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Research article

Weak (p, q)-Jordan centralizer and derivation on rings and algebras

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Abstract: In the present paper, the authors discuss two new concepts that will be known as a weak (p, q)-Jordan centralizer and a weak (p, q)-Jordan derivation on an arbitrary ring R and they prove that every weak (p, q)-Jordan derivation is a derivation on any prime ring R. Furthermore, every weak (p, q)-Jordan centralizer is a centralizer on a semiprime ring R. Later, they discuss the continuity of weak (p, q)-Jordan centralizer on a semisimple Banach algebra and prove that every weak (p, q) -Jordan centralizer on a semisimple Banach algebra is a linear continuous operator. Moreover, these results are validated with actual examples.

Keywords: weak (p, q)-Jordan derivations; weak (p, q)-Jordan centralizers; (semi) prime ring; right (left) (Jordan) centralizer; right (left) Jordan derivation
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1. Introduction

R represents an associative ring with unity e in the current research. A ring R is known as t-torsion free; if $ts = 0$, then $s = 0$, for all $s \in R$, and $t > 1$ is any fixed integer. R is considered a prime ring, in case $r_1 R r_2 = \{0\}$ produces either $r_1 = 0$ or $r_2 = 0$ and is referred to as a semiprime ring if $rRr = \{0\}$, then $r = 0$. In [6], Helgosen began to do research on the centralizers of the Banach algebra. Multipliers are also another term for centralizers (see [17]). Wang [16] also looked at the idea of centralization in the commutative Banach algebra. Johnson has proven that a centralizer is continuous in Banach algebra, and he is also pursuing centralizers on topological algebras (details are given in [9]). The notion of centralizers in certain types of rings was then developed by Johnson [10].

Following [8], an additive mapping $T : R \to R$ is recognized as a left centralizer if it is additive and $T(r_1 r_2) = T(r_1) r_2$ holds for all $r_1, r_2 \in R$. If $T(r_1 r_2) = r_1 T(r_2)$ for all pairs $r_1, r_2 \in R$, then T is known as a right centralizer. If T on a ring R is a left and right centralizer, then it is referred to as a centralizer.

Next, if T satisfies $T(r^2) = rT(r)$ and $T(r^2) = T(r)r$ for each $r \in R$ then it is said to be a Jordan right centralizer and a Jordan left centralizer, respectively, and accordingly the Jordan centralizer is defined.

Zalar [18] states that the Jordan centralizer and the centralizer are identical on a 2-torsion-free semiprime ring. Subsequently, Vukman [13] has demonstrated an astonishing discovery that states that if T is additive and fulfills the algebraic equation

$$2T(r^2) = rT(r) + T(r)r,$$

then T is a centralizer on a 2-torsion-free semiprime ring. Later, in [15], Vukman gives a new notion that is known as a (p, q)-Jordan centralizer on an arbitrary ring R and is defined as follows: if R is a ring, and $p \geq 0,\ q \geq 0$ are two fixed integers having $p + q \neq 0$. A mapping $T : R \to R$ is known as a (p, q)-Jordan centralizer, in case T is additive and

$$(p + q)T(r^2) = prT(r) + qT(r)r$$

for each $r \in R$. He established a result that, on a prime R with $charR \neq 6pq(p + q)$, every (p, q)-Jordan centralizer is a centralizer. Further, Kosi-Ulbl and Vukman extended this outcome on semiprime rings under certain restrictions in [11]. Some recent work on centralizers on certain rings and algebras is presented in [1–3, 5, 12].

Motivated by the definition of the (p, q)-Jordan centralizer, in this paper we present a new notion that will become known as a weak (p, q)-Jordan centralizer. We define it as considering R to be a ring and, $p \geq 0,\ q \geq 0$ to be two fixed integers possessing the condition that $p+q \neq 0$. An additive mapping $T : R \to R$ will be recognized as a weak (p, q)-Jordan centralizer, if

$$(p + q)T(r^{2t}) = pT(r^t)r^t + qr^tT(r^t),\ \forall\ r \in R$$

and prove that every weak (p, q)-Jordan centralizer under certain conditions on a semiprime ring R, is a centralizer.

