psi^4(\alpha)+\psi^4(\theta). \tag{3.16}$$

As a result, the mean is provided in Eq (3.12), and the variance may be determined via Eqs (3.12) and (3.14).

$$\sigma_X^2 = \psi(1,\, \alpha)+\psi(1,\, \theta)-2\psi(\alpha)\psi(\theta)+\psi^2(\alpha)+\psi^2(\theta)-(\psi(\alpha)-\psi(\theta))^2$$
$$= \psi(1,\, \alpha)+\psi(1,\, \theta)-2\psi(\alpha)\psi(\theta)+\psi^2(\alpha)+\psi^2(\theta)-\psi^2(\alpha)-\psi^2(\theta)+2\psi(\alpha)\psi(\theta)$$
$$= \psi(1,\, \alpha)+\psi(1,\, \theta). \tag{3.17}$$

The skewness (Sk) and kurtosis (Ku) can now be determined using the first four moments obtained in Eqs (3.12) and (3.14)–(3.16) as follows:

$$Sk = \frac{E\left(X^3\right)-3E(X)\sigma_X^2-\{E(X)\}^3}{\sigma_X^3}, \tag{3.18}$$

and

$$Ku = \frac{E\left(X^4\right)-4E(X)E\left(X^3\right)+6E\left(X^2\right)\{E(X)\}^2-3\{E(X)\}^4}{\sigma_X^4}. \tag{3.19}$$

Table 2 presents some properties of the proposed NE-Beta distribution for various combinations of α and θ. As α increases, the mean of the distribution shifts higher, while variance generally decreases, indicating reduced dispersion. For fixed α, an increase in θ leads to a lower mean and variance, concentrating the distribution around lower values. The variation in skewness and kurtosis with both parameters underscores the NE-Beta distribution's flexibility in modeling different data shapes and tails. This ability to adapt to various data characteristics highlights its advantage over the Beta-I distribution, which is limited to capturing data strictly within the [0, 1] range. Figure 4 shows the 3D plots of the mean, variance, kurtosis, and skewness for the NE-Beta distribution across various parameter combinations.

Table 2. The mean, variance, skewness, and kurtosis for the NE-Beta distribution for various parameter combinations.

α	θ	Mean	Variance	Skewness	Kurtosis
0.5	0.5	0.0000	8.6743	0.0000	0.6419
0.5	1	-1.3146	5.8245	-0.4793	0.7303
0.5	1.5	-1.9135	5.0130	-0.6781	0.6708
0.5	2	-2.2881	4.6426	-0.7545	0.6345
0.5	2.5	-2.5587	4.4221	-0.7883	0.5985
0.5	3	-2.7697	4.2704	-0.8039	0.5622
0.5	3.5	-2.9423	4.1567	-0.8106	0.5269
0.5	4	-3.0883	4.0663	-0.8125	0.4934
0.5	4.5	-3.2146	3.9916	-0.8116	0.4617
0.5	5	-3.3258	3.9281	-0.8091	0.4319
1	0.5	1.3146	5.8245	0.4793	0.7303
1	1	0.0000	3.2791	0.0000	1.0975
1	1.5	-0.6130	2.5725	-0.3642	1.0841
1	2	-0.9991	2.2809	-0.5552	1.1820
1	2.5	-1.2793	2.1247	-0.6676	1.2798
1	3	-1.4987	2.0277	-0.7402	1.3571
1	3.5	-1.6789	1.9616	-0.7904	1.4153
1	4	-1.8317	1.9137	-0.8268	1.4586
1	4.5	-1.9642	1.8772	-0.8542	1.4908
1	5	-2.0813	1.8484	-0.8753	1.5145
1.5	0.5	1.9135	5.0130	0.6781	0.6708
1.5	1	0.6130	2.5725	0.3642	1.0841
1.5	1.5	0.0000	1.8695	0.0000	0.8028
1.5	2	-0.3863	1.5797	-0.2135	0.7594
1.5	2.5	-0.6667	1.4251	-0.3478	0.7993
1.5	3	-0.8863	1.3296	-0.4393	0.8578
1.5	3.5	-1.0667	1.2650	-0.5055	0.9161
1.5	4	-1.2196	1.2185	-0.5555	0.9692
1.5	4.5	-1.3524	1.1833	-0.5946	1.0159
1.5	5	-1.4696	1.1559	-0.6260	1.0567

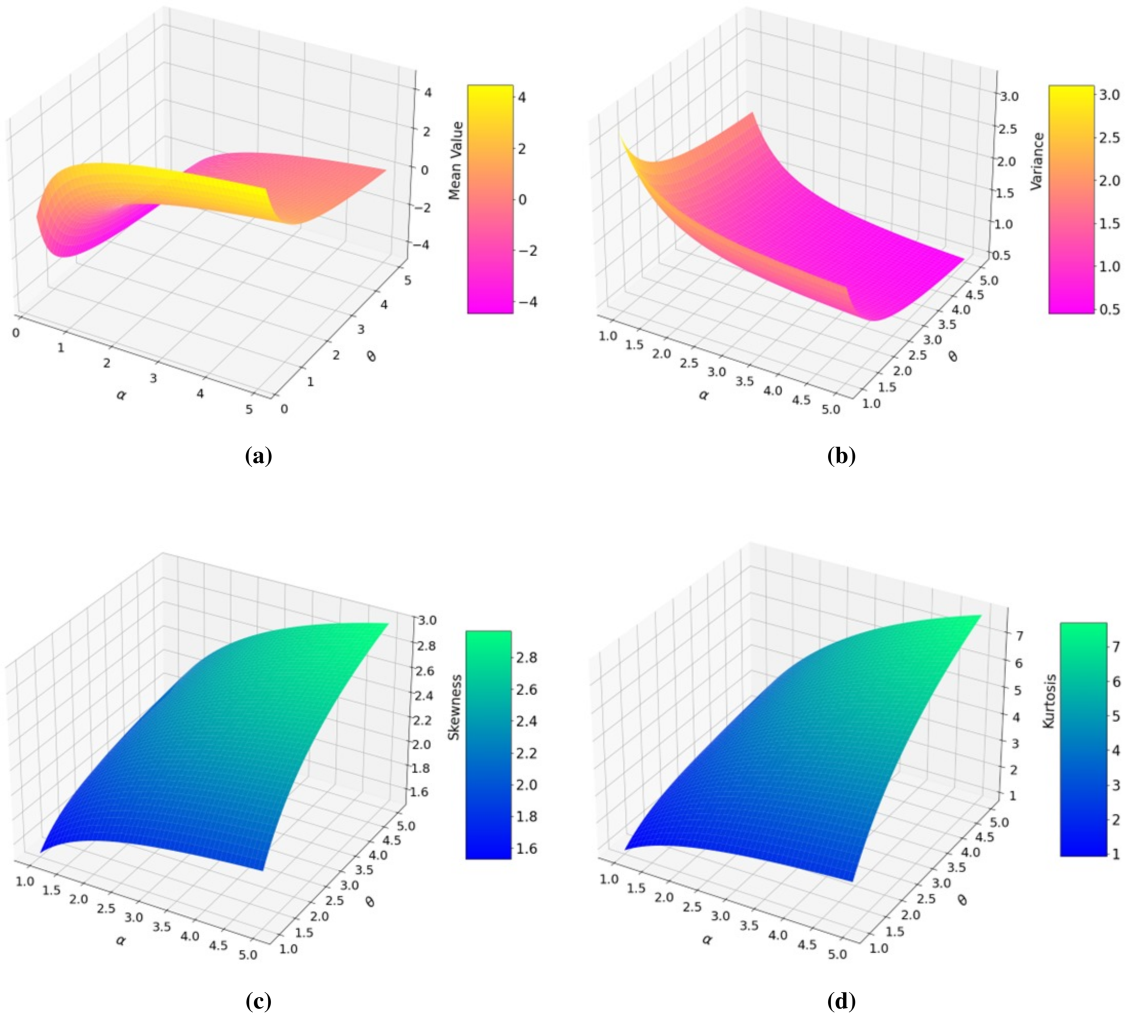


Figure 4. 3D plots of mean, variance, skewness, and kurtosis for the NE-Beta distribution across various parameter combinations.

