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Research article

Centralized solution in max-min fuzzy relation inequalities

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Abstract: In recent years, the peer-to-peer (P2P) educational information sharing system was modeled by a system of fuzzy relation inequalities (FRIs) with addition-min or max-min composition. The max-min FRIs system was applicable to the P2P network considering the highest download traffic among the terminals. Moreover, every solution to such a max-min FRIs system corresponds exactly to one feasible flow control scheme. To embody the stability of a given feasible scheme, we introduce the concept of the widest symmetrical interval solution (WSIS), regarding the corresponding solution in the max-min FRIs system. Some effective procedures are proposed to find the WSIS regarding a provided solution. In addition, aiming to find the most stable feasible scheme, we further define the concept of a centralized solution. Some effective procedures are also proposed to find the centralized solution regarding the max-min FRIs system. Some numerical examples are provided, respectively, to demonstrate our proposed resolution procedures. Our obtained centralized solution will provide decision support for system administrators considering the stability of the feasible scheme.

Keywords: max-min composition operation; P2P network system; fuzzy relation inequality; widest symmetrical interval solution; centralized solution
Mathematics Subject Classification: 90C70, 90C90

1. Introduction

In the past few decades, the research on fuzzy relation systems, including equation systems and inequality systems, has developed rapidly and comprehensively. The concept of fuzzy relation equation (FRE) was first proposed by E. Sanchez [1]. In [1], the composition operation was *max-min*. A sufficient and necessary condition for the solvable (having at least one solution) FRE system composed of max-min was displayed in [1]. Moreover, for the solvable system, the full solution set contains a finite set of lower solutions and a greatest solution. It was widely acknowledged that numerous issues related to body knowledge can be addressed as FRE problems [2]. In an FRE system, the initial composition max-min was soon later generalized to *max-T*, where T represents a triangular norm [3–6]. The structure of the solution set in a solvable max-T FRE system is the same as that in a solvable max-min one. The identification of solvability and the determination of the solution set are the central and foundational aspects of the research on FRE. Many scholars devoted their time to searching all the solutions to an FRE system, with the max-T composition [7–11].

J. Drewniak mentioned the fuzzy relation inequality (FRI) with max-min for the first time [12]. The max-min FRIs were completely solved in [13], using the so-called conservative path approach. In [13], the authors also studied a kind of optimization problem, in which the objective function was expressed in a latticized linear form and the constraint was the max-min FRIs. F. Guo et al. also proposed the FRI path approach for solving the minimal solutions in a max-min FRIs system [14].

In 2012, J.-X. Li et al. [15] employed a new composition operator, i.e., *addition-min*, in a system of FRIs. The addition-min of FRIs

$$\sum_{1\leq j\leq n} a_{ij} \wedge x_j \geq b_i, \quad i = 1, 2, \cdots, m, \tag{1.1}$$

were introduced for describing the peer-to-peer (P2P) educational information sharing system. In such a P2P network system, n terminals are represented by the notations $T_1, \cdots, T_n$ as shown in Figure 1.

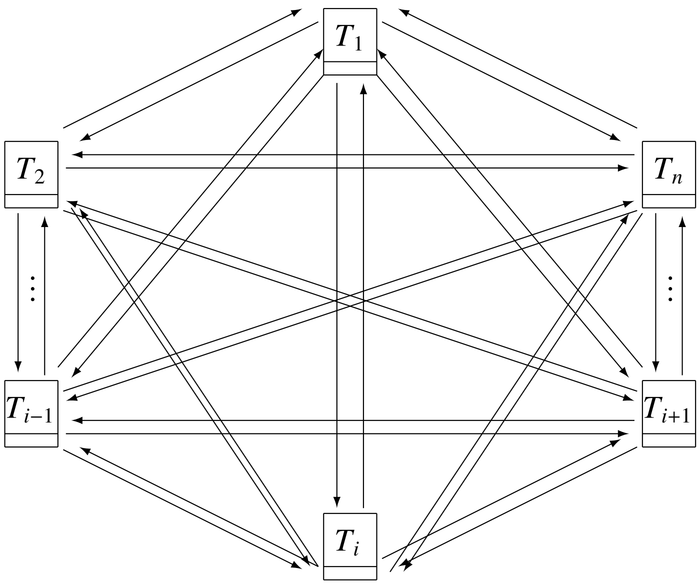

Figure 1. Peer-to-Peer educational information sharing system.

In system (1.1), the variable x_j represents the quality level on which the terminal T_j sends out (shares) its local data. The parameter a_{ij} represents the bandwidth. The download traffic requirement of T_i is requested to be no less than b_i. Obviously, the total download traffic of the ith terminal T_i is satisfied if and only if it holds that

$$\sum_{1\leq j\leq n} (a_{ij} \wedge x_j) \geq b_i. \tag{1.2}$$

Satisfying the download traffic requirements of all the terminals, the above system (1.1) is naturally generated and established.

Some detailed properties, particularly the number of minimal solutions and the convexity of the solution set, were extensively discussed in [16]. It was highlighted in [16] that the addition-min FRIs system often has infinitely many minimal solutions. Thus, it is difficult to compute its complete solution set. Of course, when the scale of the problem is small enough, it is still possible to find out all the minimal solutions, as well as the complete solution set [17]. The min-max programming and the leximax programming were formulated and studied subject to the addition-min FRIs system [18–20]. Considering the random line fault [21] or the distinguishing quality levels on which the terminals send out their local files [22, 23], some variants of the addition-min FRIs were also investigated.

As pointed out above, characterizing the P2P network system by the addition-min of the FRIs system (1.1), the authors only considered the total download traffic of the ith terminal. However, when considering the highest download traffic, the above system (1.1) is ineffective. In fact, the highest download traffic of the ith terminal T_i should be

$$\bigvee_{1\leq j\leq n}(a_{ij}\wedge x_j)\geq b_i. \tag{1.3}$$

As a consequence, adopting the highest download traffic, the P2P educational information sharing system was characterized by the max-min FRIs below [24–27],

$$\bigvee_{1\leq j\leq n}(a_{ij}\wedge x_j)\geq b_i, \quad i=1,2,\cdots,m. \tag{1.4}$$

G. Xiao et al. [25] discussed the classification of the solution set of (1.4), while X. Yang [26] attempted to find the approximate solution for the unsolvable system (1.4). Considering the rigid requirement, exactly equal to b_i, system (1.4) turns out to be

$$\bigvee_{1\leq j\leq n}(a_{ij}\wedge x_j)= b_i, \quad i=1,2,\cdots,m. \tag{1.5}$$

Different resolution methods were proposed for solving the approximate solutions of the above system (1.5) [28, 29] when it was unsolvable. In the solvable case, three types of geometric programming problems subject to system (1.5) were solved [30–32]. Considering the flexible requirement, [33–35] assumed the highest download traffic of T_i to be no less than c_i and no more than d_i, and established the following max-min FRIs with bidirectional constraints.

$$c_i\leq(b_{i1}\wedge y_1)\vee(b_{i2}\wedge y_2)\vee\cdots\vee(b_{in}\wedge y_n)\leq d_i, \quad i=1,2,\cdots,m. \tag{1.6}$$

After standardization, it is assumed that $c_i, b_{ij}, y_j, d_i\in[0,1]$, $\forall i\in\mathbb{I}$, $j\in\mathbb{J}$, where

$$\mathbb{I}=\{1,\ldots,m\}, \quad \mathbb{J}=\{1,\ldots,n\}.$$

Similar to system (1.1) for describing the P2P educational information sharing system, any solution of system (1.6) stands for a feasible flow control scheme for the P2P network system. A given solution is indeed a feasible scheme prepared in advance. However, in the actual implementation process, the values of components in the given solution might encounter some temporary adjustment or uncontrollable variation. Thus, we consider the tolerable variation for a given solution in this work. The widest symmetrical interval solution will be defined and investigated. It embodies the tolerable variation for a given solution. Moreover, we will further define the centralized solution for system (1.6). The centralized solution is indeed the solution with the biggest tolerable variation. As is well known, a feasible scheme is considered to be more stable if it is able to bear a bigger tolerable variation. As a result, the centralized solution would be considered to be the most stable feasible scheme in the P2P network system.

The remaining sections are arranged as follows. Section 2 presents the basic foundations on the max-min FRIs system (1.6). In Section 3, we propose an effective approach for solving the widest symmetrical interval solution regarding a provided solution. In Section 4, we further provide a resolution approach for searching the centralized solution for system (1.6). Some numerical examples are enumerated for illustrating our presented resolution approaches. Section 5 concludes the work.

2. Preliminaries

For convenience, we abbreviate

$$(b_{i1} \wedge y_1) \vee (b_{i2} \wedge y_2) \vee \cdots \vee (b_{in} \wedge y_n) = (b_{i1}, \cdots, b_{in}) \circ (y_1, \cdots, y_n),$$

for any $i \in \mathbb{I}$. Furthermore, the above max-min FRIs system (1.6) could be rewritten in its abbreviation form as

$$c \leq B \circ y \leq d, \tag{2.1}$$

where $c = (c_1, \cdots, c_m)$, $B = (b_{ij})_{m\times n}$, $y = (y_1, \cdots, y_n)$ and $d = (d_1, \cdots, d_m)$. Accordingly, the set of all solutions to system (1.6) or system (2.1) is exactly

$$\mathbb{S}(B, c, d) = \{y \in \mathbb{V} \mid c \leq B \circ y \leq d\}. \tag{2.2}$$

where $\mathbb{V} = [0, 1]^n$.

Definition 1. (Consistent) [33–35] System (1.6) is called consistent if it has a solution, i.e., $\mathbb{S}(B, c, d) \neq \emptyset$. Conversely, $\mathbb{S}(B, c, d) = \emptyset$; we call it inconsistent.

Definition 2. [33–35] $\hat{v} \in \mathbb{V}$ is a maximum solution if $\hat{v} \in \mathbb{S}(B, c, d)$ and $\hat{v} \geq v$ hold for any $v \in \mathbb{S}(B, c, d)$. $v^m \in \mathbb{V}$ is a minimal solution if $v^m \in \mathbb{S}(B, c, d)$ and $v \leq v^m \Rightarrow v = v^m$ holds for any $v \in \mathbb{S}(B, c, d)$.

Define the operator "@" as follows:

$$b_{ij}@d_i = \begin{cases} 1, & \text{if } b_{ij} \leq d_i, \\ d_i, & \text{if } b_{ij} > d_i. \end{cases} \tag{2.3}$$

Let $\hat{v} = (\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_n)$, where

$$\hat{v}_j = \bigwedge_{i\in\mathbb{I}} b_{ij}@d_i. \tag{2.4}$$

Theorem 1. *[33–35] $\mathbb{S}(B, c, d) \neq \emptyset$ if and only if $\hat{v} \in \mathbb{S}(B, c, d)$.*

Proposition 1. *[33–35] If* $\mathbb{S}(B,c,d) \neq \emptyset$ *and* $v \in \mathbb{S}(B,c,d)$ *is an arbitrary solution, then we have* $v \le \hat{v}$.

According to Theorem 1 and Proposition 1, when (1.6) is a consistent system, $\hat{v}$ is always its maximum solution.

Proposition 2. *[33–35] Let* $u, v \in \mathbb{S}(B,c,d)$ *be two solutions of system* (1.6)*, with* $u \le v$*. Then for any* $x \in [u,v]$*, it holds* $x \in \mathbb{S}(B,c,d)$*, i.e.,* x *is also a solution of* (1.6).