More precisely, any additive mapping T will be a centralizer, with certain limitations on torsion on R, if it satisfies $(p + q)T(r^{2t}) = pT(r^t)r^t + qr^tT(r^t)$ for all $t \in R$. It is clear that for $t = 1$, weak (p, q)-Jordan centralizer and (p, q)-Jordan centralizer are identical and $(1, 1)$-Jordan centralizer is centralizer (see [13]). Further, the $(1, 0)$-Jordan centralizer will be the Jordan left centralizer, and the $(0, 1)$-Jordan centralizer is the Jordan right centralizer. That means a weak (p, q)-Jordan centralizer covers the concepts of Jordan right (left) centralizer and (p, q)-Jordan centralizer as well. An analogous result has been proved that T will be a centralizer on certain torsion conditions if it fulfils $(p + q)T(r^{t+1}) = pT(r)r^t + qr^tT(r)$ and T is additive, for all $t \in R$ and $t, p, q \geq 1$ are some fixed integers.

We require the following result in order to retrieve the conclusion from the main theorems:

Lemma 1.1 ([11]). *Let $p, q \geq 1$ be fixed integers, and R is a semiprime ring that is $pq(p + q)$-torsion free. Then, every additive mapping $T : R \to R$ is a centralizer if it holds that $(p + q)T(r^2) = prT(r) + qT(r)r,\ for\ all\ r \in R$.*

2. Weak (p, q)-Jordan centralizer

Theorem 2.1. *Let $q, p, t \geq 1$ be fixed integers, and R be a $\{(2t - 1)!,\ pq(p + q)\}$-torsion free semiprime ring. Then, every additive mapping $T : R \to R$ will be a centralizer if it satisfies the algebraic condition $(p + q)T(r^{2t}) = pT(r^t)r^t + qr^tT(r^t)$ for every $r \in R$.*

Proof. We have

$$(p+q)T(r^{2t}) = pT(r^t)r^t + qr^tT(r^t)\ \forall\ r \in R. \tag{2.1}$$

Replace r by $r+\alpha e$ in (2.1) to obtain the following:

$(p+q)T\Big(r^{2t} + \binom{2t}{1}\alpha r^{2t-1}e + \cdots + \binom{2t}{2t-2}\alpha^{2t-2}r^2e + \binom{2t}{2t-1}\alpha^{2t-1}re + \alpha^{2t}e\Big) = pT\Big(r^t + \binom{t}{1}\alpha r^{t-1}e + \cdots + \binom{t}{t-2}\alpha^{t-2}r^2e + \binom{t}{t-1}\alpha^{t-1}re + \alpha^te\Big)\Big(r^t + \binom{t}{1}\alpha r^{t-1}e + \cdots + \binom{t}{t-2}\alpha^{t-2}r^2e + \binom{t}{t-1}\alpha^{t-1}re + \alpha^te\Big) + q\Big(r^t + \binom{t}{1}\alpha r^{t-1}e + \cdots + \binom{t}{t-2}\alpha^{t-2}r^2e + \binom{t}{t-1}\alpha^{t-1}re + \alpha^te\Big)T\Big(r^t + \binom{t}{1}\alpha r^{t-1}e + \cdots + \binom{t}{t-2}\alpha^{t-2}r^2e + \binom{t}{t-1}\alpha^{t-1}re + \alpha^te\Big),\ \forall\ r \in R$ and α is any positive integer.

