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Research article

Novel theorems on constant angle ruled surfaces with Sasai's interpretation

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Abstract: In the present study, we investigate constant-angle ruled surfaces constructed by the motion of the elements of each of the modified orthogonal frames along a base curve in three-dimensional Euclidean 3-space. These surfaces are studied and classified based on the constant-angle property. In these regards, we obtain the characterizations of the minimal and developable ruled surfaces by using partial differential equations. Also, we give the conditions for these surfaces to be Weingarten surfaces.

Keywords: constant-angle ruled surfaces; differential equations; modified orthogonal frame; Weingarten surfaces; partial differential equations
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1. Introduction

The constant-angle ruled surfaces represent an essential concept within mathematical surfaces. These surfaces possess a specific mathematical rule at each point: the tangent lines at every point maintain a constant angle with the normal vector. This property is the primary trait that distinguishes constant-angle ruled surfaces from others. Numerous disciplines, including physics, engineering, and architecture, benefit from a comprehension of constant-angle ruled surfaces. The properties of these surfaces are particularly crucial in determining the geometric features of designs and structures. By bridging the gap between mathematics and real-world applications, the understanding of constant-angle ruled surfaces facilitates the resolution of challenging issues. The potential applications of these special surfaces in both mathematics and physics have been the subject of extensive research by a number of authors in recent years. For example, Paolo and Scala used the Hamilton-Jacobi equation to examine the characteristics of surfaces with constant angles in [1]. Their research is to comprehend the behavior of surfaces with constant-angles when the direction vector becomes singular along a particular line or point. A necessary contribution to this field was given by Munteanu and Nistor by classifying surfaces where the unit normal vector maintains a constant angle with a fixed direction vector in Euclidean 3-space [2]. Subsequent research has been conducted on developable and constant-angle surfaces, shedding light on their properties and characteristics [3, 4]. Additionally, A. T. Ali explored constant-angle ruled surfaces formed by Frenet frame vectors in [5]. Latterly, the theory of constant-angle surfaces has been extended to encompass other ambient spaces. For instance, in [6, 7], researchers examined these surfaces in the context of Minkowski space E_1^3. Furthermore, in [8–12], various alternative approaches and perspectives to the concept of constant-angle surfaces were presented within the Lorentzian frame. The Frenet frame is the most popular tool for studying curves and surfaces. However, it is insufficient for any curve in analytic space whose curvatures have distinct zero points because the principal normal and binormal vectors may be discontinuous at the curvature's zero points. Sasai [13] introduced the modified orthogonal frame (MOF) to address this issue and derived a derivative formula that is analogous to the Frenet-Serret equation. Currently, the MOF with non-zero curvature (MOFC) and with non-zero torsion (MOFT) of a space curve were presented in Minkowski 3-space by Bükcü and Karacan [14]. After this development, the modified orthogonal frames attracted a lot of attention, and various studies were devoted to searching for novelties brought by these frames. For instance, some special curves, the evolution of curves, ruled surfaces, Hasimoto surfaces, and tubular surfaces were investigated by means of the modified orthogonal frames in recent studies [15–26].

In light of recent developments encapsulated above, the constant-angle ruled surfaces have been investigated with MOFs in Euclidean 3-space. These surfaces have been classified based on the constant-angle property. In these regards, the characterizations for the developable and minimal constant-angle ruled surfaces have been presented. Also, the conditions for these surfaces to be Weingarten surfaces have been given.

2. Preliminaries

Let α be a unit speed moving space curve with the arc-length parameter s in Euclidean 3-space E^3. If t, n, and b denote the tangent, principal normal, and binormal unit vectors at any point $\alpha(s)$ of α, respectively. Then a moving frame occurs satisfying the Frenet derivative equations $t' = \kappa n$, $n' = -\kappa t + \tau b$, $b' = -\tau n$ where κ and τ are the curvature and the torsion of α, respectively.

In the cases that the principal normal and binormal vectors in the Frenet frame of any space curve are discontinuous at the points where the curvature is zero, Sasai's interpretation can be put into use, i.e., his modified orthogonal frame comes onto the stage [13]. Even though the Frenet frame can display a change at the points $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ for any $\varepsilon > 0$ provided that $\kappa(s_0) = 0$, two types of orthogonal frames can be proposed. The first one is called modified orthogonal frame with non-vanishing curvature (MOFC) for $\kappa(s) \neq 0$ at such points, and the second one is modified orthogonal frame with non-vanishing torsion (MOFT) for $\tau(s) \neq 0$.

Let the curvature κ of a general analytic curve α be non-zero everywhere; then the elements of MOFC of a curve are defined as

$$T = \frac{d\alpha}{ds}, \quad N = \frac{dT}{ds}, \quad B = T \times N,$$

where "×" represents the vector product. At non-zero points of κ, there are the relations between the

MOFC and the Frenet frame as

$$T = t, \quad N = \kappa n, \quad B = \kappa b.$$

Therefore, the derivative formulas for the elements of the MOFC are

$$T' = N, \;\; N' = -\kappa^2 T + \frac{\kappa'}{\kappa} N + \tau B, \;\; B' = -\tau N + \frac{\kappa'}{\kappa} B, \tag{2.1}$$

where the prime denotes differentiation with respect to the affine arc-length parameter s and $\tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}$ is the torsion of the space curve α [13]. Also, the MOFC provides

$$\langle T, T \rangle = 1, \;\; \langle N, N \rangle = \langle B, B \rangle = \kappa^2, \;\; \langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0.$$

Secondly, assume that the torsion τ of a general analytic curve α is nonzero everywhere. Then the relations between MOFT and the Frenet frame are

$$T = t, \quad N = \tau n, \quad B = \tau b.$$

In this case, the derivative formulas for MOFT hold:

$$T' = \frac{\kappa}{\tau} N, \;\; N' = -\kappa\tau T + \frac{\tau'}{\tau} N + \tau B, \;\; B' = -\tau N + \frac{\tau'}{\tau} B, \tag{2.2}$$

where $\langle T, T \rangle = 1$, $\langle N, N \rangle = \langle B, B \rangle = \tau^2$, $\langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0$, [14].

