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Research article

Geometric analysis on warped product semi-slant submanifolds of a locally metallic Riemannian space form

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Abstract: In this paper, we study warped product semi-slant submanifolds of locally metallic Riemannian manifolds. A Chen-type inequality for such submanifolds is derived. We construct a non trivial example of such classes of submanifolds. We also provide several applications of the obtained inequality.

Keywords: submanifolds; warped product manifolds; semi-slant submanifolds; metallic Riemannian manifolds
Mathematics Subject Classification: 53C40, 53C42, 53C42

1. Introduction

A metallic structure is a polynomial structure as defined by Goldberg et al. in [1, 2], with the structural polynomial $Q(J) = J^2 - pJ - qI$. The positive solution of the equation

$$x^2 - px - q = 0,$$

is named a member of the metallic means family [3–5], where p, q are positive integers. These numbers are denoted by:

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2},$$

are also called (p, q)-metallic numbers.

The esteemed members of the metallic family are elegantly categorized as follows [6]:

(1) The golden structure $\sigma = \frac{1+\sqrt{5}}{2}$ for $p = q = 1$, entwined with the ratio of two consecutive classical Fibonacci numbers.
(2) The copper structure $\sigma_{1,2} = 2$ with $p = 1$ and $q = 2$.
(3) The nickel structure $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$ if $p = 1$ and $q = 3$.
(4) The silver structure $\sigma_{2,1} = 1 + \sqrt{2}$ if $p = 2$ and $q = 1$, enchanted by the ratio of two consecutive Pell numbers.
(5) The bronze structure $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$ with $p = 3$ and $q = 1$.
(6) The subtle structure $\sigma_{4,1} = 2 + \sqrt{5}$ if $p = 4$ and $q = 1$, and so forth.

The members of the metallic means family share an important mathematical property that constitutes a bridge between mathematics and design; e.g., the silver mean has been used in describing fractal geometry [7]. Some members of the metallic means family (golden mean and silver mean) appeared already in sacred art of Egypt, Turkey, India, China, and other ancient civilizations [8]. The members of the metallic means family are closely related to quasiperiodic dynamics [9].

The notion of metallic structure on a Riemannian manifold was introduced in [6]. A polynomial structure on a manifold M is called a metallic structure if it is determined by a $(1, 1)$ tensor field J, which satisfies the equation

$$J^2 = pJ + qI,$$

where p, q are positive integers and I is the identity operator on the Lie algebra $\chi(M)$ of the vector fields on M. We say that a Riemannian metric g is J-compatible if:

$$g(JX, Y) = g(X, JY)$$

for all $X, Y \in \chi(M)$, which means that J is a self-adjoint operator with respect to g. This condition is equivalent in our framework with:

$$g(JX, JY) = pg(X, JY) + qg(X, Y).$$

A Riemannian manifold (M, g) endowed with a metallic structure J so that the Riemannian metric g is J-compatible is named a metallic Riemannian manifolds and (g, J) is called a metallic Riemannian structure on M.

A locally metallic Riemannian manifold $(\bar{M}, g, J)$ is a manifold that has a metallic Riemannian structure such that J is parallel with respect to the Levi–Civita connection $\bar{\nabla}$ on $\bar{M}$, that is, $\bar{\nabla} J = 0$. Hence, we have:

$$g((\bar{\nabla}_X J)Y, Z) = g(Y, (\bar{\nabla}_X J)Z).$$

Warped products can be seen as a natural generalization of Cartesian products with distance in one of the factors in skewed. This concept appeared in mathematics starting with Nash's studies [10], who proved an embedding theorem that states that every Riemannian manifold can be isometrically embedded into some Euclidean space. Also, Nash's theorem shows that every warped product can be embedded as a Riemannian submanifold in some Euclidean space. Hretcanu and Blaga worked on the existence problem of proper warped product bi-slant submanifolds in locally metallic Riemannian manifolds [11]. They investigated the existence of various types of warped products, including warped product CR submanifolds in locally metallic Riemannian manifolds. They proved that there is no proper CR warped product of the form $M_T \times_f M_\perp$ where M_T and $M_\perp$ are invariant and anti-invariant submanifolds, respectively, in a locally metallic Riemannian manifold. In [12], Alqahtani et al. derived a relation for the squared norm of the second fundamental form in terms of the components of the gradient of the warping function for CR-submanifolds of the form $M_\perp \times_f M_T$.

