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Research article

Qualitative study of Caputo Erdélyi-Kober stochastic fractional delay differential equations

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Abstract: We present new results on the well-posedness and time regularity of solutions to stochastic fractional delay differential equations (SFDDEs) using the Caputo-Erdélyi-Kober fractional derivative. Additionally, we prove the averaging principle. We establish all results in the $\mathfrak{p}$th moment, which generalizes the case $\mathfrak{p} = 2$. First, by applying fixed-point theory (FPT), we prove that the solution exists, is unique, and continuously depends on the initial values as well as the fractional derivative. Second, we establish a smoothness theorem for the solution and demonstrate that the solution of the original system converges to the averaged system in the $\mathfrak{p}$th moment. Finally, to support our theoretical findings, we present illustrative examples.

Keywords: Caputo Erdélyi-Kober derivatives; stochastic processes; existence and uniqueness; inequalities
Mathematics Subject Classification: 34A07, 34A08, 60G22

1. Introduction

There are many different kinds of fractional derivatives; examples include the Riemann-Liouville (RL), Caputo fractional derivative, Grünwald-Letnikov, Caputo-Fabrizio, conformable, Katugampola, Hadamard, and Erdélyi-Kober fractional derivatives [1, 2].

Various authors have utilized the Caputo-Erdélyi-Kober fractional derivative (CEKFD) in their research on fractional differential equations (FDEs). For example, Arioua and Titraoui [3] studied the existence and uniqueness (Ex-Un) of solutions to FDEs using FPT with CEKFD. Bouteraa et al. [4] presented theoretical results for FDEs with CEKFD. Another study [5] explored solutions to FDEs using the Galerkin-Hermite method in the context of CEKFD. Zhang [6] investigated the well-posedness of FDEs with CEKFD. Fan and Zhang [7] examined the existence, uniqueness, and stability of FDEs with CEKFD. In Luchko and Trujillo [8], various properties of CEKFD were defined. Boumaaza and Benchohra [9] contributed new findings on the Ex-Un of solutions to FDEs using FPT with CEKFD.

The CEKFD of order $\mathfrak{n}-1 < \mathfrak{c} \leq \mathfrak{n}, \mathfrak{n} \in \mathbb{N}$ for a continuous function (Con-F) $\mathfrak{u}(\varsigma) : [\alpha,\ \infty) \to \mathbb{R}$ is defined as follows [10]:

$$\mathcal{D}^{\mathfrak{c}}_{\alpha+;\vartheta,\lambda}\gimel(\varsigma) = \vartheta\frac{\varsigma^{-\vartheta\lambda}}{\Gamma(\mathfrak{n}-\mathfrak{c})}\int_{\alpha}^{\varsigma}\mathfrak{u}^{\vartheta-1}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{n}-\mathfrak{c}-1}\Big(\frac{1}{\vartheta\mathfrak{u}^{\vartheta-1}}\mathcal{D}\Big)^{\mathfrak{n}}\mathfrak{u}^{\vartheta(\lambda+\mathfrak{c})}\gimel(\mathfrak{u})d\mathfrak{u}, \tag{1.1}$$

where $\varsigma > \alpha \geq 0, \vartheta > 0$, and $\lambda \in \mathbb{R}$.

The Erdélyi-Kober fractional integral of order $\mathfrak{c} > 0$ for the Con-F $\mathfrak{u}(\varsigma) : [\alpha,\ \infty) \to \mathbb{R}$ is as follows [10]:

$$\mathcal{I}^{\mathfrak{c}}_{\alpha;\vartheta,\lambda}\gimel(\varsigma) = \vartheta\frac{\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}\mathfrak{u}^{\vartheta(\lambda+1)-1}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\gimel(\mathfrak{u})d\mathfrak{u}. \tag{1.2}$$

The relationship between the Erdélyi-Kober fractional integral and the CEKFD, for $0 < \mathfrak{c} \leq 1$, is given as follows [10]:

$$\mathcal{I}^{\mathfrak{c}}_{\alpha;\vartheta,\lambda}\mathcal{D}^{\mathfrak{c}}_{\alpha+;\vartheta,\lambda}\gimel(\varsigma) = \gimel(\varsigma) - \alpha^{\vartheta(\lambda+\lambda)}\gimel(\alpha)\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}. \tag{1.3}$$

In recent years, numerous scholars have explored diverse aspects of fractional stochastic differential equations (FSDEs). Batiha et al. [11] introduced a novel method for solving FSDEs, deriving approximate solutions and comparing them with those obtained through other techniques. Chen et al. [12] examined the stability and Ex-Un of solutions to FSDEs, employing the Euler-Maruyama method for their computations. Moualkia and Xu [13] conducted a theoretical study on variable-order FSDEs, deriving approximate solutions and evaluating their precision against other approaches. Ali et al. [14] analyzed a coupled system of FSDEs, focusing on solution Ex-Un and stability. Li et al. [15] investigated the stability of FSDE systems, exploring the interplay between fractional calculus, stochastic processes, and time delays. Their work enhances the understanding of system stability and evaluates numerical solution methods and different stability types for FSDEs. In [16], researchers performed Ex-Un and stability analyses for FSDEs involving conformable derivatives. The authors of [17] demonstrated the convergence of the Euler-Maruyama method for FSDEs, applied it to obtain solutions, and presented stability findings. Meanwhile, [18] discussed solution Ex-Un for FSDEs with Lévy noise using the Picard scheme. Li et al. [19] studied Hilfer FSDEs with delay, establishing Ex-Un via the Picard method and analyzing finite-time stability using various inequalities. For additional insights into FSDEs, refer to [20–22].

The averaging principle (A-P) serves as a powerful tool for analyzing complex systems. By replacing intricate time-dependent equations with their averaged counterparts, this approach simplifies analysis and reduces computational complexity. The validity of the A-P hinges on identifying conditions under which the averaged system accurately represents the original dynamics. Several researchers have contributed to the development of the A-P from different perspectives. For instance, Zou et al. [23] derived the A-P for FSDEs with impulses, while Zou and Luo [24] extended these findings to SFDDEs involving the Caputo operator. Additionally, the authors of [25] investigated the A-P for neutral FSDEs with Caputo derivatives. Further studies have explored and validated the A-P across various contexts, as seen in [26–28].

SFDDEs are FDEs that incorporate fractional derivatives to describe memory effects, time delays to capture time-lagged interactions, and stochastic processes to model randomness or noise. These equations are particularly suitable for systems where past conditions, delay effects, and random variations have a significant impact on dynamics. This research study uses the FPT to determine the Ex-Un results of the solution to SFDDEs. Next, we present the continuous dependence results by assuming that the coefficients correspond to the global Lipschitz condition. Additionally, various inequalities are employed to establish regularity and A-P results. Finally, examples are included to support the results derived from this study. The primary tools used in our proofs include the Burkholder-Davis-Gundy inequality (Bu-Da-Gu-Ineq) [29], Jensen's inequality (Jen-Ineq) [30], and Hölder's inequality (Höl-Ineq) [31].

Remark 1.1. *SFDDEs with CEKFD operators present fundamentally greater challenges than classical fractional models due to the complex interplay of memory effects, stochasticity, and time delays, which complicates theoretical analyses of well-posedness, solution regularity, and A-P. The intrinsic connection between CEKFD operators and probabilistic properties further necessitates careful consideration of functional spaces, noise structures, and solution methodologies, making these systems distinctly more difficult to analyze than their traditional counterparts.*

Listed below are the main contributions of our study:

1) To the best of our knowledge, no research has been conducted on the well-posedness, regularity, and A-P of SFDDEs in the sense of the CEKFD. This study is the first to establish significant results regarding SFDDEs in the sense of CEKFD.
2) While existing results for FSDEs predominantly focus on mean-square convergence, our work establishes these findings in the more general $\mathfrak{p}$th moment framework. This approach naturally extends the established theory on well-posedness, regularity, and A-P for SFDDEs, with the mean-square case $\mathfrak{p} = 2$ emerging as a special instance of our broader analysis.
3) In this work, we establish theoretical results for FSDEs incorporating delay terms, extending the existing framework to account for time-lagged system dynamics.

We examined the following SFDDEs in this research driven by Brownian motions [32].

$$\begin{cases} \mathcal{D}^{\mathfrak{c}}_{\alpha+;\vartheta,\lambda}\beth(\varsigma) = \mathfrak{f}(\varsigma, \beth(\varsigma), \beth(\varsigma - \mathfrak{r})) + \mathfrak{g}(\varsigma, \beth(\varsigma), \beth(\varsigma - \mathfrak{r}))\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma}, \\ \beth(\varsigma) = \sigma(\varsigma),\ -\mathfrak{r} \leq \varsigma \leq \alpha, \\ \beth(\alpha) = \sigma, \end{cases} \tag{1.4}$$

where σ is a constant, $\mathfrak{r} \in \mathbb{R}$ is the delay time, $\sigma(\varsigma)$ is the history function for all $\varsigma \in [-\mathfrak{r}, \alpha]$, and $\mathcal{D}^{\mathfrak{c}}_{\alpha+;\vartheta,\lambda}$ represent CEKFD with $\mathfrak{c} \in (\frac{1}{2}, 1]$, $\vartheta > 0$, $\varsigma > \alpha \geq 0$, and $\lambda \in \mathbb{R}$. The $\mathfrak{f} : [\alpha, \theta] \times \mathbb{R}^{\rho} \times \mathbb{R}^{\rho} \to \mathbb{R}^{\rho}$ and $\mathfrak{g} : [\alpha, \theta] \times \mathbb{R}^{\rho} \times \mathbb{R}^{\rho} \to \mathbb{R}^{\rho\times\mathfrak{b}}$ are measurable Con-Fs. The stochastic process $(\mathrm{w}_{\varsigma})_{\varsigma\in[0,\infty)}$ follows a standard Brownian trajectory within the $\mathfrak{b}$-dimensional complete probability space $(\Omega, \mathbb{F}, \mathcal{P})$.

The remainder of this paper is organized as follows: Section 2 presents a preliminary definition and key assumptions. In Section 3, we establish our main results regarding the well-posedness and regularity of SFDDEs with CEKFD operators. Section 4 develops the A-P for these systems. Section 5 provides illustrative examples that demonstrate our theoretical results. Finally, Section 6 concludes the paper with a summary of our findings and their implications.

2. Preliminaries

We now introduce a definition and assumptions that support the findings of this study.

Definition 2.1. *For $\mathfrak{p} \geq 2$ and $\varsigma \in [\alpha, \infty)$, assume $\mathbb{A}^{\mathfrak{p}}_{\varsigma} = \mathrm{L}^{\mathfrak{p}}(\Omega, \mathbb{F}, \mathcal{P})$ consists of all $\mathbb{F}_{\varsigma}$-measurable with $\mathfrak{p}^{th}$-integrable $\beth = (\beth_1, \beth_2, \cdots, \beth_\rho)^T : \Omega \to \mathbb{R}^\rho$ as*

$$\|\beth\|_{\mathfrak{p}} = \left(\sum_{\jmath=1}^{\rho} \mathfrak{E}(|\beth_\jmath|^{\mathfrak{p}})\right)^{\frac{1}{\mathfrak{p}}}.$$

The $\beth(\varsigma) : [\alpha, \theta] \to \mathrm{L}^{\mathfrak{p}}(\Omega, \mathbb{F}, \mathcal{P})$ is a $\mathbb{F}-$adapted process when $\beth(\varsigma) \in \mathbb{A}^{\mathfrak{p}}_{\varsigma}$ and $\varsigma \geq 0$. Now, we obtain the equivalent integral form of Eq (1.4). First, apply the Erdélyi-Kober fractional integral $\mathcal{I}^{\mathfrak{c}}_{\alpha;\vartheta,\lambda}$ to both sides of Eq (1.4). So, we have

$$\mathcal{I}^{\mathfrak{c}}_{\alpha;\vartheta,\lambda}\mathcal{D}^{\mathfrak{c}}_{\alpha+;\vartheta,\lambda}\beth(\varsigma) = \mathcal{I}^{\mathfrak{c}}_{\alpha;\vartheta,\lambda}\mathfrak{f}(\varsigma, \beth(\varsigma), \beth(\varsigma - \mathfrak{r})) + \mathcal{I}^{\mathfrak{c}}_{\alpha;\vartheta,\lambda}\mathfrak{g}(\varsigma, \beth(\varsigma), \beth(\varsigma - \mathfrak{r}))\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma}. \tag{2.1}$$

Now, by utilizing Eq. (1.3), we obtain

$$\begin{aligned}\beth(\varsigma) &= \alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)} + \frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\mathfrak{u}^{\vartheta\lambda+\vartheta-1}\mathfrak{f}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u} - \mathfrak{r}))\mathrm{d}\mathfrak{u} \\ &\quad + \frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\mathfrak{u}^{\vartheta\lambda+\vartheta-1}\mathfrak{g}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u} - \mathfrak{r}))\mathrm{dw}(\mathfrak{u}).\end{aligned} \tag{2.2}$$

For $\mathfrak{f}$ and $\mathfrak{g}$, assume the following:

1) $(\mathbb{H}_1)$ When $\forall \ell_1, \ell_2, \zeta_1, \zeta_2 \in \mathbb{R}^\rho$ there are $\mathscr{L}_1$ and $\mathscr{L}_2$ such as

$$\|\mathfrak{f}(\varsigma, \ell_1, \ell_2) - \mathfrak{f}(\varsigma, \zeta_1, \zeta_2)\|_{\mathfrak{p}} \leq \mathscr{L}_1(\|\ell_1 - \zeta_1\|_{\mathfrak{p}} + \|\ell_2 - \zeta_2\|_{\mathfrak{p}}).$$

$$\|\mathfrak{g}(\varsigma, \ell_1, \ell_2) - \mathfrak{g}(\varsigma, \zeta_1, \zeta_2)\|_{\mathfrak{p}} \leq \mathscr{L}_2(\|\ell_1 - \zeta_1\|_{\mathfrak{p}} + \|\ell_2 - \zeta_2\|_{\mathfrak{p}}).$$

2) $(\mathbb{H}_2)$ For $\mathfrak{f}(\varsigma, \alpha, \alpha)$ and the $\mathfrak{g}(\varsigma, \alpha, \alpha)$, we have

$$\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\mathfrak{f}(\varsigma, \alpha, \alpha)\|_{\mathfrak{p}} < \psi, \; \underset{\varsigma\in[\alpha,\theta]}{esssup}\|\mathfrak{g}(\varsigma, \alpha, \alpha)\|_{\mathfrak{p}} < \psi.$$

Now assume

1) $(\mathbb{H}_3)$: When $\forall \ell_1, \ell_2, \zeta_1, \zeta_2, \ell, \zeta \in \mathbb{R}^\rho$, $\varsigma \in [\alpha, \theta]$ there is $\mathscr{L}_3 > 0$ such as:

$$\begin{aligned}\|\mathfrak{f}(\varsigma, \ell_1, \ell_2) &- \mathfrak{f}(\varsigma, \zeta_1, \zeta_2)\| \vee \|\mathfrak{g}(\varsigma, \ell_1, \ell_2) - \mathfrak{g}(\varsigma, \zeta_1, \zeta_2)\| \\ &\leq \mathscr{L}_3(\|\ell_1 - \zeta_1\| + \|\ell_2 - \zeta_2\|).\end{aligned}$$

2) $(\mathbb{H}_4)$: When $\forall \ell_1, \ell_2, \zeta_1, \zeta_2, \ell, \zeta \in \mathbb{R}^{\rho}$, $\varsigma \in [\alpha, \theta]$ there is $\mathscr{L}_4 > 0$ such as satisfy the following:

$$\|\mathfrak{f}(\varsigma, \ell, \zeta)\| \vee \|\mathfrak{g}(\varsigma, \ell, \zeta)\| \leq \mathscr{L}_4(1 + \|\ell\| + \|\zeta\|).$$

3) $(\mathbb{H}_5)$: Functions $\widetilde{\mathfrak{f}}$ and $\widetilde{\mathfrak{g}}$ exist and for $\theta_1 \in [\alpha, \theta]$, $\varsigma \in [\alpha, \theta]$, and $\mathfrak{p} \geq 2$, we have as follows:

$$\frac{1}{\theta_1}\int_0^{\theta_1} \|\mathfrak{f}(\varsigma, \ell, \zeta) - \widetilde{\mathfrak{f}}(\ell, \zeta)\|^{\mathfrak{p}} d\varsigma \leq \aleph_1(\theta_1)(1 + \|\ell\|^{\mathfrak{p}} + \|\zeta\|^{\mathfrak{p}}),$$

$$\frac{1}{\theta_1}\int_0^{\theta_1} \|\mathfrak{g}(\varsigma, \ell, \zeta) - \widetilde{\mathfrak{g}}(\ell, \zeta)\|^{\mathfrak{p}} d\varsigma \leq \aleph_2(\theta_1)(1 + \|\ell\|^{\mathfrak{p}} + \|\zeta\|^{\mathfrak{p}}),$$

where $\lim_{\theta_1 \to \infty} \aleph_1(\theta_1) = 0$, $\lim_{\theta_1 \to \infty} \aleph_2(\theta_1) = 0$ and $\aleph_1(\theta_1)$, $\aleph_2(\theta_1)$ are positively bound functions.

3. Well-posedness

First, we present the important results regarding well-posedness for SFDDEs.