The following basic definitions for any surface $\Phi(s, v)$ in Euclidean 3-space are well-known.

Definition 2.1. *Let $\Phi(s, v)$ be a surface in Euclidean 3-space.*

i. The unit normal vector of $\Phi(s, v)$ is defined by $U(s, v) = \frac{\Phi_s \times \Phi_v}{\|\Phi_s \times \Phi_v\|}$, where the tangent vectors of $\Phi(s, v)$ are $\Phi_s = \frac{\partial \Phi}{\partial s}$ and $\Phi_v = \frac{\partial \Phi}{\partial v}$.

ii. The coefficients of the first fundamental form $\mathbb{I}(s, v) = E ds^2 + 2F ds dv + G dv^2$ of $\Phi(s, v)$ are defined by

$$E(s, v) = \langle \Phi_s, \Phi_s \rangle, \;\; F(s, v) = \langle \Phi_s, \Phi_v \rangle, \;\; G(s, v) = \langle \Phi_v, \Phi_v \rangle.$$

iii. The coefficients of the second fundamental form $\mathbb{II}(s, v) = e ds^2 + 2f ds dv + g dv^2$ of $\Phi(s, v)$ are defined by

$$e(s, v) = \langle U, \Phi_{ss} \rangle, \;\; f(s, v) = \langle U, \Phi_{sv} \rangle, \;\; g(s, v) = \langle U, \Phi_{vv} \rangle.$$

iv. The Gaussian and the mean curvatures of $\Phi(s, v)$ are defined by

$$K(s, v) = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H(s, v) = \frac{Eg - 2Ef + Ge}{2(EG - F^2)},$$

respectively.

v. A smooth surface $\Phi(s, v)$ satisfying a functional relationship between the Gaussian curvature K and the mean curvature H is called a Weingarten surface [27].

Proposition 2.1. *Let $\Phi(s, v)$ be a surface in Euclidean 3-space.*

i. If $\Phi(s, v)$ has zero Gaussian curvature everywhere, it is developable.

ii. If $\Phi(s, v)$ has zero mean curvature everywhere, it is minimal.

iii. If $K_s H_v - K_v H_s = 0$, then $\Phi(s, v)$ is a Weingarten surface [27].

3. Constant-angle ruled surface with Sasai's interpretation

Let a ruled surface be generated by a family of straight lines along an analytical curve $\sigma(s)$, called the base curve. Its parametric equation is presented by

$$\Phi(s, v) = \sigma(s) + v\Upsilon(s). \tag{3.1}$$

Here, f, g, and h are smooth functions of s. Let the director curve $\Upsilon(s)$ be a linear combination of the modified Frenet vectors introduced by Sasai as $\Upsilon(s) = fT + gN + hB$ in the case of the Frenet frame failing where $\kappa(s_0) = 0$ at any point. In that regard, we refer to the frames MOFC and MOFT of the base curve for $\kappa(s) \neq 0$ and $\tau(s) \neq 0$ at each point $s \in (s_0 - \varepsilon, s_0 + \varepsilon) \setminus \{s_0\}$, respectively. In the following two subsections, we examine each case separately.

3.1. Constant-angle ruled surfaces with MOFC

Let the curvature of the base curve σ be non-zero everywhere. By using differential equations formulas (2.1) for MOFC, the partial differential equations of the surface $\Phi(s, v)$ represented by (3.1) are obtained as

$$\Phi_s = \left(1 - v\left(g\kappa^2 - f'\right)\right)T + v\left(f - h\tau + g' + \frac{\kappa' g}{\kappa}\right)N + v\left(g\tau + h' + \frac{h\kappa'}{\kappa}\right)B$$

and

$$\Phi_v = fT + gN + hB.$$

The cross-product of the above tangent vector fields is found as

$$\begin{aligned}\Phi_s \times \Phi_v = {} & v\left(h\left(f - h\tau + g'\right) - g\left(g\tau + h'\right)\right)T \\ & -\left(h - v\left(f\left(g\tau + h'\right) + h\left(g\kappa^2 + f' + \frac{f\kappa'}{\kappa}\right)\right)\right)N \\ & +\left(g + v\left(g\left(f' - g\kappa^2\right) - f\left(f - h\tau + g' + \frac{g\kappa'}{\kappa}\right)\right)\right)B.\end{aligned} \tag{3.2}$$

By a straightforward computation, the normal vector of $\Phi(s, v)$ is

$$U = U_1T + U_2N + U_3B,$$

where

$$U_1 = U_{11} + vU_{12},\ U_2 = U_{21} + vU_{22},\ U_3 = U_{31} + vU_{32}. \tag{3.3}$$

If Eqs (3.2) and (3.3) are compared, the following Eq (3.4) is obtained:

$$\begin{cases} U_{11} = 0,\ U_{12} = h\left(f - h\tau + g'\right) - g\left(g\tau + h'\right), \\ U_{21} = -h,\ U_{22} = f\left(g\tau + h'\right) + h\left(g\kappa^2 - f' + f\dfrac{\kappa'}{\kappa}\right), \\ U_{31} = g,\ U_{32} = g\left(f' - g\kappa^2\right) - f\left(f - h\tau + g' + \dfrac{g\kappa'}{\kappa}\right). \end{cases} \tag{3.4}$$

3.1.1. Constant-angle ruled surface parallel to tangent vector

If we assume that the normal vector U of the surface is parallel to the tangent vector T of the base curve $\sigma(s)$ according to the MOFC, then we have the following conditions:

$$U_1 \neq 0, \ U_2 = U_3 = 0. \tag{3.5}$$

Considering Eq (3.4), $f = g = h = 0$ is obtained by the common solution of Eq (3.5). This is a contradiction because of $\Upsilon(s) \neq 0$ and $U_1 \neq 0$. So, we can give the following theorem.