In this paper we study warped product semi-slant submanifolds of locally metallic Riemannian manifolds of type $M_\theta \times_f M_T$, where M_θ and M_T are proper slant and invariant submanifolds, respectively, of locally metallic Riemannian manifolds. We construct a non-trivial example of such type of submanifolds and establish, Chen-type inequality. Similar type inequalities can be found in the work of Mustafa et al. on warped product submanifolds in Kenmotsu manifold [13]; Ali et al. [14, 15] studied on semi slant submanifolds on cosymplectic manifolds. Another work of Ali et al. [16] is worth mentioning on Kähler manifolds. Also, Li et al. [17, 18] studied on generalized Sasakian space forms and in generalized complex space forms are good references for literature.

2. Preliminaries

Let $\bar{M}$ be a smooth manifold of dimension m. The metallic structure J is a $(1,1)$ tensor field defined by the equation

$$J^2 = pJ + qI,$$

where $p, q \in \mathbb{N}$ and I is the identity operator on the space of all vector fields on $\bar{M}$, denoted by $\Gamma(TM)$. Let M be an isometrically immersed submanifold in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$. Let T_xM be the tangent space of M at a point $x \in M$, and $T_x^\perp M$ is the normal space of M at x. The tangent space $T_x\bar{M}$ can be decomposed into the direct sum $T_x\bar{M} = T_xM \oplus T_x^\perp M$, for any $x \in M$. Let i_* be the differential of the immersion $i : M \to \bar{M}$. Then the induced Riemannian metric g on M is given by $g(X, Y) = \bar{g}(i_*X, i_*Y)$, for any $X, Y \in \Gamma(TM)$. For the simplification of the notations, in the rest of the paper we shall denote by X the vector field i_*X, for any $X \in \Gamma(TM)$.

Let $TX = (JX)^T$ and $NX = (JX)^\perp$, respectively, be the tangential and normal components of JX, for any $X \in \Gamma(TM)$ and $tV = (JV)^T, nV = (JV)^\perp$ be the tangential and normal components of JV, for any $V \in \Gamma(T^\perp M)$. Then we get:

$$(i) JX = TX + NX, (ii) JV = tV + nV,$$

for any $X \in \Gamma(TM), V \in \Gamma(T^\perp M)$.

The maps T and n are $\bar{g}$-symmetric:

$$(i)\bar{g}(TX, Y) = \bar{g}(X, TY), (ii)\bar{g}(nU, V) = \bar{g}(U, nV),$$

and

$$\bar{g}(NX, V) = \bar{g}(X, tV),$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.

If M is a submanifold in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$, then:

$$(i) T^2X = pTX + qX - tNX, (ii) pNX = NTX + nNX,$$
$$(iii) n^2V = pnV + qV - NtV, (iv) ptV = TtV + tnV,$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

If $p = q = 1$ then M is said to be a submanifold of golden Riemannian manifold $(\bar{M}, \bar{g}, J)$.

Let $\bar{\nabla}$ and ∇ be the Levi–Civita connections on $(\bar{M}, \bar{g})$ and on its submanifold (M, g), respectively. The Gauss and Weingarten formulas are given by:

$$(i) \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), (ii) \bar{\nabla}_X V = -A_V X + \nabla^{\perp} V,$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_V is the shape operator, which are related by:

$$\bar{g}(h(X, Y), V) = \bar{g}(A_V X, Y).$$

A $(1, 1)$-tensor field F on an m-dimensional Riemannian manifold $(\bar{M}, g)$ is known as an almost product structure [19] if it satisfies $F^2 = I$ and $F \neq \pm I$. Furthermore, $(\bar{M}, g)$ is said to be an almost product Riemannian manifold when the almost product structure F agrees with $g(FX, Y) = g(X, FY)$, for any $X, Y \in \Gamma(T\bar{M})$.

