Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(4): 7684–7705.
DOI: 10.3934/math.2025352
Received: 15 January 2025
Revised: 11 March 2025
Accepted: 17 March 2025
Published: 01 April 2025

Research article

An η-Hermitian solution to a two-sided matrix equation and a system of matrix equations over the skew-field of quaternions

Mahmoud S. Mehany[1,2,*] **and Faizah D. Alanazi**[3]

[1] Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, 11566, Egypt

[2] School of Science and Technology, NOVA University, Lisbon, Cairo Campus, New Cairo, 2024, Egypt

[3] Department of Mathematics, College of Science, Northern Border University, Arar, Saudi Arabia

* **Correspondence:** Email: mahmoudmahany2002@edu.asu.edu.eg, mahmoud2006@shu.edu.cn.

Abstract: This study investigated η-Hermitian solutions to a two-sided matrix equation and a system of matrix equations over the skew-field of quaternions. We employed solvability conditions to identify the general solution for this system. This method deduces the Moore-Penrose inverse, and the rank equalities for the system's coefficients. We utilized these strategies to develop an algorithm that can compute general solutions. We also used these algorithms in numerical instances to verify the theoretical results.

Keywords: linear matrix equations; MATLAB software; nonstandard involution; generalized inverses; matrix rank

Mathematics Subject Classification: 15A05, 15A09, 15A36, 15A63, 68W30

1. Introduction

The Schrödinger equation in quaternionic quantum mechanics was presented in [1]. The eigenvalue problem of a Hermitian quaternion matrix in quantum chemistry was examined in [2]. Some algebraic approaches for quaternion least squares problems in quaternionic quantum mechanics were analyzed in [3]. Specific types of quaternionic linear matrix equations require further research. Let $\phi : \mathbb{H} \to \mathbb{H}$ be a nonstandard involution [4]. If $S \in \mathbb{H}^{m\times n}$, then $(S)_\phi$ is an $n \times m$ matrix, which is found by applying ϕ entrywise to the transpose S. $S \in \mathbb{H}^{n\times n}$ is called a ϕ-Hermitian matrix if $(S)_\phi = S$. Took et al. investigated an instance of a ϕ-Hermitian matrix, namely the η-Hermitian matrix [5]. S is called η-Hermitian if $S^{\eta^*} := -\eta S^*\eta = S$, where $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. η-Hermitian matrices have applications in linear modeling and statistical signal processing [5–8]. In this paper, we established the solvability conditions and η-Hermitian solution for the system

$$\begin{aligned} A_1X_1A_1^{\eta^*} + C_1Y_1C_1^{\eta^*} + F_1Z_1F_1^{\eta^*} + H_1WH_1^{\eta^*} &= E_1, \\ A_2X_2A_2^{\eta^*} + C_2Y_2C_2^{\eta^*} + F_2Z_2F_2^{\eta^*} + H_2WH_2^{\eta^*} &= E_2, \\ X_1 = X_1^{\eta^*}, X_2 = X_2^{\eta^*},\ Y_1 = Y_1^{\eta^*},\ Y_2 = Y_2^{\eta^*},\ Z_1 = Z_1^{\eta^*},\ Z_2 = Z_2^{\eta^*},\ W = W^{\eta^*}&. \end{aligned} \tag{1.1}$$

As a special case of the system (1.1), we construct an η-Hermitian solution to the two-sided quaternion matrix equation

$$\begin{aligned} A_1X_1A_1^{\eta^*} + C_1Y_1C_1^{\eta^*} + F_1Z_1F_1^{\eta^*} + H_1WH_1^{\eta^*} &= E_1, \\ X_1 = X_1^{\eta^*},\ Y_1 = Y_1^{\eta^*},\ Z_1 = Z_1^{\eta^*},\ W = W^{\eta^*}&. \end{aligned} \tag{1.2}$$

In 1844, William Hamilton scouted the quaternions [9]. Quaternions have implementations in assorted domains of mathematics analogous with computation, geometry, and algebra (see, e.g., [10–13]). The study and analysis of Sylvester's matrix equations and their generalizations have many applications in graph theory [14], output feedback control [15], neural networks [16], and robust control [17].

In order to achieve (1.1) and (1.2), we present the following two-sided quaternionic matrix equations and establish the necessary and sufficient conditions for being solvable:

$$\grave{A}_1\grave{X}_1\grave{B}_1 + \grave{C}_1\grave{X}_2\grave{D}_1 + A_1X_1B_1 + C_1X_2D_1 = E_1, \tag{1.3}$$