Theorem 3.1. *There is no constant-angle ruled surface parallel to the tangent vector direction satisfying the conditions of Eq (3.5).*

3.1.2. Constant-angle ruled surface parallel to modified principal normal vector

Suppose that the normal vector U of the surface $\Phi(s, v)$ is parallel to the direction of the modified principal normal vector N of the base curve $\sigma(s)$ according to the MOFC; then there are the conditions:

$$U_2 \neq 0, \ U_1 = U_3 = 0. \tag{3.6}$$

Since $U_{31} = 0$, g vanishes. In that case, from Eq (3.4), we get the equations:

$$\begin{cases} U_{11} = 0, \ U_{12} = h(f - h\tau), \\ U_{21} = -h, \ U_{22} = fh' - h\left(f' - f\dfrac{\kappa'}{\kappa}\right), \\ U_{31} = 0, \ U_{32} = -f(f - h\tau). \end{cases}$$

Then there are two cases as follows:

Case (1): $f = h\tau$, $g = 0$, and $h \neq 0$. From this case, it is easy to see that the conditions given by (3.6) and then the constant-angle ruled surface is rewritten in the form

$$\Phi_{n1}^{c}(s, v) = \sigma(s) + vh(\tau T + B),$$

such that $f = h\tau$. By using the equations given in (2.1), the partial derivatives of the equation of the surface $\Phi_{n1}^{c}(s, v)$ are

$$(\Phi_{n1}^{c})_s = (1 + v(h\tau)')T + v\left(h' + \frac{h\kappa'}{\kappa}\right)B$$

and

$$(\Phi_{1}^{c})_v = h(\tau T + B).$$

By a straightforward computation, the normal vector of the surface $\Phi_{n1}^{c}(s, v)$ is calculated as

$$U = \frac{\left(\Phi_{n1}^{c}\right)_s \times \left(\Phi_{n1}^{c}\right)_v}{\left\|\left(\Phi_{n1}^{c}\right)_s \times \left(\Phi_{n1}^{c}\right)_v\right\|} = N.$$

Theorem 3.2. *Let $\Phi_{n1}^{c}(s,v)$ be a constant-angle ruled surface with MOFC; then the Gaussian and mean curvatures are*

$$K_{n1}^{c}=0 \text{ and } H_{n1}^{c}=\frac{\kappa\left(1+\tau^{2}\right)}{2\left(\kappa-vh\left(\tau\kappa'-\kappa\tau'\right)\right)},$$

respectively.

Proof. Let $\Phi_{n1}^{c}(s,v)$ be a constant-angle ruled surface. The coefficients of the first and second fundamental forms of $\Phi_{n1}^{c}(s,v)$ are

$$\begin{aligned}
E_{n1}^{c}&=\left\langle(\Phi_{n1}^{c})_{s},(\Phi_{n1}^{c})_{s}\right\rangle=v^{2}\left(h'+\frac{h\kappa'}{\kappa}\right)^{2}+\left(1+v\tau h'+vh\tau'\right)^{2},\\
F_{n1}^{c}&=\left\langle(\Phi_{n1}^{c})_{s},(\Phi_{n1}^{c})_{v}\right\rangle=h\left(v\left(h'+\frac{h\kappa'}{\kappa}\right)+\tau\left(1+v\tau h'+vh\tau'\right)\right),\\
G_{n1}^{c}&=\left\langle(\Phi_{n1}^{c})_{v},(\Phi_{n1}^{c})_{v}\right\rangle=h^{2}\left(1+\tau^{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
e_{n1}^{c}&=\left\langle(\Phi_{n1}^{c})_{ss},U\right\rangle=v^{2}\left(h'+\frac{h\kappa'}{\kappa}\right)^{2}+\left(1+v\tau h'+vh\tau'\right)^{2},\\
f_{n1}^{c}&=\left\langle(\Phi_{n1}^{c})_{sv},U\right\rangle=h\left(v\left(h'+\frac{h\kappa'}{\kappa}\right)+\tau\left(1+v\tau h'+vh\tau'\right)\right),\\
g_{n1}^{c}&=\left\langle(\Phi_{n1}^{c})_{vv},U\right\rangle=h^{2}\left(1+\tau^{2}\right),
\end{aligned}$$

respectively, since

$$\begin{aligned}
(\Phi_{n1}^{c})_{s}&=\left(1+v\tau h'+vh\tau'\right)T+v\left(h'+\frac{h\kappa'}{\kappa}\right)B,\ \ (\Phi_{n1}^{c})_{v}=h\tau T+hB,\\
(\Phi_{n1}^{c})_{ss}&=v\left(2h'\tau'+\tau h''+h\tau''\right)T+\left(1-\frac{vh\tau\kappa'}{\kappa}+vh\tau'\right)N+v\left(\frac{2h'\kappa'}{\kappa}+h''+\frac{h\kappa''}{\kappa}\right)B,\\
(\Phi_{n1}^{c})_{sv}&=\left(\tau h'+h\tau'\right)T+\left(h'+\frac{h\kappa'}{\kappa}\right)B,\ \ (\Phi_{n1}^{c})_{vv}=0.
\end{aligned}$$