By means of any metallic structure J on $\bar{M}$ one obtains two almost product structures on $\bar{M}$ [6]

$$F_1 = \frac{2}{2\sigma - p} J - \frac{p}{2\sigma - p} I,$$
$$F_2 = -\frac{2}{2\sigma - p} J + \frac{p}{2\sigma - p} I.$$

Also, any almost product structure F on $\bar{M}$ determines two metallic structures

$$J_1 = \frac{p}{2} I + \frac{2\sigma - p}{2} F, \quad J_2 = \frac{p}{2} I - \frac{2\sigma - p}{2} F.$$

Let $(\bar{M} = M_1(c_1) \times M_2(c_2), F)$ be a locally Riemannian product manifold, where M_1 and M_2 have constant sectional curvatures c_1 and c_2, respectively. Then, the Riemannian curvature tensor R of $\bar{M} = M_1(c_1) \times M_2(c_2)$ is

$$\begin{aligned} R(X, Y)Z =& \frac{1}{4}(c_1 + c_2)[g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY] \\ &+ \frac{1}{4}(c_1 - c_2)[g(FY, Z)X - g(FXZ)Y + g(Y, Z)FX - g(X, Z)FY]. \end{aligned}$$

In view of the above and the expressions for F_1 and F_2, we achieve

$$\begin{aligned} \bar{R}(X, Y, Z, W) =& \frac{1}{4}(c_1 + c_2)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &+ \frac{1}{4}(c_1 + c_2)\{\frac{4}{(2\sigma - p)^2}[g(JY, Z)g(JX, W) - g(JX, Z)g(JY, W)] \end{aligned}$$

$$
\begin{aligned}
&+ \frac{p^2}{(2\sigma - p)^2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\
&+ \frac{2p}{(2\sigma - p)^2}[g(JX,Z)g(Y,W) + g(X,Z)g(JY,W) - g(JY,Z)g(X,W) \\
&- g(Y,Z)g(JX,W)]\} \pm \frac{1}{2}(c_1 - c_2)\{\frac{1}{2\sigma - p}[g(Y,Z)g(JX,W) \\
&- g(X,Z)g(JY,W)] + \frac{1}{2\sigma - p}[g(JY,Z)g(X,W) - g(JX,Z)g(Y,W)] \\
&+ \frac{p}{2\sigma - p}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]\}.
\end{aligned}
$$

Definition 2.1. *A submanifold M in a metallic Riemannian manifold $(\bar{M}, \bar{g}, J)$ is called a slant submanifold if the angle $\theta(X_x)$ between JX_x and T_xM is constant for any $x \in M$ and $X_x \in T_xM$. In such a case, $\theta = \theta(X_x)$ is called the slant angle of M in $\bar{M}$ and it verifies,*

$$
\cos\theta = \frac{\bar{g}(JX, TX)}{\|JX\|\|TX\|} = \frac{\|TX\|}{\|JX\|}.
$$

The immersion $i : M \to \bar{M}$ is named the slant immersion of M in $\bar{M}$.

Remark 2.1. *The invariant and anti-invariant submanifolds in metallic Riemannian manifolds $(\bar{M}, \bar{g}, J)$ are particular cases of slant submanifolds with the slant angles $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold M in $\bar{M}$, which is neither invariant nor anti-invariant, is called a proper slant submanifold, and the immersion i is called a proper slant immersion.*

Moreover, if M is a slant submanifold of a metallic Riemannian manifold $(\bar{M}, g, J)$ with a slant angle θ, the following relationships hold:

$$
g(TX, TY) = \cos^2\theta[pg(X, TY) + qg(X, Y)]
$$

and

$$
g(NX, NY) = \sin^2\theta[pg(X, TY) + qg(X, Y)],
$$

for all $X, Y \in \Gamma(TM)$.

Furthermore, we have the additional relation

$$
T^2 = \cos^2\theta(pT + qI),
$$

Now, the warped product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) with warping function f is denoted as $M_1 \times_f M_2$ and is the product manifold $M_1 \times M_2$ equipped with the metric $g = g_1 + f^2 g_2$.

3. Example of semi-slant submanifolds of metallic Riemannian manifold

Let $M = \mathbb{R}^2$ be a submanifold of $\bar{M} = \mathbb{R}^4$ defined by the immersion i as follows:

$$
i(\alpha_1, \alpha_2) = (\alpha_1 \sin\alpha_2, \alpha_1 \cos\alpha_2, \alpha_1, \alpha_1),
$$

where σ is the metallic number defined as $\sigma = \frac{p+\sqrt{p^2+4q}}{2}$. We also consider the metallic structure J of $\bar{M}$ as:

$$J(x_1, x_2, x_3, x_4) = (\sigma x_1, \sigma x_2, \sigma x_3, \bar{\sigma} x_4),$$

where $\bar{\sigma} = \frac{p-\sqrt{p^2+4q}}{2}$ and $J^2 = pJ + qI$.

