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Research article

Existence, uniqueness, continuous dependence on the data for the product of n-fractional integral equations in Orlicz spaces

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Abstract: In this manuscript, the measure of noncompactness, the fixed-point theorem, as well as fractional calculus, are used to carry out the analysis of the solvability of a product of n-quadratic Erdélyi-Kober ($\mathcal{EK}$) fractional-type integral equations in Orlicz spaces L_φ. Several qualitative properties of the solution for the studied problem are established, such as the existence, monotonicity, uniqueness, and continuous dependence on the data. We conclude with some examples that illustrate our hypothesis.

Keywords: fixed-point theorem ($\mathcal{FPT}$); Orlicz spaces; Erdélyi-Kober ($\mathcal{EK}$) fractional operator; measure of noncompactness.
Mathematics Subject Classification: 45G10, 47H10, 47H30, 47N20

1. Introduction

Various fields of science depend on the use of fractional calculus, such as economics, biology, electrical circuits, engineering, physics, traffic models, viscoelasticity, earthquakes, electrochemistry, and fluid dynamics [1–5].

The study of fractional integration in Orlicz spaces was started by O'Neil in 1965 (see [6]), but there are a few results considering the Erdélyi-Kober ($\mathcal{EK}$) operators in Orlicz spaces [7].

The Orlicz spaces weight on conditions that enable us to discuss ($\mathcal{EK}$) fractional integral operators with singular kernels or operators with strong nonlinearities (for example, exponential growth), and then discontinuous solutions are expected [8, 9]. This requires us to examine the solutions to the considered problem not in Lebesgue spaces but in specific Orlicz spaces. Moreover, these issues have important implications for the study of equivalent differential equations in Orlicz spaces or Sobolev-Orlicz spaces, which are closely related to these issues [10, 11]. This may be inspired by statistical physics and physics models [12, 13]. For example, the thermodynamics problem

$$\psi(\nu) + \int_I k(\nu, s) \cdot e^{\psi(s)}\, ds = 0$$

has exponential nonlinearity [14].

The product of two or more than two integral equations through ($\mathcal{EK}$) fractional operators can be applied effectively in neutron transport [15], the kinetic theory of gases [16], radiative transfer, and the traffic theory [17]. Consequently, it is worthwhile to investigate the product of more than two operators, so we propose to develop a mathematical basis for this theory, especially for fractional operators (see [18, 19]).

The purpose of this paper is to analyze and demonstrate the solutions to the integral equation

$$\psi(\nu) \;=\; g(\nu) + \prod_{i=1}^{n} f_i\Big(\nu,\ \frac{\beta_i h_{1_i}(\nu, \psi(\nu))}{\Gamma(\alpha_i)} \cdot \int_0^{\nu} \frac{s^{\beta_i - 1} h_{2_i}(s, \psi(s))}{(\nu^{\beta_i} - s^{\beta_i})^{1-\alpha_i}}\, ds\Big),\ 0 < \alpha_i < 1,\ \beta_i > 0, \tag{1.1}$$

for $\nu \in [0, \rho]$ in the Orlicz spaces L_φ.

It is generally impossible to determine the solutions to all nonlinear integral equations analytically, except using numerical methods. Consequently, indirect procedures should be used to obtain information about the qualitative behavior of integral equation solutions when there is no analytical expression for the solutions.

Therefore, we establish and present assumptions that allow us to solve and study the integral Eq (1.1) under general growth conditions in L_φ. As a result, we examine some qualitative properties of the solutions for the problem (1.1), such as existence, monotonicity, and uniqueness, as well as continuous dependence on the data in the spaces L_φ.

Our method covers and generalizes different types of fractional integrals that have been examined separately and encourages us to recall some of them.

In particular, the existence and the uniqueness theorems of continuous solutions of the SI models

$$\psi(\nu) = k\Big(p_1(\nu) + \int_0^{\nu} w_1(\nu - s)\psi(s)\, ds\Big)\Big(p_2(\nu) + \int_0^{\nu} w_2(\nu - s)\psi(s)\, ds\Big),\ \nu > 0 \tag{1.2}$$

were presented in [20]. As the model (1.2) shows the spread of diseases without permanent immunity and with discontinuous data functions, it is appropriate to examine it in Orlicz spaces "L_φ".

The authors in [19] generalized the model (1.2) and examined the existence and uniqueness theorems of the continuous solutions to the equation

$$\psi(\nu) = \prod_{i=1}^{n} \Big(g_i(\nu) + \int_a^{\nu} u_i(\nu, s, \psi(s))\, ds\Big),\ \nu \in [a, b].$$

The author in [21] demonstrated and presented some basic properties of the Riemann-Liouville type fractional integral operator and explored equation solutions

$$\psi(v) = f(v) + G(\psi)(v)\int_0^v \frac{(v-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,\psi(s))\,ds, \ \ 0<\alpha<1, \ v\in[0,d]$$

in Orlicz spaces L_φ.

The author in [22] demonstrated and studied fundamental features of the Hadamard-type fractional operator within L_φ-spaces and utilized them to solve the equation:

$$\psi(v) = G_2(\psi)(v) + \frac{G_1(\psi)(v)}{\Gamma(\alpha)}\int_1^v \Big(\log\frac{v}{s}\Big)^{\alpha-1}\frac{G_2(\psi)(s)}{s}\,ds, \ \ v\in[1,e], \ 0<\alpha<1.$$

The authors in [7] showed the basic characteristics of the Erdélyi-Kober fractional operators in Lebesgue and Orlicz spaces and used them to analyze the problem:

$$\psi(v) = f(v) + f_1(v,\psi(v)) + f_2\Big(v, \frac{\beta h_1(v,\psi(v))}{\Gamma(\alpha)}\cdot\int_0^v \frac{t^{\beta-1}h_2(s,\psi(s))}{(v^\beta - s^\beta)^{1-\alpha}}\,ds\Big), \ \ v\in[0,d],$$

where $0<\alpha<1$ & $\beta>0$ in the indicated spaces.

In [23], the author demonstrated some fixed-point theorems and applied them in solving the equation

$$\psi(v) = \prod_{i=1}^n\Big(g_i(v) + \int_0^v U_i(v,s,\psi(s))\,ds\Big), v\in[a,b]$$

in some ideal spaces ($L_p, p>1$ and Orlicz spaces L_φ).

In [24], the existence of solutions for the product of n-integral equations acting on distinct Orlicz spaces

$$\psi(v) = \prod_{i=1}^n\Big(g_i(v) + \lambda_i h_i(v,\psi(v))\int_a^b K_i(v,s)f_i(s,\psi(s))\,ds\Big), v\in I=[a,b]$$

were discussed in $L_\varphi(I)$, when the studied generating N-function verifies Δ', Δ_2, and Δ_3 conditions.

The existence and the uniqueness results for the abstract product of n-quadratic Hadamard-type integral equations

$$\psi(v) = \prod_{i=1}^n\Big(h_i(v) + G_{2_i}(\psi)(v) + \frac{G_{1_i}(\psi)(v)}{\Gamma(\alpha_i)}\int_0^v \log\Big(\frac{v}{s}\Big)^{\alpha-1}\frac{G_{3_i}(\psi)(v)}{s}\,ds\Big), \ v\in[1,e], \ \alpha_i\in(0,1)$$

were discussed in arbitrary Orlicz spaces L_φ [25], where $G_{j_i}, \ j=1,2,3$ are known operators.