If the above relations are substituted in the formulas

$$K_{n1}^{c}=\frac{e_{n1}^{c}g_{n1}^{c}-{f_{n1}^{c}}^{2}}{E_{n1}^{c}G_{n1}^{c}-{F_{n1}^{c}}^{2}}=0 \text{ and } H_{n1}^{c}=\frac{1}{2}\frac{E_{n1}^{c}g_{n1}^{c}-2F_{n1}^{c}f_{n1}^{c}+G_{n1}^{c}e_{n1}^{c}}{E_{n1}^{c}G_{n1}^{c}-{F_{n1}^{c}}^{2}},$$

then the Gaussian and mean curvatures are found as in the hypothesis. □

Corollary 3.1. *Let $\Phi_{n1}^{c}(s,v)$ be a constant-angle ruled surface with MOFC; then the constant-angle surface is*

i. developable surface,
ii. not minimal surface,
iii. Weingarten surface.

Case (2): $f = 0$, $g = 0$, $h \neq 0$, and $\tau = 0$. In this case, the conditions given by (3.6) are satisfied, and then we have obtained a constant-angle ruled surface which takes the form

$$\Phi^c_{n2}(s,v) = \sigma(s) + vhB,$$

where the base curve $\sigma(s)$ is planar. The partial derivatives of the surfaces $\Phi^c_{n2}(s,v)$ using (2.1) are found as

$$(\Phi^c_{n2})_s = T + v\left(h' + \frac{h\kappa'}{\kappa}\right)B$$

and

$$(\Phi^c_{n2})_v = hB.$$

By a straightforward computation, the normal vector of the surface $\Phi^c_{n2}(s,v)$ is

$$U = -N.$$

Theorem 3.3. *Let $\Phi^c_{n2}(s,v)$ be a constant-angle ruled surface; then the Gaussian curvature and mean curvature are*

$$K^c_{n2} = 0 \text{ and } H^c_{n2} = 0,$$

respectively.

Proof. This theorem is proved in a manner akin to that of Theorem 3.2. □

Corollary 3.2. *Let $\Phi^c_{n2}(s,v)$ be a constant-angle ruled surface with MOFC; then $\Phi^c_{n2}(s,v)$ is*

i. developable surface,
ii. minimal surface,
iii. Weingarten surface.

3.1.3. Constant-angle ruled surface parallel to modified binormal vector

Let the normal vector U of the surface $\Phi(s,v)$ be parallel to the modified binormal vector B of the base curve $\sigma(s)$ according to the MOFC; then we have the following conditions:

$$U_3 \neq 0, \ U_1 = U_2 = 0. \tag{3.7}$$

Since $U_{21} = 0$, h vanishes. In that case, from Eq (3.4), we get the following equations:

$$\begin{cases} U_{11} = 0, \ U_{12} = -g^2\tau, \\ U_{21} = 0, \ U_{22} = fg\tau, \\ U_{31} = g, \ U_{32} = g\left(f' - g\kappa^2\right) - f\left(f + g' + \dfrac{g\kappa'}{\kappa}\right). \end{cases}$$

Then, there are the cases below that satisfy the conditions in (3.7).

Case (1): $f \neq 0$, $g = 0$, and $h = 0$. From this case, we have

$$\begin{cases} U_{11} = U_{12} = U_{21} = U_{22} = 0, \\ U_{31} = 0, \ U_{32} = -f^2. \end{cases}$$

These mean that the constant-angle ruled surface takes the form

$$\Phi_{b1}^{c}(s,v) = \sigma(s) + vfT.$$

Also, this case is achieved whenever the base curve $\sigma(s)$ is planar. The normal vector of the surface $\Phi_{b1}^{c}(s,v)$ is obtained as

$$U = -B,$$

since

$$(\Phi_{b1}^{c})_s = (1 + vf')T + vfN \text{ and } (\Phi_{b1}^{c})_v = fT.$$

Theorem 3.4. *Let $\Phi_{b1}^{c}(s,v)$ be a constant-angle ruled surface; then the Gaussian and mean curvatures are*

$$K_{b1}^{c} = 0 \text{ and } H_{b1}^{c} = -\frac{\tau}{2vf},$$

respectively.

Proof. This theorem is proved in a similar manner to the proof of Theorem 3.2. □

Corollary 3.3. *Let $\Phi_{b1}^{c}(s,v)$ be a constant-angle ruled surface; then the constant-angle surface is*

i. *developable surface,*
ii. *minimal surface if and only if the base curve is planar,*
iii. *Weingarten surface.*

Case (2): $f = 0$, $g \neq 0$, $h = 0$, and $\tau = 0$. In this case,

$$\begin{cases} U_{11} = U_{12} = U_{21} = U_{22} = 0, \\ U_{31} = g, \ U_{32} = -g^2\kappa^2, \end{cases}$$

and then it is found that the constant-angle ruled surface is presented in the form

$$\Phi_{b2}^{c}(s,v) = \sigma(s) + vgN,$$

where the base curve $\sigma(s)$ is planar. The normal vector of the surface $\Phi_{b2}^{c}(s,v)$ is obtained as

$$U = B,$$

where

$$(\Phi_{b2}^{c})_s = \left(1 - vg\kappa^2\right)T + v\left(h' + \frac{h\kappa'}{\kappa}\right)N \text{ and } (\Phi_{b2}^{c})_v = gN.$$

Theorem 3.5. *Let $\Phi_{b2}^{c}(s,v)$ be a constant-angle ruled surface; then the Gaussian and mean curvatures are*

$$K_{b2}^{c} = 0 \text{ and } H_{b2}^{c} = 0,$$

respectively.