Theorem 2. *[33–35] Let system* (1.6) *be consistent. Then the solution set is*

$$\mathbb{S}(B,c,d) = \bigcup_{v^m \in \mathbb{S}^m(B,c,d)} [v^m, \hat{v}], \tag{2.5}$$

where $\mathbb{S}^m(B,c,d)$ *is the collection of all minimal solutions, while* $\hat{v}$ *is the maximum one.*

3. Widest symmetrical interval solution regarding the provided solution v

In this section, we posit the hypothesis that

$$v = (v_1, v_2, \cdots, v_n) \in V$$

is one of the given solutions in the max-min FRIs system (1.6), i.e., $v \in \mathbb{S}(B,c,d)$. We will define the concept of the widest symmetrical interval solution regarding the provided solution v. Moreover, some effective procedures will be proposed for obtaining the widest symmetrical interval solution.

Definition 3. (***Interval solution*** & ***width***) Let u, v be two vectors in $\mathbb{V}$, with $u \le v$. If

$$[u,v] \subseteq \mathbb{S}(B,c,d),$$

then we say $[u,v]$ an interval solution of (1.6). Moreover, the following non-negative number, denoted by $w[u,v]$, i.e.,

$$w[u,v] = (v_1 - u_1) \wedge (v_2 - u_2) \wedge \cdots \wedge (v_n - u_n) \tag{3.1}$$

is said to be the width of the interval solution $[u,v]$.

Remark 1. *Let* $[u,v], [x,y] \subseteq \mathbb{S}(B,c,d)$ *be two interval solutions of* (1.6)*. If* $[u,v] \subseteq [x,y]$*, then it holds* $w[u,v] \le w[x,y]$*. However, this conclusion does not hold in reverse. For example, suppose* $[u,v] = ([0.2,0.8],[0.3,0.4],[0.4,0.9]), [x,y] = ([0.3,0.5],[0.4,0.6],[0.4,0.7])$*. It is clear that* $w[u,v] = 0.1 \le 0.2 = w[x,y]$*. However, the inclusion relation* $[u,v] \subseteq [x,y]$ *doesn't hold.*

Definition 4. (***Symmetrical interval solution (SIS) regarding*** v) Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \mathbb{V}$ be an arbitrary vector. Denote the following vectors $v + \varepsilon$ and $v - \varepsilon$, according to ε and v,

$$\begin{cases} v + \varepsilon = (v_1 + \varepsilon_1, \cdots, v_n + \varepsilon_n), \\ v - \varepsilon = (v_1 - \varepsilon_1, \cdots, v_n - \varepsilon_n). \end{cases} \tag{3.2}$$

If $[v - \varepsilon, v + \varepsilon]$ is an interval solution of system (1.6), we say $[v - \varepsilon, v + \varepsilon]$ a ***symmetrical interval solution*** regarding the provided solution v.

Obviously, if $[v - \varepsilon, v + \varepsilon]$, represented by (3.2), is a symmetrical interval solution (SIS) regarding v, then by (3.1), its width is indeed

$$w[v - \varepsilon, v + \varepsilon] = 2(\varepsilon_1 \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_n). \tag{3.3}$$

Definition 5. (***Widest symmetrical interval solution (WSIS) regarding*** v) Let $[v - \varepsilon^v, v + \varepsilon^v]$ be an SIS regarding the provided solution v, where $\varepsilon^v \in \mathbb{V}$. $[v - \varepsilon^v, v + \varepsilon^v]$ is said to be a ***widest symmetrical interval solution*** regarding v, if

$$w[v - \varepsilon^v, v + \varepsilon^v] \geq w[v - \varepsilon, v + \varepsilon] \tag{3.4}$$

holds for any SIS $[v - \varepsilon, v + \varepsilon]$ regarding v, where $\varepsilon \in \mathbb{V}$. Moreover, the width of $[v - \varepsilon^v, v + \varepsilon^v]$, i.e., $w[v - \varepsilon^v, v + \varepsilon^v]$, is said to be the ***(biggest) symmetrical width*** regarding v.

3.1. Construct a vector $\check{v}$ corresponding to the provided solution v

According to our provided solution v, define m indicator sets as

$$\mathbb{J}_i^v = \{j \in \mathbb{J} | b_{ij} \wedge v_j \geq c_i\}, \tag{3.5}$$

where $i \in \mathbb{I}$. Furthermore, define m indices as

$$j_i^v = \arg\max\{v_j | j \in \mathbb{J}_i^v\}, \tag{3.6}$$

where $i \in \mathbb{I}$. Now we find the indices $j_1^v, j_2^v, \cdots, j_m^v$. These indices enable us to further define

$$\mathbb{I}_j^v = \{i \in \mathbb{I} | j_i^v = j\}, \tag{3.7}$$

where $j \in \mathbb{J}$. Next, we could construct a vector, denoted by $\check{v} = (\check{v}_1, \check{v}_2, \cdots, \check{v}_n)$, in which

$$\check{v}_j = \begin{cases} \bigvee_{i \in \mathbb{I}_j^v} c_i, & \text{if } \mathbb{I}_j^v \neq \emptyset, \\ 0, & \text{if } \mathbb{I}_j^v = \emptyset, \end{cases} \quad j \in \mathbb{J}. \tag{3.8}$$

Proposition 3. *Assume that $v \in \mathbb{S}(B, c, d)$. Then there is $\mathbb{J}_i^v \neq \emptyset$ for any $i \in \mathbb{I}$.*

Proof. Since $v \in \mathbb{S}(B, c, d)$, according to system (1.6), we have

$$c_i \leq (b_{i1} \wedge v_1) \vee (b_{i2} \wedge v_2) \vee \cdots \vee (b_{in} \wedge v_n) \leq d_i, \quad \forall i \in \mathbb{I}.$$

Therefore, for any $i \in \mathbb{I}$, there is $j_i \in \mathbb{J}$ satisfying

$$b_{ij_i} \wedge y_{j_i} \geq c_i.$$

Observing Eq (3.5), we have $j_i \in \mathbb{J}_i^v$, i.e., $\mathbb{J}_i^v \neq \emptyset$. □

Proposition 4. *For the solution $v \in \mathbb{S}(B, c, d)$ of* (1.6) *and the vector $\check{v}$ defined by* (3.8)*, we have $\check{v} \leq v$.*

Proof. Let $j \in \mathbb{J}$ be an arbitrary indicator.

Case 1. If $\mathbb{I}_j^v = \emptyset$, it follows from (3.8) that $\check{v}_j = 0 \le v_j$.

Case 2. If $\mathbb{I}_j^v \neq \emptyset$, it follows from (3.8) that $\check{v}_j = \bigvee_{i \in \mathbb{I}_j^v} c_i$. Take arbitrarily $i \in \mathbb{I}_j^v$. By (3.7), we have

$$j_i^v = j.$$

According to (3.6), it is clear $j = j_i^v \in \mathbb{J}_i^v$. It follows from (3.5) that

$$v_j \ge b_{ij} \wedge v_j \ge c_i.$$

In view of the arbitrariness of i in $\mathbb{I}_j^v$, there is $v_j \ge \bigvee_{i \in \mathbb{I}_j^v} c_i = \check{v}_j$.

Cases 1 and 2 indicate $\check{v}_j \le v_j$, $\forall j \in \mathbb{J}$. Thus, we obtain $\check{v} \le v$ □

Theorem 3. *For the solution $v \in \mathbb{S}(B, c, d)$ of* (1.6) *and the vector $\check{v}$ defined by* (3.8)*, there is $\check{v} \in \mathbb{S}(B, c, d)$. That is to say, $\check{v}$ serves as a solution of* (1.6)*.*

Proof. Take arbitrarily $i \in \mathbb{I}$. It is clear $j_i^v \in \mathbb{J}_i^v$ by (3.6). According to (3.5), we have

$$b_{ij_i^v} \ge b_{ij_i^v} \wedge v_{j_i^v} \ge c_i. \tag{3.9}$$

Denote $j^\dagger = j_i^v$. Then we have $i \in \mathbb{I}_{j^\dagger}^v \neq \emptyset$ by (3.7). According to (3.8), we have

$$\check{v}_{j^\dagger} = \bigvee_{k \in \mathbb{I}_{j^\dagger}^v} c_k \ge c_i. \tag{3.10}$$

Considering $j^\dagger = j_i^v$, by (3.9) and (3.10) we further have

$$(b_{i1} \wedge \check{v}_1) \vee (b_{i2} \wedge \check{v}_2) \vee \cdots \vee (b_{in} \wedge \check{v}_n) \ge b_{ij^\dagger} \wedge \check{v}_{j^\dagger} \ge c_i, \quad \forall i \in \mathbb{I}. \tag{3.11}$$

Since $v \in \mathbb{S}(B, c, d)$, according to system (1.6), we have

$$c_i \le (b_{i1} \wedge v_1) \vee (b_{i2} \wedge v_2) \vee \cdots \vee (b_{in} \wedge v_n) \le d_i, \quad \forall i \in \mathbb{I}.$$

Thus,

$$b_{ij} \wedge v_j \le d_i, \quad \forall i \in \mathbb{I}, \forall j \in \mathbb{J}.$$

Following Proposition 4, it holds $\check{v}_j \le v_j$, $j \in \mathbb{J}$. Thus we get

$$b_{ij} \wedge \check{v}_j \le b_{ij} \wedge v_j \le d_i, \quad \forall i \in \mathbb{I}, \forall j \in \mathbb{J},$$

i.e.,

$$(b_{i1} \wedge \check{v}_1) \vee (b_{i2} \wedge \check{v}_2) \vee \cdots \vee (b_{in} \wedge \check{v}_n) \le d_i, \quad \forall i \in \mathbb{I}. \tag{3.12}$$

Combining Inequalities (3.11) and (3.12), it is evident to have $\check{v} \in \mathbb{S}(B, c, d)$. □

3.2. Construct the WSIS regarding the provided solution v

In the previous section, we have obtained the vector $\check{v}$ based on the provided solution v. Moreover, $\check{v}$ is found to be a solution of system (1.6), no more than v, i.e., $\check{v} \le v$. In this section, we further construct a symmetrical interval solution regarding v, based on the solutions $\check{v}, v, \hat{v}$.

Remind that $\hat{v}$ is the maximum solution, while v is a provided solution of (1.6). Besides, $\check{v}$ is as obtained following Eqs (3.5)–(3.8). Now we denote the vector $\varepsilon^v = (\varepsilon_1^v, \varepsilon_2^v, \cdots, \varepsilon_n^v)$ related to v as

$$\varepsilon_j^v = (v_j - \check{v}_j) \wedge (\hat{v}_j - v_j), \quad j \in \mathbb{J}. \tag{3.13}$$

Note that $\check{v}_j \le v \le \hat{v} \le (1, \cdots, 1)$. It could be evidently found that $0 \le \varepsilon_j^v \le 1$ for all $j \in \mathbb{J}$. Hence, we have $\varepsilon^v \in \mathbb{V}$.

Next, we provide some properties on the above-obtained vector ε^v.

Proposition 5. *Let $\varepsilon^v \in \mathbb{V}$ be the vector defined by* (3.13)*, related to the given solution v. Then there is $v + \varepsilon^v \in \mathbb{S}(B, c, d)$, i.e., $v + \varepsilon^v$ satisfies system* (1.6)*.*

Proof. Since $0 \le \varepsilon_j^v \le 1$ and

$$\varepsilon_j^v = (v_j - \check{v}_j) \wedge (\hat{v}_j - v_j) \le \hat{v}_j - v_j, \quad \forall j \in \mathbb{J}.$$

we have

$$v_j \le v_j + \varepsilon_j^v \le \hat{v}_j, \quad \forall j \in \mathbb{J}.$$

That is $v \le v + \varepsilon^v \le \hat{v}$. Since $v, \hat{v} \in \mathbb{S}(B, c, d)$, following Proposition 2 we have $v + \varepsilon^v \in \mathbb{S}(B, c, d)$. □

Proposition 6. *Let $\varepsilon^v \in \mathbb{V}$ be the vector defined by* (3.13)*, related to the given solution v. Then there is $v - \varepsilon^v \in \mathbb{S}(B, c, d)$.*

Proof. Since $0 \le \varepsilon_j^v \le 1$ and

$$\varepsilon_j^v = (v_j - \check{v}_j) \wedge (\hat{v}_j - v_j) \le v_j - \check{v}_j, \quad \forall j \in \mathbb{J},$$

we have

$$\check{v}_j \le v_j - \varepsilon_j^v \le v_j, \quad \forall j \in \mathbb{J}.$$

That is $\check{v} \le v - \varepsilon^v \le v$. According to Theorem 3 and the given condition, both v and $\check{v}$ are solutions to system (1.6). It follows from Proposition 2 that $v - \varepsilon^v \in \mathbb{S}(B, c, d)$. □

According to the above Propositions 5 and 6, one easily discovers the Corollary 1 below.