3.6. Rényi and Tsallis entropies

Rényi and Tsallis entropies quantify the distribution's uncertainty and diversity. These measures have significant applications in information theory, ecological diversity studies, and machine learning, where understanding uncertainty is pivotal for model development and optimization. The Rényi entropy of the NE-Beta distribution with the PDF as defined in Eq (2.1) can be described as:

$$R_q(X) = \frac{1}{1-q}\log\left\{\int_{-\infty}^{\infty} f_X^q(x;\ \alpha,\theta)dx\right\}, \qquad 0 < q < \infty,\ \ q \neq 1. \tag{3.20}$$

The integral part in Eq (3.20) can be expressed as:

$$\int_{-\infty}^{\infty} f_X^q(x;\ \alpha,\theta)dx = \frac{1}{\{B(\alpha,\ \theta)\}^q}\int_{-\infty}^{\infty}\left(\frac{e^{\alpha q x}}{(1+e^x)^{q(\alpha+\theta)}}\right)dx. \tag{3.21}$$

Integrating Eq (3.21) for x, we get

$$\int_{-\infty}^{\infty} f_X^q(x;\ \alpha, \theta)dx = \frac{1}{\{B(\alpha,\ \theta)\}^q} B(\alpha q,\ \theta q)\,. \tag{3.22}$$

Putting Eq (3.22) into Eq (3.20) yields the Rényi entropy for the NE-Beta distribution.

$$R_q(X) = \frac{1}{1-q}\log\left\{\frac{B(\alpha q,\ \theta q)}{\{B(\alpha,\ \theta)\}^q}\right\}, \qquad q \neq 1. \tag{3.23}$$

Table 3 presents the Rényi entropy values for the NE-Beta distribution across various parameter combinations of α and θ. The data indicates that Rényi entropy decreases as both α and θ increase. Specifically, for fixed θ, increasing α leads to a reduction in entropy, reflecting a more concentrated distribution with less uncertainty. Similarly, for fixed θ, increasing α also results in lower entropy, suggesting a distribution that becomes increasingly deterministic with fewer data variations. This trend highlights that the NE-Beta distribution can model a wide range of uncertainty levels, from high entropy (more spread out) to low Rényi (more concentrated), depending on the chosen parameter values. Figure 5 illustrates the 3D plots of Rényi entropy for the NE-Beta distribution for various parameter combinations.

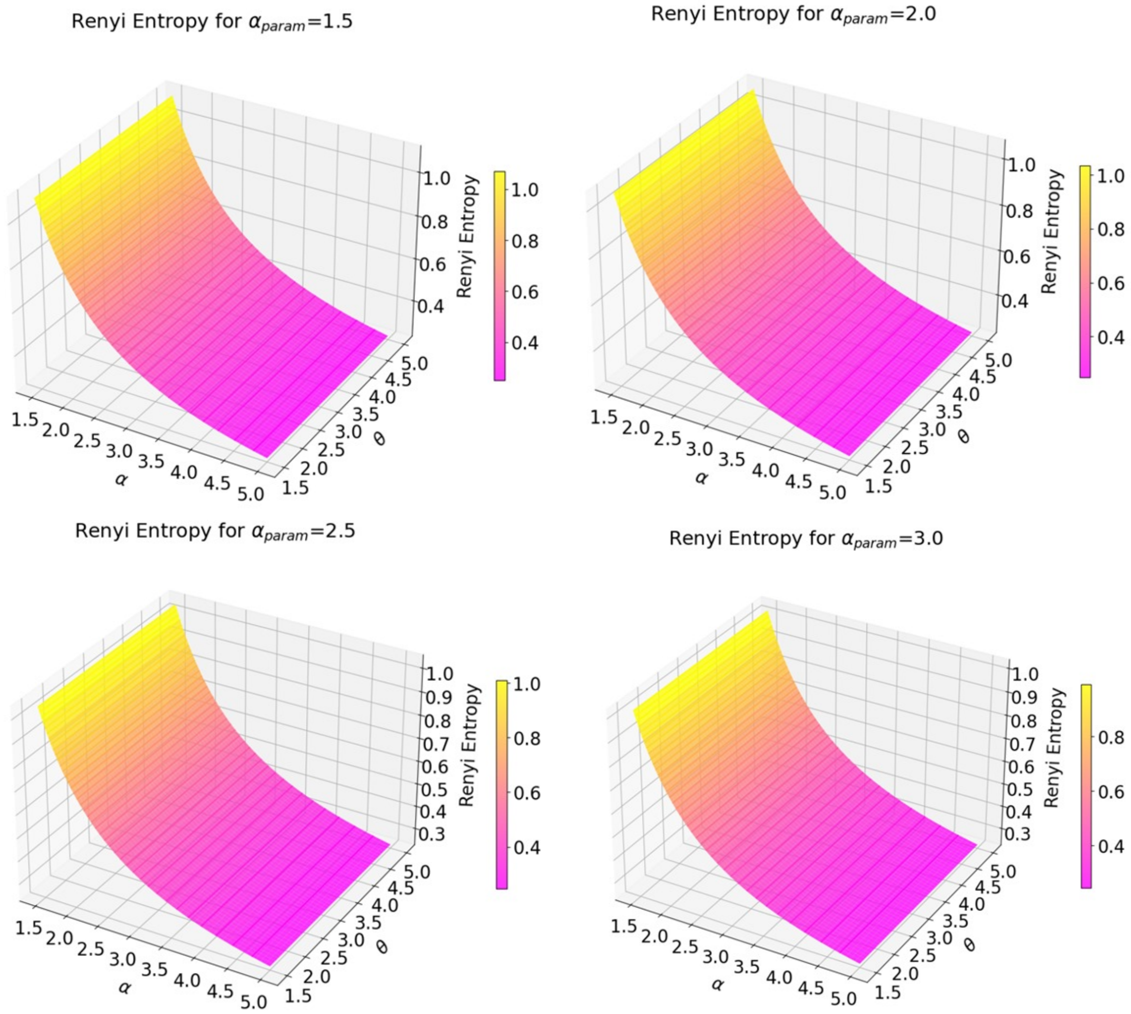


Figure 5. 3D plots of Rényi entropy for the NE-Beta distribution for various parameters.

Table 3. Rényi entropy for the NE-Beta distribution for various parameter combinations.

α	θ	Rényi entropy
1	1	1.7918
1	2	1.6094
1	3	1.5405
1	4	1.5041
1	5	1.4816
2	1	1.6094
2	2	1.3581
2	3	1.2528
2	4	1.1939
2	5	1.1562
3	1	1.5405
3	2	1.2528
3	3	1.1249
3	4	1.0508
3	5	1.0020
4	1	1.5041
4	2	1.1939
4	3	1.0508
4	4	0.9657
4	5	0.9085

The Tsallis entropy of the NE-Beta distribution can be represented using Eq (3.22) as

$$R_q(X) = \frac{\epsilon}{q-1}\left\{1 - \left(\frac{B(\alpha q,\ \theta q)}{\{B(\alpha,\ \theta)\}^q}\right)\right\}, \qquad \epsilon > 0, \quad q \neq 1. \tag{3.24}$$

Table 4 presents the Tsallis entropy for the NE-Beta distribution across various combinations of the parameters α, θ, and q. The entropy values provide insights into the uncertainty and diversity within the distribution. As α increases while holding θ constant, the Tsallis entropy generally decreases, indicating that higher α values lead to a more concentrated distribution. Similarly, for a fixed α, increasing θ also results in lower entropy values, further reflecting a trend toward more concentrated distributions. Additionally, for a given pair of α and θ, higher q values are associated with less negative entropy values, suggesting a reduction in the distribution's uncertainty. This behavior demonstrates the flexibility of the NE-Beta distribution in capturing a range of uncertainty levels by adjusting the parameters, making it suitable for various applications where modeling diversity and concentration is crucial.

Table 4. Tsallis entropy for the NE-Beta distribution for various parameter combinations.

α	θ	q	Tsallis entropy
1	1	1.1	-1.7859
1	1	2.1	-0.7934
1	1	3	-0.4728
1	1	4	-0.3283
1	1	5	-0.2496
1	2	1.1	-1.6272
1	2	2.1	-0.7638
1	2	3	-0.4661
1	2	4	-0.3267
1	2	5	-0.2492
1	3	1.1	-1.5673
1	3	2.1	-0.7510
1	3	3	-0.4629
1	3	4	-0.3258
1	3	5	-0.2490
2	1	1.1	-1.6272
2	1	2.1	-0.7638
2	1	3	-0.4661
2	1	4	-0.3267
2	1	5	-0.2492
2	2	1.1	-1.4039
2	2	2.1	-0.7123
2	2	3	-0.4519
2	2	4	-0.3225
2	2	5	-0.2479

Figure 6 illustrates the 3D plots of Tsallis entropy for the NE-Beta distribution for various parameter combinations.