Proof. The proof of this theorem follows a similar manner to the proof of Theorem 3.2. □

Corollary 3.4. *Let $\Phi_{b2}^{c}(s,v)$ be a constant-angle ruled surface; then $\Phi_{b2}^{c}(s,v)$ is*

i. developable surface,
ii. minimal surface,
iii. Weingarten surface.

Case (3): $f \neq 0$, $g \neq 0$, $h = 0$, and $\tau = 0$. In this case,

$$\begin{cases} U_{11} = U_{12} = U_{21} = U_{22} = 0, \\ U_{31} = g, \ U_{32} = g\left(f' - g\kappa^2\right) - f\left(f + g' + \dfrac{g\kappa'}{\kappa}\right), \end{cases}$$

and then the constant-angle ruled surface is represented by

$$\Phi_{b3}^{c}(s, v) = \sigma(s) + v(fT + gN),$$

where the $\sigma(s)$ is planar. The normal vector of the surface $\Phi_{b3}^{c}(s, v)$ is calculated as

$$U = B,$$

by the facts that

$$\left(\Phi_{b3}^{c}\right)_s = \left(1 - vg\kappa^2 + vf'\right)T + v\left(f + g' + \frac{g\kappa'}{\kappa}\right)N \text{ and } \left(\Phi_{b3}^{c}\right)_v = fT + gN.$$

Theorem 3.6. *Let $\Phi_{b3}^{c}(s, v)$ be a constant-angle ruled surface; then the Gaussian and mean curvatures are*

$$K_{b3}^{c} = 0 \text{ and } H_{b3}^{c} = 0,$$

respectively.

Proof. This is proved similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\Phi_{b3}^{c}(s, v)$ be a constant-angle ruled surface; then $\Phi_{b3}^{c}(s, v)$ is*

i. developable surface,
ii. minimal surface,
iii. Weingarten surface.

3.2. Constant-angle ruled surfaces with MOFT

Let the torsion of the base curve be non-zero everywhere. The partial derivatives of the surface $\Phi(s, v)$ are represented by Eq (3.1) using derivative formulas (2.2). In this section, let's investigate under what conditions the surfaces are constant-angle ruled surfaces using MOFT. Considering the derivative formulas (2.2), the partial differential equations of the surfaces $\Phi(s, v)$ presented by Eq (3.1) are

$$\Phi_s = (1 + v(-g\kappa\tau + f'))T + v\left(g' - h\tau + \frac{g\tau' + f\kappa}{\tau}\right)N + v\left(g\tau + h' + \frac{h\tau'}{\tau}\right)B$$

and

$$\Phi_v = fT + gN + hB,$$

where f, g, and h are smooth functions of s. The cross-product of the above vector fields is found as

$$\begin{aligned}\Phi_s \times \Phi_v = {}& v\left(\frac{fh\kappa}{\tau} - g^2\tau + h\left(-h\tau + g'\right) - gh'\right)T \\ &+ \left(-h + v\left(f\left(g\tau + h'\right) + h\left(g\kappa\tau - f' + \frac{f\tau'}{\tau}\right)\right)\right)N \\ &+ \left(g + v\left(-\frac{f^2\kappa}{\tau} + g\left(f' - g\kappa\tau\right) + f\left(h\tau - g' - g\frac{\tau'}{\tau}\right)\right)\right)B.\end{aligned}$$

From here, the normal vector of the surface $\Phi(s, v)$ can be given in the form

$$W = \left(W_{11} + vW_{12}\right)T + \left(W_{21} + vW_{22}\right)N + \left(W_{31} + vW_{32}\right),$$

such that

$$\begin{cases} W_{11} = 0, \ W_{12} = \dfrac{fh\kappa}{\tau} - g^2\tau + h\left(-h\tau + g'\right) - gh', \\ W_{21} = -h, \ W_{22} = f\left(g\tau + h'\right) + h\left(g\kappa\tau - f' + \dfrac{f\tau'}{\tau}\right), \\ W_{31} = g, \ W_{32} = -\dfrac{f^2\kappa}{\tau} + g\left(f' - g\kappa\tau\right) + f\left(h\tau - g' - \dfrac{g\tau'}{\tau}\right), \end{cases} \tag{3.8}$$

where $\tau \neq 0$.

3.2.1. Constant-angle ruled surface parallel to tangent vector

In this subsection, let the normal vector W of the surface $\Phi(s, v)$ be parallel to the tangent vector T of the base curve $\sigma(s)$ according to the MOFT; then we have the following conditions:

$$W_1 \neq 0, \ W_2 = W_3 = 0. \tag{3.9}$$

Case (1): Considering Eq (3.8), $f = g = h = 0$ is obtained from the solution of Eq (3.9). This situation contradicts the conditions $\Upsilon(s) \neq 0$ and $W_1 \neq 0$.