Furthermore, the noncompactness measure ($\mathcal{MNC}$) and fixed-point hypothesis ($\mathcal{FPT}$) were used to study different types of quadratic integral equations in Orlicz spaces L_φ under various sets of assumptions [26, 27].

The current manuscript is motivated and induced by the extension and generalization of the results introduced in the previous literature to prove some qualitative properties of the solutions for a product of quadratic ($\mathcal{EK}$)-fractional integral Eq (1.1), including existence, monotonicity, uniqueness, as well as continuous dependence on the data in L_φ-spaces. We use the technique of ($\mathcal{MNC}$) concerning the fixed-point hypothesis ($\mathcal{FPT}$) and the theory of fractional calculus to obtain our findings. We present a few constructed examples that support and illustrate our findings.

2. Preliminaries

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{I} = [0, \rho] \subset \mathbb{R}^+ = [0, \infty)$. Denote the Young function (YF) by $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, where

$$\varphi(\nu) = \int_0^\nu q(s)ds, \quad \text{for } \nu \geq 0$$

and $q : \mathbb{R}^+ \to \mathbb{R}^+$ is neither identically zero nor infinite and an increasing and left-continuous function on $\mathbb{R}^+$. The pair (P, Q) is said to be a complementary pair of YF if $Q(\psi) = \sup_{z\geq 0}(\psi z - P(\psi))$.

The function φ is known as the N function when it is finitely valued and verifies $\lim_{\nu\to\infty} \frac{\varphi(\nu)}{\nu} = \infty$, $\lim_{\nu\to 0} \frac{\varphi(\nu)}{\nu} = 0$, and $\Big(\varphi(\nu) > 0 \text{ if } \nu > 0, \varphi(\nu) = 0 \iff \nu = 0\Big)$.

The *Orlicz space* $L_\varphi = L_\varphi(\mathbb{I})$ is the space of all measurable functions $\psi : \mathbb{I} \to \mathbb{R}$ with the norm

$$\|\psi\|_\varphi = \inf_{\lambda>0}\left\{ \int_{\mathbb{I}} \varphi\left(\frac{\psi(s)}{\lambda}\right) ds \leq 1 \right\} < \infty.$$

It is important to recall that for any YF φ, we have $\varphi(\nu + s) \leq \varphi(\nu) + \varphi(s)$ and $\varphi(k\nu) \leq k\varphi(\nu)$, where $\nu, s \in \mathbb{R}$, and $k \in [0, 1]$.

Assume that $E_\varphi(\mathbb{I})$ is the set of all bounded functions in $L_\varphi(\mathbb{I})$ that contain absolutely continuous norms.

Moreover, we obtain $L_\varphi = E_\varphi$, if φ satisfies the Δ_2-condition, i.e.,

$$(\Delta_2)\ \exists\, \omega,\ \nu_0 \geq 0 \ \text{ such that } \varphi(2\nu) \leq \omega\varphi(\nu),\ \nu \geq \nu_0.$$

It should be noted that the classical Lebesgue spaces $L_\varphi(\mathbb{I})$ shall be considered as a particular case of Orlicz spaces $L_{\varphi_p}(\mathbb{I})$ with the corresponding N-function $\varphi_p = s^p,\ p > 1$, satisfying the above Δ_2-condition.

Proposition 2.1. *[28] Let φ be a Young function; then for any $\alpha \in (0, 1)$ and $s \in \mathbb{R}^+$, the set*

$$\mathbb{P}(s) = \left\{ \epsilon > 0 :\ \frac{1}{\|\beta s^{\beta-1}\|_\varphi} \int_0^{s^\beta \sigma^{\frac{1}{1-\alpha}}} \varphi(u^{\alpha-1})\, du \leq \sigma^{\frac{1}{1-\alpha}} \right\} \neq \emptyset, \quad \sigma > 0, \beta > 0$$

is an increasing and continuous function with $\mathbb{P}(0) = 0$.

Lemma 2.2. *[12] Assume that the function $h(\nu, \psi) :\ \mathbb{I}\times\mathbb{R} \to \mathbb{R}$ verifies Carathéodory conditions i.e.,*

(1) It is measurable in ν for any $\psi \in \mathbb{R}$.
(2) It is continuous in ψ for almost all $\nu \in J$.

The superposition operator $F_h = h(\nu, \psi) : E_{\varphi_1} \to L_\varphi = E_\varphi$ is bounded and continuous if

$$|h(\nu, \psi)| \leq a(\nu) + b\varphi^{-1}(\varphi_1(\psi)),\ \ \psi \in \mathbb{R},\ \ \nu \in \mathbb{I},$$

where $b \geq 0$, $a \in L_\varphi$ and the N-function $\varphi(\psi)$ verifies the Δ_2-condition.

Lemma 2.3. *[29] Assume that φ, φ_1, and φ_2 are arbitrary different N-functions. The following given conditions are equivalent:*

(1) *For every function* $\psi_1 \in L_{\varphi_1}$ *and* $\psi_2 \in L_{\varphi_2}$, $\psi_1 \cdot \psi_2 \in L_\varphi$.
(2) $\exists\, k > 0$ *s.t. for all measurable* ψ_1, ψ_2 *on* $\mathbb{I}$, *we have* $\|\psi_1\psi_2\|_\varphi \le k\|\psi_1\|_{\varphi_1}\|\psi_1\|_{\varphi_2}$.
(3) $\exists\, C > 0$, $s_0 \ge 0$ *s.t. for all* $v, s \ge s_0$, $\varphi\left(\frac{sv}{C}\right) \le \varphi_1(s) + \varphi_2(v)$.
(4) $\limsup_{v\to\infty} \frac{\varphi_1^{-1}(v)\varphi_2^{-1}(v)}{\varphi(v)} < \infty$.

The set $S = S(\mathbb{I})$ is the set of Lebesgue measurable functions "meas." on $(\mathbb{I})$ connected with the metric

$$d(\psi, z) = \inf_{\rho>0}[\epsilon + meas\{s : |\psi(s) - z(s)| \ge \rho\}]$$

is a complete space. Additionally, the convergence in measure on $\mathbb{I}$ is equivalent to the convergence regarding d [30]. The compactness in S is known as "compactness in measure".

Lemma 2.4. *[26] Assume that* $\Psi \subset L_\varphi(\mathbb{I})$ *is a bounded set, and* $\exists$ *a family* $(\omega_r)_{0\le r\le d} \subset \mathbb{I}$ *s.t. meas* $\omega_r = r$ *for every* $r \in [0, d]$, *and for every* $\psi \in \Psi$,

$$\psi(s_1) \ge \psi(s_2), \quad (s_1 \in \omega_r,\ s_2 \notin \omega_r).$$

Then, Ψ *is compact in measure in* $L_\varphi(\mathbb{I})$.