Corollary 1. *Let $\varepsilon^v \in \mathbb{V}$ be the vector defined by* (3.13)*, related to the given solution v. Then there is $[v - \varepsilon^v, v + \varepsilon^v] \subseteq \mathbb{S}(B, c, d)$, i.e., $[v - \varepsilon^v, v + \varepsilon^v]$ is a symmetrical interval solution regarding v.*

It is shown in Corollary 1 that $[v - \varepsilon^v, v + \varepsilon^v]$ is a symmetrical interval solution regarding v. Next, it will be further verified to be the widest one.

Proposition 7. *Let $u \in \mathbb{S}(B, c, d)$ be a solution of* (1.6) *with $u \le v$. Then it holds $w[u, v] \le w[\check{v}, v]$.*

Proof. Select arbitrarily $j \in \mathbb{J}$. Now let us examine the inequality $w[u,v] \le v_j - \check{v}_j$.

Case 1. If $\mathbb{I}_j^v = \emptyset$, then $\check{v}_j = 0 \le u_j$ by (3.8). Hence

$$\begin{aligned} w[u,v] &= \bigwedge_{k\in\mathbb{J}} (v_k - u_k) \\ &\le v_j - u_j \\ &\le v_j - \check{v}_j. \end{aligned} \tag{3.14}$$

Case 2. If $\mathbb{I}_j^v \neq \emptyset$, then $\check{v}_j = \bigvee_{i\in\mathbb{I}_j^v} c_i$ by (3.8). Accordingly, there is $i^\dagger \in \mathbb{I}_j^v$, satisfying

$$\check{v}_j = c_{i^\dagger}. \tag{3.15}$$

At the same time, by (3.7) we have

$$j_{i^\dagger}^v = j, \tag{3.16}$$

since $i^\dagger \in \mathbb{I}_j^v$. Note that $u \in \mathbb{S}(B,c,d)$. u satisfies the $i^\dagger$th inequality from (1.6), i.e.,

$$c_{i^\dagger} \le (b_{i^\dagger 1} \wedge u_1) \vee (b_{i^\dagger 2} \wedge u_2) \vee \cdots \vee (b_{i^\dagger n} \wedge u_n) \le d_{i^\dagger}.$$

As a result, there is $j^\dagger \in \mathbb{J}$ satisfying

$$b_{i^\dagger j^\dagger} \wedge u_{j^\dagger} \ge c_{i^\dagger}. \tag{3.17}$$

Considering $u \le v$, we have $b_{i^\dagger j^\dagger} \wedge v_{j^\dagger} \ge b_{i^\dagger j^\dagger} \wedge u_{j^\dagger} \ge c_{i^\dagger}$. According to (3.5), it holds that $j^\dagger \in \mathbb{J}_{i^\dagger}^v$. Moreover, Inequality (3.17) implies that

$$u_{j^\dagger} \ge b_{i^\dagger j^\dagger} \wedge u_{j^\dagger} \ge c_{i^\dagger}. \tag{3.18}$$

Observing (3.6) and (3.16), we have $j = j_{i^\dagger}^v = \arg\max\{v_l | l \in \mathbb{J}_{i^\dagger}^v\}$. Thus,

$$v_j \ge v_l, \quad \forall l \in \mathbb{J}_{i^\dagger}^v.$$

Thereby, $j^\dagger \in \mathbb{J}_{i^\dagger}^v$ indicates

$$v_j \ge v_{j^\dagger}. \tag{3.19}$$

Combining (3.15), (3.18), and (3.19), we have

$$\begin{aligned} v_j - \check{v}_j &= v_j - c_{i^\dagger} \\ &\ge v_j - u_{j^\dagger} \\ &\ge v_{j^\dagger} - u_{j^\dagger} \\ &\ge \bigwedge_{k\in\mathbb{J}} (v_k - u_k) = w[u,v]. \end{aligned} \tag{3.20}$$

According to Inequalities (3.14) and (3.20), for any j in $\mathbb{J}$, there is $v_j - \check{v}_j \ge w[u,v]$. So we have

$$w[\check{v},v] = \bigwedge_{j\in\mathbb{J}} (v_j - \check{v}_j) \ge w[u,v].$$

The proof is complete. □

Theorem 4. *Let $\varepsilon^v \in \mathbb{V}$ be the vector defined by* (3.13)*, related to the given solution v. Then $[v-\varepsilon^v, v+\varepsilon^v]$ is the widest symmetrical interval solution regarding v.*

Proof. It has been verified in Corollary 1 that $[v-\varepsilon^v, v+\varepsilon^v]$ is an SIS regarding v.

Let $[v-\varepsilon, v+\varepsilon]$ be an arbitrary SIS regarding the solution v, where $\varepsilon \in \mathbb{V} = [0,1]^n$. According to the definition of interval solution, it holds

$$[v-\varepsilon, v+\varepsilon] \subseteq \mathbb{S}(B, c, d),$$

That is to say, both $v+\varepsilon$ and $v-\varepsilon$ are solutions. Since $\hat{v}$ is maximum in $\mathbb{S}(B, c, d)$, it holds that $v+\varepsilon \le \hat{v}$. Thus, for any $j \in \mathbb{J}$,

$$v_j + \varepsilon_j \le \hat{v}_j,$$

or written as

$$\hat{v}_j - v_j \ge \varepsilon_j.$$

As a result,

$$\bigwedge_{j\in\mathbb{J}} \varepsilon_j \le \bigwedge_{j\in\mathbb{J}} (\hat{v}_j - v_j). \tag{3.21}$$

In addition, since $v-\varepsilon \in \mathbb{S}(B, c, d)$ and $v-\varepsilon \le v$, it follows from Proposition 7 that

$$w[\check{v}, v] \ge w[v-\varepsilon, v]. \tag{3.22}$$

According to (3.1) and (3.2), $w[v-\varepsilon, v] = \varepsilon_1 \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_n = \bigwedge_{j\in\mathbb{J}} \varepsilon_j$, and $w[\check{v}, v] = \bigwedge_{j\in\mathbb{J}} (v_j - \check{v}_j)$. The above inequality (3.22) turns out to be

$$\bigwedge_{j\in\mathbb{J}} (v_j - \check{v}_j) \ge \bigwedge_{j\in\mathbb{J}} \varepsilon_j. \tag{3.23}$$

Combining Inequalities (3.21), (3.23), and (3.13), it holds

$$\begin{aligned}\bigwedge_{j\in\mathbb{J}} \varepsilon_j^v &= \bigwedge_{j\in\mathbb{J}} [(\hat{v}_j - v_j) \wedge (v_j - \check{v}_j)] \\ &= [\bigwedge_{j\in\mathbb{J}} (\hat{v}_j - v_j)] \wedge [\bigwedge_{j\in\mathbb{J}} (v_j - \check{v}_j)] \\ &\ge \bigwedge_{j\in\mathbb{J}} \varepsilon_j.\end{aligned} \tag{3.24}$$

Consequently, it follows from (3.3) and (3.24)

$$w[v-\varepsilon^v, v+\varepsilon^v] = \bigwedge_{j\in\mathbb{J}} \varepsilon_j^v \ge \bigwedge_{j\in\mathbb{J}} \varepsilon_j = w[v-\varepsilon, v+\varepsilon].$$

Following Definition 5, $[v-\varepsilon^v, v+\varepsilon^v]$ is the widest symmetrical interval solution regarding v. □

3.3. Resolution algorithm for the WSIS and illustrative example

This subsection provides an algorithm to find the widest symmetrical interval solution regarding a given solution. Moreover, some numerical examples are given for verifying our proposed algorithm.

Algorithm I: solving the widest symmetrical interval solution regarding v

Step 1. Following (3.5), calculate the indicator set $\mathbb{J}_i^v$ for any $i \in \mathbb{I}$.
Step 2. Following (3.6), select the indicator j_i^v from the indicator set $\mathbb{J}_i^v$ for any $i \in \mathbb{I}$.
Step 3. Following (3.7), construct the indicator set $\mathbb{I}_j^v$ for any $j \in \mathbb{J}$.
Step 4. Following (3.8), generate the vector $\check{v} = (\check{v}_1, \check{v}_2, \cdots, \check{v}_n)$, where the $\check{v}_j$ is defined by (3.8).
Step 5. Following (3.13), calculate the vector $\varepsilon^v = (\varepsilon_1^v, \varepsilon_2^v, \cdots, \varepsilon_n^v)$, where the ε_j^v is defined by (3.13).
Step 6. Construct the widest symmetrical interval solution as $[v - \varepsilon^v, v + \varepsilon^v]$, according to (3.2).

Computational complexity of Algorithm I: In Algorithm I, Step 1 requires $2mn$ operations for calculating the indicator sets. Moreover, Steps 2–4 have an identical calculation amount as mn. Besides, Steps 5 and 6 require $3n$ and $2n$ operations, respectively. As a result, applying Algorithm I to calculate the widest symmetrical interval solution regarding v, it requires totally

$$2mn + mn + mn + mn + 3n + 2n = 5mn + 5n$$

operations. The computational complexity of Algorithm I is $O(mn)$.

Here, we have to mention that the reference [36] also proposes an algorithm to calculate the widest symmetrical interval solution regarding a given solution. However, the algorithm presented in [36] requires totally

$$\frac{1}{2}m^2n + \frac{17}{2}mn + 6n - 1$$

operations. The computational complexity is $O(m^2n)$. Obviously, our proposed algorithm I is superior to the algorithm presented in [36], considering the computational complexity.

Example 1. Consider the max-min FRIs system as follows:

$$\begin{cases} 0.4 \le (0.1 \wedge y_1) \vee (0.7 \wedge y_2) \vee (0.5 \wedge y_3) \vee (0.3 \wedge y_4) \vee (0.3 \wedge y_5) \le 0.7, \\ 0.2 \le (0.3 \wedge y_1) \vee (0.5 \wedge y_2) \vee (0.8 \wedge y_3) \vee (0.2 \wedge y_4) \vee (0.1 \wedge y_5) \le 0.6, \\ 0.2 \le (0.6 \wedge y_1) \vee (0.1 \wedge y_2) \vee (0.3 \wedge y_3) \vee (0.5 \wedge y_4) \vee (0.5 \wedge y_5) \le 0.8, \\ 0.5 \le (0.3 \wedge y_1) \vee (0.2 \wedge y_2) \vee (0.6 \wedge y_3) \vee (0.8 \wedge y_4) \vee (0.2 \wedge y_5) \le 0.7. \end{cases} \tag{3.25}$$

It is easy to check that both $v^1 = (0.8, 0.6, 0.6, 0.6, 0.3)$, $v^2 = (0.6, 0.45, 0.3, 0.65, 0.8)$ and $v^3 = (0.4, 0.6, 0.52, 0.2, 0.4)$ are solutions of system (3.25). Next, try to obtain the widest symmetrical interval solution regarding $v^t (t = 1, 2, 3)$ and calculate the (biggest) symmetrical width regarding $v^t (t = 1, 2, 3)$.

Solution: According to (2.3) and (2.4), we find

$$\hat{v} = (1, 1, 0.6, 0.7, 1).$$

as the maximum solution for (3.25) after calculation.