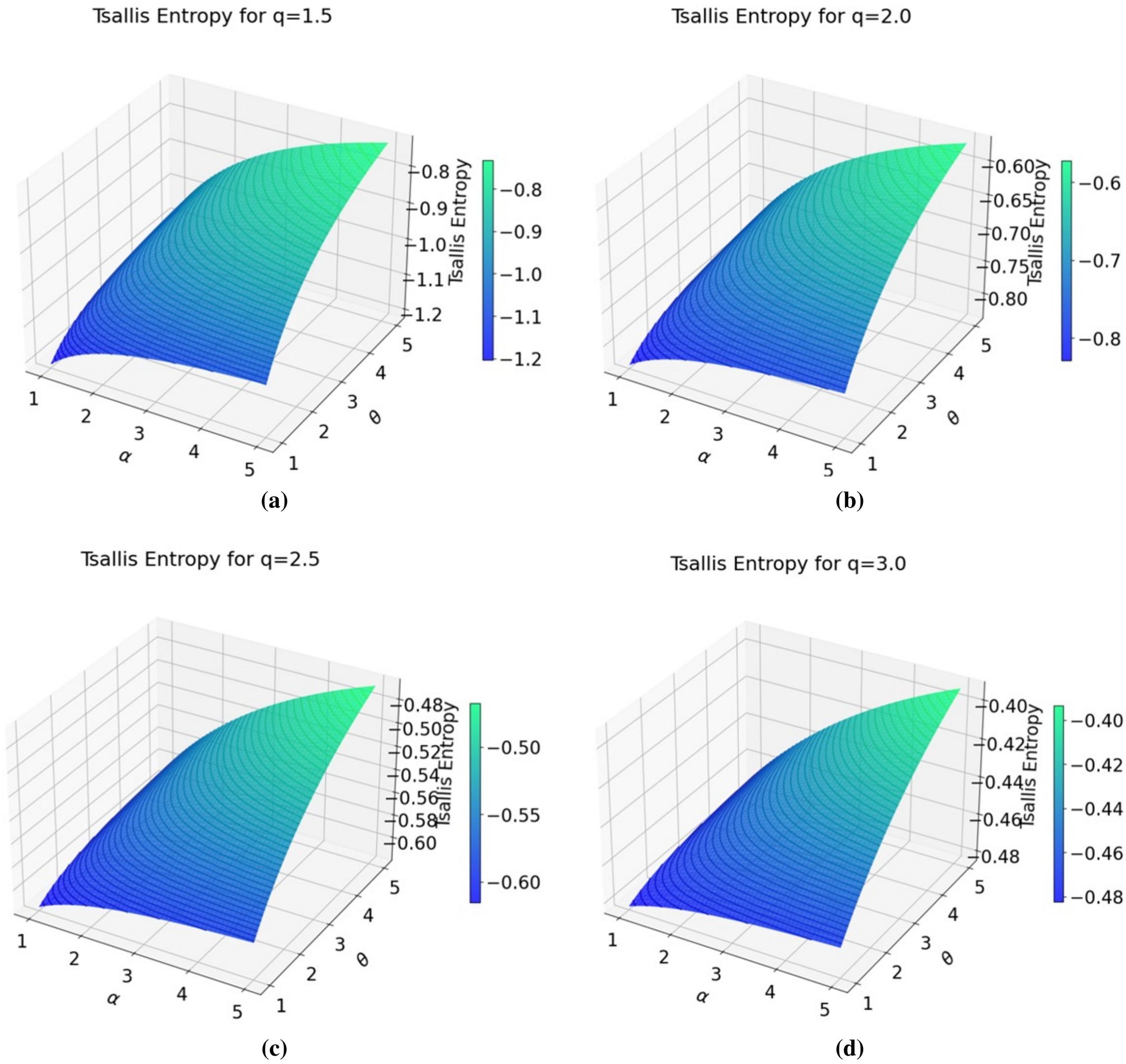


Figure 6. 3D plots of Tsallis entropy for the NE-Beta distribution for various parameter combinations.

4. Parameter estimation

This section describes the methods used to estimate the parameters of the NE-Beta distribution: least squares estimation (LSE), Cramér-von Mises (CVM), weighed least squares estimation (WLSE), and maximum likelihood estimation (MLE). The following are the key processes utilized to estimate the model parameters of the proposed distribution.

4.1. Parameter estimation using LSE

Let $x_1, x_2, x_3, \ldots, x_n$ represent random samples chosen from the NE-Beta distribution, with $F(x;\, \alpha, \theta)$ and $f(x;\, \alpha, \theta)$ denoting the CDF and PDF, respectively. Let $x_{(i)}$, for $i = 1, 2, \ldots, n$, be an ordered observation with $F(x_{(i)};\Omega)$ as the CDF, where $\Omega = (\alpha, \theta)^T$. The following should be considered when determining the parameters using the LSE. Let

$$\Phi_i(\Omega) = F(x_{(i)}; \alpha, \theta) - \left(\frac{i}{n+1}\right). \tag{4.1}$$

Given the CDF in Eq (2.2), Eq (4.1) can be expressed as follows:

$$L(\Omega) = \sum_{i=1}^{n} \Phi_i(\Omega) = \sum_{i=1}^{n} \left\{ \frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}(\alpha, \theta)}{B(\alpha, \theta)} - \left(\frac{i}{n+1}\right) \right\}. \tag{4.2}$$

The parameters for the NE-Beta distribution can now be estimated by minimizing Eq (4.2) for the parameters α and θ, as follows:

$$\frac{\partial L(\Omega)}{\partial \alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial \alpha} \left\{ \frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}(\alpha, \theta)}{B(\alpha, \theta)} \right\}, \tag{4.3}$$

and

$$\frac{\partial L(\Omega)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} \left\{ \frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}(\alpha, \theta)}{B(\alpha, \theta)} \right\}. \tag{4.4}$$

Solving for Eqs (4.3) and (4.4) yields estimates of the parameters of the NE-Beta distribution. This can be accomplished using numerical techniques such as Newton Raphson's methodology.

4.2. Parameter estimation using CVM

Suppose $x_1, x_2, x_3, \ldots, x_n$ to be random samples from the NE-Beta distribution, with the CDF and PDF as $F(x;\ \alpha, \theta)$ and $f(x;\ \alpha, \theta)$. Consider $x_{(i)}$ being an ordered observation, with the CDF as $F(x_{(i)};\Omega)$. The estimation based on the CVM can be performed using the following relations:

$$\psi_i(\Omega) = F(x_{(i)}; \alpha, \theta) - \left(\frac{2i-1}{2n}\right), \tag{4.5}$$

and

$$\begin{aligned} C(\Omega) &= \frac{1}{12n} + \sum_{i=1}^{n} \psi_i^2(\Omega) \\ &= \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}(\alpha, \theta)}{B(\alpha, \theta)} - \left(\frac{2i-1}{2n}\right) \right)^2. \end{aligned} \tag{4.6}$$

Now, the parameters of the NE-Beta distribution are estimated by minimizing Eq (4.6) for α and θ, as illustrated below.

$$\frac{\partial C(\Omega)}{\partial \alpha} = 2\sum_{i=1}^{n} \left(\frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}(\alpha, \theta)}{B(\alpha, \theta)} - \left(\frac{2i-1}{2n}\right) \right) \times \frac{\partial}{\partial \alpha} \left\{ \frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}(\alpha, \theta)}{B(\alpha, \theta)} \right\}, \tag{4.7}$$

and

$$\frac{\partial C(\Omega)}{\partial \theta} = 2\sum_{i=1}^{n} \left(\frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}(\alpha, \theta)}{B(\alpha, \theta)} - \left(\frac{2i-1}{2n}\right) \right) \times \frac{\partial}{\partial \theta} \left\{ \frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}(\alpha, \theta)}{B(\alpha, \theta)} \right\}. \tag{4.8}$$

Solving for Eqs (4.7) and (4.8) gives estimates of the parameters of the NE-Beta distribution. This can be performed using numerical techniques such as Newton Raphson's method.

4.3. Parameter estimation using WLSE

Consider the sample observations $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ with the CDF of the NE-Beta distribution as $F\left(x_{(i)}; \Omega\right)$. The parameter estimations using the WSLE technique are then attainable using the following relationships:

$$\varpi_i\left(\Omega\right) = \left(\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}\right)\left[F\left(x_{(i)}; \alpha, \theta\right) - \left(\frac{i}{n+1}\right)\right]^2. \tag{4.9}$$

Equation (4.9) can be provided as follows:

$$\omega\left(\Omega\right) = \sum_{i=1}^{n} \varpi_i\left(\Omega\right) = (n+1)^2\,(n+2) \sum_{i=1}^{n} \left\{\left(\frac{1}{i\,(n-i+1)}\right)\left[\frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}\left(\alpha,\, \theta\right)}{B\left(\alpha,\, \theta\right)} - \left(\frac{i}{n+1}\right)\right]^2\right\}. \tag{4.10}$$

To obtain the parameters of the NE-Beta distribution, minimize Eq (4.10) for α and θ, as follows:

$$\frac{\partial \omega\left(\Omega\right)}{\partial \alpha} = 2\,(n+1)^2\,(n+2) \sum_{i=1}^{n} \left\{\left(\frac{1}{i\,(n-i+1)}\right)\left[\frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}\left(\alpha,\, \theta\right)}{B\left(\alpha,\, \theta\right)} - \left(\frac{i}{n+1}\right)\right]\right\} \times \frac{\partial}{\partial \alpha}\left\{\frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}\left(\alpha,\, \theta\right)}{B\left(\alpha,\, \theta\right)}\right\}, \tag{4.11}$$

and

$$\frac{\partial \omega\left(\Omega\right)}{\partial \theta} = 2\,(n+1)^2\,(n+2) \sum_{i=1}^{n} \left\{\left(\frac{1}{i\,(n-i+1)}\right)\left[\frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}\left(\alpha,\, \theta\right)}{B\left(\alpha,\, \theta\right)} - \left(\frac{i}{n+1}\right)\right]\right\} \times \frac{\partial}{\partial \theta}\left\{\frac{B_{\frac{e^{x_{(i)}}}{1+e^{x_{(i)}}}}\left(\alpha,\, \theta\right)}{B\left(\alpha,\, \theta\right)}\right\}. \tag{4.12}$$

Solving for Eqs (4.11) and (4.12) yields parameter estimates for the NE-Beta distribution. We can achieve this by using numerical techniques, such as Newton Raphson's approach.