Definition 2.5. *[31]* The Hausdorff measure of noncompactness ($\mathcal{MNC}$) $\mu_H(\Psi)$ for a bounded set $\emptyset \ne \Psi \subset L_\varphi$ is known as

$$\mu_H(\Psi) = \inf\{r > 0 :\ \exists\, Z \subset L_\varphi \text{ s.t. } \Psi \subset\ Z\ +\ \Omega_r\ \},$$

where $\Omega_r = \{\Psi \in L_\varphi(\mathbb{I}) : \|\psi\|_\varphi \le r\}$ is the ball centered at the origin with radius r.

Remark 2.6. The above Hausdorff ($\mathcal{MNC}$) $\mu_H(\Psi)$ is suited for studying our problem because it is related to the ball Ω_r and is equivalent to the following measure of equi-integrability in $L_\varphi(\mathbb{I})$. These are useful in employing Darbo's $\mathcal{FPT}$ to get our results.

Denote a measure of equi-integrability c of $\Psi \in L_\varphi(\mathbb{I})$ by :

$$c(\Psi) \ = \ \lim_{\epsilon\to 0}\ \sup_{measD\le\epsilon}\ \sup_{\psi\in\Psi}\|\psi\cdot\chi_D\|_\varphi,$$

where $\epsilon\ >\ 0$ and χ_A points to the characteristic function $A \subset \mathbb{I}$ (see [30, 32]).

Lemma 2.7. *[26, 32] Assume that* $\emptyset \ne \Psi \subset L_\varphi$ *is a bounded set and compact in measure. Then, we get:*

$$\mu_H(\Psi) = c(\Psi).$$

Theorem 2.8. *[31] (Darbo's* $\mathcal{FPT}$*) Assume that* $\emptyset \ne \Omega \subset L_\varphi$ *is a convex, bounded, and closed set and* $T : \Omega\ \to\ \Omega$ *is a continuous operator and satisfies the contraction condition, i.e.;*

$$\mu_H(T(\Psi))\ \le\ k\mu_H(\Psi),\ \ 0 \le k < 1$$

for any $\emptyset \ne \Psi \subset \Omega$. *Then, the map* T *has at least one fixed point in* Ω.

We give and present some concepts of the Erdélyi-Kober ($\mathcal{EK}$) fractional operator in L_φ-spaces.

Definition 2.9. *[1, 4, 33]* The Erdélyi-Kober fractional ($\mathcal{EK}$) integral operator J^{α}_{β}, $\alpha > 0, \beta > 0$, of a function $\psi(\nu)$ is known as

$$J^{\alpha}_{\beta}\psi(\nu) = \frac{\beta\nu^{-\beta\alpha}}{\Gamma(\alpha)} \int_0^{\nu} \frac{s^{\beta-1}\psi(s)}{(\nu^{\beta} - s^{\beta})^{1-\alpha}}\, ds = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1}\psi(\nu s^{\frac{1}{\beta}})\, ds. \tag{2.1}$$

Remark 2.10. There are several special cases covered by Definition 2.9, which include the following:

- Put $\beta = 1$, then the ($\mathcal{EK}$) operator (2.1) reduces to ($\mathcal{RL}$) fractional operators that were discussed in [1, 2]:
$$\nu^{\alpha} J^{\alpha}_1\psi(\nu) = \frac{1}{\Gamma(\alpha)} \int_0^{\nu} \frac{\psi(s)}{(\nu - s)^{1-\alpha}}\, ds.$$
- If $\beta = 0$, the ($\mathcal{EK}$) operator (2.1) reduces to the Hadamared operator
$$\lim_{\beta\to 0} J^{\alpha}_{\beta}\psi(\nu) = \frac{1}{\Gamma(\alpha)} \int_0^{\nu} \Big(\log \frac{\nu}{s}\Big)^{\alpha-1} \frac{\psi(s)}{s}\, ds$$
that was discussed in [4, 22].
- If and $\beta = 1$, and $\alpha = 1$, then the ($\mathcal{EK}$) operator (2.1) reduces to the Hardy-Littlewood (Cesaro) operator
$$J^1_1\psi(\nu) = \frac{1}{\nu} \int_0^{\nu} \psi(s)\, ds,$$
that was discussed in [34].
- Put $\beta = 2$, then ($\mathcal{EK}$) operator (2.1) reduces to the $\mathcal{EK}$) fractional integral operator J^{α}_2 (Sneddon [35]):
$$J^{\alpha}_2\psi(\nu) = \frac{2\nu^{-2\alpha}}{\Gamma(\alpha)} \int_0^{\nu} \frac{\psi(s)}{(\nu^2 - s^2)^{1-\alpha}}\, s\, ds.$$

Proposition 2.11. *[7] For $\alpha > 0, \beta > 0$, we have:*

(1) The operator J^{α}_{β} maps nonnegative and therefore a.e. nondecreasing functions to functions with the same properties.

(2) If φ is an N-function and (P, Q) represents a complementary pair of N-functions, where P verifies $\int_0^{\nu^{\beta}} P(t^{\alpha-1})\, dt < \infty$, $\alpha \in (0, 1)$, $\beta > 0$. Then, $J^{\alpha}_{\beta} : L_Q(\mathbb{I}) \to L_{\varphi}(\mathbb{I})$ is continuous, satisfying

$$||\nu^{\beta\alpha} J^{\alpha}_{\beta}\psi||_{\varphi} \leq \frac{2}{\Gamma(\alpha)} ||k||_{\varphi} ||\psi||_Q,$$

where

$$k(\nu) = \frac{\sigma^{\frac{1}{\alpha-1}}}{||\beta s^{\beta-1}||_{\varphi}} \int_0^{\nu^{\beta}\sigma^{\frac{1}{1-\alpha}}} \varphi(t^{\alpha-1})\, dt \in E_{\varphi}(\mathbb{I}), \quad \sigma = \frac{\epsilon}{||\beta s^{\beta-1}||_{\varphi}}.$$

3. Main results

Equation (1.1) can take the form:

$$\psi = B(\psi) = g + \prod_{i=1}^{n} F_{f_i} U_i(\psi),$$

where

$$U_i(\psi) = F_{h_{1_i}}(\psi) \cdot A_i(\psi), \text{ and } A_i(\psi)(\nu) = \nu^{\beta_i \alpha_i} J^{\alpha_i}_{\beta_i} F_{h_{2_i}}(\psi)(\nu),$$

such that ${}^{\nu^\beta\alpha}J^{\alpha}_{\beta}$ is defined in Definition 2.9 and $F_{f_i}, F_{h_{j_i}}, (j = 1, 2)$ are known as the superposition operators.

Next, we will demonstrate and study the existence theorems in L_φ.

3.1. Existence of solutions.

The presented case permits us to utilize some general conditions for the studied operators.

Theorem 3.1. *For $i = 1, \cdots, n$, assume that $\varphi_i, \varphi_{1_i}, \varphi_{2_i}$, and φ are N-functions and (P_i, Q_i) is a complementary pair of N-functions, in which $Q_i, \varphi_i, \varphi_{1_i}$ verify the Δ_2 condition and $\int_0^{\nu^\beta} P_i(t^{\alpha-1})\,dt < \infty$, $\alpha_i \in (0, 1), \beta_i > 0$, and that:*

(G1) $\forall\ \psi_i \in L_{\varphi_i},\ \exists K \geq 0,$ *s.t.* $\left\| \prod_{i=1}^n \psi_i \right\|_\varphi \leq K \prod_{i=1}^n \|\psi_i\|_{\varphi_i}$.