The previous mathematical statement should be rewritten as

$$\alpha g_1(r,e) + \alpha^2 g_2(r,e) + \cdots + \alpha^{2t-1}g_{2t-1}(r,e) = 0,$$

where $g_k(r,e)$ stands for the coefficients of α^k's for all $k = 2t-1, 2t-2, \cdots, 3, 2, 1$. We obtain a system of $(2t-1)$ homogeneous linear equations. Substituting α by $2t-1, 2t-2, \cdots, 3, 2, 1$ in turn to obtain a Vandermonde matrix

$$\mathcal{V}_{\mathcal{M}} = \begin{pmatrix} 1^1 & 1^2 & \cdots & 1^{2t-1} \\ 2^1 & 2^2 & \cdots & 2^{2t-1} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ (2t-1)^1 & (2t-1)^2 & \cdots & (2t-1)^{2t-1} \end{pmatrix}.$$

As the determinant of $\mathcal{V}_{\mathcal{M}}$ equals a multiple of positive integers. Since each element of that product is smaller than $2t-1$, then $g_i(r,e) = 0$ for all $i = 2t-1, 2t-2, \cdots, 3, 2, 1$. Furthermore, setting $i = 2t-1$ to find

$g_{2t-1}(r,e) = \binom{2t}{2t-1}(p+q)T(r) - p\binom{t}{t-1}T(r) - p\binom{t}{t-1}T(e)r - q\binom{t}{t-1}T(r) - q\binom{t}{t-1}rT(e)$ for every element $r \in R$.

This is what a straightforward calculation yields.

$$(q+p)T(r) = qrT(e) + pT(e)r,\ \forall\ r \in R. \tag{2.2}$$

Next, $g_2(r,s) = 0$ implies that

$\binom{2t}{2t-2}(p+q)T(r^2) - p\binom{t}{t-2}T(e)r^2 - p\binom{t}{t-1}\binom{t}{t-1}T(r)r - p\binom{t}{t-2}T(r^2) - q\binom{t}{t-2}T(r^2) - q\binom{t}{t-1}\binom{t}{t-1}rT(r) - qr^2\binom{t}{t-2}T(e) = 0,\ \forall\ r \in R.$

After a simple computation, we arrive at

$2(2t-1)(q+p)T(r^2) = p(t-1)T(e)r^2 + 2ptT(r)r + p(t-1)T(r^2) + q(t-1)T(r^2) + 2qtrT(r) + q(t-1)r^2T(e),\ \forall\ r \in R.$

Which yields that

$$[3pt+3qt-p-q]T(r^2) = p(t-1)T(e)r^2+2ptT(r)r + 2qtrT(r)+q(t-1)r^2T(e)\ \forall\ r\in R. \tag{2.3}$$

Replacing r by r^2 in (2.2), we find

$$(p+q)T(r^2)=pT(e)r^2+qr^2T(e)\ \forall\ r\in R. \tag{2.4}$$

Multiplying from the left side by $(t-1)$, we obtain

$$(p+q)(t-1)T(r^2)=p(t-1)T(e)r^2+q(t-1)r^2T(e)\ \forall\ r\in R. \tag{2.5}$$

Using (2.5) in (2.3), we arrive at $[3pt+3qt-p-q]T(r^2) = (p+q)(t-1)T(r^2)+2ptT(r)r+2qtrT(r)\ \forall\ r\in R$. This implies that $[2pt+2qt]T(r^2) = 2ptT(r)r+2qtrT(r)\ \forall\ r\in R$. Using the torsion condition to obtain $[p+q]T(r^2)=pT(r)r+qrT(r)\ \forall\ r\in R$. Apply Lemma 1.1 to reach out at the desired conclusion.