It is straightforward to compute that the tangent bundle of M is spanned by the vectors (Z_1, Z_2), where

$$\begin{aligned}
Z_1 &= \sin\alpha_2 \frac{\partial}{\partial x_1} + \cos\alpha_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, \\
Z_2 &= \alpha_1 \cos\alpha_2 \frac{\partial}{\partial x_1} - \alpha_1 \sin\alpha_2 \frac{\partial}{\partial x_2}, \\
JZ_1 &= \sigma \sin\alpha_2 \frac{\partial}{\partial x_1} + \sigma \cos\alpha_2 \frac{\partial}{\partial x_2} + \sigma \frac{\partial}{\partial x_3} + \bar{\sigma} \frac{\partial}{\partial x_4}, \\
JZ_2 &= \sigma\alpha_1 \cos\alpha_2 \frac{\partial}{\partial x_1} - \sigma\alpha_1 \sin\alpha_2 \frac{\partial}{\partial x_2}.
\end{aligned}$$

Now, we define two vector spaces, D_θ and D_T, where $D_\theta = Span\{Z_1\}$ is the proper slant distribution with slant angle $\cos\theta = \frac{(\sigma+p)}{\sqrt{3}\sqrt{\sigma^2+p^2+2q}}$ and $D_T = Span\{Z_2\}$ is the invariant distribution, which are preserved by the action of J. Hence, the Riemannian metric of the warped product semi-slant submanifold M is given by the following:

$$g = 3d\alpha_1^2 + \alpha_1^2 d\alpha_2^2.$$

Thus, M is a warped product semi-slant submanifold of type $M_\theta \times_f M_T$ with a warping function $f = \alpha_1$.

4. Main results

Let M be a Riemannian (or pseudo-Riemannian) manifold. The Ricci curvature Ric and scalar curvature ρ of M are defined as:

$$\begin{aligned}
Ric &= \sum_q R(X, E_q)E_q, \\
\bar{\rho}(TM) &= \sum_{1 \le q \ne s \le m} K(E_q \wedge E_s),
\end{aligned}$$

where $K(E_q \wedge E_s)$ is the sectional curvature of the plane spanned by E_q and E_s. We also have

$$\bar{\rho}(G_k) = \sum_{1 \le q \ne s \le m} K(E_q \wedge E_s),$$

where G_k is the k-plane section of TM spanned by the orthonormal basis $\{E_1, E_2, \cdots, E_k\}$.

In this section, we study warped product submanifolds of locally metallic Riemannian space form $\bar{M}(c)$ which is of the form $M_\theta \times_f M_T$, where M_θ and M_T are proper slant and invariant submanifolds of $\bar{M}(c)$, respectively, with $dim\bar{M} = m, dimM = n, dimM_\theta = 2d_1 = n_1, dimM_T = 2d_2 = n_2$. We also assume D_θ and D_T are the corresponding distributions.

We consider the basis of D_θ and D_T as follows:

$$D_\theta = \langle e_1 = e_1^*, \cdots, e_{d_1} = e_{d_1}^*, e_{d_1+1} = \frac{\sec\theta T e_1^*}{\sqrt{q}}, \cdots, e_{2d_1} = \frac{\sec\theta T e_{d_1}^*}{\sqrt{q}}\rangle,$$
$$D_T = \langle e_{2d_1+1} = \bar{e}_1, \cdots, e_{2d_1+d_2} = \bar{e}_{d_2}, e_{2d_1+d_2+1} = \frac{J\bar{e}_1}{\sqrt{q}}, \cdots, e_{2d_1+d_2} = \frac{J\bar{e}_{d_2}}{\sqrt{q}}\rangle.$$

Let $i : M_\theta^{n_1} \times_f M_T^{n_2}$ be an isometric immersion of a warped product $M_\theta \times_f M_T$ into a Riemannian space form $\bar{M}(c)$, then

$$K(X \wedge Z) = \frac{1}{f}[(\nabla_X X)f - X^2 f],$$

for unit vectors $X \in \Gamma(M_1)$ and $Z \in \Gamma(M_2)$.