(G2) $\exists\ k_{1_i} > 0$ *s.t. for* $\psi_1 \in L_{\varphi_{1_i}}(\mathbb{I})$ *and* $\psi_2 \in L_{\varphi_{2_i}}(\mathbb{I})$ *we obtain* $\|\psi_1\psi_2\|_{\varphi_i} \leq k_{1_i}\|\psi_1\|_{\varphi_{1_i}}\|\psi_2\|_{\varphi_{2_i}}$.

(C1) $g \in E_\varphi(\mathbb{I})$ *is a.e. nondecreasing on* $\mathbb{I}$.

(C2) $h_{1_i}, h_{2_i}, f_i : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ *satisfy Carathéodory conditions, and* $(s, \psi) \to f_i(s, \psi)$, $(s, \psi) \to h_{1_i}(s, \psi)$, $(s, \psi) \to h_{2_i}(s, \psi)$ *are nondecreasing.*

(C3) $\exists\ e_i, d_{1_i}, d_{2_i}$ *and functions* $b_{1_i} \in E_{\varphi_{1_i}}(\mathbb{I})$, $b_{2_i} \in E_{Q_i}(\mathbb{I})$, *and* $a_i \in E_{\varphi_i}(\mathbb{I})$ *s.t.*

$$|f_i(s, \psi)| \leq a_i(s) + e_i\|\psi\|_{\varphi_i}, \text{ and}$$

$$|h_{1_i}(s, \psi)| \leq b_{1_i}(s) + d_{1_i}\varphi_{1_i}^{-1}\Big(\varphi(\psi)\Big), \ |h_{2_i}(s, \psi)| \leq b_{2_i}(s) + d_{2_i}Q_i^{-1}\Big(\varphi(\psi)\Big).$$

(C4) Assume that for a.e. $\nu \in \mathbb{I}$, $\exists\ \epsilon > 0$, *s.t.*

$$k_i(\nu) = \frac{\sigma_i^{\frac{1}{\alpha_i - 1}}}{\|\beta_i s^{\beta_i - 1}\|_{\varphi_i}} \int_0^{\nu^\beta \sigma_i^{\frac{1}{1-\alpha_i}}} P_i(u^{\alpha_i - 1})\,du \in E_{\varphi_{2_i}}(\mathbb{I}), \ \sigma_i = \frac{\epsilon}{\|\beta_i s^{\beta_i - 1}\|_{\varphi_i}}.$$

(C5) Assume that, $\exists\ r > 0$ *on* $I_0 = [0, \rho_0] \subset \mathbb{I}$ *verifying*

$$\prod_{i=1}^n \left(\|a_i\|_{\varphi_i} + \frac{2e_i k_{1_i}}{\Gamma(\alpha_i)} \|k_i\|_{\varphi_{2_i}} \Big(\|b_{1_i}\|_{\varphi_{1_i}} + d_{1_i} r \Big)\Big(\|b_2\|_{Q_i} + d_{2_i} r \Big) \right) \leq \frac{r - \|g\|_\varphi}{K}.$$

and

$$K r^n \prod_{i=1}^n \left(\frac{2e_i k_1 i d_{1_i}}{\Gamma(\alpha_i)} \|k_i\|_{\varphi_{2_i}} \Big(\|b_{2_i}\|_{Q_i} + d_{2_i} \cdot r \Big) \right) < 1.$$

Then, there exists a.e. nondecreasing solution $\psi \in E_\varphi(I_0)$ *of (1.1) on* $I_0 \subset \mathbb{I}$.

Proof. **Step I.** We will show that the operator B is well defined on E_φ i.e., $B : E_\varphi(\mathbb{I}) \to E_\varphi(\mathbb{I})$ is continuous.

For $i = 1, \cdots, n$, Lemma 2.2 and assumptions (C2), (C3) imply that $F_{h_{1_i}} : E_\varphi(\mathbb{I})) \to L_{\varphi_{1_i}}(\mathbb{I})$, $F_{h_{2_i}} : E_\varphi(\mathbb{I})) \to L_{Q_i}(\mathbb{I})$ and $F_{f_i} : E_\varphi(\mathbb{I})) \to E_\varphi(\mathbb{I}))$ are continuous. Proposition 2.11_2 gives us that $A_i =$

${}^{\nu}\!\!{}^{\beta_i\alpha_i}J^{\alpha_i}_{\beta_i}F_{h_{2_i}}\ :\ E_\varphi(\mathbb{I}) \to E_{\varphi_{2_i}}(\mathbb{I})$ and are continuous. Assumption (G2) implies that $U_i : E_\varphi(\mathbb{I}) \to E_{\varphi_i}(\mathbb{I})$ and by assumptions $(C1)$, and $(G1)$ $B : E_\varphi(\mathbb{I}) \to E_\varphi(\mathbb{I})$ and are continuous.

Step II. We should construct the ball $\Omega_r(E_\varphi) = \{\psi \in L_\varphi : \|\psi\|_\varphi \leq r\}$, where r is given in assumption (C5) for the operator B acts on.

For arbitrary $\psi \in \Omega_r(E_\varphi)$ and by using Proposition 2.11_2 and our assumptions, we get

$$\begin{aligned}
\|F_{f_i}U_i(\psi)\|_{\varphi_i} &\leq \|a_i\|_{\varphi_i} + e_i\|U_i(\psi)\|_{\varphi_i}\\
&\leq \|a_i\|_{\varphi_i} + e_ik_{1_i}\|F_{h_{1_i}}(\psi)\|_{\varphi_{1_i}}\|A_i(\psi)\|_{\varphi_{2_i}}\\
&\leq \|a_i\|_{\varphi_i} + e_ik_{1_i}\Big\|b_{1_i} + d_{1_i}\varphi_{1_i}^{-1}\Big(\varphi(|\psi|)\Big)\Big\|_{\varphi_{1_i}} \cdot \|\nu^{\beta_i\alpha_i}J^{\alpha_i}_{\beta_i}F_{h_{2_i}}(\psi)\|_{\varphi_{2_i}}\\
&\leq \|a_i\|_{\varphi_i} + e_ik_{1_i}\Big(\|b_{1_i}\|_{\varphi_{1_i}} + d_{1_i}\Big\|\varphi_{1_i}^{-1}\Big(\varphi(|\psi|)\Big)\Big\|_{\varphi_{1_i}}\Big)\\
&\times \frac{2}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{2_i}\|_{Q_i} + d_{2_i}\Big\|Q_i^{-1}\Big(\varphi(|\psi|)\Big)\Big\|_{Q_i}\Big)\\
&\leq \|a_i\|_{\varphi_i} + + \frac{2e_ik_{1_i}}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{1_i}\|_{\varphi_{1_i}} + d_{1_i}\|\psi\|_\varphi\Big)\Big(\|b_{2_i}\|_{Q_i} + d_{2_i}\|\psi\|_\varphi\Big)\\
&\leq \|a_i\|_{\varphi_i} + \frac{2e_ik_{1_i}}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{1_i}\|_{\varphi_{1_i}} + d_{1_i}r\Big)\Big(\|b_2\|_{Q_i} + d_{2_i}r\Big),
\end{aligned}$$