We have $\hbar_\sigma : \mathcal{H}^{\mathfrak{p}}(\alpha, \theta) \to \mathcal{H}^{\mathfrak{p}}(\alpha, \theta)$ with $\hbar_\sigma(\beth(\alpha)) = \sigma$. Then, we get

$$\begin{aligned}\hbar_\sigma(\beth(\varsigma)) = \alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)} &+ \frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\mathfrak{u}^{\vartheta\lambda+\vartheta-1}\mathfrak{f}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r}))d\mathfrak{u} \\ &+ \frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\mathfrak{u}^{\vartheta\lambda+\vartheta-1}\mathfrak{g}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r}))dw(\mathfrak{u}).\end{aligned} \tag{3.1}$$

The following lemma is very important to prove various results.

$$\left\|\beth_1 + \beth_2\right\|_{\mathfrak{p}}^{\mathfrak{p}} \leq 2^{\mathfrak{p}-1}(\|\beth_1\|_{\mathfrak{p}}^{\mathfrak{p}} + (\|\beth_2\|_{\mathfrak{p}}^{\mathfrak{p}}), \ \forall \beth_1, \beth_2 \in \mathbb{R}^{\rho}. \tag{3.2}$$

Lemma 3.1. *Suppose* $(\mathbb{H}_1)$ *and* $(\mathbb{H}_2)$ *are valid. Then* $\hbar_\sigma$ *is well-defined.*

Proof. For $\beth(\varsigma) \in \mathcal{H}^{\mathfrak{p}}[\alpha, \theta]$ and $\varsigma \in [\alpha, \theta]$, the following results are derived using Eqs (3.1) and (3.2).

$$\begin{aligned}\left\|\hbar_\sigma(\beth(\varsigma))\right\|_{\mathfrak{p}}^{\mathfrak{p}} \leq & 2^{\mathfrak{p}-1}\left\|\alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}\right\|_{\mathfrak{p}}^{\mathfrak{p}} + \frac{2^{2\mathfrak{p}-2}(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{\mathfrak{p}}}{\Gamma^{\mathfrak{p}}(\mathfrak{c})}\left\|\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\mathfrak{f}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r}))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\right\|_{\mathfrak{p}}^{\mathfrak{p}} \\ &+ \frac{2^{2\mathfrak{p}-2}(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{\mathfrak{p}}}{\Gamma^{\mathfrak{p}}(\mathfrak{c})}\left\|\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\mathfrak{g}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r}))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u})\right\|_{\mathfrak{p}}^{\mathfrak{p}}.\end{aligned} \tag{3.3}$$

By Höl-Ineq, we have

$$\begin{aligned}&\left\|\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\mathfrak{f}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r}))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\right\|_{\mathfrak{p}}^{\mathfrak{p}} \leq \sum_{\jmath=1}^{\rho}\mathfrak{E}\Big(\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\left|\mathfrak{f}_\jmath(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r}))\right|\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big)^{\mathfrak{p}} \\ &\leq \sum_{\jmath=1}^{\rho}\mathfrak{E}\Big(\Big(\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\frac{(\mathfrak{c}-1)\mathfrak{p}}{(\mathfrak{p}-1)}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}d\mathfrak{u}\Big)^{\mathfrak{p}-1}\int_\alpha^\varsigma \left|\mathfrak{f}_\jmath(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r}))\right|^{\mathfrak{p}}d\mathfrak{u}\Big)\end{aligned}$$

$$\leq \sum_{\jmath=1}^{\rho} \mathfrak{E}\Big(\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}\Big)^{\mathfrak{p}-1}\Big(\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\frac{(\mathfrak{c}-1)\mathfrak{p}}{(\mathfrak{p}-1)}}(\mathfrak{u}^{\vartheta-1})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}d\mathfrak{u}\Big)^{\mathfrak{p}-1}\int_{\alpha}^{\varsigma}\big|\mathfrak{f}_{\jmath}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))\big|^{\mathfrak{p}}d\mathfrak{u}\Big)$$
$$\leq \sum_{\jmath=1}^{\rho} \mathfrak{E}\Big(\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}\Big)^{\mathfrak{p}-1}\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta-1})^{\frac{1}{\mathfrak{p}-1}}\Big)^{\mathfrak{p}-1}\Big(\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\frac{(\mathfrak{c}-1)\mathfrak{p}}{(\mathfrak{p}-1)}}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\Big)^{\mathfrak{p}-1}\int_{\alpha}^{\varsigma}\big|\mathfrak{f}_{\jmath}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))\big|^{\mathfrak{p}}d\mathfrak{u}\Big)$$
$$\leq \mathbb{C}^{\mathfrak{p}-1}\mathbb{V}^{\mathfrak{p}-1}((\varsigma^{\vartheta}-\alpha^{\vartheta})^{\frac{(\mathfrak{c}\mathfrak{p}-1)}{\mathfrak{p}-1}})^{\mathfrak{p}-1}\Big(\frac{\mathfrak{p}-1}{\vartheta(\mathfrak{c}\mathfrak{p}-1)}\Big)^{\mathfrak{p}-1}\int_{\alpha}^{\varsigma}\big\|\mathfrak{f}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))\big\|_{\mathfrak{p}}^{\mathfrak{p}}d\mathfrak{u}, \tag{3.4}$$

where $\mathbb{C}=\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}$ and $\mathbb{V}=\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta-1})^{\frac{1}{\mathfrak{p}-1}}$.

From $(\mathbb{H}_1)$, we obtain

$$\begin{aligned}\big\|\mathfrak{f}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))\big\|_{\mathfrak{p}}^{\mathfrak{p}} \leq& 2^{\mathfrak{p}-1}\Big(\big\|\mathfrak{f}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))-\mathfrak{f}(\mathfrak{u},\alpha,\alpha)\big\|_{\mathfrak{p}}^{\mathfrak{p}}+\big\|\mathfrak{f}(\mathfrak{u},\alpha,\alpha)\big\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)\\ \leq& 2^{\mathfrak{p}-1}\Big(2^{\mathfrak{p}-1}\mathscr{L}_1^{\mathfrak{p}}\Big(\big\|\beth(\mathfrak{u})\big\|_{\mathfrak{p}}^{\mathfrak{p}}+\big\|\beth(\mathfrak{u}-\mathfrak{r})\big\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)+\big\|\mathfrak{f}(\mathfrak{u},\alpha,\alpha)\big\|_{\mathfrak{p}}^{\mathfrak{p}}\Big).\end{aligned} \tag{3.5}$$

Consequently, we get

$$\begin{aligned}\int_{\alpha}^{\varsigma}\big\|\mathfrak{f}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))\big\|_{\mathfrak{p}}^{\mathfrak{p}}d\mathfrak{u} \leq& 2^{2\mathfrak{p}-2}\mathscr{L}_1^{\mathfrak{p}}\Big(\Big(\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup}\|\beth(\mathfrak{u})\|_{\mathfrak{p}}\Big)^{\mathfrak{p}}+\Big(\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup}\big\|\beth(\mathfrak{u}-\mathfrak{r})\big\|_{\mathfrak{p}}\Big)^{\mathfrak{p}}\Big)\\ &\int_{\alpha}^{\varsigma}1d\mathfrak{u}+2^{\mathfrak{p}-1}\int_{\alpha}^{\varsigma}\big\|\mathfrak{f}(\mathfrak{u},\alpha,\alpha)\big\|_{\mathfrak{p}}^{\mathfrak{p}}d\mathfrak{u}\\ &\leq 2^{2\mathfrak{p}-2}\theta\mathscr{L}_1^{\mathfrak{p}}\Big(\big\|\beth(\mathfrak{u})\big\|_{\mathcal{H}^{\mathfrak{p}}}^{\mathfrak{p}}+\big\|\beth(\mathfrak{u}-\mathfrak{r})\big\|_{\mathcal{H}^{\mathfrak{p}}}^{\mathfrak{p}}\Big)\\ &+2^{\mathfrak{p}-1}\int_{\alpha}^{\varsigma}\big\|\mathfrak{f}(\mathfrak{u},\alpha,\alpha)\big\|_{\mathfrak{p}}^{\mathfrak{p}}d\mathfrak{u}.\end{aligned} \tag{3.6}$$

Based on Eqs (3.4) and (3.6), we establish

$$\begin{aligned}\Big\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\mathfrak{f}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big\|_{\mathfrak{p}}^{\mathfrak{p}} \leq& \mathbb{C}^{\mathfrak{p}-1}\mathbb{V}^{\mathfrak{p}-1}(\varsigma^{\frac{\vartheta(\mathfrak{c}\mathfrak{p}-1)}{\mathfrak{p}-1}})^{\mathfrak{p}-1}\\ &\Big(\frac{\mathfrak{p}-1}{\vartheta(\mathfrak{c}\mathfrak{p}-1)}\Big)^{\mathfrak{p}-1}2^{\mathfrak{p}-1}\Big(2^{\mathfrak{p}-1}\mathscr{L}_1^{\mathfrak{p}}\theta\Big(\big\|\beth(\mathfrak{u})\big\|_{\mathcal{H}^{\mathfrak{p}}}^{\mathfrak{p}}+\big\|\beth(\mathfrak{u}-\mathfrak{r})\big\|_{\mathcal{H}^{\mathfrak{p}}}^{\mathfrak{p}}\Big)+\int_{\alpha}^{\varsigma}\big\|\mathfrak{f}(\mathfrak{u},\alpha,\alpha)\big\|_{\mathfrak{p}}^{\mathfrak{p}}d\mathfrak{u}\Big).\end{aligned} \tag{3.7}$$

By applying $(\mathbb{H}_2)$ to Eq (3.7), we derive

$$\begin{aligned}\Big\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\mathfrak{f}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big\|_{\mathfrak{p}}^{\mathfrak{p}} \leq& \mathbb{C}^{\mathfrak{p}-1}\mathbb{V}^{\mathfrak{p}-1}(\varsigma^{\frac{\vartheta(\mathfrak{c}\mathfrak{p}-1)}{\mathfrak{p}-1}})^{\mathfrak{p}-1}\\ &\Big(\frac{\mathfrak{p}-1}{\vartheta(\mathfrak{c}\mathfrak{p}-1)}\Big)^{\mathfrak{p}-1}2^{\mathfrak{p}-1}\Big(2^{\mathfrak{p}-1}\mathscr{L}_1^{\mathfrak{p}}\theta\Big(\big\|\beth(\mathfrak{u})\big\|_{\mathcal{H}^{\mathfrak{p}}}^{\mathfrak{p}}+\big\|\beth(\mathfrak{u}-\mathfrak{r})\big\|_{\mathcal{H}^{\mathfrak{p}}}^{\mathfrak{p}}\Big)+\theta\psi^{\mathfrak{p}}\Big).\end{aligned} \tag{3.8}$$

According to the Bu-Da-Gu-Ineq and Höl-Ineq, we derive

$$
\begin{aligned}
&\Big\| \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{\mathfrak{c}-1} \mathfrak{g}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r})) \mathfrak{u}^{\vartheta\lambda+\vartheta-1} dw(\mathfrak{u}) \Big\|_{\mathfrak{p}}^{\mathfrak{p}} \\
&= \sum_{\jmath=1}^{\rho} \mathfrak{E} \Big| \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{\mathfrak{c}-1} (\mathfrak{g}_{\jmath}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r})) \mathfrak{u}^{\vartheta\lambda+\vartheta-1} dw(\mathfrak{u}) \Big|^{\mathfrak{p}} \\
&\leq \sum_{\jmath=1}^{\rho} C_{\mathfrak{p}} \mathfrak{E} \Big| \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} \Big| \mathfrak{g}_{\jmath}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r})) \Big|^{2} (\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2} d\mathfrak{u} \Big|^{\frac{\mathfrak{p}}{2}} \\
&\leq \sum_{\jmath=1}^{\rho} C_{\mathfrak{p}} \mathfrak{E} \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} \Big| \mathfrak{g}_{\jmath}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r})) \Big|^{\mathfrak{p}} (\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2} d\mathfrak{u} \\
&\Big(\int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} (\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2} d\mathfrak{u} \Big)^{\frac{\mathfrak{p}-2}{2}} \\
&\leq \sum_{\jmath=1}^{\rho} C_{\mathfrak{p}} \mathfrak{E} \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} \Big| \mathfrak{g}_{\jmath}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r})) \Big|^{\mathfrak{p}} (\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2} d\mathfrak{u} \\
&\Big(\sup_{\alpha < \mathfrak{u} \leq \varsigma} (\mathfrak{u}^{\vartheta\lambda})^{2} \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} (\mathfrak{u}^{\vartheta-1})^{2} d\mathfrak{u} \Big)^{\frac{\mathfrak{p}-2}{2}} \\
&\leq \sum_{\jmath=1}^{\rho} C_{\mathfrak{p}} \mathfrak{E} \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} \Big| \mathfrak{g}_{\jmath}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r})) \Big|^{\mathfrak{p}} (\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2} d\mathfrak{u} \\
&\Big(\sup_{\alpha < \mathfrak{u} \leq \varsigma} (\mathfrak{u}^{\vartheta\lambda})^{2} \sup_{\alpha < \mathfrak{u} \leq \varsigma} \mathfrak{u}^{\vartheta-1} \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} \mathfrak{u}^{\vartheta-1} d\mathfrak{u} \Big)^{\frac{\mathfrak{p}-2}{2}} \\
&\leq \mathbf{G}^{\frac{\mathfrak{p}-2}{2}} \mathbf{U}^{\frac{\mathfrak{p}-2}{2}} C_{\mathfrak{p}} \Big(\frac{(\varsigma^{\vartheta} - \alpha^{\vartheta})^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)} \Big)^{\frac{\mathfrak{p}-2}{2}} \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} \big\| \mathfrak{g}(\mathfrak{u}, \beth(\mathfrak{u}, \beth(\mathfrak{u}-\mathfrak{r})) \big\|_{\mathfrak{p}}^{\mathfrak{p}} (\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2} d\mathfrak{u}, \qquad (3.9)
\end{aligned}
$$

where $\mathbf{G} = \sup\limits_{\alpha<\mathfrak{u}\leq\varsigma} (\mathfrak{u}^{\vartheta\lambda})^2$, $\mathbf{U} = \sup\limits_{\alpha<\mathfrak{u}\leq\varsigma} \mathfrak{u}^{\vartheta-1}$, and $C_{\mathfrak{p}} = \left(\frac{\mathfrak{p}^{\mathfrak{p}+1}}{2(\mathfrak{p}-1)^{\mathfrak{p}-1}}\right)^{\frac{\mathfrak{p}}{2}}$.

By utilizing $(\mathbb{H}_1)$ and $(\mathbb{H}_2)$, we obtain the following result:

$$
\begin{aligned}
\big\| \mathfrak{g}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r})) \big\|_{\mathfrak{p}}^{\mathfrak{p}} &\leq 2^{2\mathfrak{p}-2} \mathscr{L}_{2}^{\mathfrak{p}} \Big(\big\| \beth(\mathfrak{u}) \big\|_{\mathfrak{p}}^{\mathfrak{p}} + \big\| \beth(\mathfrak{u}-\mathfrak{r}) \big\|_{\mathfrak{p}}^{\mathfrak{p}} \Big) + 2^{\mathfrak{p}-1} \big\| \mathfrak{g}(\mathfrak{u}, 0, 0) \big\|_{\mathfrak{p}}^{\mathfrak{p}} \\
&\leq 2^{2\mathfrak{p}-2} \mathscr{L}_{2}^{\mathfrak{p}} \Big(\big\| \beth(\mathfrak{u}) \big\|_{\mathfrak{p}}^{\mathfrak{p}} + \big\| \beth(\mathfrak{u}-\mathfrak{r}) \big\|_{\mathfrak{p}}^{\mathfrak{p}} \Big) + 2^{\mathfrak{p}-1} \psi^{\mathfrak{p}}. \qquad (3.10)
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} \big\| \mathfrak{g}(\mathfrak{u}, \beth(\mathfrak{u}), \beth(\mathfrak{u}-\mathfrak{r})) \big\|_{\mathfrak{p}}^{\mathfrak{p}} (\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2} d\mathfrak{u} &\leq 2^{2\mathfrak{p}-2} \mathscr{L}_{2}^{\mathfrak{p}} \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} \Big(\Big(\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup} \big\| \beth(\mathfrak{u}) \big\|_{\mathfrak{p}} \Big)^{\mathfrak{p}} \\
&+ \Big(\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup} \big\| \beth(\mathfrak{u}-\mathfrak{r}) \big\|_{\mathfrak{p}} \Big)^{\mathfrak{p}} \Big) (\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2} d\mathfrak{u} + 2^{\mathfrak{p}-1} \psi^{\mathfrak{p}} \int_{\alpha}^{\varsigma} (\varsigma^{\vartheta} - \mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2} (\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2} d\mathfrak{u}
\end{aligned}
$$

$$\leq 2^{2\mathfrak{p}-2}\mathscr{L}_2^{\mathfrak{p}} \sup_{\alpha<\mathfrak{u}\leq\varsigma} (\mathfrak{u}^{\vartheta\lambda})^2 \int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Bigg(\Big(\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup}\big\|\beth(\mathfrak{u})\big\|_{\mathfrak{p}}\Big)^{\mathfrak{p}}$$
$$+\Big(\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup}\big\|\beth(\mathfrak{u}-\mathfrak{r})\big\|_{\mathfrak{p}}\Big)^{\mathfrak{p}}\Bigg)(\mathfrak{u}^{\vartheta-1})^2 d\mathfrak{u} + 2^{\mathfrak{p}-1}\psi^{\mathfrak{p}} \sup_{\alpha<\mathfrak{u}\leq\varsigma} (\mathfrak{u}^{\vartheta\lambda})^2 \int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}(\mathfrak{u}^{\vartheta-1})^2 d\mathfrak{u}$$
$$\leq 2^{2\mathfrak{p}-2}\mathscr{L}_2^{\mathfrak{p}} \sup_{\alpha<\mathfrak{u}\leq\varsigma} (\mathfrak{u}^{\vartheta\lambda})^2 \sup_{\alpha<\mathfrak{u}\leq\varsigma} \mathfrak{u}^{\vartheta-1} \int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Bigg(\Big(\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup}\big\|\beth(\mathfrak{u})\big\|_{\mathfrak{p}}\Big)^{\mathfrak{p}}$$
$$+\Big(\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup}\big\|\beth(\mathfrak{u}-\mathfrak{r})\big\|_{\mathfrak{p}}\Big)^{\mathfrak{p}}\Bigg)\mathfrak{u}^{\vartheta-1} d\mathfrak{u} + 2^{\mathfrak{p}-1}\psi^{\mathfrak{p}} \sup_{\alpha<\mathfrak{u}\leq\varsigma} (\mathfrak{u}^{\vartheta\lambda})^2 \sup_{\alpha<\mathfrak{u}\leq\varsigma} \mathfrak{u}^{\vartheta-1} \int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\mathfrak{u}^{\vartheta-1} d\mathfrak{u}$$
$$= \mathbf{GU}\frac{2^{\mathfrak{p}-1}(\varsigma^\vartheta - \alpha^\vartheta)^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\left(2^{\mathfrak{p}-1}\mathscr{L}_2^{\mathfrak{p}}\Big(\|\beth(\mathfrak{u})\|_{\mathcal{H}_{\mathfrak{p}}}^{\mathfrak{p}} + \|\beth(\mathfrak{u}-\mathfrak{r})\|_{\mathcal{H}_{\mathfrak{p}}}^{\mathfrak{p}}\Big) + \psi^{\mathfrak{p}}\right). \tag{3.11}$$

So, from above, we have

$$\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\big\|\mathfrak{g}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))\big\|_{\mathfrak{p}}^{\mathfrak{p}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2 d\mathfrak{u} \leq \frac{2^{\mathfrak{p}-1}(\varsigma^\vartheta - \alpha^\vartheta)^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\mathbf{GU}$$
$$\left(2^{\mathfrak{p}-1}\mathscr{L}_2^{\mathfrak{p}}\Big(\|\beth(\mathfrak{u})\|_{\mathcal{H}_{\mathfrak{p}}}^{\mathfrak{p}} + \|\beth(\mathfrak{u}-\mathfrak{r})\|_{\mathcal{H}_{\mathfrak{p}}}^{\mathfrak{p}}\Big) + \psi^{\mathfrak{p}}\right). \tag{3.12}$$

By applying Eq (3.12) to Eq (3.9), we obtain the following result:

$$\Big\|\int_\alpha^\varsigma (\lambda(\varsigma)-\lambda(\mathfrak{u}))^{\mathfrak{c}-1}\mathfrak{g}(\mathfrak{u},\beth(\mathfrak{u}),\beth(\mathfrak{u}-\mathfrak{r}))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u})\Big\|_{\mathfrak{p}}^{\mathfrak{p}} \leq \mathbf{G}^{\frac{\mathfrak{p}}{2}}\mathbf{U}^{\frac{\mathfrak{p}}{2}}C_{\mathfrak{p}}\left(\frac{(\varsigma^\vartheta - \alpha^\vartheta)^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\right)^{\frac{\mathfrak{p}-2}{2}}$$
$$\frac{2^{\mathfrak{p}-1}\varsigma^{\vartheta(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\left(2^{\mathfrak{p}-1}\mathscr{L}_2^{\mathfrak{p}}\Big(\|\beth(\mathfrak{u})\|_{\mathcal{H}_{\mathfrak{p}}}^{\mathfrak{p}} + \|\beth(\mathfrak{u}-\mathfrak{r})\|_{\mathcal{H}_{\mathfrak{p}}}^{\mathfrak{p}}\Big) + \psi^{\mathfrak{p}}\right). \tag{3.13}$$

By putting Eqs (3.8) and (3.13) into Eq (3.3), we find that $\|\hbar_\sigma(\beth(\varsigma))\|_{\mathcal{H}\mathfrak{p}} < \infty$. So, the $\hbar_\sigma$ is well-defined. Now, we establish the following result, which we use to prove the Ex-Un.