where $\grave{A}_1 \in \mathbb{H}_{r_1}^{m_1\times n_1}$, $\grave{B}_1 \in \mathbb{H}_{q_1}^{p_1\times q_1}$, $\grave{C}_1 \in \mathbb{H}_{m_1}^{m_1\times s_1}$, $\grave{D}_1 \in \mathbb{H}_{r_2}^{l_1\times q_1}$, $A_1 \in \mathbb{H}_{r_3}^{m_1\times n_2}$, $B_1 \in \mathbb{H}_{r_4}^{p_2\times q_1}$, $C_1 \in \mathbb{H}_{r_5}^{m_1\times s_2}$, $D_1 \in \mathbb{H}_{s_3}^{l_2\times q_1}$, and $E_1 \in \mathbb{H}_{s_4}^{m_1\times q_1}$ are given matrices and $\grave{X}_1 \in \mathbb{H}_{s_5}^{n_1\times p_1}$, $\grave{X}_2 \in \mathbb{H}_{s_6}^{s_1\times l_1}$, $X_1 \in \mathbb{H}_{r_6}^{n_2\times p_2}$, and $X_2 \in \mathbb{H}_{r_7}^{s_2\times l_2}$ are unknowns. Thus, we utilize (1.3) to examine the solvability conditions for the two-sided system of matrix equations, emphasizing matrix rank equalities and Moore-Penrose inverses of matrix coefficients, which are essential for deriving (1.1):

$$A_iX_iB_i + C_iY_iD_i + F_iZ_iG_i + H_iWJ_i = E_i,\ \ (i = 1, 2). \tag{1.4}$$

A variety of established results on the consistency conditions and general solutions of linear matrix equations in the literature, particularly the discoveries related to the Sylvester-type matrix problem, enhance this work, especially the matrix equation

$$A_1X_1B_1 + C_1X_2D_1 = E_1, \tag{1.5}$$

which was first studied by Baksalary and Kala (1980) in [18]. In 1991, Özgüler gave us its analysis over a principal ideal domain [19]. Tian (2000) drove its necessary and sufficient conditions to be solvable in matrix rank equalities over an arbitrary field [20]. Wang (2004) presented the consistency conditions and the general solution in terms of the Moore-Penrose inverses over an arbitrary regular ring [21]. He (2019) [22] investigated a proper generalization of the Sylvester-type matrix equation (1.5), namely:

$$A_iX_i + Y_iD_i + F_iZ_iG_i + H_iWJ_i = E_i,\ (i = 1, 2), \tag{1.6}$$

which is solvable if and only if the following system is solvable:

$$A_iX_iB_i + C_iX_{i+1}D_i = E_i, (i = 1, 2). \tag{1.7}$$

The necessary and sufficient conditions for the system (1.1) to be solvable are carried out. An expression of the general solution indication of the Moore-Penrose inverses terms is given, in the case where the solvability conditions are met. If we choose the quaternion matrix coefficients, so that $C_i = B_i = I$, $(i = 1, 2)$, where I represents the identity matrices with feasible size, it generates the proper special case (1.6) of (1.1) and hence its equivalent system (1.7).

Sylvester [23] investigated the classical Sylvester matrix equation. Since then, the generalizations of Sylvester systems of matrix equations over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are established, where $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ represent the real number field, complex number field, and quaternion skew-field, respectively. For instance, Mehany et al. [24–26], Xie et al. [27], Liu et al. [28, 29], Xu et al. [30], He et al. [31, 32], Kyrchei et al. [33, 34], Rehman et al. [35], Bayoumi [36], and Mitra [37] have studied some generalized Sylvester systems of matrix equations that include some special cases of (1.4).

This article structures the subsequent sections as follows: Section 2 presents the initial and fundamental background findings. In Section 3, we employ the Moore-Penrose inverses and rank equalities of the quaternion matrices to satisfy the necessary and sufficient conditions for the system (1.3) to be solvable. Furthermore, we present a formula for the general solution of the system when it is solvable. In Section 4, we achieve our main goal by obtaining the necessary and sufficient conditions and the $\phi-$Hermitian solution to (1.1). Section 5 includes algorithms and numerical examples that illustrate the principal findings. This paper concludes with succinct findings in Section 6.

2. Preliminaries

We denote the real quaternion algebra by

$$\mathbb{H} = \{t_0 + t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ t_0, t_1, t_2, t_3 \in \mathbb{R}\}.$$

$\mathbb{H}_r^{m\times n}$ is the set of all matrices of size $m \times n$ with rank r over $\mathbb{H}$. S^* represents the conjugate transpose of S. The Moore-Penrose inverse of a given matrix S over $\mathbb{H}$ is denoted by $S^\dagger$ and defined to be the unique solution Y of the system $SYS = S$, $YSY = Y$, $(YS)^* = YS$, and $(SY)^* = SY$. Moreover, $L_S = I - S^\dagger S$ and $R_S = I - SS^\dagger$ stand for the two projectors along S. Furthermore, $L_S = (L_S)^* = (L_S)^2 = L_S^\dagger$, $R_S = (R_S)^2 = (R_S)^* = R_S^\dagger$.

Here, we define involutions, analyze their matrix representations, and categorize them as standard or nonstandard. In addition, we present certain algebraic features of the quaternion matrix nonstandard involution A_ϕ.