where $\Big\|\varphi_{1_i}^{-1}\Big(\varphi(|\psi|)\Big)\Big\|_{\varphi_{1_i}} \leq \|\psi\|_\varphi$ and $\Big\|Q_i^{-1}\Big(\varphi(\psi)\Big)\Big\|_{Q_i} \leq \|\psi\|_\varphi$. Recalling assumptions (G1) and (C1), we have

$$\begin{aligned}
\|B(\psi)\|_\varphi &\leq \|g\|_\varphi + \Big\|\prod_{i=1}^n F_{f_i}U_i(\psi)\Big\|_\varphi\\
&\leq \|g\|_\varphi + K\prod_{i=1}^n\Big(\|a_i\|_{\varphi_i} + \frac{2e_ik_{1_i}}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{1_i}\|_{\varphi_{1_i}} + d_{1_i}r\Big)\Big(\|b_2\|_{Q_i} + d_{2_i}r\Big)\Big) \leq r.
\end{aligned}$$

Therefore, assumption (C5) indicates that $B : \Omega_r(E_\varphi) \to E_\varphi$ is continuous.

Step III. We should construct subset $\omega_r \subset \Omega_r$, and investigate the properties of ω_r.

Assume that $\omega_r \subset \Omega_r$ contains all a.e. monotonic (nondecreasing) functions on I_0. The set $\emptyset \neq \omega_r$ is bounded, closed, compact in measure, and convex in $L_\varphi(I_0)$ [27].

Step IV. We shall check the monotonicity and continuity of the operator B on ω_r.

Take $\psi \in \omega_r$, then ψ is a.e. nondecreasing on I_0 and, consequently, for $i = 1, 2, \cdots, n$, the operators $F_{f_i}, F_{h_{1_i}}$ and $F_{h_{2_i}}$ are also a.e. nondecreasing on I_0. By Proposition 2.11_1, A_i is a.e. nondecreasing on I_0, then $U_i = F_{h_{1_i}}A_i$ are also, a.e., nondecreasing. Using assumptions (C1) and (G1), we obtain $B : \omega_r \to \omega_r$ is continuous.

Step V. Now, we show that B satisfies the contraction condition with respect to $\mathcal{MNC}\ \mu_H$.

Suppose there is a set $D \subset I_0$, with meas $D \leq \varepsilon,\ \varepsilon > 0$. Therefore, for $\psi \in \Psi$ and $\emptyset \neq \Psi \subset \omega_r$, we have:

$$\begin{aligned}
\|F_{f_i}U_i(\psi)\cdot\chi_D\|_{\varphi_i} &\leq \|a_i\cdot\chi_D\|_{\varphi_i} + e_i\|F_{h_{1_i}}A_i(\psi)\cdot\chi_D\|_{\varphi_i}\\
&\leq \|a_i\cdot\chi_D\|_{\varphi_i} + e_ik_{1_i}\|F_{h_{1_i}}(\psi)\cdot\chi_D\|_{\varphi_{1_i}}\|A_i(\psi)\cdot\chi_D\|_{\varphi_{2_i}}\\
&\leq \|a_i\cdot\chi_D\|_{\varphi_i} + e_ik_{1_i}\Big\|\Big(b_{1_i} + d_{1_i}\varphi_{1_i}^{-1}\Big(\varphi(|\psi|)\Big)\Big)\cdot\chi_D\Big\|_{\varphi_{1_i}} \cdot \Big\|\nu^{\beta_i\alpha_i}J^{\alpha_i}_{\beta_i}F_{h_{2_i}}(\psi)\Big\|_{\varphi_{2_i}}
\end{aligned}$$

$$\leq \|a_i\cdot\chi_D\|_{\varphi_i}+e_ik_{1_i}\Big(\|b_{1_i}\cdot\chi_D\|_{\varphi_{1_i}}+d_{1_i}\Big\|\varphi_{1_i}^{-1}\Big(\varphi(|\psi|)\Big)\cdot\chi_D\Big\|_{\varphi_{1_i}}\Big)$$
$$\times\frac{2}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{2_i}\|_{Q_i}+d_{2_i}\Big\|Q_i^{-1}\Big(\varphi(|\psi|)\Big)\Big\|_{Q_i}\Big)$$
$$\leq \|a_i\cdot\chi_D\|_{\varphi_i}+\frac{2e_ik_{1_i}}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{1_i}\cdot\chi_D\|_{\varphi_{1_i}}+d_{1_i}\|\psi\cdot\chi_D\|_{\varphi}\Big)\Big(\|b_{2_i}\|_{Q_i}+d_{2_i}\cdot r\Big).$$

Therefore,

$$\|B\cdot\chi_D\|_\varphi\leq\|g\cdot\chi_D\|_\varphi+\Big\|\prod_{i=1}^n F_{f_i}U_i(\psi)\Big\|_\varphi$$
$$\leq\|g\cdot\chi_D\|_\varphi+K\prod_{i=1}^n\Bigg(\|a_i\cdot\chi_D\|_\varphi+\frac{2e_ik_{1_i}}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{1_i}\cdot\chi_D\|_{\varphi_{1_i}}+d_{1_i}\|\psi\cdot\chi_D\|_\varphi\Big)\Big(\|b_{2_i}\|_{Q_i}+d_{2_i}\cdot r\Big)\Bigg).$$

Since $g, a_i\in E_\varphi$, $b_{1_i}\in E_{\varphi_{1_i}}$, then we have

$$\lim_{\varepsilon\to0}\{\sup_{meas\ D\leq\varepsilon}[\sup_{\psi\in\Psi}\{\|g\cdot\chi_D\|_\varphi\}]\}=0$$

and

$$\lim_{\varepsilon\to0}\{\sup_{meas\ D\leq\varepsilon}[\sup_{\psi\in\Psi}\{\|a_i\cdot\chi_D\|_\varphi+\frac{2e_ik_{1_i}}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\|b_{1_i}\cdot\chi_D\|_{\varphi_{1_i}}\}]\}=0.$$

By using the formula of $c(Y)$, we obtain

$$c(B(\Psi))\leq Kr^n\prod_{i=1}^n\Bigg(\frac{2e_ik_{1_i}d_{1_i}}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{2_i}\|_{Q_i}+d_{2_i}\cdot r\Big)\Bigg)c(\Psi).$$

Based on the previously established properties, we may apply Lemma 2.7 to obtain

$$\mu_H(B(\Psi))\leq Kr^n\prod_{i=1}^n\Bigg(\frac{2e_ik_{1_i}d_{1_i}}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{2_i}\|_{Q_i}+d_{2_i}\cdot r\Big)\Bigg)\mu_H(\Psi).$$

The above inequality with $Kr^n\prod_{i=1}^n\Bigg(\frac{2e_ik_{1_i}d_{1_i}}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_{2_i}}\Big(\|b_{2_i}\|_{Q_i}+d_{2_i}\cdot r\Big)\Bigg)<1$ allows us to apply Theorem 2.8. That ends the proof. □

3.1.1. Uniqueness of the solution

We may prove and discuss the uniqueness of the solutions of Eq (1.1).