Lemma 3.2. *If* $\mathfrak{c} \in (\frac{1}{2}, 1], \vartheta > 0$, *and* $\lambda \in \mathbb{R}$, *then* $\forall\varsigma \in [\alpha,\theta]$, *the following inequality holds:*

$$\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\exp(\eta\mathfrak{u}^\vartheta)\mathfrak{u}^{\vartheta-1}d\mathfrak{u} \leq \frac{\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\eta^{\mathfrak{c}}}\exp(\eta\varsigma^\vartheta).$$

Proof. From Eq (1.2), we obtain the following:

$$\begin{aligned}\mathbb{I}_\varsigma^{\lambda,\mathfrak{c}}\exp(\eta\varsigma^\vartheta) &= \frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\exp(\eta\mathfrak{u}^\vartheta)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\\ &\leq \frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\sup_{\alpha<\mathfrak{u}\leq\varsigma}\mathfrak{u}^{\vartheta\lambda}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\exp(\eta\mathfrak{u}^\vartheta)\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\\ &= \frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\mathbb{J}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\exp(\eta\mathfrak{u}^\vartheta)\mathfrak{u}^{\vartheta-1}d\mathfrak{u}.\end{aligned}$$

Accordingly, we get

$$\mathbb{I}_{\varsigma}^{\lambda,\mathfrak{c}} \exp(\eta\varsigma^{\vartheta}) \leq \frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\mathbb{J}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\exp(\eta\mathfrak{u}^{\vartheta})\mathfrak{u}^{\vartheta-1}d\mathfrak{u}.$$

By applying the variable substitution $\mathrm{z} = \varsigma^{\vartheta} - \mathfrak{u}^{\vartheta}$, we obtain

$$\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\mathbb{J}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\exp(\eta\mathfrak{u}^{\vartheta})\mathfrak{u}^{\vartheta-1}d\mathfrak{u} = \frac{\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\exp(\eta\varsigma^{\vartheta})\mathbb{J} \int_{0}^{\varsigma^{\vartheta}} \mathrm{z}^{\mathfrak{c}-1}\exp(-\eta\mathrm{z})d\mathrm{z}. \tag{3.14}$$

By now applying the substitution $\mathcal{Y} = \eta\mathrm{z}$ in Eq (3.14), we obtain

$$\begin{aligned}\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\mathbb{J}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\exp(\eta\mathfrak{u}^{\vartheta})\mathfrak{u}^{\vartheta-1}d\mathfrak{u} &= \frac{\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\eta^{\mathfrak{c}}\Gamma(\mathfrak{c})}\exp(\eta\varsigma^{\vartheta})\mathbb{J}\int_{0}^{\eta\varsigma^{\vartheta}}\mathcal{Y}^{\mathfrak{c}-1}\exp(-\mathcal{Y})d\mathcal{Y}\\ &\leq \frac{\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\eta^{\mathfrak{c}}\Gamma(\mathfrak{c})}\exp(\eta\varsigma^{\vartheta})\mathbb{J}\int_{0}^{\infty}\mathcal{Y}^{\mathfrak{c}-1}\exp(-\mathcal{Y})d\mathcal{Y}\\ &= \frac{\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\eta^{\mathfrak{c}}}\mathbb{J}\exp(\eta\varsigma^{\vartheta}).\end{aligned}$$

Thus, we have

$$\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\exp(\eta\mathfrak{u}^{\vartheta})\mathfrak{u}^{\vartheta-1}d\mathfrak{u} \leq \frac{\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\eta^{\mathfrak{c}}}\exp(\eta\varsigma^{\vartheta}). \tag{3.15}$$

Theorem 3.1. *If* $(\mathbb{H}_1)$ *and* $(\mathbb{H}_2)$ *are satisfied, then the Eq* (1.4) *with* $\beth(\alpha) = \sigma$ *has a unique solution.*

Proof. Taking $\eta > 0$, we have

$$\vartheta\eta^{2\mathfrak{c}-1} > 2\delta\Gamma(2\mathfrak{c}-1), \tag{3.16}$$

where

$$\begin{aligned}\delta = \frac{2^{\mathfrak{p}-1}(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{\mathfrak{p}}}{\Gamma^{\mathfrak{p}}(\mathfrak{c})}\Big(&2^{\mathfrak{p}-1}\mathbf{GUQ}^{\mathfrak{p}-1}F^{\mathfrak{p}-1}\frac{\mathscr{L}_1^{\mathfrak{p}}(\varsigma^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1)}(\mathfrak{p}-1)^{\mathfrak{p}-1}}{(\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1))^{\mathfrak{p}-1}}\\ &+2^{\mathfrak{p}-1}\mathbf{G}^{\frac{\mathfrak{p}}{2}}\mathbf{U}^{\frac{\mathfrak{p}}{2}}\Big(\frac{(\varsigma^{\vartheta}-\alpha^{\vartheta})^{\vartheta(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\Big)^{\frac{\mathfrak{p}-2}{2}}\mathscr{L}_2^{\mathfrak{p}}C_{\mathfrak{p}}\Big).\end{aligned} \tag{3.17}$$

The weighted norm $\|\cdot\|_{\eta}$ is

$$\|\beth(\varsigma)\|_{\eta} = \underset{\varsigma\in[\alpha,\theta]}{esssup}\Big(\frac{\|\beth(\varsigma)\|_{\mathfrak{p}}^{\mathfrak{p}}}{\exp(\eta\varsigma^{\vartheta})}\Big)^{\frac{1}{\mathfrak{p}}},\ \forall\beth(\varsigma)\in\mathcal{H}^{\mathfrak{p}}([\alpha,\theta]). \tag{3.18}$$

For $\beth(\varsigma)$ and $\widetilde{\beth}(\varsigma)$, we get

$$\|\hbar_\sigma(\daleth(\varsigma)) - \hbar_\sigma(\widetilde{\daleth}(\varsigma))\|_{\mathfrak{p}}^{\mathfrak{p}} \leq$$
$$\frac{2^{\mathfrak{p}-1}(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{\mathfrak{p}}}{\Gamma^{\mathfrak{p}}(\mathfrak{c})}\Big\|\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\daleth(\mathfrak{u}),\daleth(\mathfrak{u}-\mathfrak{r})) - \mathfrak{f}(\mathfrak{u},\widetilde{\daleth}(\mathfrak{u}),\widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big\|_{\mathfrak{p}}^{\mathfrak{p}}+$$
$$\frac{2^{\mathfrak{p}-1}(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{\mathfrak{p}}}{\Gamma^{\mathfrak{p}}(\mathfrak{c})}\Big\|\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathfrak{u},\daleth(\mathfrak{u}),\daleth(\mathfrak{u}-\mathfrak{r})) - \mathfrak{g}(\mathfrak{u},\widetilde{\daleth}(\mathfrak{u}),\widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u})\Big\|_{\mathfrak{p}}^{\mathfrak{p}}. \quad (3.19)$$

Using the Höl-Ineq and $(\mathbb{H}_1)$, we obtain

$$\begin{aligned}
&\Big\|\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\daleth(\mathfrak{u}),\daleth(\mathfrak{u}-\mathfrak{r})) - \mathfrak{f}(\mathfrak{u},\widetilde{\daleth}(\mathfrak{u}),\widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big\|_{\mathfrak{p}}^{\mathfrak{p}}\\
&=\sum_{\jmath=1}^{\rho}\mathfrak{E}\Big|\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{f}_\jmath(\mathfrak{u},\daleth(\mathfrak{u}),\daleth(\mathfrak{u}-\mathfrak{r})) - \mathfrak{f}_\jmath(\mathfrak{u},\widetilde{\daleth}(\mathfrak{u}),\widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big|^{\mathfrak{p}}\\
&\leq\sum_{\jmath=1}^{\rho}\mathfrak{E}\Bigg(\Big(\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\frac{(\mathfrak{c}-1)(\mathfrak{p}-2)}{\mathfrak{p}-1}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{\frac{\mathfrak{p}-2}{\mathfrak{p}-1}}d\mathfrak{u}\Big)^{\mathfrak{p}-1}\\
&\quad\Big(\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{f}_\jmath(\mathfrak{u},\daleth(\mathfrak{u}),\daleth(\mathfrak{u}-\mathfrak{r})) - \mathfrak{f}_\jmath(\mathfrak{u},\widetilde{\daleth}(\mathfrak{u}),\widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\Big|^{\mathfrak{p}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}\Big)\Bigg)\\
&\leq\sum_{\jmath=1}^{\rho}\mathfrak{E}\Bigg(\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}-2}{\mathfrak{p}-1}}\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\frac{(\mathfrak{c}-1)(\mathfrak{p}-2)}{\mathfrak{p}-1}}(\mathfrak{u}^{\vartheta-1})^{\frac{\mathfrak{p}-2}{\mathfrak{p}-1}}d\mathfrak{u}\Big)^{\mathfrak{p}-1}\\
&\quad\Big(\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{f}_\jmath(\mathfrak{u},\daleth(\mathfrak{u}),\daleth(\mathfrak{u}-\mathfrak{r})) - \mathfrak{f}_\jmath(\mathfrak{u},\widetilde{\daleth}(\mathfrak{u}),\widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\Big|^{\mathfrak{p}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}\Big)\Bigg)\\
&\leq\sum_{\jmath=1}^{\rho}\mathfrak{E}\Bigg(\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}-2}{\mathfrak{p}-1}}\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta-1})^{\frac{1}{1-\mathfrak{p}}}\sum_{\jmath=1}^{\rho}\mathfrak{E}\Big(\Big(\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\frac{(\mathfrak{c}-1)(\mathfrak{p}-2)}{\mathfrak{p}-1}}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\Big)^{\mathfrak{p}-1}\\
&\quad\Big(\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{f}_\jmath(\mathfrak{u},\daleth(\mathfrak{u}),\daleth(\mathfrak{u}-\mathfrak{r})) - \mathfrak{f}_\jmath(\mathfrak{u},\widetilde{\daleth}(\mathfrak{u}),\widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\Big|^{\mathfrak{p}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}\Big)\Bigg)\\
&\leq 2^{\mathfrak{p}-1}F^{\mathfrak{p}-1}\mathbf{Q}^{\mathfrak{p}-1}\frac{\mathscr{L}_1^{\mathfrak{p}}(\varsigma^\vartheta - \alpha^\vartheta)^{(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1)}(\mathfrak{p}-1)^{\mathfrak{p}-1}}{(\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1))^{\mathfrak{p}-1}}\\
&\quad\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big(\big\|\daleth(\mathfrak{u}) - \widetilde{\daleth}(\mathfrak{u}))\big\|_{\mathfrak{p}}^{\mathfrak{p}} + \big\|\daleth(\mathfrak{u}-\mathfrak{r}) - \widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\big\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)(\mathfrak{u}^{\vartheta-1})^2d\mathfrak{u}.\\
&\leq 2^{\mathfrak{p}-1}F^{\mathfrak{p}-1}\mathbf{Q}^{\mathfrak{p}-1}\frac{\mathscr{L}_1^{\mathfrak{p}}(\varsigma^\vartheta - \alpha^\vartheta)^{(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1)}(\mathfrak{p}-1)^{\mathfrak{p}-1}}{(\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1))^{\mathfrak{p}-1}}\\
&\quad\mathbf{GU}\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big(\big\|\daleth(\mathfrak{u}) - \widetilde{\daleth}(\mathfrak{u}))\big\|_{\mathfrak{p}}^{\mathfrak{p}} + \big\|\daleth(\mathfrak{u}-\mathfrak{r}) - \widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\big\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)\mathfrak{u}^{\vartheta-1}d\mathfrak{u}. \qquad (3.20)
\end{aligned}$$

where $F = \sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}-2}{\mathfrak{p}-1}}$, $\mathbf{Q} = \sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta-1})^{\frac{1}{1-\mathfrak{p}}}$.

Hence, we have

$$\Big\|\int_\alpha^\varsigma(\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\daleth(\mathfrak{u}),\daleth(\mathfrak{u}-\mathfrak{r})) - \mathfrak{f}(\mathfrak{u},\widetilde{\daleth}(\mathfrak{u}),\widetilde{\daleth}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big\|_{\mathfrak{p}}^{\mathfrak{p}}$$

$$\leq 2^{\mathfrak{p}-1}\mathbf{GUQ}^{\mathfrak{p}-1}F^{\mathfrak{p}-1}\frac{\mathscr{L}_1^{\mathfrak{p}}(\varsigma^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1)}(\mathfrak{p}-1)^{\mathfrak{p}-1}}{(\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1))^{\mathfrak{p}-1}}$$
$$\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\Big(\big\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u}))\big\|_{\mathfrak{p}}^{\mathfrak{p}}+\big\|\mathfrak{I}(\mathfrak{u}-\mathfrak{r})-\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r}))\big\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)\mathfrak{u}^{\vartheta-1}\mathrm{d}\mathfrak{u}. \tag{3.21}$$

Thus, by utilizing $(\mathbb{H}_1)$ and the Bu-Da-Gu-Ineq, we obtain:

$$\Big\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathfrak{u},\mathfrak{I}(\mathfrak{u}),\mathfrak{I}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{g}(\mathfrak{u},\widetilde{\mathfrak{I}}(\mathfrak{u}),\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}\mathrm{dw}(\mathfrak{u})\Big\|_{\mathfrak{p}}^{\mathfrak{p}}$$
$$=\sum_{\jmath=1}^{\rho}\mathfrak{E}\Big|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{g}_{\jmath}(\mathfrak{u},\mathfrak{I}(\mathfrak{u}),\mathfrak{I}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{g}_{\jmath}(\mathfrak{u},\widetilde{\mathfrak{I}}(\mathfrak{u}),\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}\mathrm{dw}(\mathfrak{u})\Big|^{\mathfrak{p}}$$
$$\leq\sum_{\jmath=1}^{\rho}C_{\mathfrak{p}}\mathfrak{E}\Big|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\big|\mathfrak{g}_{\jmath}(\mathfrak{u},\mathfrak{I}(\mathfrak{u}),\mathfrak{I}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{g}_{\jmath}(\mathfrak{u},\widetilde{\mathfrak{I}}(\mathfrak{u}),\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r}))\big|^{2}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2\mathrm{d}\mathfrak{u}\Big|^{\frac{\mathfrak{p}}{2}}$$
$$\leq\sum_{\jmath=1}^{\rho}C_{\mathfrak{p}}\mathfrak{E}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\big|\mathfrak{g}_{\jmath}(\mathfrak{u},\mathfrak{I}(\mathfrak{u}),\mathfrak{I}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{g}_{\jmath}(\mathfrak{u},\widetilde{\mathfrak{I}}(\mathfrak{u}),\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r}))\big|^{\mathfrak{p}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2\mathrm{d}\mathfrak{u}$$
$$\Big(\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2\mathrm{d}\mathfrak{u}\Big)^{\frac{\mathfrak{p}-2}{2}}$$
$$\leq\sum_{\jmath=1}^{\rho}C_{\mathfrak{p}}\mathfrak{E}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\big|\mathfrak{g}_{\jmath}(\mathfrak{u},\mathfrak{I}(\mathfrak{u}),\mathfrak{I}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{g}_{\jmath}(\mathfrak{u},\widetilde{\mathfrak{I}}(\mathfrak{u}),\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r}))\big|^{\mathfrak{p}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2\mathrm{d}\mathfrak{u}$$
$$\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}(\mathfrak{u}^{\vartheta-1})^2\mathrm{d}\mathfrak{u}\Big)^{\frac{\mathfrak{p}-2}{2}}$$
$$\leq\sum_{\jmath=1}^{\rho}C_{\mathfrak{p}}\mathfrak{E}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\big|\mathfrak{g}_{\jmath}(\mathfrak{u},\mathfrak{I}(\mathfrak{u}),\mathfrak{I}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{g}_{\jmath}(\mathfrak{u},\widetilde{\mathfrak{I}}(\mathfrak{u}),\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r}))\big|^{\mathfrak{p}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2\mathrm{d}\mathfrak{u}$$
$$\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\sup_{\alpha<\mathfrak{u}\leq\varsigma}\mathfrak{u}^{\vartheta-1}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\mathfrak{u}^{\vartheta-1}\mathrm{d}\mathfrak{u}\Big)^{\frac{\mathfrak{p}-2}{2}}$$
$$\leq 2^{\mathfrak{p}-1}\mathbf{G}^{\frac{\mathfrak{p}-2}{2}}\mathbf{U}^{\frac{\mathfrak{p}-2}{2}}\Big(\frac{(\varsigma^{\vartheta}-\alpha^{\vartheta})^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\Big)^{\frac{\mathfrak{p}-2}{2}}$$
$$\mathscr{L}_2^{\mathfrak{p}}C_{\mathfrak{p}}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\Big(\big\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\mathfrak{p}}^{\mathfrak{p}}+\big\|\mathfrak{I}(\mathfrak{u}-\mathfrak{r})-\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r})\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2\mathrm{d}\mathfrak{u}$$
$$\leq 2^{\mathfrak{p}-1}\mathbf{G}^{\frac{\mathfrak{p}-2}{2}}\mathbf{U}^{\frac{\mathfrak{p}-2}{2}}\Big(\frac{(\varsigma^{\vartheta}-\alpha^{\vartheta})^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\Big)^{\frac{\mathfrak{p}-2}{2}}\mathscr{L}_2^{\mathfrak{p}}C_{\mathfrak{p}}\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}$$
$$\Big(\big\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\mathfrak{p}}^{\mathfrak{p}}+\big\|\mathfrak{I}(\mathfrak{u}-\mathfrak{r})-\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r})\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)(\mathfrak{u}^{\vartheta-1})^2\mathrm{d}\mathfrak{u}$$
$$\leq 2^{\mathfrak{p}-1}\mathbf{G}^{\frac{\mathfrak{p}-2}{2}}\mathbf{U}^{\frac{\mathfrak{p}-2}{2}}\Big(\frac{(\varsigma^{\vartheta}-\alpha^{\vartheta})^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\Big)^{\frac{\mathfrak{p}-2}{2}}\mathscr{L}_2^{\mathfrak{p}}C_{\mathfrak{p}}\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\sup_{\alpha<\mathfrak{u}\leq\varsigma}\mathfrak{u}^{\vartheta-1}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}$$
$$\Big(\big\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\mathfrak{p}}^{\mathfrak{p}}+\big\|\mathfrak{I}(\mathfrak{u}-\mathfrak{r})-\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r})\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)\mathfrak{u}^{\vartheta-1}\mathrm{d}\mathfrak{u}. \tag{3.22}$$