Case (2): Considering Eq (3.8), $f \neq 0$ and $g = h = \kappa = 0$ are obtained from the solution of Eq (3.9). This situation contradicts the condition $W_1 \neq 0$. So, we can give the following theorem:

Theorem 3.7. *Let the normal vector W of a surface $\Phi(s, v)$ be parallel to the tangent vector of the MOFT; then there is no constant-angle ruled surface parallel to the tangent vector that satisfies conditions Eq (3.9).*

3.2.2. Constant-angle ruled surface parallel to modified principal normal vector

Let the normal vector W of the surface $\Phi(s, v)$ be parallel to the modified normal vector N of the base curve $\sigma(s)$ according to the MOFT; then we have the following conditions:

$$W_2 \neq 0, \ W_1 = W_3 = 0. \tag{3.10}$$

Since $W_{31}=0$, g vanishes. In that case, from Eq (3.8), we get the following equations:

$$\begin{cases} W_{11}=0,\ W_{12}=h\left(\dfrac{f\kappa}{\tau}-h\tau\right), \\ W_{21}=-h,\ W_{22}=fh'-h\left(f'-\dfrac{f\tau'}{\tau}\right), \\ W_{31}=0,\ W_{32}=-f\left(\dfrac{f\kappa}{\tau}-h\tau\right), \end{cases}$$

for $\tau\neq 0$. Then, considering the conditions specified in Eq (3.10), there are some situations as follows:

Case (1): $g=0$, $f\kappa=h\tau^2$, and $h\neq 0$, From this case, we have

$$\begin{cases} W_{11}=0,\ W_{12}=0, \\ W_{21}=-h,\ W_{22}=\dfrac{hh'\tau^2}{\kappa}-h\left(\left(\dfrac{h\tau^2}{\kappa}\right)'-\dfrac{h\tau^2\tau'}{\kappa\tau}\right), \\ W_{31}=0,\ W_{32}=0, \end{cases}$$

where $\kappa\neq 0$ in addition to $\tau\neq 0$. Hence, the conditions specified in Eq (3.10) are satisfied, and then we see that the constant-angle ruled surface takes the form

$$\Phi_{n1}^{t}(s,v)=\sigma(s)+vh\left(\frac{\tau^2}{\kappa}T+B\right),$$

where $f\kappa=h\tau^2$. The partial derivatives of the surface $\Phi_{n1}^{t}(s,v)$ using (2.2) are

$$(\Phi_{n1}^{t})_s=\left(\frac{\kappa^2-vh\tau^2\kappa'+v\kappa\tau(\tau h'+2h\tau')}{\kappa^2}\right)T+v\left(h'+\frac{h\tau'}{\tau}\right)B \text{ and } (\Phi_{n1}^{t})_v=h\left(\frac{\tau^2}{\kappa}T+B\right).$$

The normal vector of the surface $\Phi_{n1}^{t}(s,v)$ is found as

$$W=\frac{\left(\Phi_{n1}^{t}\right)_s\times\left(\Phi_{n1}^{t}\right)_v}{\left\|\left(\Phi_{n1}^{t}\right)_s\times\left(\Phi_{n1}^{t}\right)_v\right\|}=-N.$$

Theorem 3.8. *Let $\Phi_{n1}^{t}(s,v)$ be a constant-angle ruled surface; then the Gaussian and mean curvatures of $\Phi_{n1}^{t}(s,v)$ are*

$$K_{n1}^{t}=0 \text{ and } H_{n1}^{t}=\frac{\kappa\left(\kappa^2+\tau^4\right)}{2\tau\left(vh\tau^2\kappa'-\kappa\left(\kappa+vh\tau\tau'\right)\right)},$$

respectively, where $\tau\neq 0$.

Proof. Let $\Phi_{n1}^{t}(s,v)$ be a constant-angle ruled surface. The coefficients of the first and second fundamental forms, respectively, are determined

$$E_{n1}^{t}=\left\langle(\Phi_{n1}^{t})_s,(\Phi_{n1}^{t})_s\right\rangle=v^2\left(h'+\frac{h\tau'}{\tau}\right)^2+\frac{\left(\kappa^2-vh\tau^2\kappa'+v\kappa\tau(\tau h'+2h\tau')\right)^2}{\kappa^4},$$

$$F_{n1}^{t}=\left\langle(\Phi_{n1}^{t})_s,(\Phi_{n1}^{t})_v\right\rangle=h\left(v\left(h'+\frac{h\tau'}{\tau}\right)+\frac{\tau^2\left(\kappa^2-vh\tau^2\kappa'+v\kappa\tau(\tau h'+2h\tau')\right)}{\kappa^3}\right),$$

$$G_{n1}^{t}=\left\langle(\Phi_{n1}^{t})_v,(\Phi_{n1}^{t})_v\right\rangle=h^2\left(1+\frac{\tau^4}{\kappa^2}\right),$$

and

$$e^t_{n1} = \left\langle (\Phi^t_{n1})_{ss}, U \right\rangle = -\frac{\kappa}{\tau} + \frac{vh\tau\kappa'}{\kappa} - vh\tau',$$
$$f^t_{n1} = \left\langle (\Phi^t_{n1})_{sv}, U \right\rangle = 0,$$
$$g^t_{n1} = \left\langle (\Phi^t_{n1})_{vv}, U \right\rangle = 0,$$

by the aid of the equations

$$(\Phi^t_{n1})_s = \left(\frac{\kappa^2 - vh\tau^2\kappa' + v\kappa\tau\left(\tau h' + 2h\tau'\right)}{\kappa^2}\right)T + v\left(h' + \frac{h\tau'}{\tau}\right)B,\ \ (\Phi^t_{n1})_v = \frac{h\tau^2}{\kappa}T + hB,$$
$$(\Phi^t_{n1})_{ss} = v\left(\frac{2h\tau^2\kappa'^2}{\kappa^3} - \frac{2\tau^2 h'\kappa' + h\tau^2\kappa'' + 4h\tau\kappa'\tau'}{\kappa^2} + \frac{4\tau h'\tau' + 2h\tau'^2 + \tau^2 h'' + 2h\tau\tau''}{\kappa}\right)T$$
$$+\left(\frac{\kappa}{\tau} - \frac{vh\tau\kappa'}{\kappa} + vh\tau'\right)N + v\left(\frac{2h'\tau'}{\tau} + h'' + \frac{h\tau''}{\tau}\right)B,$$
$$(\Phi^t_{n1})_{sv} = \left(\frac{\tau\left(-h\tau\kappa' + \kappa\left(\tau h' + 2h\tau'\right)\right)}{\kappa^2}\right)T + \left(h' + \frac{h\tau'}{\tau}\right)B,\ \ (\Phi^t_{n1})_{vv} = 0.$$