Theorem 2.2. *Every additive mapping T from a $\{t!,\ pq(p+q)\}$-torsion-free semiprime ring R to R will be a centralizer if it satisfies the algebraic condition $(p+q)T(r^{t+1}) = pT(r)r^t+qr^tT(r)$ for all $t\in R$, where $t,p,q\geq 1$ are some fixed integers.*

Proof. Given that

$$(p+q)T(r^{t+1}) = pT(r)r^t+qr^tT(r) \text{ for every } r\in R. \tag{2.6}$$

In (2.6), replace r with $r+\beta e$ to obtain

$(p+q)T\Big(r^{t+1}+\binom{t+1}{1}\beta r^te+\cdots+\binom{t+1}{t-1}\beta^{t-1}r^2e+\binom{t+1}{t}\beta^t re+\beta^{t+1}e\Big) = p[T(r)+\beta T(e)]\Big(r^t+\binom{t}{1}\beta r^{t-1}e+\cdots+\binom{t}{t-2}\beta^{t-2}r^2e+\binom{t}{t-1}\beta^{t-1}re+\beta^te\Big)+q\Big(r^t+\binom{t}{1}\beta r^{t-1}e+\cdots+\binom{t}{t-2}\beta^{t-2}r^2e+\binom{t}{t-1}\beta^{t-1}re+\beta^te\Big)[T(r)+\beta T(e)],\ \forall\ r\in R$ and β is any positive integer.

Encounter the preceding statement with

$$\beta h_1(r,e)+\beta^2h_2(r,e)+\cdots+\beta^th_t(r,e)=0,$$

where $h_k(r,e)$ signifies the coefficients of β^k's for all $k=1,2,\cdots,t$. If we replace β with $1,2,\cdots,t$, we have such a system of t homogeneous linear equations that correspond to a Vandermonde matrix

$$\mathcal{N}=\begin{pmatrix} 1^1 & 1^2 & \cdots & 1^t \\ 2^1 & 2^2 & \cdots & 2^t \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ t^1 & t^2 & \cdots & t^t \end{pmatrix}.$$

Since $|\mathcal{N}|$ equals the multiple of positive integers and each element of that multiple is smaller than t, then each coefficient of the power of β is zero. Moreover, set $i = t$ and apply the torsion restriction on R to obtain

$$\begin{aligned}\binom{t+1}{t}(p+q)T(r) &= p\binom{t}{t-1}T(e)r + pT(r) \\ &\quad + q\binom{t}{t-1}rT(e) + qT(r), \text{ for each } r \in R.\end{aligned}$$

Simple computation yielded the following:

$$(q+p)T(r) = qrT(e) + pT(e)r \ \forall\ r \in R. \tag{2.7}$$

Later, $h_2(r, e) = 0$ gives that

$$\begin{aligned}\binom{t+1}{t-1}(p+q)T(r^2) &= p\binom{t}{t-2}T(e)r^2 + p\binom{t}{t-1}T(r)r \\ &\quad + q\binom{t}{t-2}r^2T(e) + q\binom{t}{t-1}rT(r) \ \forall\ r \in R.\end{aligned}$$

We obtain the following expression after a simple calculation:

$$(p+q)(t+1)tT(r^2) = (t-1)t[pT(e)r^2 + qr^2T(e)] + 2t[pT(r)r + qrT(r)] \ \forall\ r \in R.$$

Using (2.7), we find

$$\begin{aligned}2(p+q)(t+1)tT(r^2) &= (t-1)t(p+q)T(r^2) \\ &\quad + 2t[pT(r)r + qrT(r)] \ \forall\ r \in R.\end{aligned} \tag{2.8}$$

Further, some calculations with the torsion condition of R, we arrive at $[p+q]T(r^2) = pT(r)r + qrT(r) \ \forall\ r \in R$. Apply Lemma 1.1 to find the required result.
The example demonstrates that the above results are of great importance.

Example 2.1. *Examine a ring* $R = \left\{ \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} \mid r, s, t \in 2\mathbb{Z}_4 \right\}$, *where* $\mathbb{Z}_4$ *has the usual meaning. Let us define a mapping* $T : R \to R$ *by* $T\left[\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}\right] = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$. *The semiprimeness hypothesis has significance for both major theorems, as it is evident that* T *satisfies the algebraic equations of the theorems, but* T *is not a centralizer.*

3. Conclusion on semisimple Banach algebra

It is worth nothing that every semisimple Banach algebra is a semiprime ring. So, the question is: what will be the algebraic structure of the additive mapping that satisfies the algebraic equation present in Theorem 2.1? The conclusion is that additive mapping will be a linear continuous operator. To obtain the main conclusion, the following lemma is required.