So, from above

$$\begin{aligned}&\Big\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathfrak{u},\mathfrak{I}(\mathfrak{u}),\mathfrak{I}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{g}(\mathfrak{u},\widetilde{\mathfrak{I}}(\mathfrak{u}),\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}\mathrm{dw}(\mathfrak{u})\Big\|_{\mathfrak{p}}^{\mathfrak{p}}\\&\leq 2^{\mathfrak{p}-1}\mathbf{G}^{\frac{\mathfrak{p}}{2}}\mathbf{U}^{\frac{\mathfrak{p}}{2}}\Big(\frac{(\varsigma^{\vartheta}-\alpha^{\vartheta})^{\vartheta(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\Big)^{\frac{\mathfrak{p}-2}{2}}\mathscr{L}_{2}^{\mathfrak{p}}C_{\mathfrak{p}}\\&\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\Big(\big\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\mathfrak{p}}^{\mathfrak{p}}+\big\|\mathfrak{I}(\mathfrak{u}-\mathfrak{r})-\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r})\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)\mathfrak{u}^{\vartheta-1}\mathrm{du}.\end{aligned}\tag{3.23}$$

Thus, $\forall\varsigma\in[\alpha,\theta]$, we have

$$\|\hbar_{\sigma}(\mathfrak{I}(\varsigma))-\hbar_{\sigma}(\widetilde{\mathfrak{I}}(\varsigma))\big\|_{\mathfrak{p}}^{\mathfrak{p}}\leq\delta\int_{\alpha}^{\varsigma}\Big(\big\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\mathfrak{p}}^{\mathfrak{p}}+\big\|\mathfrak{I}(\mathfrak{u}-\mathfrak{r})-\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r})\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\mathfrak{u}^{\vartheta-1}\mathrm{du},\tag{3.24}$$

It follows that we derive

$$\begin{aligned}\frac{\|\hbar_{\sigma}\mathfrak{I}(\varsigma)-\hbar_{\sigma}\widetilde{\mathfrak{I}}(\varsigma)\|_{\mathfrak{p}}^{\mathfrak{p}}}{\exp(\eta\varsigma^{\vartheta})}\leq&\frac{1}{\exp(\eta\varsigma^{\vartheta})}\delta\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\\&\Big(\exp(\eta\mathfrak{u}^{\vartheta})\frac{\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\mathfrak{p}}^{\mathfrak{p}}}{\exp(\eta\mathfrak{u}^{\vartheta})}+\exp(\eta(\mathfrak{u}-\mathfrak{r})^{\vartheta})\frac{\|\mathfrak{I}(\mathfrak{u}-\mathfrak{r})-\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r})\|_{\mathfrak{p}}^{\mathfrak{p}}}{\exp(\eta(\mathfrak{u}-\mathfrak{r})^{\vartheta})}\Big)\mathfrak{u}^{\vartheta-1}\mathrm{du}\\&\leq\frac{1}{\exp(\eta\varsigma^{\vartheta})}\delta\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\Big(\exp(\eta\mathfrak{u}^{\vartheta})\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup}\Big(\frac{\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\mathfrak{p}}^{\mathfrak{p}}}{\exp(\eta\mathfrak{u}^{\vartheta})}\Big)\\&+\exp(\eta(\mathfrak{u}-\mathfrak{r})^{\vartheta})\underset{\mathfrak{u}\in[\alpha,\theta]}{esssup}\Big(\frac{\|\mathfrak{I}(\mathfrak{u}-\mathfrak{r})-\widetilde{\mathfrak{I}}(\mathfrak{u}-\mathfrak{r})\|_{\mathfrak{p}}^{\mathfrak{p}}}{\exp(\eta(\mathfrak{u}-\mathfrak{r})^{\vartheta})}\Big)\Big)\mathfrak{u}^{\vartheta-1}\mathrm{du}\\&\leq\frac{\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\eta}^{\mathfrak{p}}}{\exp(\eta\varsigma^{\vartheta})}\delta\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\Big(\exp(\eta\mathfrak{u}^{\vartheta})+\exp(\eta(\mathfrak{u}-\mathfrak{r})^{\vartheta})\Big)\mathfrak{u}^{\vartheta-1}\mathrm{du}\\&\leq\frac{2\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\eta}^{\mathfrak{p}}}{\exp(\eta\varsigma^{\vartheta})}\delta\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\exp(\eta\mathfrak{u}^{\vartheta})\mathfrak{u}^{\vartheta-1}\mathrm{du}.\end{aligned}\tag{3.25}$$

Now, replace $\mathfrak{c}$ by $2\mathfrak{c}-1$ in Eq (3.15).

$$\frac{\vartheta\varsigma^{-\vartheta(\lambda+2\mathfrak{c}-1)}}{\Gamma(2\mathfrak{c}-1)}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\exp(\eta\mathfrak{u}^{\vartheta})\mathfrak{u}^{\vartheta-1}\mathrm{du}\leq\frac{\varsigma^{-\vartheta(\lambda+2\mathfrak{c}-1)}}{\eta^{2\mathfrak{c}-1}}\exp(\eta\varsigma^{\vartheta}).$$

The above inequality can be written as

$$\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\exp(\eta\mathfrak{u}^{\vartheta})\mathfrak{u}^{\vartheta-1}\mathrm{du}\leq\frac{\Gamma(2\mathfrak{c}-1)}{\vartheta\eta^{2\mathfrak{c}-1}}\exp(\eta\varsigma^{\vartheta}).$$

We get the required result from Eq (3.25).

$$\|\hbar_{\sigma}(\mathfrak{I}(\varsigma))-\hbar_{\sigma}(\widetilde{\mathfrak{I}}(\varsigma))\|_{\eta}\leq\Big(\frac{2\delta\Gamma(2\mathfrak{c}-1)}{\vartheta\eta^{2\mathfrak{c}-1}}\Big)^{\frac{1}{\mathfrak{p}}}\|\mathfrak{I}(\mathfrak{u})-\widetilde{\mathfrak{I}}(\mathfrak{u})\|_{\eta}.\tag{3.26}$$

From Eqs (3.16), we get $\frac{2\delta\Gamma(2\mathfrak{c}-1)}{\vartheta\eta^{2\mathfrak{c}-1}}<1$.

Theorem 3.2. *$\xi_{\mathfrak{c}}(\varsigma,\sigma)$ is a solution that is Con-D on $\mathfrak{c}$, then*

$$\lim_{\mathfrak{c}\to\tilde{\mathfrak{c}}} \underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\tilde{\mathfrak{c}}}(\varsigma,\sigma)\|_{\mathfrak{p}}=0. \tag{3.27}$$

Proof. Assume $\mathfrak{c},\tilde{\mathfrak{c}}\in(\frac{1}{2},1]$. Then

$$\begin{aligned}
&\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\tilde{\mathfrak{c}}}(\varsigma,\sigma)=\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{f}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\Big)\\
&\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}+\int_\alpha^\varsigma\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}-\frac{\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}}{\Gamma(\tilde{\mathfrak{c}})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\tilde{\mathfrak{c}}-1}\Big)\mathfrak{f}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\\
&+\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{g}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u})\\
&+\int_\alpha^\varsigma\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}-\frac{\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}}{\Gamma(\tilde{\mathfrak{c}})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\tilde{\mathfrak{c}}-1}\Big)\mathfrak{g}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u}).
\end{aligned}\tag{3.28}$$

We extract the subsequent outcome from Eq (3.28) by employing Eq (3.2).

$$\begin{aligned}
&\Big\|\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\tilde{\mathfrak{c}}}(\varsigma,\sigma)\Big\|_{\mathfrak{p}}^{\mathfrak{p}}\le 2^{\mathfrak{p}-1}\delta\Big(\Big\|\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\tilde{\mathfrak{c}}}(\varsigma,\sigma)\Big\|_{\mathfrak{p}}^{\mathfrak{p}}+\Big\|\xi_{\mathfrak{c}}(\varsigma-\mathfrak{r},\sigma)-\xi_{\tilde{\mathfrak{c}}}(\varsigma-\mathfrak{r},\sigma)\Big\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)\\
&\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}+2^{2\mathfrak{p}-2}\\
&\Big\|\int_\alpha^\varsigma\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}-\frac{\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}}{\Gamma(\tilde{\mathfrak{c}})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\tilde{\mathfrak{c}}-1}\Big)\mathfrak{f}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big\|_{\mathfrak{p}}^{\mathfrak{p}}+2^{2\mathfrak{p}-2}\\
&\Big\|\int_\alpha^\varsigma\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}-\frac{\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}}{\Gamma(\tilde{\mathfrak{c}})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\tilde{\mathfrak{c}}-1}\Big)\mathfrak{g}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u})\Big\|_{\mathfrak{p}}^{\mathfrak{p}}.
\end{aligned}\tag{3.29}$$

Suppose the following:

$$\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda)=\Big|\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}-\frac{\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}}{\Gamma(\tilde{\mathfrak{c}})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\tilde{\mathfrak{c}}-1}\Big|\mathfrak{u}^{\vartheta\lambda+\vartheta-1}. \tag{3.30}$$

By applying Höl-Ineq, $(\mathbb{H}_1)$, $(\mathbb{H}_2)$, and Eq (3.30), we obtain

$$\begin{aligned}
&\Big\|\int_\alpha^\varsigma\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}-\frac{\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}}{\Gamma(\tilde{\mathfrak{c}})}(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\tilde{\mathfrak{c}}-1}\Big)\mathfrak{f}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big\|_{\mathfrak{p}}^{\mathfrak{p}}\\
&\le\sum\nolimits_{\iota=1}^m\mathfrak{E}\Big(\int_\alpha^\varsigma\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda)\Big|\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\Big|d\mathfrak{u}\Big)^{\mathfrak{p}}\\
&\le\sum\nolimits_{\iota=1}^m\mathfrak{E}\Big(\Big(\int_\alpha^\varsigma(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}\Big)^{\mathfrak{p}-1}\int_\alpha^\varsigma\Big|\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\Big|^{\mathfrak{p}}d\mathfrak{u}\Big)\\
&\le\Big(\int_\alpha^\varsigma(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^2\Big)^{\frac{\mathfrak{p}}{2}}\Big(\int_\alpha^\varsigma 1d\mathfrak{u}\Big)^{\frac{\mathfrak{p}-2}{2}}\int_\alpha^\varsigma\Big\|\mathfrak{f}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\Big\|_{\mathfrak{p}}^{\mathfrak{p}}d\mathfrak{u}\\
&\le\Big(\int_\alpha^\varsigma(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^2\Big)^{\frac{\mathfrak{p}}{2}}(\theta-\alpha)^{\frac{\mathfrak{p}-2}{2}}\int_\alpha^\varsigma 2^{\mathfrak{p}-1}\Big(2^{\mathfrak{p}-1}\mathscr{L}_1^{\mathfrak{p}}(\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}}+\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}})+\|\mathfrak{f}(\mathfrak{u},0)\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)
\end{aligned}$$

$$\leq\Big(\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^2\Big)^{\frac{\mathfrak{p}}{2}}(\theta-\alpha)^{\frac{\mathfrak{p}}{2}}2^{\mathfrak{p}-1}$$
$$\Big(2^{\mathfrak{p}-1}\mathscr{L}_1^{\mathfrak{p}}(\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}}+\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}})+\psi^{\mathfrak{p}}\Big). \tag{3.31}$$

Now, by Bu-Da-Gu-Ineq, Eq (3.30), ($\mathbb{H}_1$), and ($\mathbb{H}_2$), we have

$$\Big\|\int_{\alpha}^{\varsigma}\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}-\frac{\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}}{\Gamma(\tilde{\mathfrak{c}})}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\tilde{\mathfrak{c}}-1}\Big)\mathfrak{g}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}\mathrm{dw}(\mathfrak{u})\Big\|_{\mathfrak{p}}^{\mathfrak{p}}$$
$$=\sum_{\iota=1}^{m}\mathfrak{E}\Big|\int_{\alpha}^{\varsigma}\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda)\mathfrak{g}_{\jmath}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\mathrm{dw}(\mathfrak{u})\Big|^{\mathfrak{p}}$$
$$\leq\sum_{\iota=1}^{m}C_{\mathfrak{p}}\mathfrak{E}\Big|\int_{\alpha}^{\varsigma}\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda)^2\Big|\mathfrak{g}_{\jmath}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\Big|^2\mathrm{dw}(\mathfrak{u})\Big|^{\frac{\mathfrak{p}}{2}}$$
$$\leq\sum_{\iota=1}^{m}C_{\mathfrak{p}}\mathfrak{E}\Big[\Big(\int_{\alpha}^{\varsigma}\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda)^2\Big|\mathfrak{g}_{\jmath}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\Big|^{\mathfrak{p}}\mathrm{du}\Big)^{\frac{2}{\mathfrak{p}}}\Big(\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^2\mathrm{du}\Big)^{\frac{\mathfrak{p}-2}{\mathfrak{p}}}\Big]^{\frac{\mathfrak{p}}{2}}$$
$$=C_{\mathfrak{p}}\int_{\alpha}^{\varsigma}\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda)^2\Big\|\mathfrak{g}(\mathfrak{u},\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma),\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma))\Big\|_{\mathfrak{p}}^{\mathfrak{p}}\mathrm{du}\Big(\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^2\mathrm{du}\Big)^{\frac{\mathfrak{p}-2}{\mathfrak{p}}}$$
$$\leq C_{\mathfrak{p}}\Big(\int_{\alpha}^{\varsigma}\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda)^2\mathrm{du}\Big)^{\frac{\mathfrak{p}}{2}}2^{\mathfrak{p}-1}\Big(2^{\mathfrak{p}-1}\mathscr{L}_2^{\mathfrak{p}}(\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}}+\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}})+\psi^{\mathfrak{p}}\Big). \tag{3.32}$$

Hence, we derive the following:

$$\frac{\Big\|\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\tilde{\mathfrak{c}}}(\varsigma,\sigma)\Big\|_{\mathfrak{p}}^{\mathfrak{p}}}{\exp(\eta\varsigma^{\vartheta})}\leq\frac{1}{\exp(\eta\varsigma^{\vartheta})}2^{\mathfrak{p}-1}\delta\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}\mathfrak{u}^{\vartheta-1}$$
$$\Big(\frac{\Big\|\xi_{\mathfrak{c}}(\mathfrak{u},\sigma)-\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\Big\|_{\mathfrak{p}}^{\mathfrak{p}}}{\exp(\eta\mathfrak{u}^{\vartheta})}\exp(\eta\mathfrak{u}^{\vartheta})+\frac{\Big\|\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma)-\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\Big\|_{\mathfrak{p}}^{\mathfrak{p}}}{\exp(\eta(\mathfrak{u}-\mathfrak{r})^{\vartheta})}\exp(\eta(\mathfrak{u}-\mathfrak{r})^{\vartheta})\Big)\mathrm{du}$$
$$+2^{3\mathfrak{p}-3}\Big(2^{\mathfrak{p}-1}\mathscr{L}_1^{\mathfrak{p}}(\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}}+\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}})+\psi^{\mathfrak{p}}\Big)\Big(\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^2\mathrm{du}\Big)^{\frac{\mathfrak{p}}{2}}\theta^{\frac{\mathfrak{p}}{2}}$$
$$+2^{3\mathfrak{p}-3}\Big(2^{\mathfrak{p}-1}\mathscr{L}_2^{\mathfrak{p}}(\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}}+\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}})+\psi^{\mathfrak{p}}\Big)C_{\mathfrak{p}}\Big(\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^2\mathrm{du}\Big)^{\frac{\mathfrak{p}}{2}}$$
$$\leq\frac{2^{\mathfrak{p}}\delta\Gamma(2\mathfrak{c}-1)}{\vartheta\eta^{2\mathfrak{c}-1}}\Big\|\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\tilde{\mathfrak{c}}}(\varsigma,\sigma)\Big\|_{\eta}^{\mathfrak{p}}$$
$$+2^{3\mathfrak{p}-3}\Big(2^{\mathfrak{p}-1}\mathscr{L}_1^{\mathfrak{p}}(\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}}+\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}})+\psi^{\mathfrak{p}}\Big)\Big(\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^2\mathrm{du}\Big)^{\frac{\mathfrak{p}}{2}}\theta^{\frac{\mathfrak{p}}{2}}$$
$$+2^{3\mathfrak{p}-3}\Big(2^{\mathfrak{p}-1}\mathscr{L}_2^{\mathfrak{p}}(\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}}+\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}})+\psi^{\mathfrak{p}}\Big)C_{\mathfrak{p}}\Big(\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^2\mathrm{du}\Big)^{\frac{\mathfrak{p}}{2}}. \tag{3.33}$$

We derive the result by using Lemma 3.2.

$$\Big(1-\frac{2^{\mathfrak{p}-1}\delta\Gamma(2\mathfrak{c}-1)}{\eta^{2\mathfrak{c}-1}}\Big)\Big\|\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\tilde{\mathfrak{c}}}(\varsigma,\sigma)\Big\|_{\eta}^{\mathfrak{p}}\leq 2^{3\mathfrak{p}-3}\Big(2^{\mathfrak{p}-1}\mathscr{L}_1^{\mathfrak{p}}(\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}}+\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}})$$

$$+\psi^{\mathfrak{p}}\Big)\Big(\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^{2}d\mathfrak{u}\Big)^{\frac{\mathfrak{p}}{2}}(\theta-\alpha)^{\frac{\mathfrak{p}}{2}}+2^{3\mathfrak{p}-3}\Big(2^{\mathfrak{p}-1}\mathscr{L}_{2}^{\mathfrak{p}}(\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}}+$$

$$\underset{\varsigma\in[\alpha,\theta]}{esssup}\|\xi_{\tilde{\mathfrak{c}}}(\mathfrak{u}-\mathfrak{r},\sigma)\|_{\mathfrak{p}}^{\mathfrak{p}})+\psi^{\mathfrak{p}}\Big)C_{\mathfrak{p}}\Big(\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^{2}d\mathfrak{u}\Big)^{\frac{\mathfrak{p}}{2}}. \tag{3.34}$$

Now, we prove the following:

$$\lim_{\tilde{\mathfrak{c}}\to\mathfrak{c}}\sup_{\varsigma\in[\alpha,\theta]}\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^{2}d\mathfrak{u}=0. \tag{3.35}$$

We possess the following:

$$\begin{aligned}
&\int_{\alpha}^{\varsigma}(\Phi(\varsigma,\mathfrak{u},\mathfrak{c},\tilde{\mathfrak{c}},\lambda))^{2}d\mathfrak{u}=(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{2}\\
&\quad\int_{\alpha}^{\varsigma}\frac{(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}}{\Gamma^{2}(\mathfrak{c})}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2}d\mathfrak{u}+(\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})})^{2}\int_{\alpha}^{\varsigma}\frac{(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\tilde{\mathfrak{c}}-2}}{\Gamma^{2}(\tilde{\mathfrak{c}})}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2}d\mathfrak{u}\\
&-2\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}\int_{\alpha}^{\varsigma}\frac{(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}+\tilde{\mathfrak{c}}-2}}{\Gamma(\mathfrak{c})\Gamma(\tilde{\mathfrak{c}})}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{2}d\mathfrak{u}\\
&\leq(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{2}\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{2}\sup_{\alpha<\mathfrak{u}\leq\varsigma}\mathfrak{u}^{\vartheta-1}\int_{\alpha}^{\varsigma}\frac{(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\mathfrak{c}-2}}{\Gamma^{2}(\mathfrak{c})}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\\
&+(\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})})^{2}\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{2}\sup_{\alpha<\mathfrak{u}\leq\varsigma}\mathfrak{u}^{\vartheta-1}\int_{\alpha}^{\varsigma}\frac{(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{2\tilde{\mathfrak{c}}-2}}{\Gamma^{2}(\tilde{\mathfrak{c}})}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\\
&-2\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{2}\sup_{\alpha<\mathfrak{u}\leq\varsigma}\mathfrak{u}^{\vartheta-1}\int_{\alpha}^{\varsigma}\frac{(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}+\tilde{\mathfrak{c}}-2}}{\Gamma(\mathfrak{c})\Gamma(\tilde{\mathfrak{c}})}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\\
&=(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{2}\mathbf{GU}\Big(\frac{(\varsigma^{\vartheta}-\alpha^{\vartheta})^{(2\mathfrak{c}-1)}}{(2\mathfrak{c}-1)}\Big)\frac{1}{\Gamma^{2}(\mathfrak{c})}+(\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})})^{2}\mathbf{GU}\Big(\frac{(\varsigma^{\vartheta}-\alpha^{\vartheta})^{(2\tilde{\mathfrak{c}}-1)}}{\vartheta(2\tilde{\mathfrak{c}}-1)}\Big)\frac{1}{\Gamma^{2}(\tilde{\mathfrak{c}})}\\
&-\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}\vartheta\varsigma^{-\vartheta(\lambda+\tilde{\mathfrak{c}})}\mathbf{GU}\frac{2(\varsigma^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{c}+\tilde{\mathfrak{c}}-1)}}{\vartheta(\mathfrak{c}+\tilde{\mathfrak{c}}-1)\Gamma(\mathfrak{c})\Gamma(\tilde{\mathfrak{c}})}.
\end{aligned} \tag{3.36}$$

When we apply $\lim_{\tilde{\mathfrak{c}}\to\mathfrak{c}}\sup_{\varsigma\in[\alpha,\theta]}$ in Eq (3.36), we prove the required result, Eq (3.35). It thereby demonstrated the necessary outcome.