If we consider the local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ such that $\{e_1, e_2, \cdots, e_{n_1}\}$ and $\{e_{n_1+1}, e_{n_1+2}, \cdots, e_n\}$ are tangents to M_θ and M_T, respectively, then we have

$$\sum_{i=1}^{n} K(e_i \wedge e_j) = \frac{\Delta f}{f},$$
$$\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i \wedge e_j) = \frac{n_2(\nabla f)}{f}.$$

For the submanifold M, the Gauss equation is defined as

$$\bar{R}(U, V, Z, W) = R(U, V, Z, W) + g(h(U, Z), h(V, W)) - g(h(U, W), h(V, Z)),$$

for any $U, V, Z, W \in \Gamma(TM)$, where $\bar{R}$ and R are the curvature tensors on $\bar{M}$ and M, respectively.

Theorem 4.1. *Let $i : M = M_\theta \times_f M_T \to \bar{M}$ be an M_θ-minimal isometric immersion from an n-dimensional warped product semi-slant submanifold M into an m-dimensional locally metallic product space form*

$$\bar{M} = (\bar{M}_1(c_1) \times \bar{M}_2(c_2), g, \phi).$$

Then

$$||h||^2 \geq 2(\bar{\rho}(TM) - \bar{\rho}(TM_T) - \bar{\rho}(TM_\theta) - \frac{n_2\nabla f}{f}).$$

The equality holds if and only if M_θ is totally geodesic and M_T is totally umbilical in $\bar{M}$.

Proof. The proof is similar to [14, Theorem 5.1]. □

Using the result of the above theorem, we construct a Chen-type inequality for warped product semi-slant submanifolds of locally metallic Riemannian manifolds as follows:

Theorem 4.2. *Let $i : M = M_\theta \times_f M_T \to \bar{M}$ be an M_θ-minimal isometric immersion from an n-dimensional warped product semi-slant submanifold M into an m-dimensional locally metallic product space form*

$$\bar{M} = (\bar{M}_1(c_1) \times \bar{M}_2(c_2), g, \phi).$$

Then

$$\|h\|^2 \geq \alpha + 2l_2\|\nabla lnf\|^2 - 2l_2\Delta(lnf),$$

where

$$\begin{aligned}\alpha = &\frac{1}{4}(c_1+c_2)n(n-1) + \frac{1}{4}(c_1+c_2)\{\frac{4}{(2\sigma-p)^2}[p^2\cos^4\theta\frac{n_1}{2}(\frac{n_1}{2}-1) + \frac{p^2}{2}n_1n_2\cos^2\theta\\ &+ p^2\frac{n_2}{2}(\frac{n_2}{2}-1) - \frac{n_1^2}{2}q\cos^2\theta - \frac{n_2^2}{2}q] + \frac{p^2}{(2\sigma-p)^2}n(n-1) - \frac{4p}{(2\sigma-p)^2}[\frac{1}{4}pn_1^2\cos^2\theta\\ &+ \frac{1}{2}pn_1n_2 + p\cos^2\theta\frac{n_1}{2}(\frac{n_1}{2}-1) + \frac{1}{2}pn_1n_2\cos^2\theta + \frac{1}{4}pn_2^2 + p\frac{n_2}{2}(\frac{n_2}{2}-1)]]\}\\ &\pm\frac{1}{2}(c_1-c_2)\{\frac{2}{2\sigma-p}[\frac{1}{4}pn_1^2\cos^2\theta + \frac{1}{2}pn_1n_2 + p\cos^2\theta\frac{n_1}{2}(\frac{n_1}{2}-1) + \frac{1}{2}pn_1n_2\cos^2\theta\\ &+ \frac{1}{4}pn_2^2 + p\frac{n_2}{2}(\frac{n_2}{2}-1)] - \frac{p}{2\sigma-p}n(n-1)\} - \{\frac{1}{4}(c_1+c_2)n_1(n_1-1)\\ &+ \frac{1}{4}(c_1+c_2)\{\frac{4}{(2\sigma-p)^2}[p^2\cos^4\theta\frac{n_1}{2}(\frac{n_1}{2}-1) - \frac{1}{2}n_1^2q\cos^2\theta] + \frac{p^2}{(2\sigma-p)^2}n_1(n_1-1)\\ &- \frac{4p}{(2\sigma-p)^2}[\frac{1}{4}pn_1^2\cos^2\theta + p\cos^2\theta\frac{n_1}{2}(\frac{n_1}{2}-1)]]\} \pm \frac{1}{2}(c_1-c_2)\{\frac{2}{2\sigma-p}[\frac{1}{4}pn_1^2\cos^2\theta\\ &+ p\cos^2\theta\frac{n_1}{2}(\frac{n_1}{2}-1)] - \frac{p}{2\sigma-p}n_1(n_1-1)\}\} - \{\frac{1}{4}(c_1+c_2)n_2(n_2-1) + \frac{1}{4}(c_1+c_2)\\ &\times\{\frac{4}{(2\sigma-p)^2}[p^2\frac{n_2}{2}(\frac{n_2}{2}-1) - \frac{1}{2}qn_2^2] + \frac{p^2}{(2\sigma-p)^2}n_2(n_2-1)\\ &- \frac{4p}{(2\sigma-p)^2}[\frac{1}{4}pn_2^2 + \frac{1}{2}pn_2(\frac{n_2}{2}-1)]]\} \pm \frac{1}{2}(c_1-c_2)\{\frac{2}{2\sigma-p}[\frac{1}{4}pn_2^2 + \frac{1}{2}pn_2(\frac{n_2}{2}-1)]\\ &- \frac{p}{2\sigma-p}n_2(n_2-1)\}\}.\end{aligned}$$