4.4. Parameter Estimation using MLE

Let $X_1, X_2, \ldots, X_n$ denote the random sample of size n which is drawn from the NE-Beta model with vector parameter $\Phi = (\alpha,\ \theta)^T$ and observed values $x_1, x_2 \ldots x_n$. The estimate of the parameter can be derived by obtaining the likelihood function of Eq (2.1) as:

$$L\left(\Phi\right) = \left(\frac{1}{B\left(\alpha,\, \theta\right)}\right)^n \prod_{i=1}^{n} \left(\frac{e^{\alpha x_i}}{\left(1+e^{x_i}\right)^{\alpha+\theta}}\right). \tag{4.13}$$

The log-likelihood function of Eq (4.13) is determined as

$$LL = n \log\left(\frac{\Gamma\left(\alpha+\theta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\theta\right)}\right) + \alpha \sum_{i=1}^{n} x_i - (\alpha+\theta) \sum_{i=1}^{n} \log\left(1+e^{x_i}\right). \tag{4.14}$$

The parameter estimations can be obtained by considering the partial derivatives of Eq (4.14) with respect to the parameters α and θ.

$$\frac{\partial LL}{\partial \alpha} = n\psi\left(\alpha+\theta\right) - n\psi\left(\alpha\right) + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \log\left(1+e^{x_i}\right), \tag{4.15}$$

and

$$\frac{\partial LL}{\partial \theta} = n\psi\left(\alpha+\theta\right) - n\psi\left(\theta\right) - \sum_{i=1}^{n} \log\left(1+e^{x_i}\right), \tag{4.16}$$

where $\psi(t) = \frac{d}{dt} \ln \Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}$. Taking the results of the partial derivatives of Eqs (4.15) and (4.16) to zero and solving them as nonlinear equations, we can obtain the MLEs for the parameters α and θ. However, Eqs (4.15) and (4.16) cannot be solved analytically, thus statistical software can be used to solve them numerically using iterative methods.

5. Application

5.1. Monte Carlo simulation

This section presents a simulation study to investigate the performances of the parameters of the NE-Beta distribution. The simulation study can be conducted utilizing the quantile function obtained in Eq (3.2) by employing various estimation approaches such as LSE, CVM, WLSE, and MLE. The simulation was carried out and the numerical results are evaluated using the following steps:

1) Generate a random sample of size n taken from the NE-Beta distribution.
2) The NE-Beta distribution parameters are computed using the LSE approach.
3) Steps 1 through 2 should be repeated 1000 times.
4) We compute the averages of the mean, bias, and mean squared error (MSE) for the NE-Beta distribution with various sample sizes (n = 5, 10, 20, 30, 50, and 100) and parameter values ($\alpha = 1$ and $\theta = 1$).

The CVM, WLSE, and MLE techniques, which are based on the LSE procedure, are used to determine the parameters of the NE-Beta distribution. Tables 5 and 6 present the simulation findings for $\alpha = 1$ and $\theta = 1$.

Table 5. The mean, bias, and MSE for the NE-Beta distribution.

			LSE			CVM	
n	Parameter	Mean	Bias	MSE	Mean	Bias	MSE
5	α	4.2542	3.2542	337.9146	6.2886	5.2886	657.6480
	θ	4.3120	3.3120	288.1962	6.1745	5.1745	498.9125
10	α	1.3409	0.3409	1.3983	1.7383	0.7383	7.1976
	θ	1.3377	0.3377	1.4357	1.7498	0.7498	10.1371
30	α	1.0283	0.0283	0.1058	1.1148	0.1148	0.1445
	θ	1.0384	0.0384	0.1117	1.1262	0.1262	0.1541
50	α	1.0097	0.0097	0.0575	1.0604	0.0604	0.0696
	θ	1.0184	0.0184	0.0602	1.0695	0.0695	0.0730
100	α	1.0044	0.0044	0.0251	1.0288	0.0288	0.0277
	θ	1.0068	0.0068	0.0261	1.0313	0.0313	0.0291

Table 6. The Mean, Bias as well as MSE for NE-Beta distribution.

		WLSE			MLE		
n	Parameter	Mean	Bias	MSE	Mean	Bias	MSE
5	α	1.6940	0.6940	17.6434	2.2972	1.2972	26.1344
	θ	1.5526	0.5526	11.8531	2.3243	1.3243	24.6300
10	α	1.1313	0.1313	1.0930	1.3540	0.3540	0.8149
	θ	1.1293	0.1293	1.0130	1.3595	0.3595	0.9090
30	α	1.0237	0.0237	0.0961	1.1000	0.1000	0.0959
	θ	1.0336	0.0336	0.0989	1.1084	0.1084	0.0971
50	α	1.0150	0.0150	0.0516	1.0574	0.0574	0.0478
	θ	1.0235	0.0235	0.0526	1.0630	0.0630	0.0485
100	α	1.0102	0.0102	0.0214	1.0292	0.0292	0.0196
	θ	1.0130	0.0130	0.0224	1.0316	0.0316	0.0208

The mean, bias, and MSE of the parameters using LSE, CVM, WLSE, and MLE are presented in Tables 5 and 6. These tables show that as the sample size increases, the means approach true parameter values, while the bias and MSE of each estimate drop and converge to zero.

Table 7 displays the MSE summary for the results from Tables 5 and 6. As the sample size is 5, the MSE utilizing the WLSE technique provided the optimal result regardless of the parameters α and θ, followed by MLE, LSE, and CVM. With a sample size of 10, the MLE approach produced lower MSE values for all parameters α and θ. As the sample size increased, all of the MSEs utilizing various estimation methodologies reduced and approached zero, but MLE produced a better estimate.

Table 7. MSE summary for different methods of estimation.

n	MSE (α, θ)	LSE	CVM	WLSE	MLE
5	α	337.9146	657.6480	17.6434	26.1344
	θ	288.1962	498.9125	11.8531	24.6300
10	α	1.3983	7.1976	1.0930	0.8149
	θ	1.4357	10.1371	1.0130	0.9090
30	α	0.1058	0.1445	0.0961	0.0959
	θ	0.1117	0.1541	0.0989	0.0971
50	α	0.0575	0.0696	0.0516	0.0478
	θ	0.0602	0.0730	0.0526	0.0485
100	α	0.0251	0.0277	0.0213	0.0196
	θ	0.0261	0.0291	0.0222	0.0208

Figures 7 and 8 demonstrate that the MSE for all estimation methods decreases with increasing sample size. This trend shows the increased accuracy and efficiency of estimation methods with larger datasets.