Theorem 3.2. *Assume the assumptions of Theorem 3.1 are verified but replace assumption (C3) with:*

(C6) There exist positive constants e_i, d_{1_i}, d_{2_i} and functions $a_i\in E_{\varphi_i}(\mathbb{I})$, $b_{1_i}\in E_{\varphi_{1_i}}(\mathbb{I})$, and $b_{2_i}\in E_{Q_i}(\mathbb{I})$, s.t.

$$|f_i(s,0)|\leq a_i(s),\ \ |h_{j_i}(s,0)|\leq b_{j_i}(s),\ \ j=1,2,$$
$$|f_i(s,\psi)-f_i(s,z)|\leq e_i\|\psi-z\|_\varphi,\ \ |h_{1_i}(s,\psi)-h_{1_i}(s,z)|\leq d_{1_i}\varphi_{1_i}^{-1}\Big(\varphi(|\psi-z|)\Big),$$

$$\textit{and}\ \ |h_{2_i}(s,\psi)-h_{2_i}(s,z)|\le d_{2_i}Q_i^{-1}\Big(\varphi(|\psi-z|)\Big),\ \ \psi,z\in\omega_r,$$

where ω_r is as in Theorem 3.1 for $i=1,\cdots,n$, in addition, let

$$C=\sum_{j=1}^{n}\left[\frac{2e_jk_{1_j}\|k_j\|_{\varphi_{2_j}}}{\Gamma(\alpha_j)}\Big(d_{1_j}\Big(\|b_{2_j}\|_{Q_j}+d_{2_j}r\Big)+d_{2_j}\Big(\|b_{1_j}\|_{\varphi_{1_j}}+d_{1_j}r\Big)\Big)\Big(r-\|g\|_\varphi\Big)\right]<1, \tag{3.1}$$

where r is given in assumption (C5). Then (1.1) has a unique solution $\psi\in L_\varphi$ in ω_r.

Proof. By applying assumption (C6), we obtain

$$\begin{aligned}\Big||h_{1_i}(s,\psi)|-|h_{1_i}(s,0)|\Big| &\le |h_1(s,\psi)-h_1(s,0)|\le d_{1_i}\varphi_{1_i}^{-1}\Big(\varphi(\psi)\Big)\\ \Rightarrow |h_1(s,\psi)| &\le |h_{1_i}(s,0)|+d_{1_i}\varphi_{1_i}^{-1}\Big(\varphi(\psi)\Big)\le b_{1_i}(s)+d_{1_i}\varphi_{1_i}^{-1}\Big(\varphi(\psi)\Big).\end{aligned}$$

Similarly, $|h_{2_i}(s,\psi)|\le b_{2_i}(s)+d_{2_i}Q_i^{-1}\Big(\varphi(\psi)\Big)$ and $|f_i(s,\psi)|\le a_i(s)+e_i\|\psi\|_\varphi$. Thus, Theorem 3.1 implies that there exists a.e. nondecreasing solution $\psi\in E_\varphi$ of (1.1) in ω_r.

Next, let $\psi,\ z\in\omega_r$ be two distinct solutions of Eq (1.1); then by using assumption (C6), we obtain

$$\begin{aligned}|\psi-z| &= \Big|\prod_{i=1}^n B_i(\psi)-\prod_{i=1}^n B_i(z)\Big|\\ &\le \Big|\prod_{i=1}^n B_i(\psi)-B_1(z)\prod_{i=2}^n B_i(\psi)\Big|+\Big|B_1(z)\prod_{i=2}^n B_i(\psi)-B_1(z)B_2(z)\prod_{i=3}^n B_i(\psi)\Big|\\ &\quad+\cdots+\Big|B_n(\psi)\prod_{i=1}^{n-1}B_i(z)-\prod_{i=1}^n B_i(z)\Big|\\ &\le |B_1(\psi)-B_1(z)|\cdot\prod_{i=2}^n|B_i(\psi)|+|B_1(z)|\cdot|B_2(\psi)-B_2(z)|\cdot\prod_{i=3}^n B_i(\psi)\\ &\quad+\ \cdots\ +\ |B_n(\psi)-B_n(z)|\cdot\prod_{i=1}^{n-1}|B_i(z)|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\psi-z\|_\varphi\le K\Big\|B_1(\psi)-B_1(z)\Big\|_{\varphi_1}\prod_{i=2}^n\|B_i(\psi)\|_{\varphi_i}+K\|B_1(z)\|_{\varphi_1}\Big\|B_2(\psi)-B_2(z)\Big\|_{\varphi_2}\prod_{i=3}^n\|B_i(\psi)\|_{\varphi_i}\\ &+\ldots+K\Big\|B_n(\psi)-B_n(z)\Big\|_{\varphi_n}\prod_{i=1}^{n-1}\|B_i(z)\|_{\varphi_i}.\end{aligned} \tag{3.2}$$

To estimate inequality (3.2), we use Proposition 2.11, for $j=1,\cdots,n$, to calculate the following:

$$\begin{aligned}&\Big\|B_j(\psi)-B_j(z)\Big\|_{\varphi_j}=\Big\|F_{f_i}U_i(\psi)-F_{f_i}U_i(z)\Big\|_{\varphi_j}\\ \le\ & e_i\Big\|F_{h_{1_j}}(\psi)A_j(\psi)-F_{h_{1_j}}(z)A_j(z)\Big\|_{\varphi_j}\end{aligned}$$