(i) Find the WSIS regarding $v^1 = (0.8, 0.6, 0.6, 0.6, 0.3)$.

Steps 1 and 2. Following (3.5) and (3.6), we have

$$\begin{aligned}
\mathbb{J}_1^{v^1} &= \{j \in \mathbb{J} | b_{1j} \wedge v_j^1 \geq c_1\} = \{j \in \mathbb{J} | b_{1j} \wedge v_j^1 \geq 0.4\} = \{2, 3\},\\
\mathbb{J}_2^{v^1} &= \{j \in \mathbb{J} | b_{2j} \wedge v_j^1 \geq c_2\} = \{j \in \mathbb{J} | b_{2j} \wedge v_j^1 \geq 0.2\} = \{1, 2, 3, 4\},\\
\mathbb{J}_3^{v^1} &= \{j \in \mathbb{J} | b_{3j} \wedge v_j^1 \geq c_3\} = \{j \in \mathbb{J} | b_{3j} \wedge v_j^1 \geq 0.2\} = \{1, 3, 4, 5\},\\
\mathbb{J}_4^{v^1} &= \{j \in \mathbb{J} | b_{4j} \wedge v_j^1 \geq c_4\} = \{j \in \mathbb{J} | b_{4j} \wedge v_j^1 \geq 0.5\} = \{3, 4\},
\end{aligned}$$

and

$$\begin{aligned}
j_1^{v^1} &= \arg\max\{v_j^1 | j \in \mathbb{J}_1^{v^1}\} = \arg\max\{v_2^1, v_3^1\} = 2 \text{ or } 3,\\
j_2^{v^1} &= \arg\max\{v_j^1 | j \in \mathbb{J}_2^{v^1}\} = \arg\max\{v_1^1, v_2^1, v_3^1, v_4^1\} = 1,\\
j_3^{v^1} &= \arg\max\{v_j^1 | j \in \mathbb{J}_3^{v^1}\} = \arg\max\{v_1^1, v_3^1, v_4^1, v_5^1\} = 1,\\
j_4^{v^1} &= \arg\max\{v_j^1 | j \in \mathbb{J}_4^{v^1}\} = \arg\max\{v_3^1, v_4^1\} = 3 \text{ or } 4.
\end{aligned}$$

Step 3. If we take $j_1^{v^1} = 2$, $j_2^{v^1} = 1$, $j_3^{v^1} = 1$, $j_4^{v^1} = 4$ and combine (3.7), we can obtain that

$$\begin{aligned}
\mathbb{I}_1^{v^1} &= \{i \in \mathbb{I} | j_i^{v^1} = 1\} = \{2, 3\},\\
\mathbb{I}_2^{v^1} &= \{i \in \mathbb{I} | j_i^{v^1} = 2\} = \{1\},\\
\mathbb{I}_3^{v^1} &= \{i \in \mathbb{I} | j_i^{v^1} = 3\} = \emptyset,\\
\mathbb{I}_4^{v^1} &= \{i \in \mathbb{I} | j_i^{v^1} = 4\} = \{4\},\\
\mathbb{I}_5^{v^1} &= \{i \in \mathbb{I} | j_i^{v^1} = 5\} = \emptyset.
\end{aligned}$$

Step 4. Following (3.8), we find

$$\begin{aligned}
\breve{v}_1^1 &= \bigvee_{i \in \mathbb{I}_1^{v^1}} c_i = c_2 \vee c_3 = 0.2,\\
\breve{v}_2^1 &= \bigvee_{i \in \mathbb{I}_2^{v^1}} c_i = c_1 = 0.4,\\
\breve{v}_4^1 &= \bigvee_{i \in \mathbb{I}_4^{v^1}} c_i = c_4 = 0.5,\\
\breve{v}_3^1 &= \breve{v}_5^1 = 0.
\end{aligned}$$

Step 5. As a result, we get the vector $\breve{v}^1 = (0.2, 0.4, 0, 0.5, 0)$. Moreover,

$$\begin{aligned}
\varepsilon_1^{v^1} &= (\hat{v}_1 - v_1^1) \wedge (v_1^1 - \breve{v}_1^1) = (1 - 0.8) \wedge (0.8 - 0.2) = 0.2 \wedge 0.6 = 0.2,\\
\varepsilon_2^{v^1} &= (\hat{v}_2 - v_2^1) \wedge (v_2^1 - \breve{v}_2^1) = (1 - 0.6) \wedge (0.6 - 0.4) = 0.4 \wedge 0.2 = 0.2,\\
\varepsilon_3^{v^1} &= (\hat{v}_3 - v_3^1) \wedge (v_3^1 - \breve{v}_3^1) = (0.6 - 0.6) \wedge (0.6 - 0) = 0 \wedge 0.6 = 0,\\
\varepsilon_4^{v^1} &= (\hat{v}_4 - v_4^1) \wedge (v_4^1 - \breve{v}_4^1) = (0.7 - 0.6) \wedge (0.6 - 0.5) = 0.1 \wedge 0.1 = 0.1,\\
\varepsilon_5^{v^1} &= (\hat{v}_5 - v_5^1) \wedge (v_5^1 - \breve{v}_5^1) = (1 - 0.3) \wedge (0.3 - 0) = 0.7 \wedge 0.3 = 0.3.
\end{aligned}$$

Thus,

$$\varepsilon^{v^1} = (0.2, 0.2, 0, 0.1, 0.3).$$

Step 6. Based on the above-obtained ε^{v^1}, we further obtain the widest symmetrical interval solution regarding v^1 as

$$[v^1 - \varepsilon^{v^1}, v^1 + \varepsilon^{v^1}] = ([0.6, 1], [0.4, 0.8], 0.6, [0.5, 0.7], [0, 0.6]).$$

The (biggest) symmetrical width regarding v^1 is

$$\begin{aligned} w[v^1 - \varepsilon^{v^1}, v^1 + \varepsilon^{v^1}] &= 2(\varepsilon_1^{v^1} \wedge \varepsilon_2^{v^1} \wedge \varepsilon_3^{v^1} \wedge \varepsilon_4^{v^1} \wedge \varepsilon_5^{v^1}) \\ &= 2(0.2 \wedge 0.2 \wedge 0 \wedge 0.1 \wedge 0.3) = 0. \end{aligned}$$

In fact, since $v_3^1 = \hat{v}_3^1 = 0.6$, it can be concluded from equations (3.3) and (3.13) that there must be $w[v^1 - \varepsilon^{v^1}, v^1 + \varepsilon^{v^1}] = 0$.

(ii) Find the WSIS regarding $v^2 = (0.6, 0.45, 0.3, 0.65, 0.8)$.

Steps 1 and 2. Following (3.5) and (3.6), we find

$$\begin{aligned} \mathbb{J}_1^{v^2} &= \{j \in \mathbb{J} | b_{1j} \wedge v_j^2 \geq c_1\} = \{j \in \mathbb{J} | b_{1j} \wedge v_j^2 \geq 0.4\} = \{2\}, \\ \mathbb{J}_2^{v^2} &= \{j \in \mathbb{J} | b_{2j} \wedge v_j^2 \geq c_2\} = \{j \in \mathbb{J} | b_{2j} \wedge v_j^2 \geq 0.2\} = \{1, 2, 3, 4\}, \\ \mathbb{J}_3^{v^2} &= \{j \in \mathbb{J} | b_{3j} \wedge v_j^2 \geq c_3\} = \{j \in \mathbb{J} | b_{3j} \wedge v_j^2 \geq 0.2\} = \{1, 3, 4, 5\}, \\ \mathbb{J}_4^{v^2} &= \{j \in \mathbb{J} | b_{4j} \wedge v_j^2 \geq c_4\} = \{j \in \mathbb{J} | b_{4j} \wedge v_j^2 \geq 0.5\} = \{4\}, \end{aligned}$$

and

$$\begin{aligned} j_1^{v^2} &= \arg\max\{v_j^2 | j \in \mathbb{J}_1^{v^2}\} = \arg\max\{v_2^2\} = 2, \\ j_2^{v^2} &= \arg\max\{v_j^2 | j \in \mathbb{J}_2^{v^2}\} = \arg\max\{v_1^2, v_2^2, v_3^2, v_4^2\} = 4, \\ j_3^{v^2} &= \arg\max\{v_j^2 | j \in \mathbb{J}_3^{v^2}\} = \arg\max\{v_1^2, v_3^2, v_4^2, v_5^2\} = 5, \\ j_4^{v^2} &= \arg\max\{v_j^2 | j \in \mathbb{J}_4^{v^2}\} = \arg\max\{v_4^2\} = 4. \end{aligned}$$

Step 3. Following (3.7), we can obtain that

$$\begin{aligned} \mathbb{I}_1^{v^2} &= \{i \in \mathbb{I} | j_i^{v^2} = 1\} = \emptyset, \\ \mathbb{I}_2^{v^2} &= \{i \in \mathbb{I} | j_i^{v^2} = 2\} = \{1\}, \\ \mathbb{I}_3^{v^2} &= \{i \in \mathbb{I} | j_i^{v^2} = 3\} = \emptyset, \\ \mathbb{I}_4^{v^2} &= \{i \in \mathbb{I} | j_i^{v^2} = 4\} = \{2, 4\}, \\ \mathbb{I}_5^{v^2} &= \{i \in \mathbb{I} | j_i^{v^2} = 5\} = \{3\}. \end{aligned}$$

Step 4. According to (3.8), we further obtain

$$\begin{aligned} \check{v}_1^2 &= \check{v}_3^2 = 0, \\ \check{v}_2^2 &= \bigvee_{i \in \mathbb{I}_2^{v^2}} c_i = c_1 = 0.4, \\ \check{v}_4^2 &= \bigvee_{i \in \mathbb{I}_4^{v^2}} c_i = c_2 \vee c_4 = 0.5, \\ \check{v}_5^2 &= \bigvee_{i \in \mathbb{I}_5^{v^2}} c_i = c_3 = 0.2. \end{aligned}$$

Step 5. As a result, we obtain the vector $\check{v}^2 = (0, 0.4, 0, 0.5, 0.2)$. Moreover,

$$\begin{aligned}
\varepsilon_1^{v^2} &= (\hat{v}_1 - v_1^2) \wedge (v_1^2 - \check{v}_1^2) = (1 - 0.6) \wedge (0.6 - 0) = 0.4 \wedge 0.6 = 0.4,\\
\varepsilon_2^{v^2} &= (\hat{v}_2 - v_2^2) \wedge (v_2^2 - \check{v}_2^2) = (1 - 0.45) \wedge (0.45 - 0.4) = 0.55 \wedge 0.05 = 0.05,\\
\varepsilon_3^{v^2} &= (\hat{v}_3 - v_3^2) \wedge (v_3^2 - \check{v}_3^2) = (0.6 - 0.3) \wedge (0.3 - 0) = 0.3 \wedge 0.3 = 0.3,\\
\varepsilon_4^{v^2} &= (\hat{v}_4 - v_4^2) \wedge (v_4^2 - \check{v}_4^2) = (0.7 - 0.65) \wedge (0.65 - 0.5) = 0.05 \wedge 0.15 = 0.05,\\
\varepsilon_5^{v^2} &= (\hat{v}_5 - v_5^2) \wedge (v_5^2 - \check{v}_5^2) = (1 - 0.8) \wedge (0.8 - 0.2) = 0.2 \wedge 0.6 = 0.2.
\end{aligned}$$

Thus,

$$\varepsilon^{v^2} = (0.4, 0.05, 0.3, 0.05, 0.2).$$

Step 6. Based on the above-obtained ε^{v^2}, we further obtain the widest symmetrical interval solution regarding v^2 as

$$[v^2 - \varepsilon^{v^2}, v^2 + \varepsilon^{v^2}] = ([0.2, 1], [0.4, 0.5], [0, 0.6], [0.6, 0.7], [0.6, 1]).$$