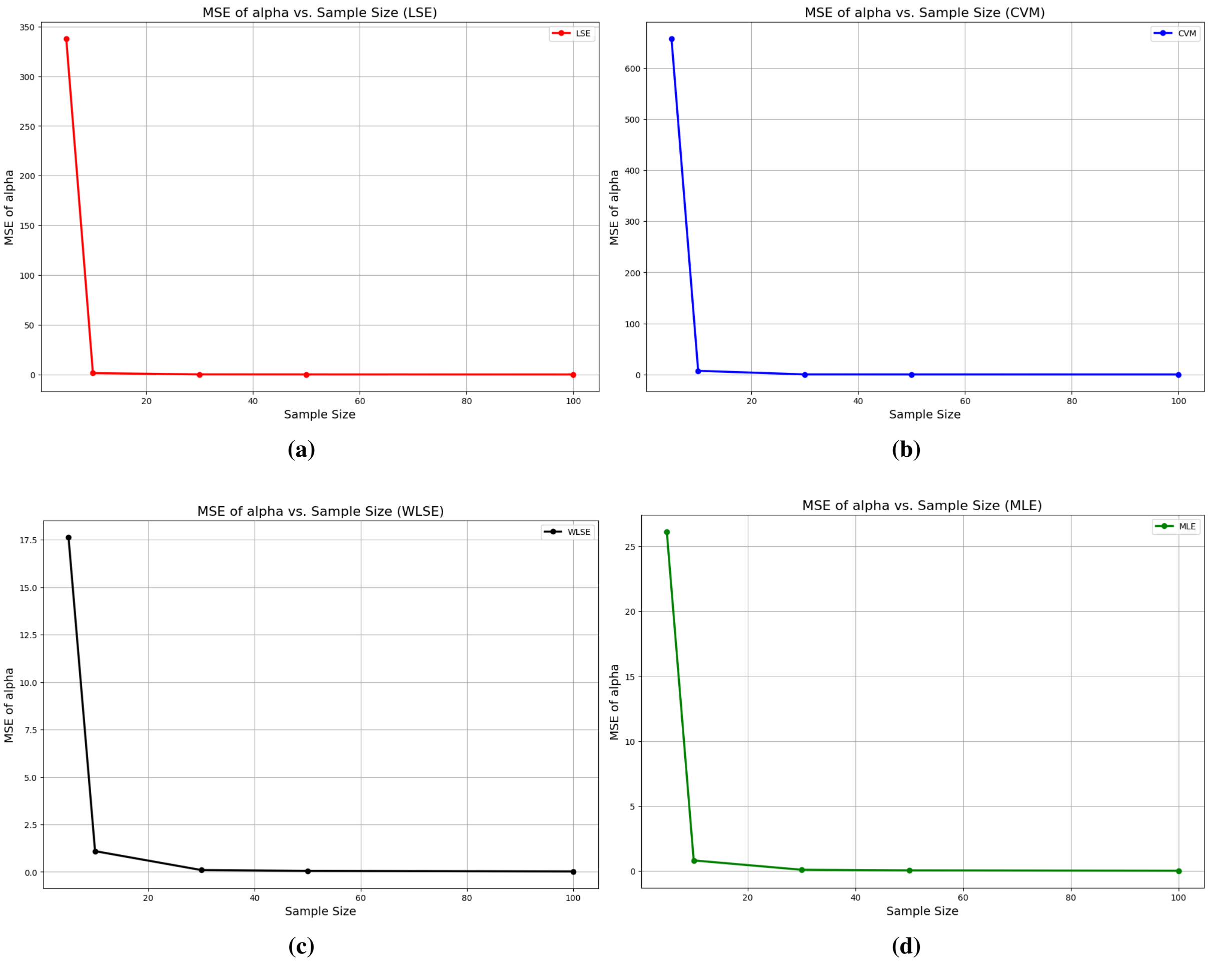
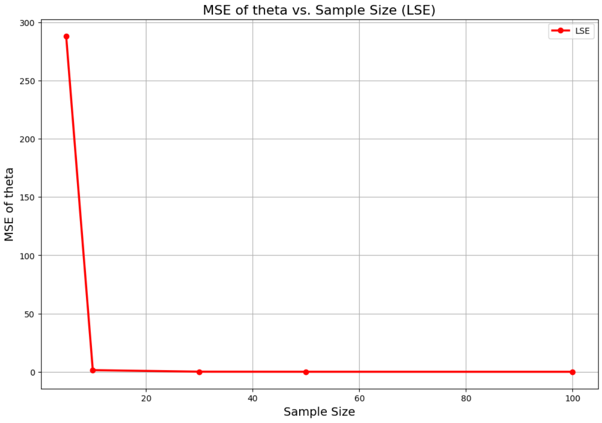
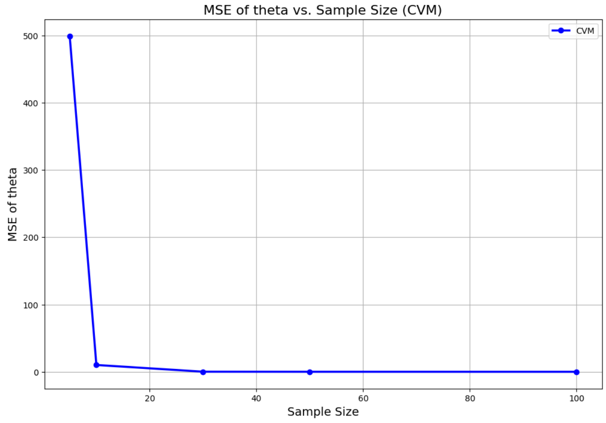


Figure 7. Performance of parameter α for the NE-Beta distribution.



(a)



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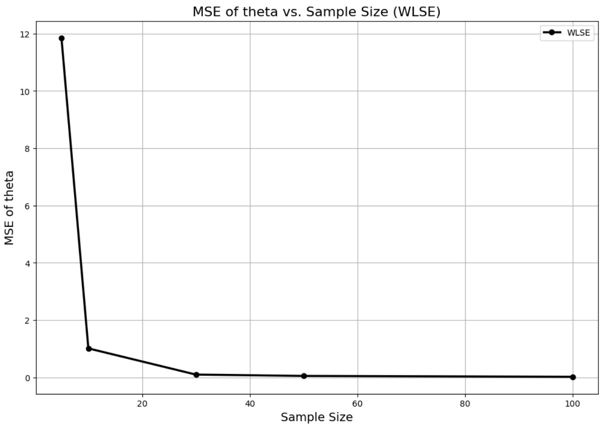

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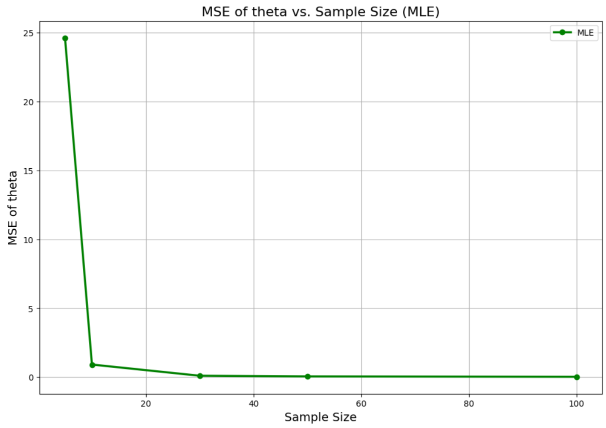


(d)

Figure 8. Performance of parameter θ for the NE-Beta distribution.

5.2. *Interdisciplinary data applications*

The flexibility and robustness of the proposed NE-Beta distribution allow it to be applied across various disciplines, offering significant advancements in fields as diverse as finance, biomedicine, engineering, and environmental science. In finance, the distribution is particularly suited for modeling the heavy tails and skewness observed in returns data, providing more accurate risk assessments and portfolio management strategies. In biomedicine, it can effectively model survival times and the distribution of biological measurements, aiding in the development of predictive models for patient outcomes and treatment efficacy. Engineering applications, particularly in reliability analysis, benefit from the distribution's ability to accurately model failure times, which is critical in the design and maintenance of systems. Finally, in environmental science, the distribution is used to analyze flood data, predict extreme events, and assess environmental risks, contributing to better disaster management and sustainable development practices. These diverse applications will emphasize the broad applicability and interdisciplinary impact of the NE-Beta distribution in addressing complex, real-world problems across multiple fields.

5.2.1. Numerical and graphical data presentations

Different data sets across various disciplines are considered in this study to check the accuracy of the proposed New Beta distribution with unbounded supports against the other existing distributions that has the same range of intervals. These include: the logistic, normal, log-gamma, extreme value, and student's t distributions. The data sets considered are presented as follows:

Financial data: The first data set covers the daily Nigerian exchange rate between the naira and USD (United States Dollar), and it spans the period from 4 January, 2021, to 1 February, 2024. This data is accessible at the CBN (Central Bank of Nigeria) using the following link: https://www.cbn.gov.ng/rates/exrate.html. The returns for the data were computed and considered by using the following relation:

$$Return_t = \left(\frac{Original\_Data_t}{Original\_Data_{t-1}} - 1\right) \times 100. \tag{5.1}$$

The return values show the percentage change in the exchange rate from one period to another. These returns are critical in determining the volatility and overall trend of the exchange rate over time.

Biomedical data: The second data was studied in [32]. It involves the survival times (in days) of 73 patients suffering from acute bone cancer.

Engineering data: The third data set consisted of failure times of 50 components (per 1000 h). This data was studied and analyzed in [33].

Hydrological data: The fourth data set included 141 observations of maximum flood data, which were investigated in [34].

Table 8 presents the descriptive statistics for the four datasets, illustrating several characteristics such as skewness and kurtosis, which differ across each dataset. These metrics reveal unique distributional patterns in the data, offering valuable insights into their behavior and guiding the selection of appropriate statistical models for analysis.

Table 8. Some descriptive summaries for the four datasets.