Lemma 3.1 ([18]). *Let* $\mathcal{S}$ *be a semisimple Banach algebra. Then every additive mapping* $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ *will be a continuous linear operator if it satisfies the functional condition* $\mathcal{T}(a^2) = \mathcal{T}(a)a$ *for every* $a \in \mathcal{S}$.

More precisely, we obtain the following result:

Theorem 3.1. *Let $\mathcal{S}$ be a semisimple Banach algebra. Then every linear mapping $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ will be continuous if it satisfies the algebraic condition $(p+q)\mathcal{T}(a^{2t}) = p\mathcal{T}(a^t)a^t + qa^t\mathcal{T}(a^t)$ for every $a \in \mathcal{S}$, where t, q, p are some fixed positive integers.*

Proof. In $\mathcal{S}$, $\mathcal{T}$ will act as a centralizer, as semisimple Banach algebra is a semiprime ring. Therefore, from Lemma 3.1, $\mathcal{T}$ is a continuous linear operator.

4. Weak (p, q)-Jordan derivation

This section starts with the study of a derivation, defined as an additive mapping $\mathcal{D} : R \to R$, is recognized as a derivation if it is additive and satisfies $\mathcal{D}(s_1 s_2) = \mathcal{D}(s_1)s_2 + s_1\mathcal{D}(s_2)$ for each $s_1, s_2 \in R$, and particularly, $\mathcal{D}$ is known as a Jordan derivation if $\mathcal{D}(r^2) = \mathcal{D}(r)r + r\mathcal{D}(r)$ for all $r \in R$. It is clear that every derivation is a Jordan derivation, but in general the converse does not hold. In [7], Herstein affirms that every Jordan derivation is a derivation on a prime ring of characteristic different from 2. A generalization of this outcome is given by Cusack [4]. Vukman [14] gave a notion of a (p, q)-Jordan derivation motivated by the ideal of Jordan derivation, which is defined as; let R be a ring and $p \geq 0,\ q \geq 0$ be two fixed integers with $p + q \neq 0$. An additive mapping $\mathcal{D} : R \to R$ is recognized as a (p, q)-Jordan derivation in case $(p + q)\mathcal{D}(r^2) = 2pr\mathcal{D}(r) + 2q\mathcal{D}(r)r,\ \forall\ r \in R$. He established a result that every (p, q)-Jordan derivation is a derivation of any prime ring R having some conditions.

Intrigued by the notion of (p, q)-Jordan derivation, in this paper we define a new concept that will be known as weak (p, q)-Jordan derivation as follows: Let R be a ring and $p,\ q \geq 0$ be two fixed integers having the condition that $p + q \neq 0$. An additive mapping $\mathcal{D} : R \to R$ will be known as a weak (p, q)-Jordan derivation, if $(p + q)\mathcal{D}(r^{2t}) = 2p\mathcal{D}(r^t)r^t + 2qr^t\mathcal{D}(r^t),\ \forall\ r \in R$. Obviously, for $t = 1$, weak (p, q)-Jordan derivation and (p, q)-Jordan derivation are identical, and on a 2-torsion-free ring, $(1, 1)$-Jordan derivation will be a derivation. Later, $(1, 0)$-Jordan derivation will become Jordan left derivation, and $(0, 1)$-Jordan derivation will be a Jordan right derivation. That means a weak (p, q)-Jordan derivation embraces the idea of Jordan left (right) derivation and (p, q)-Jordan derivation both. The authors prove that every weak (p, q)-Jordan derivation under certain conditions on a prime ring R, is a derivation. More precisely, it is substantiated that every additive mapping $\mathcal{D}$ is derivation with some torsion condition of R if it satisfies $(p + q)\mathcal{D}(r^{2t}) = 2p\mathcal{D}(r^t)r^t + 2qr^t\mathcal{D}(r^t)$ for all $r \in R$, and $t \geq 1$ is a fixed integer.