The equality holds if and only if M_θ is totally geodesic and M_T is totally umbilical in $\bar{M}$.

Proof. Now, for metallic Riemannian space form $\bar{M} = (\bar{M}_1(c_1) \times \bar{M}_2(c_2), g, \phi)$, we have

$$\begin{aligned}\bar{R}(X,Y,Z,W) = &\frac{1}{4}(c_1+c_2)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &+ \frac{1}{4}(c_1+c_2)\{\frac{4}{(2\sigma-p)^2}[g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W)]\\ &+ \frac{p^2}{(2\sigma-p)^2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &+ \frac{2p}{(2\sigma-p)^2}[g(JX,Z)g(Y,W) + g(X,Z)g(JY,W) - g(JY,Z)g(X,W)\end{aligned}$$

$$
\begin{aligned}
&- g(Y,Z)g(JX,W)]\} \pm \frac{1}{2}(c_1 - c_2)\{\frac{1}{2\sigma - p}[g(Y,Z)g(JX,W)\\
&- g(X,Z)g(JY,W)] + \frac{1}{2\sigma - p}[g(JY,Z)g(X,W) - g(JX,Z)g(Y,W)]\\
&+ \frac{p}{2\sigma - p}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]\}.
\end{aligned}
$$

Putting $X = W = e_i$ and $Y = Z = e_j$ and taking summation over $1 \le i \ne j \le n$, we obtain

$$
\begin{aligned}
2\bar{\rho}(TM) = {} & \frac{1}{4}(c_1 + c_2)n(n-1) + \frac{1}{4}(c_1 + c_2)\{\frac{4}{(2\sigma - p)^2}[p^2 \cos^4\theta \frac{n_1}{2}(\frac{n_1}{2} - 1) + \frac{p^2}{2} n_1 n_2 \cos^2\theta\\
& + p^2 \frac{n_2}{2}(\frac{n_2}{2} - 1) - \frac{n_1^2}{2} q \cos^2\theta - \frac{n_2^2}{2} q] + \frac{p^2}{(2\sigma - p)^2} n(n-1) - \frac{4p}{(2\sigma - p)^2}[\frac{1}{4} p n_1^2 \cos^2\theta\\
& + \frac{1}{2} p n_1 n_2 + p \cos^2\theta \frac{n_1}{2}(\frac{n_1}{2} - 1) + \frac{1}{2} p n_1 n_2 \cos^2\theta + \frac{1}{4} p n_2^2 + p \frac{n_2}{2}(\frac{n_2}{2} - 1)]\}\\
& \pm \frac{1}{2}(c_1 - c_2)\{\frac{2}{2\sigma - p}[\frac{1}{4} p n_1^2 \cos^2\theta + \frac{1}{2} p n_1 n_2 + p \cos^2\theta \frac{n_1}{2}(\frac{n_1}{2} - 1) + \frac{1}{2} p n_1 n_2 \cos^2\theta\\
& + \frac{1}{4} p n_2^2 + p \frac{n_2}{2}(\frac{n_2}{2} - 1)] - \frac{p}{2\sigma - p} n(n-1)\},
\end{aligned}
$$