Statistic	Financial data	Biomedical data	Engineering data	Hydrological data
Observations	748	73	50	141
Minimum	-26.7696	0.090	0.0360	0.2083
Maximum	20.4322	86.010	15.0800	6.1876
Mean	-0.1311	3.755	3.3430	1.4706
Median	-0.0024	1.5700	1.4140	0.9714
Standard deviation	2.9148	10.5986	4.1815	1.5943
First quartile	-0.0416	0.9200	0.2075	0.5914
Third quartile	0.0000	2.750	4.4988	1.9698
Skewness	-1.5173	6.6596	1.3745	1.5943
Kurtosis	32.7223	47.3686	0.9229	2.2632

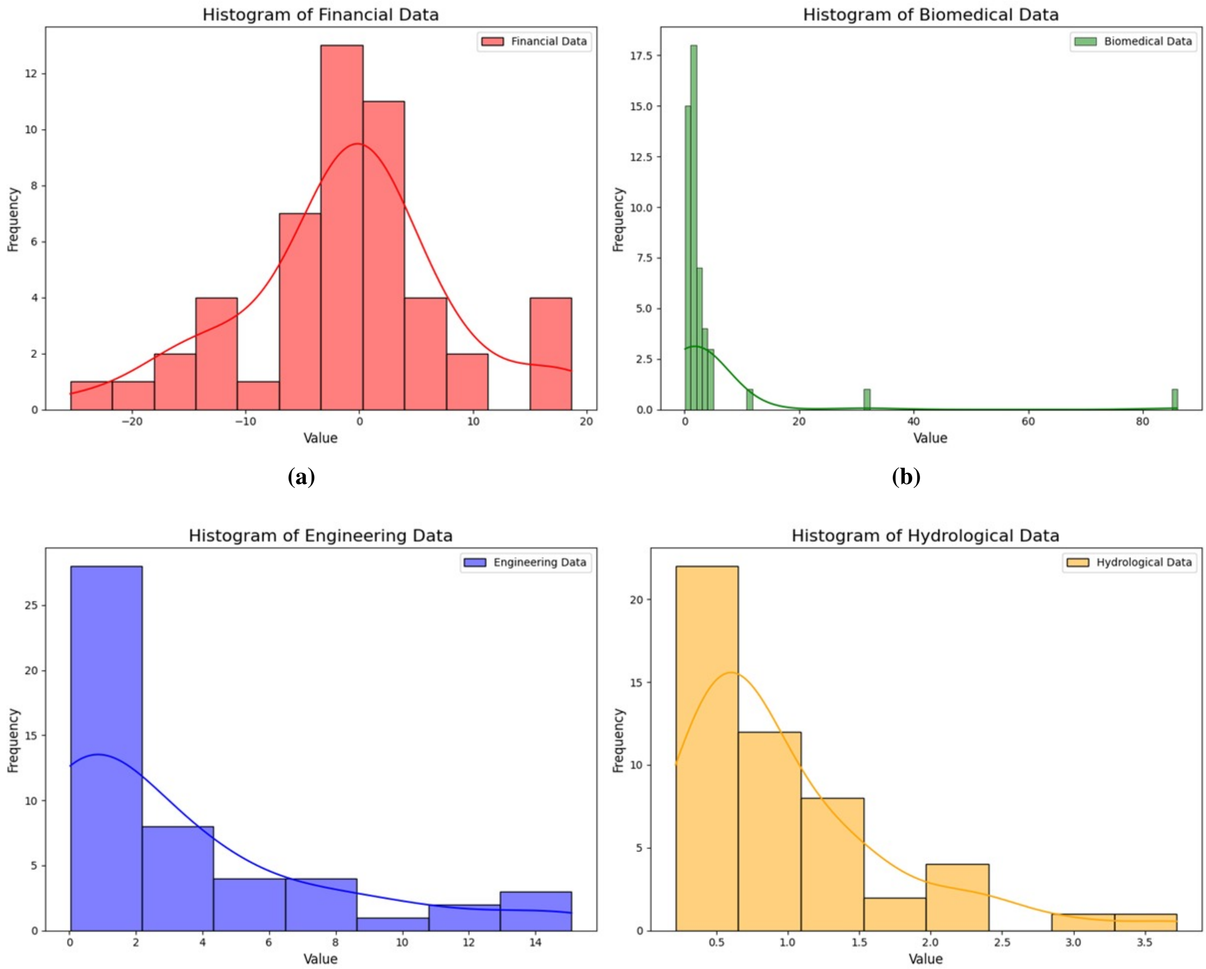
For the financial dataset, the summary statistics reveal a distribution with significant negative skewness and extremely high kurtosis. The returns range from a minimum of -26.77 to a maximum of 20.43, with a mean of -0.1311 and a median of -0.0024. The standard deviation of 2.9148 reflects considerable variability in returns. The negative skewness (-1.5173) indicates a distribution with a tail on the left, while the extremely high kurtosis (32.7223) suggests the presence of outliers and a peaked distribution.

In the biomedical dataset, the statistics show a highly right-skewed and sharply peaked distribution. The data ranges from a minimum of 0.090 days to a maximum of 86.010 days, with a mean of 3.755 and a median of 1.5700. The standard deviation of 10.5986 reflects significant variability in survival times. The extremely high positive skewness (6.6596) and kurtosis (47.3686) suggest a distribution heavily skewed to the right, with a concentration of values at the lower end and a long tail extending toward higher survival times.

For the engineering dataset, the dataset shows positive skewness and moderate kurtosis. The values range from a minimum of 0.0360 to a maximum of 15.0800, with a mean of 3.3430 and a median of 1.4140. The standard deviation of 4.1815 indicates a broad spread in the data. The positive skewness (1.3745) suggests a right-skewed distribution, while the moderate kurtosis (0.9229) indicates that the distribution is less peaked and has thinner tails compared to a normal distribution.

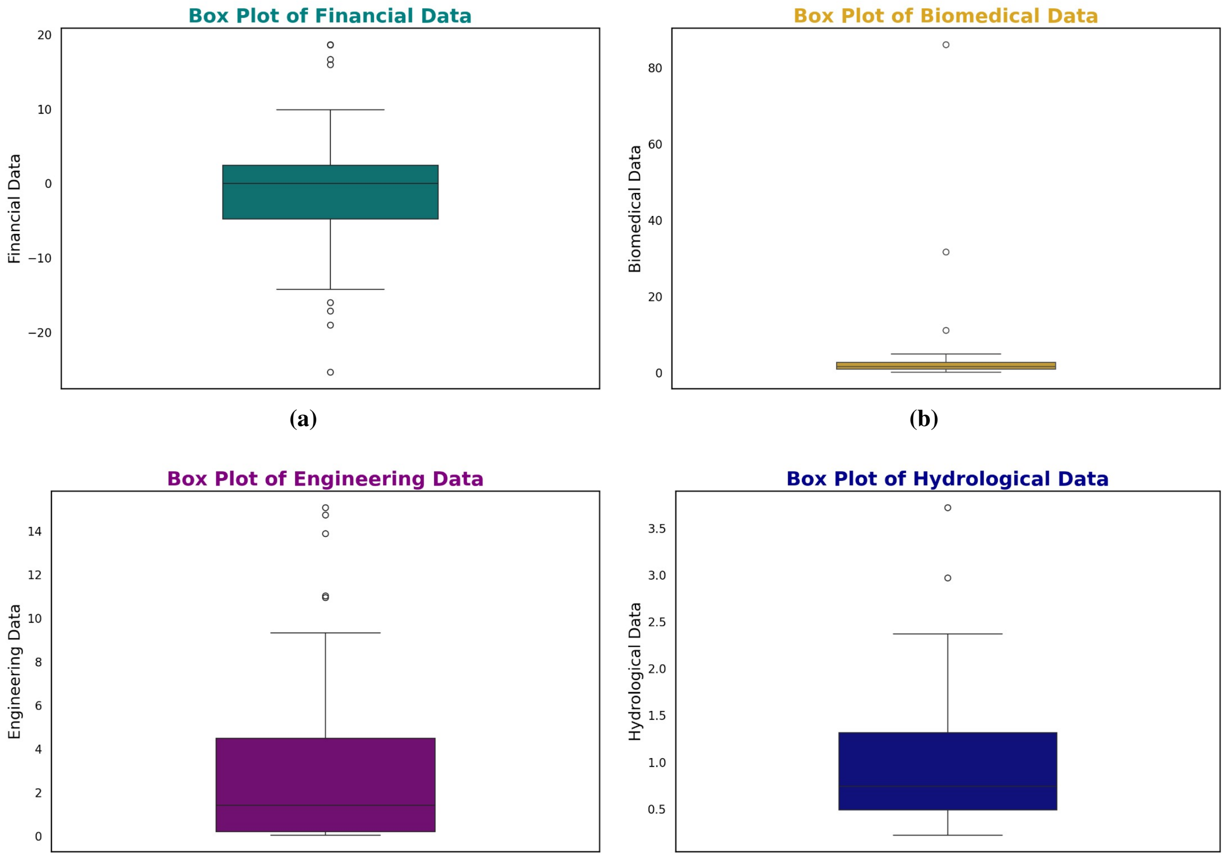
Lastly, the hydrological dataset exhibits positive skewness and moderately high kurtosis. The values range from a minimum of 0.2083 to a maximum of 6.1876, with a mean of 1.4706 and a median of 0.9714. The standard deviation is 1.5943. The positive skewness (1.5943) indicates a right-skewed distribution, while the moderately high kurtosis (2.2632) suggests a distribution with a somewhat peaked shape and fatter tails than a normal distribution, likely influenced by extreme flood events.