Theorem 3.3. *For $\sigma,\Psi\in\mathbb{A}_{0}^{\mathfrak{p}}$, we have*

$$\|\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\mathfrak{c}}(\varsigma,\Psi)|_{\mathfrak{p}}\leq\mathscr{L}\|\sigma-\Psi\|_{\mathfrak{p}},\ \forall\ \varsigma\in[0,\ \theta]. \tag{3.37}$$

Proof. As we have

$$\begin{aligned}
&\xi_{\mathfrak{c}}(\mathfrak{u},\sigma)\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\mathfrak{c}}(\varsigma,\Psi)=\alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}-\alpha^{\vartheta(\mathfrak{c}+\lambda)}\Psi\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}\\
&+\frac{1}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{f}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\Psi),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\\
&+\frac{1}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}(\mathfrak{g}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{g}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\Psi),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi)))\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u}).
\end{aligned} \tag{3.38}$$

By applying Eq (3.2) to Eq (3.38), we obtain the following:

$$
\begin{aligned}
&\left\|\xi_{\mathfrak{c}}(\varsigma,\sigma)-\xi_{\mathfrak{c}}(\varsigma,\Psi)\right\|_{\mathfrak{p}}^{\mathfrak{p}} \leq 2^{\mathfrak{p}-1}\left\|\alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}-\alpha^{\vartheta(\mathfrak{c}+\lambda)}\Psi\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}\right\|_{\mathfrak{p}}^{\mathfrak{p}}+\frac{2^{2\mathfrak{p}-2}(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{\mathfrak{p}}}{\Gamma^{\mathfrak{p}}(\mathfrak{c})}\\
&\left\|\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{f}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\Psi),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\right\|_{\mathfrak{p}}^{\mathfrak{p}}+\frac{2^{2\mathfrak{p}-2}(\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})})^{\mathfrak{p}}}{\Gamma^{\mathfrak{p}}(\mathfrak{c})}\\
&\left\|\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma)))-\mathfrak{g}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u})\right\|_{\mathfrak{p}}^{\mathfrak{p}}.
\end{aligned}\tag{3.39}
$$

By Höl-Ineq and $(\mathbb{H}_1)$.

$$
\begin{aligned}
&\left\|\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{f}(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\Psi),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\right\|_{\mathfrak{p}}^{\mathfrak{p}}\\
&=\sum_{\jmath=1}^{\rho}\mathfrak{E}\left|\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\Psi),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\right|^{\mathfrak{p}}\\
&\leq\sum_{\jmath=1}^{\rho}\mathfrak{E}\Bigg(\Big(\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\frac{(\mathfrak{c}-1)(\mathfrak{p}-2)}{\mathfrak{p}-1}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{\frac{\mathfrak{p}-2}{\mathfrak{p}-1}}d\mathfrak{u}\Big)^{\mathfrak{p}-1}\\
&\Big(\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\Psi),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi))\Big|(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}\Big)\Bigg)\\
&\leq\sum_{\jmath=1}^{\rho}\mathfrak{E}\Bigg(\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}-2}{\mathfrak{p}-1}}\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\frac{(\mathfrak{c}-1)(\mathfrak{p}-2)}{\mathfrak{p}-1}}(\mathfrak{u}^{\vartheta-1})^{\frac{\mathfrak{p}-2}{\mathfrak{p}-1}}d\mathfrak{u}\Big)^{\mathfrak{p}-1}\\
&\Big(\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\Psi),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi))\Big|(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}\Big)\Bigg)\\
&\leq\sum_{\jmath=1}^{\rho}\mathfrak{E}\Bigg(\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}-2}{\mathfrak{p}-1}}\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta-1})^{\frac{1}{1-\mathfrak{p}}}\sum_{\jmath=1}^{\rho}\mathfrak{E}\Bigg(\Big(\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{\frac{(\mathfrak{c}-1)(\mathfrak{p}-2)}{\mathfrak{p}-1}}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\Big)^{\mathfrak{p}-1}\\
&\Big(\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\sigma),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma))-\mathfrak{f}_\jmath(\mathfrak{u},\xi_{\mathfrak{c}}(\mathfrak{u},\Psi),\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi))\Big|(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}\Big)\Bigg)\\
&\leq 2^{\mathfrak{p}-1}F^{\mathfrak{p}-1}\mathbf{Q}^{\mathfrak{p}-1}\frac{\mathscr{L}_1^{\mathfrak{p}}\varsigma^{\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1)}(\mathfrak{p}-1)^{\mathfrak{p}-1}}{(\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1))^{\mathfrak{p}-1}}\\
&\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big(\left\|\xi_{\mathfrak{c}}(\mathfrak{u},\sigma)-\xi_{\mathfrak{c}}(\mathfrak{u},\Psi)\right\|_{\mathfrak{p}}^{\mathfrak{p}}+\left\|\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma)-\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi)\right\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)(\mathfrak{u}^{\vartheta-1})^2d\mathfrak{u}.\\
&\leq 2^{\mathfrak{p}-1}F^{\mathfrak{p}-1}\mathbf{Q}^{\mathfrak{p}-1}\frac{\mathscr{L}_1^{\mathfrak{p}}\varsigma^{\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1)}(\mathfrak{p}-1)^{\mathfrak{p}-1}}{(\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1))^{\mathfrak{p}-1}}\\
&\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\sup_{\alpha<\mathfrak{u}\leq\varsigma}\mathfrak{u}^{\vartheta-1}\\
&\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big(\left\|\xi_{\mathfrak{c}}(\mathfrak{u},\sigma)-\xi_{\mathfrak{c}}(\mathfrak{u},\Psi)\right\|_{\mathfrak{p}}^{\mathfrak{p}}+\left\|\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\sigma)-\xi_{\mathfrak{c}}(\mathfrak{u}-\mathfrak{r},\Psi)\right\|_{\mathfrak{p}}^{\mathfrak{p}}\Big)\mathfrak{u}^{\vartheta-1}d\mathfrak{u}.
\end{aligned}\tag{3.40}
$$

Hence, we have

$$\Big\| \int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \sigma), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma)) - \mathfrak{f}(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \Psi), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\Big\|_\mathfrak{p}^\mathfrak{p}$$
$$\leq 2^{\mathfrak{p}-1}\mathbf{GUQ}^{\mathfrak{p}-1}F^{\mathfrak{p}-1}\frac{\mathscr{L}_1^\mathfrak{p}\varsigma^{\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1)}(\mathfrak{p}-1)^{\mathfrak{p}-1}}{(\vartheta(\mathfrak{p}\mathfrak{c}-2\mathfrak{c}+1))^{\mathfrak{p}-1}}$$
$$\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big(\big\|\xi_\mathfrak{c}(\mathfrak{u}, \sigma) - \xi_\mathfrak{c}(\mathfrak{u}, \Psi)\big\|_\mathfrak{p}^\mathfrak{p} + \big\|\xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma) - \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi)\big\|_\mathfrak{p}^\mathfrak{p}\Big)\mathfrak{u}^{\vartheta-1}d\mathfrak{u}. \tag{3.41}$$

Now, utilizing ($\mathbb{H}_1$), Höl-Ineq, and Bu-Da-Gu-Ineq, we derive

$$\Big\| \int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \sigma), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma)) - \mathfrak{g}(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \Psi), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathrm{w}(\mathfrak{u})\Big\|_\mathfrak{p}^\mathfrak{p}$$
$$= \sum_{\jmath=1}^{\rho} \mathfrak{E}\Big| \int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{\mathfrak{c}-1}\Big(\mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \sigma), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma)) - \mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \Psi), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathrm{w}(\mathfrak{u})\Big|^\mathfrak{p}$$
$$\leq \sum_{\jmath=1}^{\rho} C_\mathfrak{p}\mathfrak{E}\Big| \int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \sigma), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma)) - \mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \Psi), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi))\Big|^2(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}\Big|^{\frac{\mathfrak{p}}{2}}$$
$$\leq \sum_{\jmath=1}^{\rho} C_\mathfrak{p}\mathfrak{E}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \sigma), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma)) - \mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \Psi), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi))\Big|^\mathfrak{p}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}$$
$$\Big(\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}\Big)^{\frac{\mathfrak{p}-2}{2}}$$
$$\leq \sum_{\jmath=1}^{\rho} C_\mathfrak{p}\mathfrak{E}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \sigma), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma)) - \mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \Psi), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi))\Big|^\mathfrak{p}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}$$
$$\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}(\mathfrak{u}^{\vartheta-1})^2d\mathfrak{u}\Big)^{\frac{\mathfrak{p}-2}{2}}$$
$$\leq \sum_{\jmath=1}^{\rho} C_\mathfrak{p}\mathfrak{E}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big|\mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \sigma), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma)) - \mathfrak{g}_\jmath(\mathfrak{u}, \xi_\mathfrak{c}(\mathfrak{u}, \Psi), \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi))\Big|^\mathfrak{p}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}$$
$$\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\sup_{\alpha<\mathfrak{u}\leq\varsigma}\mathfrak{u}^{\vartheta-1}\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\Big)^{\frac{\mathfrak{p}-2}{2}}$$
$$\leq 2^{\mathfrak{p}-1}\mathbf{G}^{\frac{\mathfrak{p}-2}{2}}\mathbf{U}^{\frac{\mathfrak{p}-2}{2}}\mathscr{L}_2^\mathfrak{p}C_\mathfrak{p}\Big(\frac{(\varsigma^\vartheta-\alpha^\vartheta)^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\Big)^{\frac{\mathfrak{p}-2}{2}}$$
$$\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big(\big\|\xi_\mathfrak{c}(\mathfrak{u}, \sigma) - \xi_\mathfrak{c}(\mathfrak{u}, \Psi)\big\|_\mathfrak{p}^\mathfrak{p} + \big\|\xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma) - \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi)\big\|_\mathfrak{p}^\mathfrak{p}\Big)(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2d\mathfrak{u}$$
$$\leq 2^{\mathfrak{p}-1}\mathbf{G}^{\frac{\mathfrak{p}-2}{2}}\mathbf{U}^{\frac{\mathfrak{p}-2}{2}}\mathscr{L}_2^\mathfrak{p}C_\mathfrak{p}\Big(\frac{(\varsigma^\vartheta-\alpha^\vartheta)^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\Big)^{\frac{\mathfrak{p}-2}{2}}$$
$$\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\int_\alpha^\varsigma (\varsigma^\vartheta - \mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big(\big\|\xi_\mathfrak{c}(\mathfrak{u}, \sigma) - \xi_\mathfrak{c}(\mathfrak{u}, \Psi)\big\|_\mathfrak{p}^\mathfrak{p} + \big\|\xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \sigma) - \xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r}, \Psi)\big\|_\mathfrak{p}^\mathfrak{p}\Big)(\mathfrak{u}^{\vartheta-1})^2d\mathfrak{u}$$
$$\leq 2^{\mathfrak{p}-1}\mathbf{G}^{\frac{\mathfrak{p}-2}{2}}\mathbf{U}^{\frac{\mathfrak{p}-2}{2}}\mathscr{L}_2^\mathfrak{p}C_\mathfrak{p}\Big(\frac{(\varsigma^\vartheta-\alpha^\vartheta)^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\Big)^{\frac{\mathfrak{p}-2}{2}}$$

$$\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^2\sup_{\alpha<\mathfrak{u}\leq\varsigma}\mathfrak{u}^{\vartheta-1}\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta))^{2\mathfrak{c}-2}\Big(\big\|\xi_\mathfrak{c}(\mathfrak{u},\sigma)-\xi_\mathfrak{c}(\mathfrak{u},\Psi)\big\|_\mathfrak{p}^\mathfrak{p}+\big\|\xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r},\sigma)-\xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r},\Psi)\big\|_\mathfrak{p}^\mathfrak{p}\Big)d\mathfrak{u}$$

$$=2^{\mathfrak{p}-1}\mathbf{G}^{\frac{\mathfrak{p}}{2}}\mathbf{U}^{\frac{\mathfrak{p}}{2}}\mathscr{L}_2^\mathfrak{p}C_\mathfrak{p}\Big(\frac{(\varsigma^\vartheta-\alpha^\vartheta)^{(2\mathfrak{c}-1)}}{\vartheta(2\mathfrak{c}-1)}\Big)^{\frac{\mathfrak{p}-2}{2}}$$

$$\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big(\big\|\xi_\mathfrak{c}(\mathfrak{u},\sigma)-\xi_\mathfrak{c}(\mathfrak{u},\Psi)\big\|_\mathfrak{p}^\mathfrak{p}+\big\|\xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r},\sigma)-\xi_\mathfrak{c}(\mathfrak{u}-\mathfrak{r},\Psi)\big\|_\mathfrak{p}^\mathfrak{p}\Big)\mathfrak{u}^{\vartheta-1}d\mathfrak{u}. \tag{3.42}$$

By putting Eqs (3.40) and (3.42) in Eq (3.39).

$$\begin{aligned}\big\|\xi_\mathfrak{c}(\varsigma,\sigma)-\xi_\mathfrak{c}(\varsigma,\Psi)\big\|_\mathfrak{p}^\mathfrak{p}\leq{}&2^{\mathfrak{p}-1}\big\|\alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}-\alpha^{\vartheta(\mathfrak{c}+\lambda)}\Psi\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}\big\|_\mathfrak{p}^\mathfrak{p}\\&+2^{\mathfrak{p}-1}\delta\int_\alpha^\varsigma(\varsigma^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\Big(\big\|\xi_\mathfrak{c}(\mathfrak{u},\sigma)-\xi_\mathfrak{c}(\mathfrak{u},\Psi)\big\|_\mathfrak{p}^\mathfrak{p}\Big)\mathfrak{u}^{\vartheta-1}d\mathfrak{u}.\end{aligned} \tag{3.43}$$

From Eq (3.43), we get

$$\begin{aligned}\big\|\xi_\mathfrak{c}(\varsigma,\sigma)-\xi_\mathfrak{c}(\varsigma,\Psi)\big\|_\eta^\mathfrak{p}\leq{}&2^{\mathfrak{p}-1}\frac{1}{\exp(\eta\varsigma^\vartheta)}\big\|\alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}-\alpha^{\vartheta(\mathfrak{c}+\lambda)}\Psi\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}\big\|_\mathfrak{p}^\mathfrak{p}\\&+\Big(\frac{2^\mathfrak{p}\delta\Gamma(2\mathfrak{c}-1)}{\vartheta\eta^{2\mathfrak{c}-1}}\Big)\big\|\xi_\mathfrak{c}(\varsigma,\sigma)-\xi_\mathfrak{c}(\varsigma,\Psi)\big\|_\eta^\mathfrak{p}\end{aligned} \tag{3.44}$$

From Eq (3.44), we obtain

$$\big\|\xi_\mathfrak{c}(\varsigma,\sigma)-\xi_\mathfrak{c}(\varsigma,\Psi)\big\|_\mathfrak{p}^\mathfrak{p}\Big(1-\Big(\frac{2^\mathfrak{p}\delta\Gamma(2\mathfrak{c}-1)}{\vartheta\eta^{2\mathfrak{c}-1}}\Big)\Big)\leq2^{\mathfrak{p}-1}\frac{1}{\exp(\eta\varsigma^\vartheta)}\big\|\alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}-\alpha^{\vartheta(\mathfrak{c}+\lambda)}\Psi\varsigma^{-\vartheta(\mathfrak{c}+\lambda)}\big\|_\mathfrak{p}^\mathfrak{p}.$$

Therefore, we conclude

$$\lim_{\sigma\to\Psi}\big\|\xi_\mathfrak{c}(\varsigma,\sigma)-\xi_\mathfrak{c}(\varsigma,\Psi)\big\|_\mathfrak{p}=0.$$

The proof is so done.

4. Averaging principle result

We now establish A-P in the $\mathfrak{p}$th moment for SFDDEs within the framework of CEKFD.

Lemma 4.1. *For $\widetilde{\mathfrak{g}}$, when $\theta_1\in[\alpha,\theta]$, we get*

$$\|\widetilde{\mathfrak{g}}(\ell,\zeta)\|^\mathfrak{p}\leq\mathscr{L}_6\left(1+\|\ell\|^\mathfrak{p}+\|\zeta\|^\mathfrak{p}\right),$$

where $\mathscr{L}_6=\left(2^{\mathfrak{p}-1}\aleph_2(\theta_1)+6^{\mathfrak{p}-1}\mathscr{L}_4^\mathfrak{p}\right)$.