We require the following result in order to arrive at the conclusion of main theorems.

Lemma 4.1 ([14])**.** *Let R be a prime ring possessing $charR \neq 2pq(p + q) \mid p - q \mid$. Every additive mapping $\mathcal{D} : R \to R$ is a derivation, and R is commutative if it fulfills the algebraic condition $(p + q)\mathcal{D}(r^2) = 2pr\mathcal{D}(r) + 2q\mathcal{D}(r)r$, for all $r \in R$ with either $charR = 0$ or $charR > 3$, where $q, p \geq 1$ are fixed integers with p not the same as q.*

Theorem 4.1. *Every additive mapping $\mathcal{D}$ from any prime ring R with $charR \neq 2pq(p + q) \mid p - q \mid$ to itself will be a derivation and R is commutative if it satisfies the algebraic condition $(p + q)\mathcal{D}(r^{2t}) = 2p\mathcal{D}(r^t)r^t + 2qr^t\mathcal{D}(r^t)$ for every $r \in R$, where $p, q, t \geq 1$ are fixed integers with $p \neq q$.*

Proof. We have

$$(p + q)\mathcal{D}(r^{2t}) = 2p\mathcal{D}(r^t)r^t + 2qr^t\mathcal{D}(r^t),\ \forall\ r \in R. \tag{4.1}$$

Replacing r by $r + \varpi e$ in (4.1), we obtain

$(p + q)T\Big(r^{2t} + \binom{2t}{1}\varpi r^{2t-1}e + \cdots + \binom{2t}{2t-2}\varpi^{2t-2}r^2e + \binom{2t}{2t-1}\varpi^{2t-1}re + \varpi^{2t}e\Big) = 2p\mathcal{D}\Big(r^t + \binom{t}{1}\varpi r^{t-1}e + \cdots + \binom{t}{t-2}\varpi^{t-2}r^2e + \binom{t}{t-1}\varpi^{t-1}re + \varpi^t e\Big)\Big(r^t + \binom{t}{1}\varpi r^{t-1}e + \cdots + \binom{t}{t-2}\varpi^{t-2}r^2e + \binom{t}{t-1}\varpi^{t-1}re + \varpi^t e\Big) + 2q\Big(r^t + \binom{t}{1}\varpi r^{t-1}e + \cdots + \binom{t}{t-2}\varpi^{t-2}r^2e + \binom{t}{t-1}\varpi^{t-1}re + \varpi^t e\Big)\mathcal{D}\Big(r^t + \binom{t}{1}\varpi r^{t-1}e + \cdots + \binom{t}{t-2}\varpi^{t-2}r^2e + \binom{t}{t-1}\varpi^{t-1}re + \varpi^t e\Big)$, $\forall\ r \in R$ and ϖ is any positive integer.

If $g_k(r, e)$ stands for the coefficients of ϖ^k's for all $k = 2t-1, 2t-2, \cdots, 3, 2, 1$, then the above equation will be as follows:

$$\varpi g_1(r, e) + \varpi^2 g_2(r, e) + \cdots + \varpi^{2t-1} g_{2t-1}(r, e) = 0$$

that gives a $2t - 1$ homogeneous system of equations. Replacing ϖ with $2t - 1, 2t - 2, \cdots, 3, 2, 1$ in turn, we obtain a Vandermonde matrix of size $(2t - 1) \times (2t - 1)$. Use similar arguments as the last section; we have the coefficients of power of ϖ as 0. Furthermore, setting $i = 2t - 1$ to find

$$\binom{2t}{2t-1}(p + q)\mathcal{D}(r) = 2p\binom{t}{t-1}\mathcal{D}(r) + 2p\binom{t}{t-1}\mathcal{D}(e)r + 2q\binom{t}{t-1}\mathcal{D}(r) + 2q\binom{t}{t-1}r\mathcal{D}(e) \text{ for all } r \in R.$$