taking summation over $1 \le i \ne j \le n_1$, we obtain

$$
\begin{aligned}
2\bar{\rho}(TM_\theta) = {} & \frac{1}{4}(c_1 + c_2)n_1(n_1 - 1) + \frac{1}{4}(c_1 + c_2)\{\frac{4}{(2\sigma - p)^2}[p^2 \cos^4\theta \frac{n_1}{2}(\frac{n_1}{2} - 1) - \frac{1}{2} n_1^2 q \cos^2\theta]\\
& + \frac{p^2}{(2\sigma - p)^2} n_1(n_1 - 1) - \frac{4p}{(2\sigma - p)^2}[\frac{1}{4} p n_1^2 \cos^2\theta + p \cos^2\theta \frac{n_1}{2}(\frac{n_1}{2} - 1)]\} \pm \frac{1}{2}(c_1 - c_2)\\
& \times \{\frac{2}{2\sigma - p}[\frac{1}{4} p n_1^2 \cos^2\theta + p \cos^2\theta \frac{n_1}{2}(\frac{n_1}{2} - 1)] - \frac{p}{2\sigma - p} n_1(n_1 - 1)\},
\end{aligned}
$$

taking summation over $n_1 + 1 \le i \ne j \le n$, we obtain

$$
\begin{aligned}
2\bar{\rho}(TM_T) = {} & \frac{1}{4}(c_1 + c_2)n_2(n_2 - 1) + \frac{1}{4}(c_1 + c_2)\{\frac{4}{(2\sigma - p)^2}[p^2 \frac{n_2}{2}(\frac{n_2}{2} - 1) - \frac{1}{2} q n_2^2]\\
& + \frac{p^2}{(2\sigma - p)^2} n_2(n_2 - 1) - \frac{4p}{(2\sigma - p)^2}[\frac{1}{4} p n_2^2 + \frac{1}{2} p n_2(\frac{n_2}{2} - 1)]\}\\
& \pm \frac{1}{2}(c_1 - c_2)\{\frac{2}{2\sigma - p}[\frac{1}{4} p n_2^2 + \frac{1}{2} p n_2(\frac{n_2}{2} - 1)] - \frac{p}{2\sigma - p} n_2(n_2 - 1)\},
\end{aligned}
$$

we assume that,

$$2\bar{\rho}(T_pM) - 2\bar{\rho}(TM_\theta) - 2\bar{\rho}(TM_T) = \alpha.$$

So we have

$$\|h\|^2 \ge \alpha + 2n_2\|\nabla lnf\|^2 - 2n_2\Delta(lnf). \tag{4.1}$$

□

5. Application to Euler–Lagrange equation

In the potential theory, Dirichlet energies have significant use. If $f : M \to \mathbb{R}$ is a smooth function, then the Dirichlet energy is defined as:

$$E(f) = \frac{1}{2}\int_M \|\nabla f\|^2 dV,$$

where $E(f)$ and dV are Dirichlet energy and volume element, respectively. In Lagrangian mechanics, the Lagrangian L of a mechanical system is $T - V$, where T is the kinetic energy and V is the system's potential energy, respectively. As a generalization to smooth manifolds, the Lagrangian of the smooth function f, is determined by

$$L = \frac{1}{2}\|f\|^2.$$

The Euler–Lagrange equation for a Lagrangian L is $\Delta f = 0$. Following are a few useful results that will be needed to prove the next results:

Lemma 5.1 ([14]). *Let M be a compact, connected Riemannian manifold without boundary and f be a smooth function on M such that $\Delta f \geq 0 (\Delta f \leq 0)$. Then f is a constant function.*

Theorem 5.1. *Let $M = M_\theta \times_f M_T$ be a compact (without boundary) and connected warped product semi-slant submanifold in locally metallic Riemannian manifold $\bar{M}(c)$. If the warping function f is a solution of the Euler–Lagrange equation, then M is simply a Riemannian product with $\|h\|^2 \geq \alpha$.*

Proof. Suppose, $\Delta f = 0$, then we have $\Delta(lnf) = -\frac{1}{f^2}|\nabla f|^2 \leq 0$. By Lemma 5.1 we have f = constant, which proves that M is simply a Riemannian product. Further, from Eq (4.1), we have $\Delta \ln f \leq 0 \implies \|h\|^2 \geq \alpha$. □

6. Application to gradient Ricci soliton

Gradient Ricci solitons are important in understanding the Hamilton's Ricci flow. They are self-similar solutions of the Ricci flows and arise often as singularity models of the Ricci flow.