Proof. By utilizing $(\mathbb{H}_4)$, $(\mathbb{H}_5)$, and Eq (3.2), we obtain the following:

$$\begin{aligned}\|\widetilde{\mathfrak{g}}(\ell,\zeta)\|^\mathfrak{p}&\leq2^{\mathfrak{p}-1}\|\mathfrak{g}(\varsigma,\ell,\zeta)-\widetilde{\mathfrak{g}}(\ell,\zeta)\|^\mathfrak{p}+2^{\mathfrak{p}-1}\|\mathfrak{g}(\varsigma,\ell,\zeta)\|^\mathfrak{p}\\&\leq2^{\mathfrak{p}-1}\aleph_2(\theta_1)\left(1+\|\ell\|^\mathfrak{p}+\|\zeta\|^\mathfrak{p}\right)+2^{\mathfrak{p}-1}\mathscr{L}_4^\mathfrak{p}(1+\|\ell\|+\|\zeta\|)^\mathfrak{p}\\&\leq\left(2^{\mathfrak{p}-1}\aleph_2(\theta_1)+6^{\mathfrak{p}-1}\mathscr{L}_4^\mathfrak{p}\right)\left(1+\|\ell\|^\mathfrak{p}+\|\zeta\|^\mathfrak{p}\right).\end{aligned}$$

To prove A-P, we construct the standard form of Eq (1.4).

$$\begin{cases} \mathcal{D}^{\mathfrak{c}}_{\alpha+;\vartheta,\lambda}\gimel_{\varepsilon}(\varsigma) = \varepsilon^{\mathfrak{c}}\mathfrak{f}(\varsigma, \gimel_{\varepsilon}(\varsigma), \gimel_{\varepsilon}(\varsigma - \mathfrak{r})) + \varepsilon^{\mathfrak{c}-\frac{1}{2}}\mathfrak{g}(\varsigma, \gimel_{\varepsilon}(\varsigma), \gimel_{\varepsilon}(\varsigma - \mathfrak{r}))\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma}, \\ \gimel_{\varepsilon}(\alpha) = \sigma. \end{cases} \tag{4.1}$$

Thus, Eq (4.1) can be expressed integrally as

$$\begin{aligned} \gimel_{\varepsilon}(\varsigma) &= \alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)} + \varepsilon^{\mathfrak{c}}\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}\mathrm{u}^{\vartheta\lambda+\vartheta-1}(\varsigma^{\vartheta} - \mathrm{u}^{\vartheta})^{1-\mathfrak{c}}\mathfrak{f}(\mathrm{u}, \gimel_{\varepsilon}(\mathrm{u}), \gimel_{\varepsilon}(\mathrm{u} - \mathfrak{r}))\mathrm{du} \\ &+ \varepsilon^{\mathfrak{c}-\frac{1}{2}}\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}\mathrm{u}^{\vartheta\lambda+\vartheta-1}(\varsigma^{\vartheta} - \mathrm{u}^{\vartheta})^{1-\mathfrak{c}}\mathfrak{g}(\mathrm{u}, \gimel_{\varepsilon}(\mathrm{u}), \gimel_{\varepsilon}(\mathrm{u} - \mathfrak{r}))\mathrm{dw}(\mathrm{u}), \end{aligned} \tag{4.2}$$

for $\varepsilon \in (0, \varepsilon_0]$. The averaged representation of Eq (4.2) is as

$$\begin{aligned} \gimel^{*}_{\varepsilon}(\varsigma) &= \alpha^{\vartheta(\mathfrak{c}+\lambda)}\sigma\varsigma^{-\vartheta(\mathfrak{c}+\lambda)} + \varepsilon^{\mathfrak{c}}\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}\mathrm{u}^{\vartheta\lambda+\vartheta-1}(\varsigma^{\vartheta} - \mathrm{u}^{\vartheta})^{1-\mathfrak{c}}\widetilde{\mathfrak{f}}(\gimel^{*}_{\varepsilon}(\mathrm{u}), \gimel^{*}_{\varepsilon}(\mathrm{u} - \mathfrak{r}))\mathrm{du} \\ &+ \varepsilon^{\mathfrak{c}-\frac{1}{2}}\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}\mathrm{u}^{\vartheta\lambda+\vartheta-1}(\varsigma^{\vartheta} - \mathrm{u}^{\vartheta})^{1-\mathfrak{c}}\widetilde{\mathfrak{g}}(\gimel^{*}_{\varepsilon}(\mathrm{u}), \gimel^{*}_{\varepsilon}(\mathrm{u} - \mathfrak{r}))\mathrm{dw}(\mathrm{u}), \end{aligned} \tag{4.3}$$

where $\widetilde{\mathfrak{f}} : \mathbb{R}^{\rho} \times \mathbb{R}^{\rho} \to \mathbb{R}^{\rho}, \widetilde{\mathfrak{g}} : \mathbb{R}^{\rho} \times \mathbb{R}^{\rho} \to \mathbb{R}^{\varkappa\times\mathbf{b}}$.

Theorem 4.1. *For any given small positive number $\mho$, there exists $\varrho > 0$ and $\varepsilon_1 \in (0, \varepsilon_0]$ with $\kappa \in (0, 1)$, then*

$$\boldsymbol{\mathfrak{E}}\Big[\sup_{\varsigma\in[0,\ \varrho\varepsilon^{-\kappa}]}\left\|\gimel_{\varepsilon}(\varsigma) - \gimel^{*}_{\varepsilon}(\varsigma)\right\|^{\mathfrak{p}}\Big] \le \mho,\ \varepsilon \in (0, \varepsilon_1]. \tag{4.4}$$

Proof. By using Eqs (4.2) and (4.3), we obtain:

$$\begin{aligned} &\gimel_{\varepsilon}(\varsigma) - \gimel^{*}_{\varepsilon}(\varsigma) \\ &= \varepsilon^{\mathfrak{c}}\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta} - \mathrm{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathrm{u}, \gimel_{\varepsilon}(\mathrm{u}), \gimel_{\varepsilon}(\mathrm{u} - \mathfrak{r})) - \widetilde{\mathfrak{f}}(\gimel^{*}_{\varepsilon}(\mathrm{u}), \gimel^{*}_{\varepsilon}(\mathrm{u} - \mathfrak{r}))\Big)\mathrm{u}^{\vartheta\lambda+\vartheta-1}\mathrm{du} \\ &+ \varepsilon^{\mathfrak{c}-\frac{1}{2}}\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta} - \mathrm{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathrm{u}, \gimel_{\varepsilon}(\mathrm{u}), \gimel_{\varepsilon}(\mathrm{u} - \mathfrak{r})) - \widetilde{\mathfrak{g}}(\gimel^{*}_{\varepsilon}(\mathrm{u}), \gimel^{*}_{\varepsilon}(\mathrm{u} - \mathfrak{r}))\Big)\mathrm{u}^{\vartheta\lambda+\vartheta-1}\mathrm{dw}(\mathrm{u}). \end{aligned} \tag{4.5}$$

Via Jen-Ineq, we have as

$$\begin{aligned} \left\|\gimel_{\varepsilon}(\varsigma) - \gimel^{*}_{\varepsilon}(\varsigma)\right\|^{\mathfrak{p}} &\le 2^{\mathfrak{p}-1} \\ &\left\|\varepsilon^{\mathfrak{c}}\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta} - \mathrm{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathrm{u}, \gimel_{\varepsilon}(\mathrm{u}), \gimel_{\varepsilon}(\mathrm{u} - \mathfrak{r})) - \widetilde{\mathfrak{f}}(\gimel^{*}_{\varepsilon}(\mathrm{u}), \gimel^{*}_{\varepsilon}(\mathrm{u} - \mathfrak{r}))\Big)\mathrm{u}^{\vartheta\lambda+\vartheta-1}\mathrm{du}\right\|^{\mathfrak{p}} \\ &+ 2^{\mathfrak{p}-1} \\ &\left\|\varepsilon^{\mathfrak{c}-\frac{1}{2}}\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta} - \mathrm{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathrm{u}, \gimel_{\varepsilon}(\mathrm{u}), \gimel_{\varepsilon}(\mathrm{u} - \mathfrak{r})) - \widetilde{\mathfrak{g}}(\gimel^{*}_{\varepsilon}(\mathrm{u}), \gimel^{*}_{\varepsilon}(\mathrm{u} - \mathfrak{r}))\Big)\mathrm{u}^{\vartheta\lambda+\vartheta-1}\mathrm{dw}(\mathrm{u})\right\|^{\mathfrak{p}} \\ &\le \left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}}2^{\mathfrak{p}-1}\varepsilon^{\mathfrak{p}\mathfrak{c}} \\ &\left\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta} - \mathrm{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathrm{u}, \gimel_{\varepsilon}(\mathrm{u}), \gimel_{\varepsilon}(\mathrm{u} - \mathfrak{r})) - \widetilde{\mathfrak{f}}(\gimel^{*}_{\varepsilon}(\mathrm{u}), \gimel^{*}_{\varepsilon}(\mathrm{u} - \mathfrak{r}))\Big)\mathrm{u}^{\vartheta\lambda+\vartheta-1}\mathrm{du}\right\|^{\mathfrak{p}} \end{aligned}$$

$$+\left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}}2^{\mathfrak{p}-1}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}$$
$$\left\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathfrak{u},\beth_{\varepsilon}(\mathfrak{u}),\beth_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\widetilde{\mathfrak{g}}(\beth^{*}_{\varepsilon}(\mathfrak{u}),\beth^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathrm{w}(\mathfrak{u})\right\|^{\mathfrak{p}}. \tag{4.6}$$

Utilizing Eq (4.6) in Eq (4.4).

$$\begin{aligned}
&\mathfrak{E}\Big[\sup_{\alpha\le\varsigma\le\varphi}\left\|\beth_{\varepsilon}(\varsigma)-\beth^{*}_{\varepsilon}(\varsigma)\right\|^{\mathfrak{p}}\Big]\\
&\quad\le\left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}}2^{\mathfrak{p}-1}\varepsilon^{\mathfrak{p}\mathfrak{c}}\\
&\quad\mathfrak{E}\Big[\sup_{\alpha\le\varsigma\le\varphi}\left\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\beth_{\varepsilon}(\mathfrak{u}),\beth_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\widetilde{\mathfrak{f}}(\beth^{*}_{\varepsilon}(\mathfrak{u}),\beth^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\right\|^{\mathfrak{p}}\Big]\\
&\quad+\left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}}2^{\mathfrak{p}-1}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}\\
&\quad\mathfrak{E}\Big[\sup_{\alpha\le\varsigma\le\varphi}\left\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{g}(\mathfrak{u},\beth_{\varepsilon}(\mathfrak{u}),\beth_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\widetilde{\mathfrak{g}}(\beth^{*}_{\varepsilon}(\mathfrak{u}),\beth^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathrm{w}(\mathfrak{u})\right\|^{\mathfrak{p}}\Big]\\
&\quad=\Upsilon_1+\Upsilon_2.
\end{aligned}\tag{4.7}$$

From Υ_1, we have

$$\begin{aligned}
\Upsilon_1\le&\left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{\mathfrak{p}\mathfrak{c}}\\
&\mathfrak{E}\Big[\sup_{\alpha\le\varsigma\le\varphi}\left\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\beth_{\varepsilon}(\mathfrak{u}),\beth_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{f}(\mathfrak{u},\beth^{*}_{\varepsilon}(\mathfrak{u}),\beth^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\right\|^{\mathfrak{p}}\Big]\\
&+\left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{\mathfrak{p}\mathfrak{c}}\\
&\mathfrak{E}\Big[\sup_{\alpha\le\varsigma\le\varphi}\left\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big(\mathfrak{f}(\mathfrak{u},\beth^{*}_{\varepsilon}(\mathfrak{u}),\beth^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\widetilde{\mathfrak{f}}(\beth^{*}_{\varepsilon}(\mathfrak{u}),\beth^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\Big)\mathfrak{u}^{\vartheta\lambda+\vartheta-1}d\mathfrak{u}\right\|^{\mathfrak{p}}\Big]\\
&=\Upsilon_{11}+\Upsilon_{12}.
\end{aligned}\tag{4.8}$$

The following outcome is obtained by applying Höl-Ineq, Jen-Ineq, and ($\mathbb{H}_3$) on Υ_{11}:

$$\begin{aligned}
\Upsilon_{11}\le&\left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{\mathfrak{p}\mathfrak{c}}\left(\int_{\alpha}^{\varphi}(\varphi^{\vartheta}-\mathfrak{u}^{\vartheta})^{\frac{(\mathfrak{c}-1)\mathfrak{p}}{\mathfrak{p}-1}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}d\mathfrak{u}\right)^{\mathfrak{p}-1}\\
&\mathfrak{E}\left[\sup_{\alpha\le\varsigma\le\varphi}\int_{\alpha}^{\varsigma}\left\|\mathfrak{f}\left(\mathfrak{u},\beth_{\varepsilon}(\mathfrak{u}),\beth_{\varepsilon}(\mathfrak{u}-\mathfrak{r})\right)-\mathfrak{f}\left(\mathfrak{u},\beth^{*}_{\varepsilon}(\mathfrak{u}),\beth^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r})\right)\right\|^{\mathfrak{p}}d\mathfrak{u}\right]\\
\le&\left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{\mathfrak{p}\mathfrak{c}}\left(\sup_{\alpha\le\varsigma\le\varphi}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}\int_{\alpha}^{\varphi}(\varphi^{\vartheta}-\mathfrak{u}^{\vartheta})^{\frac{(\mathfrak{c}-1)\mathfrak{p}}{\mathfrak{p}-1}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}d\mathfrak{u}\right)^{\mathfrak{p}-1}\\
&\mathfrak{E}\left[\sup_{\alpha\le\varsigma\le\varphi}\int_{\alpha}^{\varsigma}\left\|\mathfrak{f}\left(\mathfrak{u},\beth_{\varepsilon}(\mathfrak{u}),\beth_{\varepsilon}(\mathfrak{u}-\mathfrak{r})\right)-\mathfrak{f}\left(\mathfrak{u},\beth^{*}_{\varepsilon}(\mathfrak{u}),\beth^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r})\right)\right\|^{\mathfrak{p}}d\mathfrak{u}\right]
\end{aligned}$$

$$\leq\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{\mathfrak{p}\mathfrak{c}}\left(\sup_{\alpha\leq\varsigma\leq\varphi}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}\sup_{0<\mathfrak{u}\leq\varphi}(\mathfrak{u}^{\vartheta-1})^{\frac{1}{\mathfrak{p}-1}}\int_{\alpha}^{\varphi}(\varphi^{\vartheta}-\mathfrak{u}^{\vartheta})^{\frac{(\mathfrak{c}-1)\mathfrak{p}}{\mathfrak{p}-1}}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\right)^{\mathfrak{p}-1}$$
$$\mathfrak{E}\left[\sup_{\alpha\leq\varsigma\leq\varphi}\int_{\alpha}^{\varsigma}\left\|\mathfrak{f}(\mathfrak{u},\mathfrak{I}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{f}(\mathfrak{u},\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\right\|^{\mathfrak{p}}d\mathfrak{u}\right]$$
$$\leq\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{3\mathfrak{p}-3}\varepsilon^{\mathfrak{p}\mathfrak{c}}\mathscr{L}_{3}^{\mathfrak{p}}((\varphi^{\vartheta}-\alpha^{\vartheta})^{\frac{(\mathfrak{c}\mathfrak{p}-1)}{\mathfrak{p}-1}})^{\mathfrak{p}-1}\Big(\frac{\mathfrak{p}-1}{\vartheta(\mathfrak{c}\mathfrak{p}-1)}\Big)^{\mathfrak{p}-1}$$
$$\widetilde{\mathbb{C}}^{\mathfrak{p}-1}\widetilde{\mathbb{V}}^{\mathfrak{p}-1}\Big(\mathfrak{E}\Big[\sup_{\alpha\leq\varsigma\leq\varphi}\int_{\alpha}^{\varsigma}\left\|\mathfrak{I}_{\varepsilon}(\mathfrak{u})-\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u})\right\|^{\mathfrak{p}}d\mathfrak{u}\Big]+\mathfrak{E}\Big[\sup_{\alpha\leq\varsigma\leq\varphi}\int_{\alpha}^{\varsigma}\left\|\mathfrak{I}_{\varepsilon}(\mathfrak{u}-\mathfrak{r})-\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r})\right\|^{\mathfrak{p}}d\mathfrak{u}\Big]\Big)$$
$$=\Upsilon_{11}\varepsilon^{\mathfrak{p}\mathfrak{c}}\varphi^{\vartheta(\mathfrak{c}\mathfrak{p}-1)}\Big(\int_{\alpha}^{\varphi}\mathfrak{E}\Big[\sup_{0\leq\mathfrak{q}\leq\mathfrak{u}}\left\|\mathfrak{I}_{\varepsilon}(\mathfrak{q})-\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{q})\right\|^{\mathfrak{p}}\Big]d\mathfrak{u}$$
$$+\int_{\alpha}^{\varphi}\mathfrak{E}\Big[\sup_{0\leq\mathfrak{q}\leq\mathfrak{u}}\|\mathfrak{I}_{\varepsilon}(\mathfrak{q}-\mathfrak{r})-\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{q}-\mathfrak{r})\|^{\mathfrak{p}}\Big]d\mathfrak{u}\Big),\tag{4.9}$$

where $\widetilde{\mathbb{C}}=\sup_{0<\mathfrak{u}\leq\varphi}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}$, $\widetilde{\mathbb{V}}=\sup_{0<\mathfrak{u}\leq\varphi}(\mathfrak{u}^{\vartheta-1})^{\frac{1}{\mathfrak{p}-1}}$, and $\Upsilon_{11}=\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{3\mathfrak{p}-3}\mathscr{L}_{3}^{\mathfrak{p}}\Big(\frac{\mathfrak{p}-1}{\vartheta(\mathfrak{c}\mathfrak{p}-1)}\Big)^{\mathfrak{p}-1}\widetilde{\mathbb{C}}^{\mathfrak{p}-1}\widetilde{\mathbb{V}}^{\mathfrak{p}-1}$.