Figures 9 and 10 illustrate the histograms and box plots for the four datasets, respectively. These visualizations reveal distinct skewness and kurtosis behaviors, particularly within the biomedical dataset, offering insights into their distribution characteristics and suitability for specific statistical models. It is evident that the financial data exhibits negative skewness, while the biomedical, engineering, and hydrological datasets demonstrate positive skewness. The box plots further highlight that the datasets, especially the biomedical data, have high extreme values and pronounced peak tails. This indicates that the proposed distribution, as shown in Figure 2, is well-suited for modeling these datasets and serves as a robust model for interdisciplinary data analysis.



(c) (d)

Figure 9. Histogram plots of four datasets.



(a) **(b)** **(c)** **(d)**

Figure 10. Box plots of four datasets.

This study extends the applicability of the NE-Beta distribution by demonstrating its versatility across interdisciplinary datasets, including financial returns data. Unlike classical Beta distributions, which are limited to bounded data, the NE-Beta distribution accommodates real-valued data, allowing it to capture diverse characteristics such as negative skewness and extreme kurtosis. By modeling datasets from biomedical, engineering, hydrological, and financial domains, this study highlights the NE-Beta distribution's potential as a unified and flexible framework for diverse data characteristics, establishing its originality and practical relevance.

5.2.2. Model performance evaluation

This subsection presents a comparative analysis of the proposed distribution against other existing distributions using various datasets. The evaluation is based on goodness-of-fit measurements, specifically BIC (Bayesian information criterion), HQIC (Hannan-Quinn information criterion), CAIC (consistent Akaike information criterion), and AIC (Akaike information criterion). The distribution that yields the lowest values for these criteria will be considered the best fit for the datasets, demonstrating its superiority in modeling diverse types of data.

Tables 9–12 present the goodness-of-fit measures for the financial, biomedical, engineering, and hydrological datasets, respectively. These tables include the parameter estimates (denoted as Est-I and Est-II) for the NE-Beta distribution, alongside other comparator distributions, with their corresponding standard errors in parentheses. The goodness-of-fit metrics, log-likelihood (LL), AIC, BIC, HQIC, and

CAIC are also provided for each distribution.

As shown in Tables 9 through 12, the NE-Beta, logistic, normal, extreme value, and student's t distributions were evaluated. While all models demonstrated satisfactory performance, the NE-Beta distribution consistently yielded the lowest AIC, BIC, HQIC, and CAIC values, along with the highest log-likelihood values. This indicates that the NE-Beta distribution may be considered the optimal model for fitting these diverse datasets.

Table 9. Goodness-of-fit measures for financial data.

Model	Est-I	Est-II	LL	AIC	BIC	HQIC	CAIC
NE-Beta	0.9639 (0.0456)	1.0439 (0.0505)	-1494.8350	2993.6700	3002.9070	2997.2300	2993.6860
Logistic	1.21e-09 (0.0328)	0.5580 (0.0160)	-1509.9080	3023.8160	3033.0530	3027.3760	3023.8320
Normal	4.08e-07 (0.0916)	2.5010 (0.0523)	-1881.3610	3766.7220	3775.9590	3770.2820	3766.7380
Extreme value	1.3875 (0.1857)	4.8055 (0.0922)	-2158.5430	4321.0860	4330.3230	4324.6460	4321.1020
Student's t	0.1376 (0.0052)	— —	-7438.3660	14,878.7300	14,883.3500	14,880.5100	14,878.7400

Table 10. Goodness-of-fit measures for biomedical data.

Model	Est-I	Est-II	LL	AIC	BIC	HQIC	CAIC
NE-Beta	1.3219 (0.3137)	0.3012 (0.0481)	-128.3802	260.7604	264.5844	262.2166	261.0157
Logistic	2.6197 (0.5351)	2.1796 (0.2652)	-140.1824	284.3648	288.1888	285.8210	284.6201
Normal	3.3430 (0.5845)	4.1395 (0.4143)	-141.9751	287.9502	291.7742	289.4064	288.2055
Extreme value	5.6546 (0.7735)	5.1011 (0.4900)	-154.1306	312.2612	316.0852	313.7174	312.5165
Student's t	0.7408 (0.1482)	— —	-144.2953	290.5906	292.5026	291.3187	290.6739

Table 11. Goodness-of-fit measures for engineering data.

Model	Est-I	Est-II	LL	AIC	BIC	HQIC	CAIC
NE-Beta	1.7172 (0.3517)	0.2975 (0.0390)	-185.9118	375.8236	380.4045	377.6492	375.9950
Logistic	2.1117 (0.3951)	2.2213 (0.2443)	-221.3699	446.7398	451.3207	448.5654	446.9112
Normal	3.7552 (1.2108)	10.5258 (0.8746)	-275.4119	554.8238	559.4047	556.6494	554.9952
Extreme value	11.2870 (3.0630)	24.3240 (1.5320)	-328.5893	661.1786	665.7595	663.0042	661.3500
Student's t	1.0628 (0.1851)	— —	-203.4054	408.8108	411.1013	409.7236	408.8671

Table 12. Goodness-of-fit measures for hydrological data.

Model	Est-I	Est-II	LL	AIC	BIC	HQIC	CAIC
NE-Beta	4.6141 (0.5563)	1.4075 (0.1510)	-212.1446	428.2892	434.1867	430.6857	428.3762
Logistic	1.2606 (0.0936)	0.6468 (0.0466)	-224.5021	453.0042	458.9017	455.4007	453.0912
Normal	1.4706 (0.1059)	1.2570 (0.0749)	-232.3228	468.6456	474.5431	471.0421	468.7326
Extreme value	2.1805 (0.1486)	1.6537 (0.0906)	-272.4572	548.9144	554.8119	551.3109	549.0014
Student's t	2.0701 (0.3623)	— —	-296.4461	594.8922	597.8410	596.0905	594.9210

Moreover, the superior performance of the NE-Beta distribution across multiple datasets underscores its versatility and suitability as a robust model for interdisciplinary research. This makes the proposed distribution particularly valuable for analyzing complex data across various fields. However, the limited size and scope of the datasets used in the study may also impact the generalizability of the results, suggesting that caution should be exercised when applying the NE-Beta distribution to broader contexts.

While the present research focuses on a univariate scenario, it is noteworthy to recognize the importance of multivariate extensions, particularly in describing dependencies across multiple variables. Several studies have proposed multivariate extensions of probability distributions, including the multivariate Student-t process model, which efficiently handles dependent tail-weighted degradation data [35].

6. Conclusions

The NE-Beta distribution introduced in this study represents a new advancement in the family of Beta distributions by extending their applicability to both positive and negative data. This flexibility is absent in earlier versions such as Beta-I, Beta-II, and Beta-III. The NE-Beta distribution achieves nearly symmetric or right-skewed density functions and an increasing hazard function, making it highly suitable for practical applications that require these characteristics. Expressions for the quantile function, moments, and moment-generating function for the new model have been derived. The practical value of the NE-Beta distribution was confirmed through diverse data characteristics.

7. Future study

The Future study should be based on the following:

i. To investigate the applicability of various interval estimation methods, including asymptotic, bootstrap, and Bayesian approaches, to enhance statistical inference for the NE-Beta distribution.
ii. To delve into analogous extensions of the recently proposed NE-Beta distribution to improve its application in high-dimensional statistical modeling.