If the coefficients of the first and second fundamental forms are substituted in the formulas of Gaussian and mean curvatures, the proof is completed. □

Corollary 3.6. *Let $\Phi^t_{n1}(s,v)$ be a constant-angle ruled surface with MOFT; then $\Phi^t_{n1}(s,v)$ is*

i. developable surface,
ii. minimal surface if and only if the base curve is a line,
iii. Weingarten surface.

Case (2): $f = 0$, $g = 0$, and $h \neq 0$. This situation contradicts the fact that $W_1 = 0$. So, the ruled surface cannot be a constant-angle surface.

Case (3): $f \neq 0$, $g = 0$, and $h = 0$. This situation contradicts the fact that $W_2 \neq 0$ and $W_3 = 0$. So, the ruled surface cannot be a constant-angle surface.

3.2.3. Constant-angle ruled surface parallel to modified binormal vector

Let the normal vector W of the surface $\Phi(s,v)$ be parallel to the modified binormal vector B of the base curve $\sigma(s)$ according to the MOFT; then we have the following conditions:

$$W_3 \neq 0,\ \ W_1 = W_2 = 0. \tag{3.11}$$

Since $W_{21} = 0$, h vanishes. In that case, from Eq (3.8), we get the following equations:

$$\begin{cases} W_{11} = 0,\ W_{12} = -g^2\tau, \\ W_{21} = 0,\ \ W_{22} = fg\tau, \\ W_{31} = g,\ \ W_{32} = -\dfrac{f^2\kappa}{\tau} + g\left(f' - g\kappa\tau\right) - f\left(g' + \dfrac{g\tau'}{\tau}\right), \end{cases}$$

for $\tau \neq 0$. Then, considering the conditions in Eq (3.9), there exist the following cases.

Case (1): $f \neq 0$, g, h, and $\kappa \neq 0$. From this case, we have

$$\begin{cases} W_{11} = W_{12} = W_{21} = W_{22} = 0, \\ W_{31} = 0, \;\; W_{32} = \dfrac{-f^2\kappa}{\tau}, \end{cases}$$

and we obtain the constant-angle ruled surface, which takes the form

$$\Phi^t_{b1}(s, v) = \sigma(s) + vfT.$$

By a straightforward computation, the normal vector of the surface $\Phi^t_{b1}(s, v)$ is obtained as

$$W = -B,$$

by the partial derivatives

$$(\Phi^t_{b1})_s = (1 + vf')T + \frac{vf\kappa}{\tau}N \text{ and } (\Phi^t_{b1})_v = fT.$$

Theorem 3.9. *Let $\Phi^t_{b1}(s, v)$ be a constant-angle ruled surface; then the Gaussian curvature and mean curvature of $\Phi^t_{b1}(s, v)$ are*

$$K^t_{b1} = 0 \text{ and } H^t_{b1} = -\frac{\tau^2}{2vf\kappa},$$

respectively.

Proof. The proof of this theorem is similar to the proof of Theorem 3.2. □

Corollary 3.7. *Let $\Phi^t_{b1}(s, v)$ be a constant-angle ruled surface with MOFT; then $\Phi^t_{b1}(s, v)$ is*

i. developable surface,
ii. not minimal surface,
iii. Weingarten surface.

Case (2): $f \neq 0$, $g = 0$, $h = 0$, and $\kappa = 0$. This is a contradiction because of $W_3 \neq 0$, which means that in this case, the ruled surface cannot be a constant-angle surface.

Case (3): $f = 0$, $g = 0$, $h = 0$, and $\kappa \neq 0$. This situation contradicts the fact that $\Upsilon(s) \neq 0$ and $W_3 \neq 0$. Thus, we say that the ruled surface cannot be a constant-angle surface in this case.

Example 3.1. *Let us consider a curve given by the parametric equation*

$$\sigma(s) = \left(\frac{1}{\sqrt{2}} \int_0^s \cos\left(\frac{\pi t^2}{2}\right) dt, \frac{1}{\sqrt{2}} \int_0^s \sin\left(\frac{\pi t^2}{2}\right) dt, \frac{s}{\sqrt{2}} \right),$$

which is known as the Cornu spiral or Euler spiral [16]. Also, the components $\int_0^s \cos\left(\frac{\pi t^2}{2}\right) dt$ and $\int_0^s \sin\left(\frac{\pi t^2}{2}\right) dt$ of the curve are called Fresnel integrals. The elements of the Frenet trihedron of the curve

$\sigma(s)$ are found as

$$t = \left(\frac{1}{\sqrt{2}}\cos\left(\frac{\pi s^2}{2}\right), \frac{1}{\sqrt{2}}\sin\left(\frac{\pi s^2}{2}\right), \frac{1}{\sqrt{2}}\right),\ n = \left(-\sin\left(\frac{\pi s^2}{2}\right), \cos\left(\frac{\pi s^2}{2}\right), 0\right),$$
$$b = \left(-\frac{1}{\sqrt{2}}\cos\left(\frac{\pi s^2}{2}\right), -\frac{1}{\sqrt{2}}\sin\left(\frac{\pi s^2}{2}\right), \frac{1}{\sqrt{2}}\right),\ \kappa = \frac{\pi s}{\sqrt{2}},\ \tau = \frac{\pi s}{\sqrt{2}}.$$