The following outcome is obtained by applying Höl-Ineq, Jen-Ineq, and $(\mathbb{H}_5)$ on Υ_{12}:

$$\Upsilon_{12}\leq\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{\mathfrak{p}\mathfrak{c}}\Big(\int_{\alpha}^{\varphi}(\varphi^{\vartheta}-\mathfrak{u}^{\vartheta})^{\frac{(\mathfrak{c}-1)\mathfrak{p}}{\mathfrak{p}-1}}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}d\mathfrak{u}\Big)^{\mathfrak{p}-1}$$
$$\mathfrak{E}\Big[\sup_{\alpha\leq\varsigma\leq\varphi}\int_{\alpha}^{\varsigma}\Big\|\mathfrak{f}(\mathfrak{u},\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\widetilde{\mathfrak{f}}(\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\Big\|^{\mathfrak{p}}d\mathfrak{u}\Big]$$
$$\leq\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{\mathfrak{p}\mathfrak{c}}\Big(\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta\lambda})^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}\sup_{\alpha<\mathfrak{u}\leq\varsigma}(\mathfrak{u}^{\vartheta-1})^{\frac{1}{\mathfrak{p}-1}}\int_{\alpha}^{\varphi}(\varphi^{\vartheta}-\mathfrak{u}^{\vartheta})^{\frac{(\mathfrak{c}-1)\mathfrak{p}}{\mathfrak{p}-1}}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\Big)^{\mathfrak{p}-1}$$
$$\mathfrak{E}\Big[\sup_{\alpha\leq\varsigma\leq\varphi}\int_{\alpha}^{\varsigma}\Big\|\mathfrak{f}(\mathfrak{u},\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\widetilde{\mathfrak{f}}(\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\Big\|^{\mathfrak{p}}d\mathfrak{u}\Big]$$
$$\leq\widetilde{\mathbb{C}}^{\mathfrak{p}-1}\widetilde{\mathbb{V}}^{\mathfrak{p}-1}\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{\mathfrak{p}\mathfrak{c}}((\varphi^{\vartheta}-\alpha^{\vartheta})^{\frac{(\mathfrak{c}\mathfrak{p}-1)}{\mathfrak{p}-1}})^{\mathfrak{p}-1}\Big(\frac{\mathfrak{p}-1}{\vartheta(\mathfrak{c}\mathfrak{p}-1)}\Big)^{\mathfrak{p}-1}\aleph_{1}(\varphi)$$
$$(\varphi-\alpha)(1+\mathfrak{E}\left\|\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u})\right\|^{\mathfrak{p}}+\mathfrak{E}\left\|\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r})\right\|^{\mathfrak{p}})$$
$$=\Upsilon_{12}\varepsilon^{\mathfrak{p}\mathfrak{c}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{c}\mathfrak{p}-1)}(\varphi-\alpha),\tag{4.10}$$

where $\Upsilon_{12}=\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\Big(\frac{\mathfrak{p}-1}{\vartheta(\mathfrak{c}\mathfrak{p}-1)}\Big)^{\mathfrak{p}-1}\aleph_{1}(\varphi)(1+\mathfrak{E}\left\|\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u})\right\|^{\mathfrak{p}}+\mathfrak{E}\left\|\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r})\right\|^{\mathfrak{p}})\widetilde{\mathbb{C}}^{\mathfrak{p}-1}\widetilde{\mathbb{V}}^{\mathfrak{p}-1}$.

From Υ_2, by applying Jen-Ineq, we obtain:

$$\Upsilon_{2}\leq\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}$$
$$\Big(\mathfrak{E}\Big[\sup_{\alpha\leq\varsigma\leq\varphi}\Big\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big[\mathfrak{g}(\mathfrak{u},\mathfrak{I}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\mathfrak{g}(\mathfrak{u},\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\Big]\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u})\Big\|^{\mathfrak{p}}\Big]\Big)$$
$$+\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}$$
$$\Big(\mathfrak{E}\Big[\sup_{\alpha\leq\varsigma\leq\varphi}\Big\|\int_{\alpha}^{\varsigma}(\varsigma^{\vartheta}-\mathfrak{u}^{\vartheta})^{\mathfrak{c}-1}\Big[\mathfrak{g}(\mathfrak{u},\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))-\widetilde{\mathfrak{g}}(\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}),\mathfrak{I}^{*}_{\varepsilon}(\mathfrak{u}-\mathfrak{r}))\Big]\mathfrak{u}^{\vartheta\lambda+\vartheta-1}dw(\mathfrak{u})\Big\|^{\mathfrak{p}}\Big]\Big)$$

$$=\Upsilon_{21}+\Upsilon_{22}. \tag{4.11}$$

We achieve the subsequent findings on Υ_{21} using $(\mathbb{H}_3)$, Höl-Ineq, and Bu-Da-Gu-Ineq:

$$\begin{aligned}
\Upsilon_{21} \leq & \Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}} 2^{2\mathfrak{p}-2}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}\left(2(\mathfrak{p}-1)^{1-\mathfrak{p}}\mathfrak{p}^{\mathfrak{p}+1}\right)^{\frac{\mathfrak{p}}{2}} \\
& \mathfrak{E}\left[\int_\alpha^\varphi (\varphi^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\left\|\mathfrak{g}\left(\mathfrak{u},\mathfrak{I}_\varepsilon(\mathfrak{u}),\mathfrak{I}_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)-\mathfrak{g}\left(\mathfrak{u},\mathfrak{I}^*_\varepsilon(\mathfrak{u}),\mathfrak{I}^*_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)\right\|^2(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2 d\mathfrak{u}\right]^{\frac{\mathfrak{p}}{2}} \\
& \leq \left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}} 2^{2\mathfrak{p}-2}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}\left(2(\mathfrak{p}-1)^{1-\mathfrak{p}}\mathfrak{p}^{\mathfrak{p}+1}\right)^{\frac{\mathfrak{p}}{2}} \\
& \mathfrak{E}\left[\sup_{\alpha\leq\varsigma\leq\varphi}(\mathfrak{u}^{\vartheta\lambda})^2\int_\alpha^\varphi(\varphi^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}\left\|\mathfrak{g}\left(\mathfrak{u},\mathfrak{I}_\varepsilon(\mathfrak{u}),\mathfrak{I}_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)-\mathfrak{g}\left(\mathfrak{u},\mathfrak{I}^*_\varepsilon(\mathfrak{u}),\mathfrak{I}^*_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)\right\|^2(\mathfrak{u}^{\vartheta-1})^2 d\mathfrak{u}\right]^{\frac{\mathfrak{p}}{2}} \\
& \leq \widetilde{\mathbf{G}}^{\frac{\mathfrak{p}}{2}}\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}} 2^{2\mathfrak{p}-2}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}\left(\mathfrak{p}^{\mathfrak{p}+1}2(\mathfrak{p}-1)^{1-\mathfrak{p}}\right)^{\frac{\mathfrak{p}}{2}} \\
& \mathfrak{E}\left[\int_\alpha^\varphi(\varphi^\vartheta-\mathfrak{u}^\vartheta)^{(\mathfrak{c}-1)\mathfrak{p}}\left\|\mathfrak{g}\left(\mathfrak{u},\mathfrak{I}_\varepsilon(\mathfrak{u}),\mathfrak{I}_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)-\mathfrak{g}\left(\mathfrak{u},\mathfrak{I}^*_\varepsilon(\mathfrak{u}),\mathfrak{I}^*_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)\right\|^{\mathfrak{p}}(\mathfrak{u}^{\vartheta-1})^{\mathfrak{p}} d\mathfrak{u}\right] \\
& \leq \widetilde{\mathbf{G}}^{\frac{\mathfrak{p}}{2}}\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}} 2^{3\mathfrak{p}-3}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}\mathscr{L}_3^{\mathfrak{p}}(\mathfrak{p}^{\mathfrak{p}+1}2(1-\mathfrak{p})^{\mathfrak{p}-1})^{\frac{\mathfrak{p}}{2}}\int_\alpha^\varphi(\varphi^\vartheta-\mathfrak{u}^\vartheta)^{(\mathfrak{c}-1)\mathfrak{p}} \\
& \mathfrak{E}\Big[\sup_{0\leq\mathfrak{q}\leq\mathfrak{u}}\left[\left\|\mathfrak{I}_\varepsilon(\mathfrak{q})-\mathfrak{I}^*_\varepsilon(\mathfrak{q})\right\|^{\mathfrak{p}}+\left\|\mathfrak{I}_\varepsilon(\mathfrak{q}-\mathfrak{r})-\mathfrak{I}^*_\varepsilon(\mathfrak{q}-\mathfrak{r})\right\|^{\mathfrak{p}}\right](\mathfrak{u}^{\vartheta-1})^{\mathfrak{p}} d\mathfrak{u}\Big] \\
& \leq \widetilde{\mathbf{G}}^{\frac{\mathfrak{p}}{2}}\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}} 2^{3\mathfrak{p}-3}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}\mathscr{L}_3^{\mathfrak{p}}(\mathfrak{p}^{\mathfrak{p}+1}2(1-\mathfrak{p})^{\mathfrak{p}-1})^{\frac{\mathfrak{p}}{2}}\sup_{0<\mathfrak{u}\leq\varphi}(\mathfrak{u}^{\vartheta-1})^{\mathfrak{p}-1}\int_\alpha^\varphi(\varphi^\vartheta-\mathfrak{u}^\vartheta)^{(\mathfrak{c}-1)\mathfrak{p}} \\
& \mathfrak{E}\Big[\sup_{0\leq\mathfrak{q}\leq\mathfrak{u}}\left[\left\|\mathfrak{I}_\varepsilon(\mathfrak{q})-\mathfrak{I}^*_\varepsilon(\mathfrak{q})\right\|^{\mathfrak{p}}+\left\|\mathfrak{I}_\varepsilon(\mathfrak{q}-\mathfrak{r})-\mathfrak{I}^*_\varepsilon(\mathfrak{q}-\mathfrak{r})\right\|^{\mathfrak{p}}\right]\mathfrak{u}^{\vartheta-1} d\mathfrak{u}\Big] \\
& = \Upsilon_{21}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}\Big(\int_\alpha^\varphi(\varphi^\vartheta-\mathfrak{u}^\vartheta)^{(\mathfrak{c}-1)\mathfrak{p}}\mathfrak{E}\Big[\sup_{0\leq\mathfrak{q}\leq\mathfrak{u}}\left\|\mathfrak{I}_\varepsilon(\mathfrak{q})-\mathfrak{I}^*_\varepsilon(\mathfrak{q})\right\|^{\mathfrak{p}}\Big]\mathfrak{u}^{\vartheta-1}d\mathfrak{u} \\
& + \int_\alpha^\varphi(\varphi^\vartheta-\mathfrak{u}^\vartheta)^{(\mathfrak{c}-1)\mathfrak{p}}\mathfrak{E}\Big[\sup_{\alpha\leq\mathfrak{q}\leq\mathfrak{u}}\left\|\mathfrak{I}_\varepsilon(\mathfrak{q}-\mathfrak{r})-\mathfrak{I}^*_\varepsilon(\mathfrak{q}-\mathfrak{r})\right\|^{\mathfrak{p}}\Big]\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\Big),
\end{aligned} \tag{4.12}$$

where $\widetilde{\mathbf{G}}=\sup_{0<\mathfrak{u}\leq\varphi}(\mathfrak{u}^{\vartheta\lambda})^2$, $\mathbb{B}=\sup_{0<\mathfrak{u}\leq\varphi}(\mathfrak{u}^{\vartheta-1})^{\mathfrak{p}-1}$, and $\Upsilon_{21}=2^{3\mathfrak{p}-3}\mathscr{L}_3^{\mathfrak{p}}\left(\frac{\mathfrak{p}^{\mathfrak{p}+1}}{2(\mathfrak{p}-1)^{\mathfrak{p}-1}}\right)^{\frac{\mathfrak{p}}{2}}\left(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\right)^{\mathfrak{p}}\widetilde{\mathbf{G}}^{\frac{\mathfrak{p}}{2}}\mathbb{B}$.

By Höl-Ineq and Bu-Da-Gu-Ineq.

$$\begin{aligned}
\Upsilon_{22} \leq & \Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}} 2^{2\mathfrak{p}-2}(2(\mathfrak{p}-1)^{1-\mathfrak{p}}\mathfrak{p}^{\mathfrak{p}+1})^{\frac{\mathfrak{p}}{2}}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}} \\
& \mathfrak{E}\left[\int_\alpha^\varphi\left\|\mathfrak{f}\left(\mathfrak{u},\mathfrak{I}^*_\varepsilon(\mathfrak{u}),\mathfrak{I}^*_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)-\widetilde{\mathfrak{g}}\left(\mathfrak{u},\mathfrak{I}^*_\varepsilon(\mathfrak{u}),\mathfrak{I}^*_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)\right\|^2(\varphi^\vartheta-\mathfrak{u}^\vartheta)^{2\mathfrak{c}-2}(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^2 d\mathfrak{u}\right]^{\frac{\mathfrak{p}}{2}} \\
\leq & \Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}} 2^{2\mathfrak{p}-2}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}(2(\mathfrak{p}-1)^{\mathfrak{p}-1}\mathfrak{p}^{\mathfrak{p}+1})^{\frac{\mathfrak{p}}{2}}\mathfrak{E}\Big[\int_\alpha^\varphi(\varphi^\vartheta-\mathfrak{u}^\vartheta)^{(\mathfrak{c}-1)\mathfrak{p}} \\
& \left(\left\|\mathfrak{g}\left(\mathfrak{u},\mathfrak{I}^*_\varepsilon(\mathfrak{u}),\mathfrak{I}^*_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)\right\|^{\mathfrak{p}}+\left\|\widetilde{\mathfrak{g}}\left(\mathfrak{I}^*_\varepsilon(\mathfrak{u}),\mathfrak{I}^*_\varepsilon(\mathfrak{u}-\mathfrak{r})\right)\right\|^{\mathfrak{p}}\right)(\mathfrak{u}^{\vartheta\lambda+\vartheta-1})^{\mathfrak{p}}d\mathfrak{u}\Big]
\end{aligned}$$

$$\leq\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}2^{2\mathfrak{p}-2}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}(2(\mathfrak{p}-1)^{\mathfrak{p}-1}\mathfrak{p}^{\mathfrak{p}+1})^{\frac{\mathfrak{p}}{2}}\mathfrak{E}\Big[\sup_{0<\mathfrak{u}\leq\varphi}(\mathfrak{u}^{\vartheta\lambda})^{\mathfrak{p}}\sup_{0<\mathfrak{u}\leq\varphi}(\mathfrak{u}^{\vartheta-1})^{(\mathfrak{p}-1)}$$
$$\int_{\alpha}^{\varphi}(\varphi^{\vartheta}-\mathfrak{u}^{\vartheta})^{(\mathfrak{c}-1)\mathfrak{p}}\left(\left\|\mathfrak{g}\left(\mathfrak{u},\beth_{\varepsilon}^{*}(\mathfrak{u}),\beth_{\varepsilon}^{*}(\mathfrak{u}-\mathfrak{r})\right)\right\|^{\mathfrak{p}}+\left\|\widetilde{\mathfrak{g}}\left(\beth_{\varepsilon}^{*}(\mathfrak{u}),\beth_{\varepsilon}^{*}(\mathfrak{u}-\mathfrak{r})\right)\right\|^{\mathfrak{p}}\right)\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\Big]$$
$$\leq\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}\frac{2^{3\mathfrak{p}-3}3^{\mathfrak{p}-1}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{((\mathfrak{c}-1)\mathfrak{p}+1)}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}\mathscr{L}_4^{\mathfrak{p}}\left(\mathscr{L}_4^{\mathfrak{p}}+\mathscr{L}_6\right)^{\mathfrak{p}}}{(\vartheta(\mathfrak{c}-1)\mathfrak{p}+1)}$$
$$\mathbb{B}\Theta(2(\mathfrak{p}-1)^{1-\mathfrak{p}}\mathfrak{p}^{\mathfrak{p}+1})^{\frac{\mathfrak{p}}{2}}(1+\mathfrak{E}[\left\|\beth_{\varepsilon}^{*}(\mathfrak{u})\right\|^{\mathfrak{p}}]+\mathfrak{E}[\left\|\beth_{\varepsilon}^{*}(\mathfrak{u}-\mathfrak{r})\right\|^{\mathfrak{p}}])$$
$$=\Upsilon_{22}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{((\mathfrak{c}-1)\mathfrak{p}+1)}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1},\tag{4.13}$$

where $\Theta=\sup_{0<\mathfrak{u}\leq\varphi}(\mathfrak{u}^{\vartheta-1})^{\mathfrak{p}-1}$ and

$$\Upsilon_{22}=2^{3\mathfrak{p}-3}3^{\mathfrak{p}-1}\mathscr{L}_4^{\mathfrak{p}}\left(\mathscr{L}_4^{\mathfrak{p}}+\mathscr{L}_6\right)^{\mathfrak{p}}\frac{1}{(\vartheta(\mathfrak{c}-1)\mathfrak{p}+1)}\mathbb{B}\Theta$$
$$(2(\mathfrak{p}-1)^{1-\mathfrak{p}}\mathfrak{p}^{\mathfrak{p}+1})^{\frac{\mathfrak{p}}{2}}(1+\mathfrak{E}[\left\|\beth_{\varepsilon}^{*}(\mathfrak{u})\right\|^{\mathfrak{p}}]+\mathfrak{E}[\left\|\beth_{\varepsilon}^{*}(\mathfrak{u}-\mathfrak{r})\right\|^{\mathfrak{p}}])\Big(\frac{\vartheta\varsigma^{-\vartheta(\lambda+\mathfrak{c})}}{\Gamma(\mathfrak{c})}\Big)^{\mathfrak{p}}.$$

By applying Eqs (4.8) to (4.13) in (4.7).

$$\mathfrak{E}\Big[\sup_{\alpha\leq\varsigma\leq\varphi}\left\|\beth_{\varepsilon}(\varsigma)-\beth_{\varepsilon}^{*}(\varsigma)\right\|^{\mathfrak{p}}\Big]\leq\Upsilon_{12}\varepsilon^{\mathfrak{p}\mathfrak{c}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{c}\mathfrak{p}-1)}(\varphi-\alpha)+\Upsilon_{22}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{((\mathfrak{c}-1)\mathfrak{p}+1)}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}$$
$$+\int_{\alpha}^{\varphi}\Big(\Upsilon_{11}\varepsilon^{\mathfrak{p}\mathfrak{c}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{c}\mathfrak{p}-1)}+\Upsilon_{21}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}(\varphi^{\vartheta}-\mathfrak{u}^{\vartheta})^{(\mathfrak{c}-1)\mathfrak{p}}\mathfrak{u}^{\vartheta-1}d\mathfrak{u}\Big)$$
$$\mathfrak{E}\Big[\sup_{0\leq\mathfrak{q}\leq\mathfrak{u}}\left\|\beth_{\varepsilon}(\mathfrak{q})-\beth_{\varepsilon}^{*}(\mathfrak{q})\right\|^{\mathfrak{p}}\Big]d\mathfrak{u}+\int_{\alpha}^{\varphi}\Big(\Upsilon_{11}\varepsilon^{\mathfrak{p}\mathfrak{c}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{c}\mathfrak{p}-1)}+\Upsilon_{21}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}$$
$$(\varphi^{\vartheta}-\mathfrak{u}^{\vartheta})^{(\mathfrak{c}-1)\mathfrak{p}}\mathfrak{u}^{\vartheta-1}\Big)\mathfrak{E}\Big[\sup_{\alpha\leq\mathfrak{q}\leq\mathfrak{u}}\left\|\beth_{\varepsilon}(\mathfrak{q}-\mathfrak{r})-\beth_{\varepsilon}^{*}(\mathfrak{q}-\mathfrak{r})\right\|^{\mathfrak{p}}\Big]d\mathfrak{u}.\tag{4.14}$$

From Eq (4.14).