A simple calculation gives the following:

$$p\mathcal{D}(e)r + qr\mathcal{D}(e) = 0,\ \forall\ r \in R. \tag{4.2}$$

Next, $g_2(r, s) = 0$ implies that

$$\binom{2t}{2t-2}(p + q)\mathcal{D}(r^2) - 2p\binom{t}{t-2}\mathcal{D}(e)r^2 - 2p\binom{t}{t-1}\binom{t}{t-1}\mathcal{D}(r)r - 2p\binom{t}{t-2}\mathcal{D}(r^2) - 2q\binom{t}{t-2}\mathcal{D}(r^2) - 2q\binom{t}{t-1}\binom{t}{t-1}r\mathcal{D}(r) - 2qr^2\binom{t}{t-2}\mathcal{D}(e) = 0,\ \forall\ r \in R.$$

We conclude the expression that follows after a simple computation:

$$(p + q)(2t - 1)\mathcal{D}(r^2) = p(t-1)\mathcal{D}(e)r^2 + 2pt\mathcal{D}(r)r + p(t-1)\mathcal{D}(r^2) + q(t-1)\mathcal{D}(r^2) + 2qtr\mathcal{D}(r) + q(t-1)r^2\mathcal{D}(e),\ \forall\ r \in R.$$

This gives that

$$\begin{aligned}[p + q]t\mathcal{D}(r^2) &= 2pt\mathcal{D}(r)r + 2qtr\mathcal{D}(r)\\ &\quad +(t-1)[p\mathcal{D}(e)r^2 + qr^2\mathcal{D}(e)]\ \forall\ r \in R.\end{aligned} \tag{4.3}$$

Replacing r by r^2 in (4.2), we find

$$p\mathcal{D}(e)r^2 + qr^2\mathcal{D}(e) = 0\ \forall\ r \in R. \tag{4.4}$$

Using (4.4) in (4.3), we arrive at $[p + q]t\mathcal{D}(r^2) = 2pt\mathcal{D}(r)r + 2qtr\mathcal{D}(r)\ \forall\ r \in R$. Use the torsion condition of R to obtain $[p + q]\mathcal{D}(r^2) = 2p\mathcal{D}(r)r + 2qr\mathcal{D}(r)\ \forall\ r \in R$. Apply Lemma 4.1 to arrive at the required outcome.

The following example shows that the theorem presented in this paper is not inappropriate.

Example 4.1. *Define a ring* $R = \left\{ \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \mid r_1, r_2 \in 2\mathbb{Z}_8 \right\}$, $\mathbb{Z}_8$ *has its usual meaning. Define mappings* $\mathcal{D} : R \rightarrow R$ *by* $\mathcal{D}\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & r_2 \end{pmatrix}$. *It is clear that* $\mathcal{D}$ *is not a derivation, but* $\mathcal{D}$ *satisfies the algebraic conditions in the theorem for* $p = 1,\ q = 2$, *and* $t = 1$. *Which shows that primness and charR are essential conditions for this theorem.*

5. Conclusions

The authors' concern to create a connection between the algebraic and analytical properties of additive mappings that meets a specific functional identity is the article's conclusion. We determined the setting in which, in the case of prime rings, the weak (p, q)-Jordan derivations became the derivation. Also, on semiprime rings, weak (p, q)-Jordan centralizers and the centralizers are precisely the same. Furthermore, every weak (p, q)-Jordan centralizer on a semisimple Banach algebra is a linear and continuous operator, according to the present line of research that added the flavor of continuity. Many new methods in operator theory, functional analysis, and fundamental iterative theory approaches will become accessible if linearity and continuity are established as a generic concept.

Author contributions

Every author made an equal contribution to this study. Conceptualization and formal analysis by A. Z. Ansari; validation and visualization by F. Alharbi; and Writing – original draft, review and editing by F. Shujat.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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