The (biggest) symmetrical width regarding v^2 is

$$\begin{aligned}
w[v^2 - \varepsilon^{v^2}, v^2 + \varepsilon^{v^2}] &= 2(\varepsilon_1^{v^2} \wedge \varepsilon_2^{v^2} \wedge \varepsilon_3^{v^2} \wedge \varepsilon_4^{v^2} \wedge \varepsilon_5^{v^2})\\
&= 2(0.4 \wedge 0.05 \wedge 0.3 \wedge 0.05 \wedge 0.2) = 0.1.
\end{aligned}$$

(iii) Find the WSIS regarding $v^3 = (0.4, 0.6, 0.52, 0.2, 0.4)$.
Steps 1 and 2. Following (3.5) and (3.6), we find

$$\begin{aligned}
\mathbb{J}_1^{v^3} &= \{j \in \mathbb{J} | b_{1j} \wedge v_j^3 \geq c_1\} = \{j \in \mathbb{J} | b_{1j} \wedge v_j^3 \geq 0.4\} = \{2, 3\},\\
\mathbb{J}_2^{v^3} &= \{j \in \mathbb{J} | b_{2j} \wedge v_j^3 \geq c_2\} = \{j \in \mathbb{J} | b_{2j} \wedge v_j^3 \geq 0.2\} = \{1, 2, 3, 4\},\\
\mathbb{J}_3^{v^3} &= \{j \in \mathbb{J} | b_{3j} \wedge v_j^3 \geq c_3\} = \{j \in \mathbb{J} | b_{3j} \wedge v_j^3 \geq 0.2\} = \{1, 3, 4, 5\},\\
\mathbb{J}_4^{v^3} &= \{j \in \mathbb{J} | b_{4j} \wedge v_j^3 \geq c_4\} = \{j \in \mathbb{J} | b_{4j} \wedge v_j^3 \geq 0.5\} = \{3\},
\end{aligned}$$

and

$$\begin{aligned}
j_1^{v^3} &= \arg\max\{v_j^3 | j \in \mathbb{J}_1^{v^3}\} = \arg\max\{v_2^3, v_3^3\} = 2,\\
j_2^{v^3} &= \arg\max\{v_j^3 | j \in \mathbb{J}_2^{v^3}\} = \arg\max\{v_1^3, v_2^3, v_3^3, v_4^3\} = 2,\\
j_3^{v^3} &= \arg\max\{v_j^3 | j \in \mathbb{J}_3^{v^3}\} = \arg\max\{v_1^3, v_3^3, v_4^3, v_5^3\} = 3,\\
j_4^{v^3} &= \arg\max\{v_j^3 | j \in \mathbb{J}_4^{v^3}\} = \arg\max\{v_3^3\} = 3.
\end{aligned}$$

Step 3. Following (3.7), we can obtain that

$$\begin{aligned}
\mathbb{I}_1^{v^3} &= \{i \in \mathbb{I} | j_i^{v^3} = 1\} = \emptyset,\\
\mathbb{I}_2^{v^3} &= \{i \in \mathbb{I} | j_i^{v^3} = 2\} = \{1, 2\},\\
\mathbb{I}_3^{v^3} &= \{i \in \mathbb{I} | j_i^{v^3} = 3\} = \{3, 4\},\\
\mathbb{I}_4^{v^3} &= \{i \in \mathbb{I} | j_i^{v^3} = 4\} = \emptyset,\\
\mathbb{I}_5^{v^3} &= \{i \in \mathbb{I} | j_i^{v^3} = 5\} = \emptyset,
\end{aligned}$$

Step 4. According to (3.8), we further obtain

$$\begin{aligned}\check{v}_1^3 &= \check{v}_4^3 = \check{v}_5^3 = 0,\\ \check{v}_2^3 &= \bigvee_{i\in\mathbb{I}_2^{v^3}} c_i = c_1 \vee c_2 = 0.4,\\ \check{v}_3^3 &= \bigvee_{i\in\mathbb{I}_3^{v^3}} c_i = c_3 \vee c_4 = 0.5.\end{aligned}$$

Step 5. As a result, we obtain the vector $\check{v}^3 = (0, 0.4, 0.5, 0, 0)$. Moreover,

$$\begin{aligned}\varepsilon_1^{v^3} &= (\hat{v}_1 - v_1^3) \wedge (v_1^3 - \check{v}_1^3) = (1 - 0.4) \wedge (0.4 - 0) = 0.6 \wedge 0.4 = 0.4,\\ \varepsilon_2^{v^3} &= (\hat{v}_2 - v_2^3) \wedge (v_2^3 - \check{v}_2^3) = (1 - 0.6) \wedge (0.6 - 0.4) = 0.4 \wedge 0.2 = 0.2,\\ \varepsilon_3^{v^3} &= (\hat{v}_3 - v_3^3) \wedge (v_3^3 - \check{v}_3^3) = (0.6 - 0.52) \wedge (0.52 - 0.5) = 0.08 \wedge 0.02 = 0.02,\\ \varepsilon_4^{v^3} &= (\hat{v}_4 - v_4^3) \wedge (v_4^3 - \check{v}_4^3) = (0.7 - 0.2) \wedge (0.2 - 0) = 0.5 \wedge 0.2 = 0.2,\\ \varepsilon_5^{v^3} &= (\hat{v}_5 - v_5^3) \wedge (v_5^3 - \check{v}_5^3) = (1 - 0.4) \wedge (0.4 - 0) = 0.6 \wedge 0.4 = 0.4.\end{aligned}$$

Thus,

$$\varepsilon^{v^3} = (0.4, 0.2, 0.02, 0.2, 0.4).$$

Step 6. Based on the above-obtained ε^{v^3}, we further obtain the widest symmetrical interval solution regarding v^3 as

$$[v^3 - \varepsilon^{v^3}, v^3 + \varepsilon^{v^3}] = ([0, 0.8], [0.4, 0.8], [0.5, 0.54], [0, 0.4], [0, 0.8]).$$

The (biggest) symmetrical width regarding v^3 is

$$\begin{aligned}w[v^3 - \varepsilon^{v^3}, v^3 + \varepsilon^{v^3}] &= 2(\varepsilon_1^{v^3} \wedge \varepsilon_2^{v^3} \wedge \varepsilon_3^{v^3} \wedge \varepsilon_4^{v^3} \wedge \varepsilon_5^{v^3})\\ &= 2(0.4 \wedge 0.2 \wedge 0.02 \wedge 0.2 \wedge 0.4) = 0.04.\end{aligned}$$

Then, the problem has already been solved. □

Remark 2. *In fact, the widest symmetrical interval solution regarding an identical solution might be not unique. For example, considering* $v^1 = (0.8, 0.6, 0.6, 0.6, 0.3)$ *in system* (3.25)*, we find* $j_1^{v^1} = 2$ *or* 3*,* $j_2^{v^1} = 1$*,* $j_3^{v^1} = 1$*, and* $j_4^{v^1} = 3$ *or* 4 *in Example 1. Taking* $j_1^{v^1} = 2$*,* $j_2^{v^1} = 1$*,* $j_3^{v^1} = 1$*, and* $j_4^{v^1} = 4$*, we find the widest symmetrical interval solution regarding* v^1 *as*

$$[v^1 - \varepsilon^{v^1}, v^1 + \varepsilon^{v^1}] = ([0.6, 1], [0.4, 0.8], 0.6, [0.5, 0.7], [0, 0.6]),$$

with the (biggest) symmetrical width $w[v^1 - \varepsilon^{v^1}, v^1 + \varepsilon^{v^1}] = 0$*. However, when taking* $j_1^{v^1} = 3$*,* $j_2^{v^1} = 1$*,* $j_3^{v^1} = 1$*,* $j_4^{v^1} = 4$*, the corresponding widest symmetrical interval solution is*

$$([0.6, 1], [0.2, 1], 0.6, [0.5, 0.7], [0, 0.6]).$$

The corresponding (biggest) symmetrical width is also $w([0.6, 1], [0.2, 1], 0.6, [0.5, 0.7], [0, 0.6]) = 0$*. That is to say, we find two different widest symmetrical interval solutions regarding the solution* $v^1 = (0.8, 0.6, 0.6, 0.6, 0.3)$*. But they have the same (biggest) symmetrical width.*

Example 2. Consider the max-min FRIs system as follows:

$$\begin{cases} 0.37 \leq (0.35 \wedge y_1) \vee (0.48 \wedge y_2) \vee (0.87 \wedge y_3) \vee (0.63 \wedge y_4) \vee (0.26 \wedge y_5) \vee (0.74 \wedge y_6) \leq 0.82, \\ 0.34 \leq (0.28 \wedge y_1) \vee (0.92 \wedge y_2) \vee (0.76 \wedge y_3) \vee (0.81 \wedge y_4) \vee (0.53 \wedge y_5) \vee (0.19 \wedge y_6) \leq 0.87, \\ 0.25 \leq (0.89 \wedge y_1) \vee (0.76 \wedge y_2) \vee (0.42 \wedge y_3) \vee (0.35 \wedge y_4) \vee (0.56 \wedge y_5) \vee (0.75 \wedge y_6) \leq 0.79, \\ 0.36 \leq (0.33 \wedge y_1) \vee (0.85 \wedge y_2) \vee (0.27 \wedge y_3) \vee (0.49 \wedge y_4) \vee (0.93 \wedge y_5) \vee (0.35 \wedge y_6) \leq 0.85, \\ 0.41 \leq (0.59 \wedge y_1) \vee (0.28 \wedge y_2) \vee (0.18 \wedge y_3) \vee (0.66 \wedge y_4) \vee (0.83 \wedge y_5) \vee (0.91 \wedge y_6) \leq 0.89, \\ 0.32 \leq (0.29 \wedge y_1) \vee (0.23 \wedge y_2) \vee (0.31 \wedge y_3) \vee (0.73 \wedge y_4) \vee (0.56 \wedge y_5) \vee (0.82 \wedge y_6) \leq 0.86. \end{cases} \tag{3.26}$$

It is easy to check that system (3.26) is consistent and $v = (0.65, 0.48, 0.51, 0.43, 0.39, 0.45)$ is one of its solutions. Apply Algorithm I to find the widest symmetrical interval solution regarding v and calculate the (biggest) symmetrical width regarding v.

Solution: According to (2.3) and (2.4), we find

$$\hat{v} = (0.79, 0.87, 0.82, 1, 0.85, 0.89).$$

as the maximum solution for (3.26) after calculation.