$$
\begin{aligned}
&\leq\ e_i\Big\|F_{h_{1_j}}(\psi)A_j(\psi) - F_{h_{1_j}}(z)A_j(\psi)\Big\|_{\varphi_j} + e_i\Big\|F_{h_{1_j}}(z)A_j(\psi) - F_{h_{1_j}}(z)A_j(z)\Big\|_{\varphi_j} \\
&\leq\ e_ik_{1_j}\Big\|F_{h_{1_j}}(\psi) - F_{h_{1_j}}(z)\Big\|_{\varphi_{1_j}}\Big\|A_j(\psi)\Big\|_{\varphi_{2_j}} + e_ik_{1_j}\Big\|F_{h_{1_j}}(z)\Big\|_{\varphi_{1_j}}\Big\|A_j(\psi) - A_j(z)\Big\|_{\varphi_{2_j}} \\
&\leq\ e_jk_{1_j}d_{1_j}\Big\|\varphi_{1_j}^{-1}\Big(\varphi(|\psi - z|)\Big)\Big\|_{\varphi_{1_j}}\Big\|\nu^{\beta_j\alpha_j}J_{\beta_j}^{\alpha_j}F_{h_{2_j}}(\psi)\Big\|_{\varphi_{2_j}} \\
&\quad +e_jk_{1_j}\Big\|b_{1_j} + d_{1_j}\varphi_{1_j}^{-1}\Big(\varphi(z)\Big)\Big\|_{\varphi_{1_j}}\Big\|\nu^{\beta_j\alpha_j}J_{\beta_j}^{\alpha_j}\Big|F_{h_{2_j}}(\psi) - F_{h_{2_j}}\Big|(z)\Big\|_{\varphi_{2_j}} \\
&\leq\ e_jk_{1_j}d_{1_j}\frac{2\|k_j\|_{\varphi_{2_j}}}{\Gamma(\alpha_j)}\Big(\|b_{2_j}\|_{Q_j} + d_{2_j}\|\psi\|_{\varphi}\Big)\|\psi - z\|_{\varphi} \\
&\quad +e_jk_{1_j}\Big(\|b_{1_j}\|_{\varphi_{1_j}} + d_{1_j}\|z\|_{\varphi}\Big)\frac{2\|k_j\|_{\varphi_{2_j}}}{\Gamma(\alpha_j)}\Big\|d_{2_j}Q_j^{-1}\Big(\varphi(|\psi - z|)\Big)\Big\|_{Q_j} \\
&\leq\ e_jk_{1_j}d_{1_j}\frac{2\|k_j\|_{\varphi_{2_j}}}{\Gamma(\alpha_j)}\Big(\|b_{2_j}\|_{Q_j} + d_{2_j}\|\psi\|_{\varphi}\Big)\|\psi - z\|_{\varphi} \\
&\quad +e_jk_{1_j}\Big(\|b_{1_j}\|_{\varphi_{1_j}} + d_{1_j}\|z\|_{\varphi}\Big)\frac{2d_{2_j}\|k_j\|_{\varphi_{2_j}}}{\Gamma(\alpha_j)}\|\psi - z\|_{\varphi} \\
&\leq\ \frac{2e_jk_{1_j}\|k_j\|_{\varphi_{2_j}}}{\Gamma(\alpha_j)}\Big(d_{1_j}\Big(\|b_{2_j}\|_{Q_j} + d_{2_j}r\Big) + d_{2_j}\Big(\|b_{1_j}\|_{\varphi_{1_j}} + d_{1_j}r\Big)\Big)\|\psi - z\|_{\varphi}.
\end{aligned}
\tag{3.3}
$$

From assumption (C5), we have $\prod_{i=1}^{n}\|B_i(\psi)\|_{\varphi} \leq \frac{r-\|g\|_{\varphi}}{K}$, and by substituting from (3.3) into (3.2), we obtain

$$
\begin{aligned}
\|\psi - z\|_{\varphi} \leq K\Bigg[&\frac{2e_1k_{1_1}\|k_1\|_{\varphi_{2_1}}}{\Gamma(\alpha_1)}\Big(d_{1_1}\Big(\|b_{2_1}\|_{Q_1} + d_{2_1}r\Big) + d_{2_1}\Big(\|b_{1_1}\|_{\varphi_{1_1}} + d_{1_1}r\Big)\Big)\Big(\frac{r - \|g\|_{\varphi}}{K}\Big) \\
&+\frac{2e_2k_{1_2}\|k_2\|_{\varphi_{2_2}}}{\Gamma(\alpha_2)}\Big(d_{1_2}\Big(\|b_{2_2}\|_{Q_2} + d_{2_2}r\Big) + d_{2_2}\Big(\|b_{1_2}\|_{\varphi_{1_2}} + d_{1_2}r\Big)\Big)\Big(\frac{r - \|g\|_{\varphi}}{K}\Big) \\
&+\ldots + \frac{2e_nk_{1_n}\|k_n\|_{\varphi_{2_n}}}{\Gamma(\alpha_n)}\Big(d_{1_n}\Big(\|b_{2_n}\|_{Q_n} + d_{2_n}r\Big) + d_{2_n}\Big(\|b_{1_n}\|_{\varphi_{1_n}} + d_{1_n}r\Big)\Big)\Big(\frac{r - \|g\|_{\varphi}}{K}\Big)\Bigg]\|\psi - z\|_{\varphi} \\
&= C \cdot \|\psi - z\|_{\varphi}.
\end{aligned}
$$

The above estimate with inequality (3.1) concludes the proof. □

3.2. Continuous dependence on the function g.

Here, we study how solutions of Eq (1.1) depend continuously on g.

Definition 3.3. A solution $\psi \in L_{\varphi}(\mathbb{I})$ of (1.1) is continuously dependent on the function g if $\forall\ \epsilon > 0$, $\exists\delta > o$ such that $\|g - \bar{g}\|_{\varphi} \leq \delta$ implies that $\|\psi - \bar{\psi}\|_{\varphi} \leq \epsilon$, where

$$
\bar{\psi}(\nu)\ = \bar{g}(\nu) + \prod_{i=1}^{n} f_i\Big(\nu,\ \frac{\beta_i h_{1_i}(\nu, \bar{\psi}(\nu))}{\Gamma(\alpha_i)} \cdot \int_0^{\nu} \frac{s^{\beta_i-1}h_{2_i}(s, \bar{\psi}(s))}{(\nu^{\beta_i} - s^{\beta_i})^{1-\alpha_i}}\, ds\Big),\ \ \nu \in \mathbb{I} = [0, \rho]. \tag{3.4}
$$

Theorem 3.4. *Suppose that the assumptions of Theorem 3.2 are held. Then solutions $\psi \in L_{\varphi}(\mathbb{I})$ of Eq (1.1) depend continuously on the function g.*

Proof. Let $\psi, \bar{\psi}$ be any two solutions of (1.1); then, similarly as done in Theorem 3.2, we have

$$\begin{aligned}\|\psi - \bar{\psi}\|_\varphi \leq \|g - \bar{g}\|_\varphi + \Bigg\| \prod_{i=1}^{n} f_i\Big(\nu, \frac{\beta h_{1_i}(\nu, \psi(\nu))}{\Gamma(\alpha_i)} \cdot \int_0^\nu \frac{s^{\beta_i - 1} h_{2_i}(s, \psi(s))}{(\nu^{\beta_i} - s^{\beta_i})^{1-\alpha}}\, ds\Big) \\ - \prod_{i=1}^{n} f_i\Big(\nu, \frac{\beta_i h_{1_i}(\nu, \bar{\psi}(\nu))}{\Gamma(\alpha_i)} \cdot \int_0^\nu \frac{s^{\beta_i - 1} h_{2_i}(s, \bar{\psi}(s))}{(\nu^{\beta_i} - s^{\beta_i})^{1-\alpha}}\, ds\Big)\Bigg\|_\varphi \\ \leq \|g - \bar{g}\|_\varphi + C\|\psi - \bar{\psi}\|_\varphi,\end{aligned}$$

where C is given by (3.1). Then, we obtain

$$\|\psi - \bar{\psi}\|_\varphi \leq \Big(1 - C\Big)^{-1} \|g - \bar{g}\|_\varphi.$$

Therefore, if $\|g - \bar{g}\|_\varphi \leq \delta(\epsilon)$, then $\|\psi - \bar{\psi}\|_\varphi \leq \epsilon$, where

$$\delta(\epsilon) = \epsilon \cdot (1 - C).$$

□

4. Remarks and examples

Remark 4.1. It is important to note that our solutions are not necessarily continuous, as in many of the cases that have been examined and investigated previously [1,36] or in L_p-spaces [8,37]. So we do not assume that the operators studied map the spaces $C(I)$ or L_p into themselves. Our results belong to the space L_φ, for more examples and assumptions related to the operators studied in L_φ see [9,29].

Finally, we demonstrate and clarify some examples that support our outcomes.