Here, we refer to the modified Frenet vectors presented by Sasai because the principal normal and binormal vectors are discontinuous at the neighborhood of the point $s_0 = 0$. In that regard, we find the modified Frenet vectors of σ as follows:

$$T = \left(\frac{1}{\sqrt{2}}\cos\left(\frac{\pi s^2}{2}\right), \frac{1}{\sqrt{2}}\sin\left(\frac{\pi s^2}{2}\right), \frac{1}{\sqrt{2}}\right),$$
$$N = \left(-\frac{\pi s}{\sqrt{2}}\sin\left(\frac{\pi s^2}{2}\right), \frac{\pi s}{\sqrt{2}}\cos\left(\frac{\pi s^2}{2}\right), 0\right),$$
$$B = \left(-\frac{\pi s}{2}\cos\left(\frac{\pi s^2}{2}\right), -\frac{\pi s}{2}\sin\left(\frac{\pi s^2}{2}\right), \frac{\pi s}{2}\right).$$

If we assume that $f = \frac{\pi s \sin(s)}{\sqrt{2}}$, $g = 0$, and $h = \sin(s)$, the equation of the constant-angle ruled surface parallel to the modified normal vector of σ for Case 1 with the MOFC is represented as

$$\Phi^c_{n1}(s, v) = \left(\frac{1}{\sqrt{2}}\int_0^s \cos\left(\frac{\pi t^2}{2}\right)dt, \frac{1}{\sqrt{2}}\int_0^s \sin\left(\frac{\pi t^2}{2}\right)dt, \frac{s}{\sqrt{2}} + v\pi s\sin(s)\right),$$

see Figure 1.

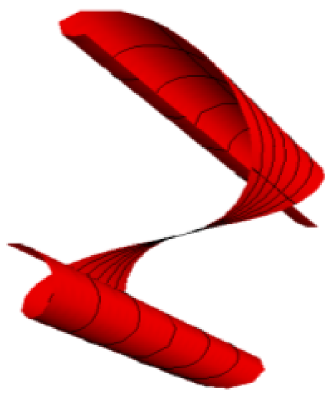

Figure 1. The graph of $\Phi^c_{n1}(s, v)$ parallel to the modified principal normal vector for $s \in (-2, 2)$ and $v \in (-1, 1)$.

Let us take that $f = \sin(s)$ and $h = g = 0$. Then the equation of the constant-angle ruled surface *parallel to the modified binormal vector of σ for Case 1 with MOFC is*

$$\Phi^{c}_{b1}(s,v)=\frac{1}{\sqrt{2}}\left(\begin{array}{c}\int\limits_{0}^{s}\cos\left(\frac{\pi t^{2}}{2}\right)dt+v\cos\left(\frac{\pi s^{2}}{2}\right)\sin(s),\\ \int\limits_{0}^{s}\sin\left(\frac{\pi t^{2}}{2}\right)dt+v\sin\left(\frac{\pi s^{2}}{2}\right)\sin(s),\ s+v\sin(s)\end{array}\right),$$

see Figure 2.

Figure 2. The graph of $\Phi^{c}_{b1}(s,v)$ parallel to modified binormal vector for $s\in(-2,2)$ and $v\in(-1,1)$.

For $f=\cos(s)\frac{\pi s}{\sqrt{2}}$ *and* $h=\cos(s)$, *the equation of the constant-angle ruled surface parallel to the modified principal normal vector of σ for Case 1 with the MOFT is represented by*

$$\Phi^{t}_{n1}(s,v)=\frac{1}{\sqrt{2}}\left(\int\limits_{0}^{s}\cos\left(\frac{\pi t^{2}}{2}\right)dt,\ \int\limits_{0}^{s}\sin\left(\frac{\pi t^{2}}{2}\right)dt,\ s\left(\sqrt{2}\pi v\cos(s)+1\right)\right),$$

see Figure 3.

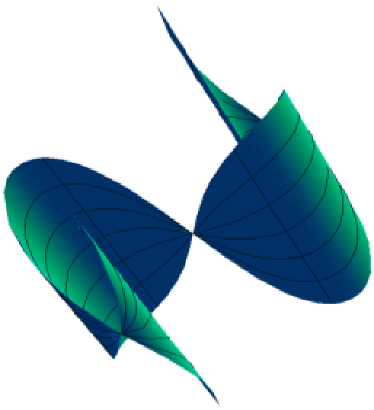

Figure 3. The graph of $\Phi^{t}_{n1}(s,v)$ parallel to modified principal normal vector for $s\in(-2,2)$ and $v\in(-1,1)$.

4. Conclusions

In this study, the modified orthogonal frames in Euclidean 3-space have been employed to investigate the constant-angle ruled surfaces. The investigation involves determining the necessary and sufficient conditions for any ruled surface to stand the angles between each modified Frenet vector of the base curve and the unit normal vector of the surface to be constant. Within this context, the conditions for such surfaces to be minimal, developable, and Weingarten surfaces have been derived. Notably, this study presents novel insights, as constant-angle ruled surfaces have not been previously examined in the context of MOFs. Finally, the study provides examples of some constant-angle surfaces, accompanied by their graphics, and offers a new perspective for future research in the field of surface theory.

Author contributions

Kemal Eren, Soley Ersoy and Mohammad N. I. Khan: Conceptualization, methodology, investigation, writing - original draft, writing - review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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