Step 1. Following (3.5), we have

$$\mathbb{J}_1^v = \{2,3,4,6\}, \quad \mathbb{J}_2^v = \{2,3,4,5\}, \quad \mathbb{J}_3^v = \{1,2,3,4,5,6\},$$
$$\mathbb{J}_4^v = \{2,4,5\}, \quad \mathbb{J}_5^v = \{1,4,6\}, \quad \mathbb{J}_6^v = \{4,5,6\}.$$

Step 2. Following (3.6), we have

$$j_1^v = \arg\max\{v_j | j \in \mathbb{J}_1^v\} = 3, \quad j_2^v = \arg\max\{v_j | j \in \mathbb{J}_2^v\} = 3,$$
$$j_3^v = \arg\max\{v_j | j \in \mathbb{J}_3^v\} = 1, \quad j_4^v = \arg\max\{v_j | j \in \mathbb{J}_4^v\} = 2,$$
$$j_5^v = \arg\max\{v_j | j \in \mathbb{J}_5^v\} = 1, \quad j_6^v = \arg\max\{v_j | j \in \mathbb{J}_6^v\} = 6.$$

Step 3. Following (3.7), we have

$$\begin{aligned} \mathbb{I}_1^v &= \{i \in \mathbb{I} | j_i^v = 1\} = \{3,5\}, \\ \mathbb{I}_2^v &= \{i \in \mathbb{I} | j_i^v = 2\} = \{4\}, \\ \mathbb{I}_3^v &= \{i \in \mathbb{I} | j_i^v = 3\} = \{1,2\}, \\ \mathbb{I}_4^v &= \{i \in \mathbb{I} | j_i^v = 4\} = \emptyset, \\ \mathbb{I}_5^v &= \{i \in \mathbb{I} | j_i^v = 5\} = \emptyset, \\ \mathbb{I}_6^v &= \{i \in \mathbb{I} | j_i^v = 6\} = \{6\}. \end{aligned}$$

Step 4. Following (3.8), we have

$$\begin{aligned} \breve{v}_1 &= \bigvee_{i \in \mathbb{I}_1^v} c_i = c_3 \vee c_5 = 0.41, \\ \breve{v}_2 &= \bigvee_{i \in \mathbb{I}_2^v} c_i = c_4 = 0.36, \\ \breve{v}_3 &= \bigvee_{i \in \mathbb{I}_3^v} c_i = c_1 \vee c_2 = 0.37, \\ \breve{v}_4 &= \breve{v}_5 = 0, \\ \breve{v}_6 &= \bigvee_{i \in \mathbb{I}_6^v} c_i = c_6 = 0.32. \end{aligned}$$

Hence, we obtain the vector $\check{v} = (0.41, 0.36, 0.37, 0, 0, 0.32)$.

Step 5. Following (3.13), we have

$$\varepsilon_1^v = (v_1 - \check{v}_1) \wedge (\hat{v}_1 - v_1) = (0.65 - 0.41) \wedge (0.79 - 0.65) = 0.24 \wedge 0.14 = 0.14.$$

In a similar way, we further have $\varepsilon_2^v = 0.12$, $\varepsilon_3^v = 0.14$, $\varepsilon_4^v = 0.43$, $\varepsilon_5^v = 0.39$, and $\varepsilon_6^v = 0.13$. Then we find the vector $\varepsilon^v = (0.14, 0.12, 0.14, 0.43, 0.39, 0.13)$.

Step 6. According to (3.2), we can obtain the widest symmetrical interval solution regarding v as

$$[v - \varepsilon^v, v + \varepsilon^v] = ([0.51, 0.79], [0.36, 0.6], [0.37, 0.65], [0, 0.86], [0, 0.78], [0.32, 0.58]).$$

Moreover, the (biggest) symmetrical width regarding v is

$$2(\varepsilon_1^v \wedge \cdots \wedge \varepsilon_6^v) = 0.14 \wedge 0.12 \wedge 0.14 \wedge 0.43 \wedge 0.39 \wedge 0.13 = 0.24.$$

□

4. Centralized solution regarding system (1.6)

In the previous section, we have studied the symmetrical interval solution having the biggest width, which is named WSIS. The width of a WSIS reflects its stability, i.e., the ability to bear the tolerable variation of a provided feasible scheme. For characterizing the most stable feasible scheme, we further define the concept of a centralized solution regarding the max-min FRI system (1.6). We will proposed an effective method for solving the centralized solution.

Definition 6. (***Centralized solution regarding system*** (1.6)) Let $v^C \in \mathbb{S}(B, c, d)$ be a solution of (1.6). The WSIS regarding v^C is $[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}]$. Then v^C is said to be a ***centralized solution*** regarding system (1.6), if

$$w[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}] \geq w[v - \varepsilon^v, v + \varepsilon^v]$$

holds for any solution $v \in \mathbb{S}(B, c, d)$, where $[v - \varepsilon^v, v + \varepsilon^v]$ is the WSIS regarding v.

4.1. Construct a vector $\check{v}$ corresponding to the maximum solution $\hat{v}$

Define the index set

$$\mathbb{J}_i^{\hat{v}} = \{j \in \mathbb{J} | b_{ij} \wedge \hat{v}_j \geq c_i\}, \tag{4.1}$$

where $i \in \mathbb{I}$, in accordance with the maximum solution $\hat{v}$. Moreover, the following m indices are induced by the above index sets:

$$j_i^{\hat{v}} = \arg\max\{\hat{v}_j | j \in \mathbb{J}_i^{\hat{v}}\}, \tag{4.2}$$

where $i \in \mathbb{I}$. In addition, based on the indices $j_1^{\hat{v}}, j_2^{\hat{v}}, \cdots, j_m^{\hat{v}}$, we could define

$$\mathbb{I}_j^{\hat{v}} = \{i \in \mathbb{I} | j_i^{\hat{v}} = j\}, \tag{4.3}$$

where $j \in \mathbb{J}$. Accordingly, we construct the vector $\check{v} = (\check{v}_1, \check{v}_2, \cdots, \check{v}_n)$ by

$$\check{v}_j = \begin{cases} \bigvee_{i \in \mathbb{I}_j^{\hat{v}}} c_i, & \text{if } \mathbb{I}_j^{\hat{v}} \neq \emptyset, \\ 0, & \text{if } \mathbb{I}_j^{\hat{v}} = \emptyset, \end{cases} \quad j \in \mathbb{J}. \tag{4.4}$$

Proposition 8. *Let* $\check{v}$ *be defined by* (4.4). *Then it holds* $\check{v} \in \mathbb{S}(B, c, d)$*, serving as a solution of* (1.6).

Proof. The given conditions indicate $c_i \in [0, 1]$, $\forall i \in \mathbb{I}$. According to (4.4), it is evident that $\check{v}_j \in [0, 1]$, $\forall j \in \mathbb{J}$.

For arbitrary $i^\dagger \in \mathbb{I}$, denote $j^\dagger = j^{\hat{v}}_{i^\dagger}$. By (4.2), it is clear $j^\dagger = j^{\hat{v}}_{i^\dagger} \in \mathbb{J}^{\hat{v}}_{i^\dagger}$. According to (4.1), it holds

$$b_{i^\dagger j^\dagger} \geq b_{i^\dagger j^\dagger} \wedge \hat{v}_{j^\dagger} \geq c_{i^\dagger}. \tag{4.5}$$

On the other hand, $j^\dagger = j^{\hat{v}}_{i^\dagger}$ indicates $i^\dagger \in \mathbb{I}^{\hat{v}}_{j^\dagger}$, by (4.3). Hence, $\mathbb{I}^{\hat{v}}_{j^\dagger} \neq \emptyset$ and by (4.4), we have $\check{v}_{j^\dagger} = \bigvee_{i \in \mathbb{I}^{\hat{v}}_{j^\dagger}} c_i$. It follows from $i^\dagger \in \mathbb{I}^{\hat{v}}_{j^\dagger}$ that

$$\check{v}_{j^\dagger} = \bigvee_{i \in \mathbb{I}^{\hat{v}}_{j^\dagger}} c_i \geq c_{i^\dagger}. \tag{4.6}$$

Combining (4.5) and (4.6), we have

$$\bigvee_{j \in \mathbb{J}} (b_{i^\dagger j} \wedge \check{v}_j) \geq b_{i^\dagger j^\dagger} \wedge \check{v}_{j^\dagger} \geq c_{i^\dagger} \wedge c_{i^\dagger} = c_{i^\dagger}. \tag{4.7}$$

Note that Inequality (4.7) holds for all $i^\dagger \in \mathbb{I}$. Therefore, $\check{v}$ is a solution in the inequalities (1.6). □

Proposition 9. *Let* $u \in \mathbb{S}(B, c, d)$ *be a solution of* (1.6). *Then it holds that* $w[\check{v}, \hat{v}] \geq w[u, \hat{v}]$.

Proof. Let j be an arbitrary index in $\mathbb{J}$. Next, we first examine that $w[u, \hat{v}] \leq \hat{v}_j - \check{v}_j$ in two cases.

Case 1. If $\mathbb{I}^{\hat{v}}_j = \emptyset$, then by (4.4), $\check{v}_j = 0$. The given condition $u \in \mathbb{S}(B, c, d)$ indicates $u_j \in [0, 1]$. Hence

$$\hat{v}_j - \check{v}_j = \hat{v}_j - u_j \leq \hat{v}_j - 0 \geq \bigwedge_{k \in \mathbb{J}} (\hat{v}_k - u_k) = w[u, \hat{v}]. \tag{4.8}$$

Case 2. If $\mathbb{I}^{\hat{v}}_j \neq \emptyset$, then by (4.4), $\check{v}_j = \bigvee_{i \in \mathbb{I}^{\hat{v}}_j} c_i$. There exists $i^* \in \mathbb{I}^{\hat{v}}_j$ such that

$$c_{i^*} = \bigvee_{i \in \mathbb{I}^{\hat{v}}_j} c_i = \check{v}_j. \tag{4.9}$$

Meanwhile, $i^* \in \mathbb{I}^{\hat{v}}_j$ indicates $j^{\hat{v}}_{i^*} = j$ by (4.3). The given condition $u \in \mathbb{S}(B, c, d)$ also indicates

$$(b_{i^*1} \wedge u_1) \vee (b_{i^*2} \wedge u_2) \vee \cdots \vee (b_{i^*n} \wedge u_n) \geq c_{i^*}. \tag{4.10}$$

There is $j^* \in \mathbb{J}$ such that

$$b_{i^* j^*} \wedge u_{j^*} \geq c_{i^*}. \tag{4.11}$$

Inequality (4.11) implies that

$$u_{j^*} \geq b_{i^* j^*} \wedge u_{j^*} \geq c_{i^*}. \tag{4.12}$$

Since $\hat{v}$ is maximum and $u \in \mathbb{S}(B, c, d)$, we have $\hat{v}_{j^*} \geq u_{j^*}$. Hence by (4.11) we further have

$$b_{i^* j^*} \wedge \hat{v}_{j^*} \geq b_{i^* j^*} \wedge u_{j^*} \geq c_{i^*}. \tag{4.13}$$

According to (4.1), it holds that $j^* \in \mathbb{J}_{i^*}^{\hat{v}}$. Considering $j_{i^*}^{\hat{v}} = j$ and $j^* \in \mathbb{J}_{i^*}^{\hat{v}}$, it follows from (4.2) that

$$\hat{v}_j \geq \hat{v}_{j^*}. \tag{4.14}$$

Using the inequalities (4.9), (4.12), and (4.14), we have

$$\hat{v}_j - \check{v}_j = \hat{v}_j - c_{i^*} \geq \hat{v}_j - u_{j^*} \geq \hat{v}_{j^*} - u_{j^*} \geq \bigwedge_{k \in \mathbb{J}} (\hat{v}_k - u_k) = w[u, \hat{v}]. \tag{4.15}$$

Considering the arbitrariness of the index j in $\mathbb{J}$, by (4.8) and (4.15) we have $\hat{v}_j - \check{v}_j \geq w[u, \hat{v}]$, $\forall j \in \mathbb{J}$. Hence it holds $w[\check{v}, \hat{v}] = \bigwedge_{j \in \mathbb{J}} (\hat{v}_j - \check{v}_j) \geq w[u, \hat{v}]$. □

4.2. Construct the centralized solution regarding system (1.6)

In accordance with the solutions $\hat{v}$ and $\check{v}$, we denote

$$v^C = \frac{\hat{v} + \check{v}}{2}, \quad \varepsilon^{v^C} = \frac{\hat{v} - \check{v}}{2}. \tag{4.16}$$

Some properties are investigated for the vectors v^C and ε^{v^C} as follows.