Author contributions

U. Panitanarak: Conceptualization, Writing original draft, Formal analysis, Software, Investigation, Methodology, Supervision; A. I. Ishaq: Validation, Resources, Writing-review and editing, Data curation, Methodology; A. A. Suleiman: Writing-review and editing, Software, Investigation, Methodology; H. Daud: Conceptualization, Formal analysis, Writing original draft, Software, Investigation, Methodology, Supervision. N. S. S. Singh: Validation, Resources, Writing-review and editing, Data curation, Methodology; A. U. Usman: Writing-review and editing, Investigation; N. Alsadat: Conceptualization, Formal analysis, Writing original draft, Software, Investigation, Methodology, Supervision; M. Elgarhy: Conceptualization, Formal analysis, Writing original draft, Software, Investigation, Methodology, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. M. K. Simon, *Probability distributions involving Gaussian random variables: A handbook for engineers and scientists*, New York: Springer, 2002. https://doi.org/10.1007/978-0-387-47694-0
2. A. N. O'Connor, *Probability distributions used in reliability engineering*, Reliability information analysis center, 2011.
3. A. I. Ishaq, U. Panitanarak, A. A. Abiodun, A. A. Suleiman, H. Daud, The generalized Odd Maxwell-Kumaraswamy distribution: Its properties and applications, *Contemp. Math.*, **5** (2024), 711–742. https://doi.org/10.37256/cm.5120242888
4. J. B. McDonald, Y. J. Xu, A generalization of the Beta distribution with applications, *J. Econometrics*, **66** (1995), 133–152. https://doi.org/10.1016/0304-4076(94)01612-4
5. M. Y. Sulaiman, W. M. Hlaing Oo, M. Abd Wahab, A. Zakaria, Application of Beta distribution model to Malaysian sunshine data, *Renew. Energy*, **18** (1999), 573–579. https://doi.org/10.1016/S0960-1481(99)00002-6
6. M. Graf, D. Nedyalkova, Modeling of income and indicators of poverty and social exclusion using the generalized Beta distribution of the second kind, *Rev. Income Wewlth*, **60** (2014), 821–842. https://doi.org/10.1111/roiw.12031
7. M. Raschke, Empirical behaviour of tests for the Beta distribution and their application in environmental research, *Stoch. Environ. Res. Risk Assess.*, **25** (2011), 79–89. https://doi.org/10.1007/s00477-010-0410-3
8. B. Kim, K. F. Reinschmidt, Probabilistic forecasting of project duration using Bayesian inference and the Beta distribution, *J. Constr. Eng. M.*, **135** (2009), 178–186. https://doi.org/10.1061/(ASCE)0733-9364(2009)135:3(178)
9. P. Chen, X. Xiao, Novel closed-form point estimators for the Beta distribution, *Stat. Theory Relat. F.*, **9** (2025), 12–33. http://doi.org/10.1080/24754269.2024.2419360
10. A. A. Suleiman, H. Daud, N. S. S. Singh, M. Othman, A. I. Ishaq, R. Sokkalingam, A novel Odd Beta prime-logistic distribution: Desirable mathematical properties and applications to engineering and environmental data, *Sustainability*, **15** (2023), 10239. https://doi.org/10.3390/su151310239
11. L. Chen, V. Singh, Generalized Beta distribution of the second kind for flood frequency analysis, *Entropy*, **19** (2017), 254. https://doi.org/10.3390/e19060254
12. A. A. Suleiman, H. Daud, N. S. S. Singh, A. I. Ishaq, M. Othman, A new Odd Beta prime-burr X distribution with applications to petroleum rock sample data and COVID-19 mortality rate, *Data*, **8** (2023), 143. https://doi.org/10.3390/data8090143
13. J. B. McDonald, Model selection: some generalized distributions, *Commun. Stat. Theor. M.*, **16** (1987), 1049–1074. https://doi.org/10.1080/03610928708829422
14. A. A. Suleiman, H. Daud, A. I. Ishaq, M. Othman, R. Sokkalingam, A. Usman, et al., The Odd Beta prime inverted Kumaraswamy distribution with application to COVID-19 mortality rate in Italy, *Eng. proc.*, **56** (2023), 218. https://doi.org/10.3390/ASEC2023-16310

15. A. A. Suleiman, H. Daud, M. Othman, N. S. S. Singh, A. I. Ishaq, R. Sokkalingam, et al., A novel extension of the Fréchet distribution: statistical properties and application to groundwater pollutant concentrations, *J. Data Sci. Insight.*, **1** (2023), 8–24.

16. A. K. Gupta, D. K. Nagar, Matrix-variate Beta distribution, *Int. J. Math. Math. Sci.*, **24** (2000), 449–459. https://doi.org/10.1155/S0161171200002398

17. L. Cardeno, D. K. Nagar, L. E. Sánchez, Beta type 3 distribution and its multivariate, *Tamsui Oxford J. Math. Sci.*, **21** (2005), 225–241.

18. A. I. Ishaq, A. A. Suleiman, H. Daud, N. S. S. Singh, M. Othman, R. Sokkaalingam, et al., Log-Kumaraswamy distribution: its features and applications, *Front. Appl. Math. Stat.*, **9** (2023), 1258961. https://doi.org/10.3389/fams.2023.1258961

19. A. I. Ishaq, A. A. Suleiman, A. Usman, H. Daud, R. Sokkalingam, Transformed log-burr III distribution: Structural features and application to milk production, *Eng. proc.*, **56** (2023), 322. https://doi.org/10.3390/ASEC2023-15289

20. A. Usman, A. I. Ishaq, A. A. Suleiman, M. Othman, H. Daud, Y. Aliyu, Univariate and bivariate log-topp-leone distribution using censored and uncensored datasets, *Comput. Sci. Math. Forum*, **7** (2023), 32. https://doi.org/10.3390/IOCMA2023-14421

21. A. A. Suleiman, H. Daud, A. A. Ishaq, M. Kayid, R. Sokkalingam, Y. Hamed, et al., A new Weibull distribution for modeling complex biomedical data, *J. Radiat. Res. Appl. Sci.*, **17** (2024), 101190. https://doi.org/10.1016/j.jrras.2024.101190

22. H. Daud, A. A. Suleiman, A. I. Ishaq, N. Alsadat, M. Elgarhy, A. Usman, et al., A new extension of the Gumbel distribution with biomedical data analysis, *J. Radiat. Res. Appl. Sci.*, **17** (2024), 101055. https://doi.org/10.1016/j.jrras.2024.101055

23. A. M. Almarashi, M. Elgarhy, A new muth generated family of distributions with applications, *J. Nonlinear Sci. Appl.*, **11** (2018), 1171–1184.

24. M. A. ul Haq, M. Elgarhy, The Odd Frechet-G family of probability distributions, *J. Stat. Appl. Probab.*, **7** (2018), 185–201. https://doi.org/10.18576/jsap/070117

25. S. Alkarni, A. Z. Afify, I. Elbatal, M. Elgarhy, The extended inverse Weibull distribution: properties and applications, *Complexity*, **2020** (2020), 3297693. https://doi.org/10.1155/2020/3297693

26. M. Elgarhy, M. Shakil, G. Kibria, Exponentiated Weibull-exponential distribution with applications, *Appl. Appl. Math.*, **12** (2017), 5.

27. S. A. Alyami, I. Elbatal, N. Alotaibi, E. M. Almetwally, H. M. Okasha, M. Elgarhy, Topp–Leone modified Weibull model: Theory and applications to medical and engineering data, *Appl. Sci.*, **12** (2022), 10431. https://doi.org/10.3390/app122010431

28. M. M. Salama, E. S. A. El-Sherpieny, A. E. A. Abd-elaziz, The length-biased weighted exponentiated inverted exponential distribution: properties and estimation, *Comput. J. Math. Stat. Sci.*, **2** (2023), 181–196. https://doi.org/10.21608/cjmss.2023.215674.1009

29. M. Shakil, T. Hussain, Z. U. Rehman, A. Khadim, M. Sirajo, J. N. Singh, et al., A Skew product distribution with applications, *Comput. J. Math. Stat.l Sci.*, **3** (2024), 432–453. https://doi.org/10.21608/cjmss.2024.283619.1049

30. M. N. Atchade, A. A. Agbahide, T. Otodji, M. J. Bogninou, A. M. Djibril, A new shifted Lomax-X family of distributions: Properties and applications to actuarial and financial data, *Comput. J. Math. Stat. Sci.*, **4** (2025), 41–71. http://doi.org/10.21608/cjmss.2024.307114.1066

31. J. N. Kapur, H. C. Saxena, *Mathematical statistics: S Chand & Co*, New Delhi, 1997.

32. H. S. Klakattawi, Survival analysis of cancer patients using a new extended Weibull distribution, *PLoS One*, **17** (2022), e0264229. https://doi.org/10.1371/journal.pone.0264229

33. A. G. Abubakari, C. C. Kandza-Tadi, E. Moyo, Modified beta inverse flexible Weibull extension distribution, *Ann. Data. Sci.*, **10** (2023), 589–617. https://doi.org/10.1007/s40745-021-00330-3

34. H. M. Alshanbari, O. H. Odhah, H. Al-Mofleh, Z. Ahmad, S. K. Khosa, A. H. El-Bagoury, A new flexible Weibull extension model: Different estimation methods and modeling an extreme value data, *Heliyon*, **9** (2023), e21704. https://doi.org/10.1016/j.heliyon.2023.e21704

35. A. Xu, G. Fang, L. Zhuang, C. Gu, A multivariate student-t process model for dependent tail-weighted degradation data, *IISE Trans.*, **2024** (2024), 1–17. http://doi.org/10.1080/24725854.2024.2389538

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