Example 4.2. Considering the N-functions $P_i(\nu) = Q_i(\nu) = \nu^2$ and $\varphi_{2_i}(\nu) = \exp|\nu| - |\nu| - 1$. We can examine that Proposition 2.11_2 is verified and the fractional operator $J_{\beta_i}^{\alpha_i} : L_{Q_i}(\mathbb{I}) \to L_{\varphi_{2_i}}(\mathbb{I})$ is continuous, where $\mathbb{I} = [0, \rho]$.

Therefore: For $i = 1, 2, \cdots, n$, and any $\alpha_i, \beta_i > 0$ and $\nu \in \mathbb{I}$, we obtain

$$k_i(\nu) = \int_0^{\nu^{\beta_i}} \varphi_i(u^{\alpha_i - 1})\, du = \int_0^{\nu^{\beta_i}} u^{2\alpha_i - 2}\, du = \frac{\nu^{\beta_i(2\alpha_i - 1)}}{2\alpha_i - 1}.$$

Moreover,

$$\int_0^\rho \varphi_{2_i}(k(\nu))\, d\nu = \int_0^\rho \Big(e^{\frac{\nu^{\beta_i(2\alpha_i - 1)}}{2\alpha_i - 1}} - \frac{\nu^{\beta_i(2\alpha_i - 1)}}{2\alpha_i - 1} - 1\Big) d\nu < \infty.$$

Then, Proposition 2.11_2 is verified. Therefore, for $\psi \in L_{Q_i}(\mathbb{I})$, we obtain $J_{\beta_i}^{\alpha_i} : L_{Q_i}(\mathbb{I}) \to L_{\varphi_{2_i}}(\mathbb{I})$ is continuous.

Remark 4.3. In the literature, many applications and significant results exist for N-functions that do not satisfy the global Δ_2-condition. Such functions often appear in models of nonlinear elasticity, statistical physics, and image processing, where exponential or logarithmic growth is relevant. Regarding the examples in the manuscript:

(1) The functions $P_i(\nu) = Q_i(\nu) = \nu^2$ satisfy the Δ_2-condition globally, as it is a simple power function with polynomial growth.
(2) The functions $\varphi_{2_i}(\nu) = \exp|\nu| - |\nu| - 1$ do not satisfy the Δ_2-condition globally due to the exponential growth. However, for small values of ν, they behave like the quadratic functions:

$$\varphi_{2_i}(\nu) \approx \frac{\nu^2}{2}, \text{ for } \nu \to 0,$$

and satisfy the Δ_2-condition locally in this regime.

Remark 4.4. For further particular cases verifying Example 4.2, see [7, 21, 22], and for more information and numerous examples of the functions φ_{2_i} and (P_i, Q_i) verifying Proposition 2.11_2, see [12].

Example 4.5. For $\alpha_i = \frac{1}{2}, \beta_i = \frac{1}{2}, i = 1, 2, \cdots, n$, and $\nu \in \mathbb{I} = [0, 1]$, we have

$$\psi(\nu) = g(\nu) + \prod_{i=1}^{n}\left(a_i(\nu) + \frac{\left(b_{1_i}(\nu) + \frac{1}{10}\varphi_{1_i}^{-1}(\varphi(\psi(\nu)))\right)}{10\sqrt{\pi}}\int_0^{\nu}\frac{b_{2_i}(s) + \frac{1}{10}Q_i^{-1}\left(\varphi(\psi(s))\right)}{2\sqrt{s}(\sqrt{\nu} - \sqrt{s})^{\frac{1}{2}}}ds\right). \tag{4.1}$$

This may be considered a particular case of Eq (1.1), where

$$|f_i(\nu, \psi)| \le a_i(\nu) + \frac{1}{10}\|\psi\|_{\varphi}, \ |h_{1_i}(\nu, \psi)| \le b_{1_i}(\nu) + \frac{1}{10}\varphi_{1_i}^{-1}(\varphi(\psi)), \text{ and } |h_{2_i}(\nu, \psi)| \le b_{2_i}(s) + \frac{1}{10}Q_i^{-1}\left(\varphi(\psi)\right)$$

with

$$\|f_i\|_{\varphi_i} \le \|a_i\|_{\varphi_i} + \frac{1}{10}\|\psi\|_{\varphi}, \ |h_{1_i}\|_{\varphi_{1_i}} \le \|b_{1_i}\|_{\varphi_{1_i}} + \frac{1}{10}\|\psi\|_{\varphi}, \text{ and } \|h_{2_i}\|_{Q_i} \le \|b_{2_i}\|_{Q_i} + \frac{1}{10}\|\psi\|_{\varphi}.$$

Thus, assumptions (C2) and (C3) are satisfied with constants $e_i = d_{1_i} = d_{2_i} = \frac{1}{10}$ and for suitable forms of the functions $b_{1_i} \in E_{\varphi_{1_i}}(\mathbb{I})$, $b_{2_i} \in E_{Q_i}(\mathbb{I})$, and $a_i \in E_{\varphi_i}(\mathbb{I})$.

For assumption (C4), see Example 4.2.

Now for a suitable nondecreasing function g satisfying assumption (C1), we can find $r > 0$ on $\mathbb{I}$, verifying

$$\prod_{i=1}^{n}\left(\|a_i\|_{\varphi_i} + \frac{2k_{1_i}}{10\sqrt{\pi}}\|k_i\|_{\varphi_{2_i}}\left(\|b_{1_i}\|_{\varphi_{1_i}} + \frac{1}{10}r\right)\left(\|b_2\|_{Q_i} + \frac{1}{10}r\right)\right) \le \frac{r - \|g\|_{\varphi}}{K}.$$

and

$$Kr^n\prod_{i=1}^{n}\left(\frac{2k_{1_i}}{100\sqrt{\pi}}\|k_i\|_{\varphi_{2_i}}\left(\|b_{2_i}\|_{Q_i} + \frac{1}{10}\cdot r\right)\right) < 1.$$

Therefore, we obtain our verifications and get that Eq

5. Conclusions

The presented paper established existence, monotonicity, and uniqueness, as well as continuous dependence on the data of a product of n-quadratic Erdélyi-Kober ($\mathcal{EK}$) fractional integral equation. The analysis used to obtain the results is the ($\mathcal{MNC}$) measure of noncompactness, the ($\mathcal{FPT}$) fixed-point theorem, as well as fractional calculus in the Orlicz spaces L_φ. Some examples are given to illustrate the hypothesis.

Open discussion and future possibilities:

(1) The authors could discuss potential extensions of their work to other fractional operators.
(2) The authors could apply their results to specific fields such as physics, engineering, or biology.
(3) The authors can focus on the corresponding problems of equivalent differential equations in Orlicz spaces or Sobolev-Orlicz spaces.
(4) The authors could check some numerical results for the considered problems.

Author contributions

Mohamed M. A. Metwali: methodology, validation, formal analysis, investigation, writing-original draft preparation, visualization; Abdulaziz M. Alotaibi: Methodology, validation, resources, investigation; writing-original draft preparation, and editing; Hala H.Taha: formal analysis, resources, investigation, writing-review and editing; Ravi P Agarwal: Conceptualization, validation, writing-review and editing, Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that there are no conflict of interest.

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