Theorem 5. *Let v^C and ε^{v^C} be defined by* (4.16). *Then there is $v^C \in \mathbb{S}(B, c, d)$. Moreover, $[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}]$ is the widest symmetrical interval solution regarding v^C.*

Proof. According to Proposition 8, it holds $\check{v} \in \mathbb{S}(B, c, d)$. $\hat{v}$ is the maximum solution. It is clear that $\hat{v} \geq \check{v}$. Then we have

$$v^C = \frac{\hat{v} + \check{v}}{2} \in [\check{v}, \hat{v}]. \tag{4.17}$$

Hence, $v^C \in \mathbb{S}(B, c, d)$ is a solution. Note that

$$\begin{cases} \check{v} = \frac{\hat{v}+\check{v}}{2} - \frac{\hat{v}-\check{v}}{2} = v^C - \varepsilon^{v^C} \in \mathbb{S}(B, c, d), \\ \hat{v} = \frac{\hat{v}-\check{v}}{2} + \frac{\hat{v}+\check{v}}{2} = v^C + \varepsilon^{v^C} \in \mathbb{S}(B, c, d). \end{cases} \tag{4.18}$$

It follows from Definitions 3 and 4 that $[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}]$ is an SIS regarding the solution v^C.

Let $[v^C - \varepsilon^\dagger, v^C + \varepsilon^\dagger]$ be an arbitrary SIS regarding v^C, where $\varepsilon^\dagger \in \mathbb{V}$. Following Definitions 3 and 4, we obtain

$$v^C - \varepsilon^\dagger, v^C + \varepsilon^\dagger \in \mathbb{S}(B, c, d). \tag{4.19}$$

Thus, $v^C - \varepsilon^\dagger \leq v^C + \varepsilon^\dagger \leq \hat{v}$. This indicates

$$w[v^C - \varepsilon^\dagger, \hat{v}] \geq w[v^C - \varepsilon^\dagger, v^C + \varepsilon^\dagger]. \tag{4.20}$$

On the other hand, since $v^C - \varepsilon^\dagger \in \mathbb{S}(B, c, d)$, we have

$$w[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}] = w[\check{v}, \hat{v}] \geq w[v^C - \varepsilon^\dagger, \hat{v}], \tag{4.21}$$

by Proposition 9. Inequalities (4.20) and (4.21) imply that $w[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}] \geq w[v^C - \varepsilon^\dagger, v^C + \varepsilon^\dagger]$.

As a result, by Definition 5 we know that $[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}]$ is a widest symmetrical interval solution regarding v^C. □

Proposition 10. *Let $u \in \mathbb{S}(B, c, d)$ be an arbitrary solution in* (1.6). *Then it holds $w[u, \hat{v}] \le w[\check{v}, \hat{v}]$.*

Proof. Let j be an arbitrary index in $\mathbb{J}$. Now we examine the inequality $\hat{v}_j - \check{v}_j \ge w[u, \hat{v}]$.

Case 1. If $\mathbb{I}_j^{\hat{v}} = \emptyset$, then $\check{v}_j = 0$ by (4.4). Hence

$$w[u, \hat{v}] = \bigwedge_{k \in \mathbb{J}} (\hat{v}_k - u_k) \le \hat{v}_j - u_j \le \hat{v}_j - 0 = \hat{v}_j - \check{v}_j. \tag{4.22}$$

Case 2. If $\mathbb{I}_j^{\hat{v}} \ne \emptyset$, then $\check{v}_j = \bigvee_{i \in \mathbb{I}_j^{\hat{v}}} c_i$ by (4.4). Obviously, there exists $i^\dagger \in \mathbb{I}_j^{\hat{v}}$, such that

$$\check{v}_j = c_{i^\dagger}. \tag{4.23}$$

At the same time, by (4.3) and $i^\dagger \in \mathbb{I}_j^{\hat{v}}$ we have

$$j_{i^\dagger}^{\hat{v}} = j. \tag{4.24}$$

Note that $u \in \mathbb{S}(B, c, d)$. Obviously, u satisfies the $i^\dagger$th inequality in (1.6), i.e.,

$$c_{i^\dagger} \le (b_{i^\dagger 1} \wedge u_1) \vee (b_{i^\dagger 2} \wedge u_2) \vee \cdots \vee (b_{i^\dagger n} \wedge u_n) \le d_{i^\dagger}.$$

As a result, there is $j^\dagger \in \mathbb{J}$ satisfying

$$b_{i^\dagger j^\dagger} \wedge u_{j^\dagger} \ge c_{i^\dagger}. \tag{4.25}$$

Considering $u \le \hat{v}$ since $\hat{v}$ is the maximum solution, we obtain $b_{i^\dagger j^\dagger} \wedge \hat{v}_{j^\dagger} \ge b_{i^\dagger j^\dagger} \wedge u_{j^\dagger} \ge c_{i^\dagger}$. According to (4.1), it holds $j^\dagger \in \mathbb{J}_{i^\dagger}^{\hat{v}}$. Moreover, Inequality (4.25) implies that

$$u_{j^\dagger} \ge b_{i^\dagger j^\dagger} \wedge u_{j^\dagger} \ge c_{i^\dagger}. \tag{4.26}$$

Observing (4.2) and (4.24), we have $j = j_{i^\dagger}^{\hat{v}} = \arg\max\{\hat{v}_l | l \in \mathbb{J}_{i^\dagger}^{\hat{v}}\}$. Thus,

$$\hat{v}_j = \hat{v}_{j_{i^\dagger}^{\hat{v}}} \ge \hat{v}_l, \quad \forall l \in \mathbb{J}_{i^\dagger}^{\hat{v}}.$$

Since $j^\dagger \in \mathbb{J}_{i^\dagger}^{v}$, it holds

$$\hat{v}_j \ge \hat{v}_{j^\dagger}. \tag{4.27}$$

Combining (4.23), (4.26), and (4.27), we have

$$\hat{v}_j - \check{v}_j = \hat{v}_j - c_{i^\dagger} \ge \hat{v}_j - u_{j^\dagger} \ge \hat{v}_{j^\dagger} - u_{j^\dagger} \ge \bigwedge_{k \in \mathbb{J}} (\hat{v}_k - u_k) = w[u, \hat{v}]. \tag{4.28}$$

Inequalities (4.22) and (4.28) contribute to $\hat{v}_j - \check{v}_j \ge w[u, \hat{v}]$, $\forall j \in \mathbb{J}$. So we have

$$w[\check{v}, \hat{v}] = \bigwedge_{j \in \mathbb{J}} (\hat{v}_j - \check{v}_j) \ge w[u, \hat{v}].$$

□

Theorem 6. *v^C and ε^{v^C} are defined by* (4.16). *Then it holds that v^C is a centralized solution regarding system* (1.6).

Proof. Let $v \in \mathbb{S}(B, c, d)$ be an arbitrary solution in (1.6). $[v - \varepsilon^v, v + \varepsilon^v]$ is the WSIS regarding v. Since $[v - \varepsilon^v, v + \varepsilon^v]$ is an interval solution of (1.6); by Definition 3, it holds

$$[v - \varepsilon^v, v + \varepsilon^v] \subseteq \mathbb{S}(B, c, d). \tag{4.29}$$

That is $v - \varepsilon^v, v + \varepsilon^v \in \mathbb{S}(B, c, d)$. According to Proposition 10,

$$w[\check{v}, \hat{v}] \geq w[v - \varepsilon^v, \hat{v}]. \tag{4.30}$$

Since $\hat{v}$ is the maximum solution, it is clear $\hat{v} \geq v + \varepsilon^v$. Thus

$$w[v - \varepsilon^v, \hat{v}] \geq w[v - \varepsilon^v, v + \varepsilon^v]. \tag{4.31}$$

Inequalities (4.30) and (4.31) imply that

$$w[\check{v}, \hat{v}] \geq w[v - \varepsilon^v, v + \varepsilon^v]. \tag{4.32}$$

By (4.16), it is clear $\check{v} = v^C - \varepsilon^{v^C}$ and $\hat{v} = v^C + \varepsilon^{v^C}$. Inequality (4.32) could be further written as

$$w[v - \varepsilon^v, v + \varepsilon^v] \leq w[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}]. \tag{4.33}$$

According to Definition 6, v^C is a centralized solution regarding system (1.6). Moreover, the WSIS regarding v^C is $[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}] = [\check{v}, \hat{v}]$. □

4.3. Resolution algorithm for the centralized solution and illustrative example

This subsection provides an algorithm to find the centralized solution regarding a given max-min system. Moreover, some numerical examples are given for verifying our proposed algorithm.

Algorithm II: solving the centralized solution regarding system (1.6)

Step 1. Following (4.1), calculate the indicator set $\mathbb{J}_i^{\hat{v}}$ for any $i \in \mathbb{I}$.
Step 2. Following (4.2), select the indicator $j_i^{\hat{v}}$ from the indicator set $\mathbb{J}_i^{\hat{v}}$ for any $i \in \mathbb{I}$.
Step 3. Following (4.3), construct the indicator set $\mathbb{I}_j^{\hat{v}}$ for any $j \in \mathbb{J}$.
Step 4. Following (4.4), generate the vector $\check{v} = (\check{v}_1, \check{v}_2, \cdots, \check{v}_n)$, where the $\check{v}_j$ is defined by (4.4).
Step 5. Following (4.16), calculate the vectors v^C and ε^{v^C}. Then we find the centralized solution regarding system (1.6) as v^C and the corresponding (biggest) symmetrical width as $2(\varepsilon_1^{v^C} \wedge \varepsilon_2^{v^C} \wedge \cdots \wedge \varepsilon_n^{v^C})$.

Computational complexity of Algorithm II: In Algorithm II, Step 1 requires $2mn$ operations for calculating the indicator sets. Moreover, Steps 2–4 have an identical calculation amount as mn. Besides, Step 5 requires $4n + 2$ operations. As a result, applying Algorithm II to calculate the widest symmetrical interval solution regarding v, it requires totally

$$2mn + mn + mn + mn + 4n = 5mn + 4n + 2$$

operations. The computational complexity of Algorithm II is also $O(mn)$.

Example 3. Consider the max-min FRIs system as follows:

$$\begin{cases} 0.3 \le (0.2 \wedge y_1) \vee (0.4 \wedge y_2) \vee (0.5 \wedge y_3) \vee (0.1 \wedge y_4) \vee (0.6 \wedge y_5) \vee (0.7 \wedge y_6) \le 0.8, \\ 0.4 \le (0.3 \wedge y_1) \vee (0.6 \wedge y_2) \vee (0.2 \wedge y_3) \vee (0.8 \wedge y_4) \vee (0.5 \wedge y_5) \vee (0.1 \wedge y_6) \le 0.9, \\ 0.1 \le (0.7 \wedge y_1) \vee (0.3 \wedge y_2) \vee (0.4 \wedge y_3) \vee (0.2 \wedge y_4) \vee (0.5 \wedge y_5) \vee (0.6 \wedge y_6) \le 0.6, \\ 0.5 \le (0.8 \wedge y_1) \vee (0.1 \wedge y_2) \vee (0.3 \wedge y_3) \vee (0.4 \wedge y_4) \vee (0.7 \wedge y_5) \vee (0.2 \wedge y_6) \le 0.9, \\ 0.2 \le (0.5 \wedge y_1) \vee (0.2 \wedge y_2) \vee (0.1 \wedge y_3) \vee (0.9 \wedge y_4) \vee (0.6 \wedge y_5) \vee (0.4 \wedge y_6) \le 0.8. \end{cases} \tag{4.34}$$

Try to find the centralized solution regarding the system (4.34).