$$\mathfrak{E}\Big[\sup_{\alpha\leq\varsigma\leq\varphi}\left\|\beth_{\varepsilon}(\varsigma)-\beth_{\varepsilon}^{*}(\varsigma)\right\|^{\mathfrak{p}}\Big]\leq\Big(\Upsilon_{12}\varepsilon^{\mathfrak{p}\mathfrak{c}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{c}\mathfrak{p}-1)}(\varphi-\alpha)+\Upsilon_{22}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{((\mathfrak{c}-1)\mathfrak{p}+1)}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}\Big)$$
$$\exp\Big(2\Upsilon_{11}\varepsilon^{\mathfrak{p}\mathfrak{c}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{c}\mathfrak{p}-1)}(\varphi-\alpha)+\frac{2\Upsilon_{21}}{(\vartheta(\mathfrak{c}-1)\mathfrak{p}+1)}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}(\varphi^{\vartheta}-\alpha^{\vartheta})^{((\mathfrak{c}-1)\mathfrak{p}+1)}(\varphi-\alpha)^{\frac{\mathfrak{p}}{2}-1}\Big).$$

This suggests that there exist $\varrho>0$ and $\kappa\in(0,1)$ for every $\varsigma\in[0,\varrho\varepsilon^{-\kappa}]\subseteq[\alpha,\theta]$.

$$\mathfrak{E}\left[\sup_{0\leq\varsigma\leq\varrho\varepsilon^{-\kappa}}\left\|\beth_{\varepsilon}(\varsigma)-\beth_{\varepsilon}^{*}(\varsigma)\right\|^{\mathfrak{p}}\right]\leq\mathcal{Z}\varepsilon^{1-\kappa},\tag{4.15}$$

where

$$\mathcal{Z}=\varepsilon^{\kappa-1}\Big(\Upsilon_{12}\varepsilon^{\mathfrak{p}\mathfrak{c}}((\varrho\varepsilon^{-\kappa})^{\vartheta}-\varphi^{\vartheta})^{(\mathfrak{c}\mathfrak{p}-1)+1}+\Upsilon_{22}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}((\varrho\varepsilon^{-\kappa})^{\vartheta}-\alpha^{\vartheta})^{((\mathfrak{c}-1)\mathfrak{p}+1)}((\varrho\varepsilon^{-\kappa})^{\vartheta}-\alpha)^{\frac{\mathfrak{p}}{2}-1}\Big)$$

$$\exp\Big(2\Upsilon_{11}\varepsilon^{\mathfrak{p}\mathfrak{c}}((\varrho\varepsilon^{-\kappa})^{\vartheta}-\alpha^{\vartheta})^{(\mathfrak{c}\mathfrak{p}-1)}((\varrho\varepsilon^{-\kappa})^{\vartheta}-\alpha)+$$
$$\frac{2\Upsilon_{21}}{(\vartheta(\mathfrak{c}-1)\mathfrak{p}+1)}\varepsilon^{(\mathfrak{c}-\frac{1}{2})\mathfrak{p}}((\varrho\varepsilon^{-\kappa})^{\vartheta}-\alpha^{\vartheta})^{((\mathfrak{c}-1)\mathfrak{p}+1)}((\varrho\varepsilon^{-\kappa})^{\vartheta}-\alpha^{\vartheta})^{\frac{\mathfrak{p}}{2}-1}\Big).$$

So, proved the required result.

5. Examples

Now, to understand the theoretical results that we established in this research work, we provide numerical examples.

Example 1. Consider the following:

$$\begin{cases} \mathcal{D}^{0.95}_{\alpha+;\vartheta,\lambda}\daleth_{\varepsilon}(\varsigma) = 6\varepsilon^{0.95}\sin^2(\varsigma)\daleth_{\varepsilon}(\varsigma-\frac{1}{2}) + \varepsilon^{0.95}\daleth_{\varepsilon}(\varsigma-\frac{1}{2})\cos^2(\varsigma)+ \\ 3\varepsilon^{0.95-\frac{1}{2}}\daleth_{\varepsilon}(\varsigma-\frac{1}{2})\cos^2(\varsigma)\sin(\daleth_{\varepsilon}(\varsigma))\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma},\ \varsigma\in[0,\pi], \\ \daleth(0)=\sigma, \end{cases} \tag{5.1}$$

where $\mathfrak{c} = 0.95$, $\alpha = 0$, $\mathfrak{r} = \frac{1}{2}$, and

$$\mathfrak{f}(\varsigma,\daleth(\varsigma),\daleth(\varsigma-\mathfrak{r})) = 6\sin^2(\varsigma)\daleth_{\varepsilon}(\varsigma-\frac{1}{2}) + \daleth_{\varepsilon}(\varsigma-\frac{1}{2})\cos^2(\varsigma),$$
$$\mathfrak{g}(\varsigma,\daleth(\varsigma),\daleth(\varsigma-\mathfrak{r})) = 3\daleth_{\varepsilon}(\varsigma-\frac{1}{2})\cos^2(\varsigma)\sin(\daleth_{\varepsilon}(\varsigma)).$$

The criteria of Ex-Un are fulfilled by $\mathfrak{f}(\varsigma,\daleth(\varsigma),\daleth(\varsigma-\mathfrak{r}))$ and $\mathfrak{g}(\varsigma,\daleth(\varsigma),\daleth(\varsigma-\mathfrak{r}))$.

The averages of $\mathfrak{f}$ and $\mathfrak{g}$ are as

$$\widetilde{\mathfrak{f}}(\daleth(\varsigma),\daleth(\varsigma-\mathfrak{r})) = \frac{1}{\pi}\int_0^{\pi}\Big(6\sin^2(\varsigma)\daleth_{\varepsilon}(\varsigma) + \daleth_{\varepsilon}(\varsigma)\cos^2\big(\frac{1}{2}\varsigma\big)\Big)\mathrm{d}\varsigma = \frac{7}{2}\daleth^*_{\varepsilon}(\varsigma-\frac{1}{2}),$$
$$\widetilde{\mathfrak{g}}(\daleth(\varsigma),\daleth(\varsigma-\mathfrak{r})) = \frac{1}{\pi}\int_0^{\pi}3\daleth_{\varepsilon}(\varsigma)\cos^2(\varsigma)\sin(\daleth_{\varepsilon}(\varsigma))\mathrm{d}\varsigma = \frac{3}{2}\daleth^*_{\varepsilon}(\varsigma-\frac{1}{2})\sin(\daleth^*_{\varepsilon}(\varsigma)).$$

The corresponding average is

$$\begin{cases} \mathcal{D}^{0.95}_{\alpha+;\vartheta,\lambda}\daleth^*_{\varepsilon}(\varsigma) = \frac{7}{2}\varepsilon^{0.95}\daleth^*_{\varepsilon}(\varsigma-\frac{1}{2}) + \frac{3}{2}\varepsilon^{0.95-\frac{1}{2}}\daleth^*_{\varepsilon}(\varsigma-\frac{1}{2})\sin(\daleth^*_{\varepsilon}(\varsigma))\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma}, \\ \daleth^*_{\varepsilon}(0)=\sigma. \end{cases}$$

Example 2. Take the following:

$$\begin{cases} \mathcal{D}^{0.90}_{\alpha+;\vartheta,\lambda}\daleth_{\varepsilon}(\varsigma) = 3\varepsilon^{0.90}\sin(\daleth_{\varepsilon}(\varsigma-\frac{1}{3}))\sin^2(\varsigma)\daleth_{\varepsilon}(\varsigma) + \varepsilon^{0.90-\frac{1}{2}}\sin(\daleth_{\varepsilon}(\varsigma))\cos(\daleth_{\varepsilon}(\varsigma)) \\ \varsigma^2\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma},\ \varsigma\in[0,\pi], \\ \daleth(0)=\sigma, \end{cases} \tag{5.2}$$

where $\mathfrak{c} = 0.90$, $\alpha = 0$, $\mathfrak{r} = \frac{1}{3}$, and

$$\mathfrak{f}(\varsigma,\daleth(\varsigma),\daleth(\varsigma-\mathfrak{r})) = 3\sin(\daleth_{\varepsilon}(\varsigma-\frac{1}{3}))\sin^2(\varsigma)\daleth_{\varepsilon}(\varsigma),$$

$$\mathfrak{g}(\varsigma, \gimel(\varsigma), \gimel(\varsigma-\mathfrak{r})) = \sin(\gimel_\varepsilon(\varsigma))\cos(\gimel_\varepsilon(\varsigma))\varsigma^2.$$

The criteria of Ex-Un are fulfilled by $\mathfrak{f}(\varsigma, \gimel(\varsigma), \gimel(\varsigma-\mathfrak{r}))$ and $\mathfrak{g}(\varsigma, \gimel(\varsigma), \gimel(\varsigma-\mathfrak{r}))$.

The averages of $\mathfrak{f}$ and $\mathfrak{g}$ are as

$$\begin{aligned}
\widetilde{\mathfrak{f}}(\gimel(\varsigma), \gimel(\varsigma-\mathfrak{r})) &= \frac{1}{\pi}\int_0^\pi 3\sin(\gimel_\varepsilon(\varsigma-\frac{1}{3}))\sin^2(\varsigma)\gimel_\varepsilon(\varsigma)\mathrm{d}\varsigma = \frac{3}{2}\sin(\gimel^*_\varepsilon(\varsigma-\frac{1}{3}))\gimel^*_\varepsilon(\varsigma),\\
\widetilde{\mathfrak{g}}(\gimel(\varsigma), \gimel(\varsigma-\mathfrak{r})) &= \frac{1}{\pi}\int_0^\pi \sin(\gimel_\varepsilon(\varsigma))\cos(\gimel_\varepsilon(\varsigma))\varsigma^2\mathrm{d}\varsigma = \sin(\gimel^*_\varepsilon(\varsigma))\cos(\gimel^*_\varepsilon(\varsigma))\frac{\pi^2}{3}.
\end{aligned}$$

The corresponding average is

$$\begin{cases}
\mathcal{D}^{0.90}_{\alpha+;\vartheta,\lambda}\gimel^*_\varepsilon(\varsigma) = \frac{3}{2}\varepsilon^{0.90}\sin(\gimel^*_\varepsilon(\varsigma-\frac{1}{3}))\gimel^*_\varepsilon(\varsigma) + \varepsilon^{0.90-\frac{1}{2}}\sin(\gimel^*_\varepsilon(\varsigma))\cos(\gimel^*_\varepsilon(\varsigma))\frac{\pi^2}{3}\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma},\\
\gimel^*_\varepsilon(0) = \sigma.
\end{cases}$$

Example 3. Examine the following:

$$\begin{cases}
\mathcal{D}^{0.95}_{\alpha+;\vartheta,\lambda}\gimel_\varepsilon(\varsigma) = \frac{1}{3}\varepsilon^{0.95}\gimel_\varepsilon(\varsigma-\frac{1}{4})\cos(\gimel_\varepsilon(\varsigma))\sin(\gimel_\varepsilon(\varsigma)) + \frac{3\pi}{4}\varepsilon^{0.95-\frac{1}{2}}\sin^3\varsigma\cos(\gimel_\varepsilon(\varsigma))\sin(\gimel_\varepsilon(\varsigma))\gimel_\varepsilon(\varsigma)\\
\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma},\ \varsigma\in[0,\pi],\\
\gimel(0) = \sigma,
\end{cases} \tag{5.3}$$

where $\mathfrak{c} = 0.95$, $\alpha = 0$, $\mathfrak{r} = \frac{1}{4}$, and

$$\begin{aligned}
\mathfrak{f}(\varsigma, \gimel(\varsigma), \gimel(\varsigma-\mathfrak{r})) &= \frac{1}{3}\gimel_\varepsilon(\varsigma-\frac{1}{4})\cos(\gimel_\varepsilon(\varsigma))\sin(\gimel_\varepsilon(\varsigma)),\\
\mathfrak{g}(\varsigma, \gimel(\varsigma), \gimel(\varsigma-\mathfrak{r})) &= \frac{3\pi}{4}\sin^3\varsigma\cos(\gimel_\varepsilon(\varsigma))\gimel_\varepsilon(\varsigma)\sin(\gimel_\varepsilon(\varsigma)).
\end{aligned}$$

$\mathfrak{f}(\varsigma, \gimel(\varsigma), \gimel(\varsigma-\mathfrak{r}))$ and $\mathfrak{g}(\varsigma, \gimel(\varsigma), \gimel(\varsigma-\mathfrak{r}))$ satisfies the needs of Ex-Un.

The averages are given below.

$$\begin{aligned}
\widetilde{\mathfrak{f}}(\gimel(\varsigma), \gimel(\varsigma-\mathfrak{r})) &= \frac{1}{\pi}\int_0^\pi \frac{1}{3}\gimel_\varepsilon(\varsigma-\frac{1}{4})\sin(\gimel_\varepsilon(\varsigma))\cos(\gimel_\varepsilon(\varsigma))\mathrm{d}\varsigma = \frac{1}{3}\gimel^*_\varepsilon(\varsigma-\frac{1}{4})\sin(\gimel^*_\varepsilon(\varsigma))\cos(\gimel^*_\varepsilon(\varsigma)),\\
\widetilde{\mathfrak{g}}(\gimel(\varsigma), \gimel(\varsigma-\mathfrak{r})) &= \frac{1}{\pi}\int_0^\pi \frac{3\pi}{4}\sin^3\varsigma\cos(\gimel_\varepsilon(\varsigma))\gimel_\varepsilon(\varsigma)\sin(\gimel_\varepsilon(\varsigma))\mathrm{d}\varsigma = \sin(\gimel^*_\varepsilon(\varsigma))\cos(\gimel^*_\varepsilon(\varsigma))\gimel^*_\varepsilon(\varsigma).
\end{aligned}$$

Thus,

$$\begin{cases}
\mathcal{D}^{0.95}_{\alpha+;\vartheta,\lambda}\gimel^*_\varepsilon(\varsigma) = \frac{1}{3}\varepsilon^{0.95}\gimel^*_\varepsilon(\varsigma-\frac{1}{4})\sin(\gimel^*_\varepsilon(\varsigma))\cos(\gimel^*_\varepsilon(\varsigma)) + \varepsilon^{0.95-\frac{1}{2}}\cos(\gimel^*_\varepsilon(\varsigma))\gimel^*_\varepsilon(\varsigma)\sin(\gimel^*_\varepsilon(\varsigma))\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma},\\
\gimel^*_\varepsilon(0) = \sigma.
\end{cases}$$

Example 4. Take the following:

$$\begin{cases}
\mathcal{D}^{0.95}_{\alpha+;\vartheta,\lambda}\gimel_\varepsilon(\varsigma) = \frac{9}{2}\varepsilon^{0.95}\sin(\gimel_\varepsilon(\varsigma))\cos(\gimel_\varepsilon(\varsigma))\exp^{-\varsigma} + \varepsilon^{0.95-\frac{1}{2}}\sin(\gimel_\varepsilon(\varsigma))\\
\gimel_\varepsilon(\varsigma-\frac{2}{3})\cos(\gimel_\varepsilon(\varsigma))\frac{\mathrm{dw}(\varsigma)}{\mathrm{d}\varsigma},\ \varsigma\in[0,\pi],\\
\gimel(0) = \sigma,
\end{cases} \tag{5.4}$$

where $\mathfrak{c} = 0.95$, $\alpha = 0$, $\mathfrak{r} = \frac{2}{3}$, and

$$\mathfrak{f}(\varsigma, \gimel(\varsigma), \gimel(\varsigma - \mathfrak{r})) = \frac{9}{2} \sin(\gimel_{\varepsilon}(\varsigma)) \cos(\gimel_{\varepsilon}(\varsigma)) \exp^{-\varsigma},$$
$$\mathfrak{g}(\varsigma, \gimel(\varsigma), \gimel(\varsigma - \mathfrak{r})) = \sin(\gimel_{\varepsilon}(\varsigma))\gimel_{\varepsilon}(\varsigma - \frac{2}{3}) \cos(\gimel_{\varepsilon}(\varsigma)).$$

The $\frac{9}{2} \sin(\gimel_{\varepsilon}(\varsigma)) \cos(\gimel_{\varepsilon}(\varsigma)) \exp^{-\varsigma}$ and $\sin(\gimel_{\varepsilon}(\varsigma))\gimel_{\varepsilon}(\varsigma - \frac{2}{3}) \cos(\gimel_{\varepsilon}(\varsigma))$ satisfies the needs of Ex-Un. So we have

$$\begin{aligned}
\widetilde{\mathfrak{f}}(\gimel(\varsigma), \gimel(\varsigma - \mathfrak{r})) &= \frac{1}{\pi} \int_0^{\pi} \left(\frac{9}{2} \sin(\gimel_{\varepsilon}(\varsigma)) \cos(\gimel_{\varepsilon}(\varsigma)) \exp^{-\varsigma} \right) d\varsigma \\
&= \frac{9}{2\pi} \sin(\gimel^*_{\varepsilon}(\varsigma)) \cos(\gimel^*_{\varepsilon}(\varsigma))(1 - \exp^{-\pi}), \\
\widetilde{\mathfrak{g}}(\gimel(\varsigma), \gimel(\varsigma - \mathfrak{r})) &= \frac{1}{\pi} \int_0^{\pi} \sin(\gimel_{\varepsilon}(\varsigma))\gimel_{\varepsilon}(\varsigma - \frac{2}{3}) \cos(\gimel_{\varepsilon}(\varsigma)) d\varsigma = \sin(\gimel^*_{\varepsilon}(\varsigma))\gimel^*_{\varepsilon}(\varsigma - \frac{2}{3}) \cos(\gimel^*_{\varepsilon}(\varsigma)).
\end{aligned}$$

So,

$$\begin{cases}
\mathcal{D}^{0.95}_{\alpha+;\vartheta,\lambda}\gimel^*_{\varepsilon}(\varsigma) = \varepsilon^{0.95} \frac{9}{2\pi} \sin(\gimel^*_{\varepsilon}(\varsigma)) \cos(\gimel^*_{\varepsilon}(\varsigma)))(1 - \exp^{-\pi}) + \\
\varepsilon^{0.95 - \frac{1}{2}} \sin(\gimel^*_{\varepsilon}(\varsigma))\gimel^*_{\varepsilon}(\varsigma - \frac{2}{3}) \cos(\gimel^*_{\varepsilon}(\varsigma)) \frac{dw(\varsigma)}{d\varsigma}, \\
\gimel^*_{\varepsilon}(0) = \sigma.
\end{cases}$$

6. Conclusions

This research establishes the Ex-Un results through FPT, continuous dependence, and time regularity when the coefficients satisfy the global Lipschitz criteria for the solution of SFDDEs. It also demonstrates the A-P using inequality and interval translation methods. Additionally, examples are provided to clarify and validate the theoretical results.

We make three significant contributions to the establishment of the Ex-Un: continuous dependence, time regularity, and A-P results. First, by proving results in the $\mathfrak{p}$th moment, we expand the outcomes for $\mathfrak{p} = 2$. Secondly, for the first time in the literature, we establish well-posedness, time regularity, and A-P results in the context of CEKFD for SFDDEs. Third, we consider SFDDEs, which represent a more generalized class of FSDEs.

Author contributions

Wedad Albalawi, Muhammad Imran Liaqat, Kottakkaran Sooppy Nisar, and Abdel-Haleem Abdel-Aty: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing–original draft preparation, Writing-review and editing, Visualization; Wedad Albalawi, Kottakkaran Sooppy Nisar, and Abdel-Haleem Abdel-Aty: Resources, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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