A complete n-dimensional Riemannian manifold (M, g) is said to be a gradient shrinking Ricci soliton if there exists a smooth function f on M such that the equation

$$Ric + \nabla^2 f = \lambda g,$$

holds for some positive constant $\lambda \in \mathbb{R}$ [20], where Ric is the Ricci tensor and $\nabla^2 f$ is the Hessian of the function f.

The function f is called the potential function of the gradient-shrinking Ricci soliton. Similarly, a gradient Ricci soliton is called steady and expanding if the real number λ is 0 and negative, respectively.

We have

$$\Delta f = -traceH^f = -\sum_{t=1}^{n} H^f(e_t, e_t) \implies \|h\|^2 \geq \alpha + 2n_2\|\nabla lnf\|^2 + 2n_2\sum_{t=1}^{n} H^f(e_t, e_t).$$

Now, if M admits a shrinking gradient Ricci soliton, then

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + Hess^{lnf}(X_1, X_2),$$

for

$$\begin{aligned} & X_1, X_2 \in \Gamma(TM_\theta) \\ \Longrightarrow \ & \sum_{t=1}^{n} Ric(e_t, e_t) = n_1 \lambda g + \sum_{t=1}^{n} Hess^{lnf}(e_t, e_t), \\ \Longrightarrow \ & \sum_{t=1}^{n} Hess^{lnf}(e_t, e_t) = \sum_{t=1}^{n} Ric(e_t, e_t) - n_1 \lambda g, \\ \Longrightarrow \ & ||h||^2 \geq \alpha + 2n_2 ||\nabla lnf||^2 + \sum_{t=1}^{n} Ric(e_t, e_t) - n_1 \lambda g. \end{aligned}$$

If $\lambda < 0$, then $||h||^2 \geq \alpha + 2n_2||\nabla lnf||^2 + \sum_{t=1}^{n} Ric(e_t, e_t)$. So we have the following:

Theorem 6.1. *Let $M = M_\theta \times_f M_T$ be a warped product semi-slant submanifold in $\bar{M}(c)$ admitting a shrinking gradient Ricci soliton. Then*

$$||h||^2 \geq \alpha + 2n_2 ||\nabla lnf||^2 + \sum_{t=1}^{n} Ric(e_t, e_t).$$

Proof. Proof follows from the above discussion. □

7. Conclusions

In conclusion, this research has delved deeply into the examination of warped product semi-slant submanifolds situated within a locally metallic Riemannian manifold. Through this study, we have crafted a Chen type inequality that is uniquely suited for these submanifolds, offering a specialized insight into their geometric properties. Additionally, we have showcased a non-trivial example belonging to this specific class of submanifolds, underscoring the richness and diversity within this geometric framework. Our exploration extends beyond theoretical considerations, as we have uncovered a multitude of practical applications stemming from the implications of the derived inequality. This work not only contributes to the theoretical understanding of these submanifolds but also lays the groundwork for further research and applications in the broader field of geometric analysis.The results established in this paper can be extended to other classes of submanifolds of locally metallic Riemannian manifolds. One may also consider the applications of the above result to other soliton structures like, Ricci-Yamabe soliton, Ricci -Bourguignon soliton or Bach soliton etc. for further investigation.

Author contributions

Conceptualization, S.K.H and B.B.; methodology, B.B and T.P. (Assistant Teacher); validation, S.K.H., M.A.K. and T.P.; formal analysis, B.B. and T.P.; investigation, B.B. and S.K.H.; writing–original draft preparation, B.B. and M.A.K.; writing–review and editing, B.B. and M.A.K.;

supervision, S.K.H.; project administration, S.K.H. and M.A.K. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used any Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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