Solution:
Notice that

$$\hat{v} = (0.6, 1, 1, 0.8, 1, 1).$$

is the maximum solution of system (4.34). Combining (4.1) and (4.2), we have

$$\begin{aligned} \mathbb{J}_1^{\hat{v}} &= \{j \in \mathbb{J} | b_{1j} \wedge \hat{v}_j \ge c_1\} = \{j \in \mathbb{J} | b_{1j} \wedge \hat{v}_j \ge 0.3\} = \{2, 3, 5, 6\}, \\ \mathbb{J}_2^{\hat{v}} &= \{j \in \mathbb{J} | b_{2j} \wedge \hat{v}_j \ge c_2\} = \{j \in \mathbb{J} | b_{2j} \wedge \hat{v}_j \ge 0.4\} = \{2, 4, 5\}, \\ \mathbb{J}_3^{\hat{v}} &= \{j \in \mathbb{J} | b_{3j} \wedge \hat{v}_j \ge c_3\} = \{j \in \mathbb{J} | b_{3j} \wedge \hat{v}_j \ge 0.1\} = \{1, 2, 3, 4, 5, 6\}, \\ \mathbb{J}_4^{\hat{v}} &= \{j \in \mathbb{J} | b_{4j} \wedge \hat{v}_j \ge c_4\} = \{j \in \mathbb{J} | b_{4j} \wedge \hat{v}_j \ge 0.5\} = \{1, 5\}, \\ \mathbb{J}_5^{\hat{v}} &= \{j \in \mathbb{J} | b_{5j} \wedge \hat{v}_j \ge c_5\} = \{j \in \mathbb{J} | b_{5j} \wedge \hat{v}_j \ge 0.2\} = \{1, 2, 4, 5, 6\}, \end{aligned}$$

and

$$\begin{aligned} j_1^{\hat{v}} &= \arg\max\{\hat{v}_j | j \in \mathbb{J}_1^{\hat{v}}\} = \arg\max\{\hat{v}_2, \hat{v}_3, \hat{v}_5, \hat{v}_6\} = 2 \text{ or } 3 \text{ or } 5 \text{ or } 6, \\ j_2^{\hat{v}} &= \arg\max\{\hat{v}_j | j \in \mathbb{J}_2^{\hat{v}}\} = \arg\max\{\hat{v}_2, \hat{v}_4, \hat{v}_5\} = 2 \text{ or } 5, \\ j_3^{\hat{v}} &= \arg\max\{\hat{v}_j | j \in \mathbb{J}_3^{\hat{v}}\} = \arg\max\{\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4, \hat{v}_5, \hat{v}_6\} = 2 \text{ or } 3 \text{ or } 5 \text{ or } 6, \\ j_4^{\hat{v}} &= \arg\max\{\hat{v}_j | j \in \mathbb{J}_4^{\hat{v}}\} = \arg\max\{\hat{v}_1, \hat{v}_5\} = 5, \\ j_5^{\hat{v}} &= \arg\max\{\hat{v}_j | j \in \mathbb{J}_5^{\hat{v}}\} = \arg\max\{\hat{v}_1, \hat{v}_2, \hat{v}_4, \hat{v}_5, \hat{v}_6\} = 2 \text{ or } 5 \text{ or } 6. \end{aligned}$$

Take $j_1^{\hat{v}} = 2$, $j_2^{\hat{v}} = 5$, $j_3^{\hat{v}} = 3$, $j_4^{\hat{v}} = 5$, $j_5^{\hat{v}} = 6$. Combining (4.3) and (4.4), we can obtain that

$$\begin{aligned} \mathbb{I}_1^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 1\} = \emptyset, \\ \mathbb{I}_2^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 2\} = \{1\}, \\ \mathbb{I}_3^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 3\} = \{3\}, \\ \mathbb{I}_4^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 4\} = \emptyset, \\ \mathbb{I}_5^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 5\} = \{2, 4\}, \\ \mathbb{I}_6^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 6\} = \{5\}, \end{aligned}$$

and

$$\check{v}_1 = 0, \quad \check{v}_2 = \bigvee_{i \in \mathbb{I}_2^{\hat{v}}} c_i = c_1 = 0.3, \quad \check{v}_3 = \bigvee_{i \in \mathbb{I}_3^{\hat{v}}} c_i = c_3 = 0.1,$$

$$\check{v}_4 = 0, \quad \check{v}_5 = \bigvee_{i \in \mathbb{I}_5^{\hat{v}}} c_i = c_2 \vee c_4 = 0.5, \quad \check{v}_6 = \bigvee_{i \in \mathbb{I}_6^{\hat{v}}} c_i = c_5 = 0.2.$$

Then we find the vector $\check{v} = (0, 0.3, 0.1, 0, 0.5, 0.2)$. In accordance with (4.16), we further have

$$v^C = \frac{\hat{v} + \check{v}}{2} = (0.3, 0.65, 0.55, 0.4, 0.75, 0.6),$$
$$\varepsilon^{v^C} = \frac{\hat{v} - \check{v}}{2} = (0.3, 0.35, 0.45, 0.4, 0.25, 0.4).$$

Then it holds that $v^C = (0.3, 0.65, 0.55, 0.4, 0.75, 0.6)$ is a centralized solution regarding system (4.34). The corresponding widest symmetrical interval solution regarding v^C is

$$[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}] = ([0, 0.6], [0.3, 1], [0.1, 1], [0, 0.8], [0.5, 1], [0.2, 1]), \tag{4.35}$$

with the (biggest) symmetrical width $w[v^C - \varepsilon^{v^C}, v^C + \varepsilon^{v^C}] = 0.5$. □

Remark 3. *In fact, the centralized solution regarding a given system with max-min FRIs might not be unique. For example, it can be easily examined that*

$$(0.35, 0.75, 0.75, 0.55, 0.75, 0.75)$$

is indeed another centralized solution regarding system (4.34).

Example 4. Consider the max-min FRIs system as follows:

$$\begin{cases} 0.37 \le (0.35 \wedge y_1) \vee (0.48 \wedge y_2) \vee (0.87 \wedge y_3) \vee (0.63 \wedge y_4) \vee (0.26 \wedge y_5) \vee (0.74 \wedge y_6) \le 0.82, \\ 0.34 \le (0.28 \wedge y_1) \vee (0.92 \wedge y_2) \vee (0.76 \wedge y_3) \vee (0.31 \wedge y_4) \vee (0.53 \wedge y_5) \vee (0.19 \wedge y_6) \le 0.87, \\ 0.25 \le (0.89 \wedge y_1) \vee (0.76 \wedge y_2) \vee (0.42 \wedge y_3) \vee (0.18 \wedge y_4) \vee (0.56 \wedge y_5) \vee (0.75 \wedge y_6) \le 0.79, \\ 0.36 \le (0.33 \wedge y_1) \vee (0.85 \wedge y_2) \vee (0.27 \wedge y_3) \vee (0.35 \wedge y_4) \vee (0.93 \wedge y_5) \vee (0.35 \wedge y_6) \le 0.85, \\ 0.41 \le (0.59 \wedge y_1) \vee (0.28 \wedge y_2) \vee (0.18 \wedge y_3) \vee (0.38 \wedge y_4) \vee (0.83 \wedge y_5) \vee (0.91 \wedge y_6) \le 0.89, \\ 0.32 \le (0.48 \wedge y_1) \vee (0.23 \wedge y_2) \vee (0.31 \wedge y_3) \vee (0.26 \wedge y_4) \vee (0.56 \wedge y_5) \vee (0.28 \wedge y_6) \le 0.86. \end{cases} \tag{4.36}$$

Try to find the centralized solution regarding system (4.36).

Solution: After calculation, the maximum solution of system (3.26) can be obtained as $\hat{v} = (0.79, 0.87, 0.82, 1, 0.85, 0.89)$. Next, we apply our proposed Algorithm II to find the centralized solution regarding (4.36).

Step 1. Following (4.1), we have

$$\mathbb{J}_1^{\hat{v}} = \{2, 3, 4, 6\}, \quad \mathbb{J}_2^{\hat{v}} = \{2, 3, 5\}, \quad \mathbb{J}_3^{\hat{v}} = \{1, 2, 3, 5, 6\},$$
$$\mathbb{J}_4^{\hat{v}} = \{2, 5\}, \quad \mathbb{J}_5^{\hat{v}} = \{1, 5, 6\}, \quad \mathbb{J}_6^{\hat{v}} = \{1, 5\}.$$

Step 2. Following (4.2), we have

$$j_1^{\hat{v}} = \arg\max\{v_j | j \in \mathbb{J}_1^{\hat{v}}\} = 4, \quad j_2^{\hat{v}} = \arg\max\{v_j | j \in \mathbb{J}_2^{\hat{v}}\} = 2,$$
$$j_3^{\hat{v}} = \arg\max\{v_j | j \in \mathbb{J}_3^{\hat{v}}\} = 6, \quad j_4^{\hat{v}} = \arg\max\{v_j | j \in \mathbb{J}_4^{\hat{v}}\} = 2,$$
$$j_5^{\hat{v}} = \arg\max\{v_j | j \in \mathbb{J}_5^{\hat{v}}\} = 6, \quad j_6^{\hat{v}} = \arg\max\{v_j | j \in \mathbb{J}_6^{\hat{v}}\} = 5.$$

Step 3. Following (4.3), we have

$$\begin{aligned}
\mathbb{I}_1^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 1\} = \emptyset, \\
\mathbb{I}_2^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 2\} = \{2, 4\}, \\
\mathbb{I}_3^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 3\} = \emptyset, \\
\mathbb{I}_4^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 4\} = \{1\}, \\
\mathbb{I}_5^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 5\} = \{6\}, \\
\mathbb{I}_6^{\hat{v}} &= \{i \in \mathbb{I} | j_i^{\hat{v}} = 6\} = \{3, 5\}.
\end{aligned}$$

Step 4. Following (4.4), we have

$$\begin{aligned}
\check{v}_1 &= 0, \\
\check{v}_2 &= \bigvee_{i \in \mathbb{I}_2^{\hat{v}}} c_i = c_2 \vee c_4 = 0.36, \\
\check{v}_3 &= 0, \\
\check{v}_4 &= \bigvee_{i \in \mathbb{I}_4^{\hat{v}}} c_i = c_1 = 0.37, \\
\check{v}_5 &= \bigvee_{i \in \mathbb{I}_5^{\hat{v}}} c_i = c_6 = 0.32, \\
\check{v}_6 &= \bigvee_{i \in \mathbb{I}_6^{\hat{v}}} c_i = c_3 \vee c_5 = 0.41.
\end{aligned}$$

Hence, we obtain the vector $\check{v} = (0, 0.36, 0, 0.37, 0.32, 0.41)$.

Step 5. Following (4.16), we can calculate the vectors v^C and ε^{v^C}. After calculation, we have

$$v^C = \frac{\hat{v} + \check{v}}{2} = (0.395, 0.615, 0.41, 0.685, 0.585, 0.65).$$

$$\varepsilon^{v^C} = \frac{\hat{v} - \check{v}}{2} = (0.395, 0.255, 0.41, 0.315, 0.265, 0.24).$$

Hence, the centralized solution regarding system (4.36) is $v^C = (0.395, 0.615, 0.41, 0.685, 0.585, 0.65)$, with the corresponding (biggest) symmetrical width $2(\varepsilon_1^{v^C} \wedge \varepsilon_2^{v^C} \wedge \cdots \wedge \varepsilon_6^{v^C}) = 0.48$. □

5. Conclusions

Recently, some researchers adopted the max-min FRIs or FREs system to model a P2P educational information sharing system. Any feasible scheme in the P2P network system was found to be a solution to the max-min system, e.g., system (1.6). As a result, we employed the concept of the widest symmetrical interval solution to represent the stability for a feasible scheme. Moreover, reflecting the most stable feasible scheme, the concept of a centralized solution of system (1.6) was further defined. Effective resolution methods were proposed for the widest symmetrical interval solution regarding a provided solution and the centralized solution regarding system (1.6), respectively. The proposed resolution methods have been demonstrated and examined through the numerical examples. In future works, one might further generalize the widest symmetrical interval solution and centralized solution to the addition-min FRIs system, or even to the classical max-t-norm fuzzy relational system.

Author contributions

Miaoxia Chen: Validation, Investigation, Writing-original draft; Guocheng Zhu: Validation, Methodology; Shayla Islam: Conceptualization, Supervision; Xiaopeng Yang: Conceptualization, Supervision, Funding acquisition, Writing-review & editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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