Definition 2.1. *[4] A map ϕ: $\mathbb{H} \to \mathbb{H}$ is said to be an antiendomorphism if $\phi(xy) = \phi(y)\phi(x)$ for all x, $y \in \mathbb{H}$, and $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in \mathbb{H}$. An antiendomorphism ϕ is called an involution if $\phi(\phi(x)) = x$ for every $x \in \mathbb{H}$.*

Lemma 2.1. *[4] Let ϕ be a nonzero antiendomorphism of $\mathbb{H}$. Then ϕ is a bijection on $\mathbb{H}$; thus, ϕ is an antiautomorphism. Additionally, ϕ is real linear, which can be expressed as a 4×4 real matrix with respect to $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Then ϕ is an involution if and only if $\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$ where either $T = -I_3$ or T is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$. Moreover, an involution ϕ*

is standard if $\phi = \begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}$. *An involution* ϕ *is nonstandard if* $\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$ *where* T *is a* 3×3 *real orthogonal symmetric matrix with eigenvalues* $1, 1, -1$.

Lemma 2.2. *[4] Let* $A, B \in \mathbb{H}^{m\times n}$, $C \in \mathbb{H}^{n\times p}$, $\alpha, \beta \in \mathbb{H}$, *and* ϕ *be a nonstandard involution over* $\mathbb{H}$. *Then*

$$\begin{cases} (1) & (\alpha A + \beta B)_\phi = A_\phi \phi(\alpha) + B_\phi \phi(\beta), \\ (2) & (A\alpha + B\beta)_\phi = \phi(\alpha) A_\phi + \phi(\beta) B_\phi, \\ (3) & (AC)_\phi = C_\phi A_\phi, \end{cases} \quad \begin{cases} (4) & (A_\phi)_\phi = A, \\ (5) & r(A) = r(A_\phi), \\ (6) & I_\phi = I,\ 0_\phi = 0, \end{cases} \quad \begin{cases} (7) & (A_\phi)^\dagger = (A^\dagger)_\phi, \\ (8) & (R_A)_\phi = L_{A_\phi}, \\ (9) & (L_A)_\phi = R_{A_\phi}. \end{cases}$$

Lemma 2.3. *[38] Let* $S \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, *and* $C \in \mathbb{H}^{l\times n}$ *be given. Then*

$$(1)\ \ r(S) + r(R_S B) = r(B) + r(R_B S) = r\begin{pmatrix} S & B \end{pmatrix},$$
$$(2)\ \ r(S) + r(C L_S) = r(C) + r(S L_C) = r\begin{pmatrix} S \\ C \end{pmatrix}.$$

Lemma 2.4. *[18–21] Let* A_1, B_1, C_1, D_1, *and* E_1 *be given. Set*

$$M_1 = R_{A_1} C_1,\ N_1 = D_1 L_{B_1},\ S = C_1 L_{M_1}.$$

Then the following are equivalent:

1) (1.5) *is solvable.*

2)

$$R_{M_1} R_{A_1} E_1 = 0,\ E_1 L_{B_1} L_{N_1} = 0,\ R_{A_1} E_1 L_{D_1} = 0,\ R_{C_1} E_1 L_{B_1} = 0.$$

3)

$$r\begin{pmatrix} A_1 & E_1 & C_1 \end{pmatrix} = r\begin{pmatrix} A_1 & C_1 \end{pmatrix},\ r\begin{pmatrix} B_1 \\ E_1 \\ D_1 \end{pmatrix} = r\begin{pmatrix} B_1 \\ D_1 \end{pmatrix},$$
$$r\begin{pmatrix} A_1 & E_1 \\ 0 & D_1 \end{pmatrix} = r\begin{pmatrix} A_1 \end{pmatrix} + r\begin{pmatrix} D_1 \end{pmatrix},\ r\begin{pmatrix} B_1 & 0 \\ E_1 & C_1 \end{pmatrix} = r\begin{pmatrix} B_1 \end{pmatrix} + r\begin{pmatrix} C_1 \end{pmatrix}.$$

In that case, the general solution of (1.5) *can be expressed as*

$$\begin{aligned} X_1 =& A_1^\dagger E_1 B_1^\dagger - A_1^\dagger C_1 M_1^\dagger E_1 B_1^\dagger - A_1^\dagger S_1 C_1^\dagger E_1 N_1^\dagger D_1 B_1^\dagger - A_1^\dagger S_1 Y_1 R_{N_1} D_1 B_1^\dagger \\ &+ L_{A_1} Y_2 + Y_3 R_{B_1}, \\ X_2 =& M_1^\dagger E_1 D_1^\dagger + S_1^\dagger S_1 C_1^\dagger E_1 N_1^\dagger + L_{M_1} L_{S_1} Y_4 + L_{M_1} Y_1 R_{N_1} + Y_5 R_{D_1}, \end{aligned}$$

where $Y_1, Y_2, ..., Y_5$ *are arbitrary matrices with fitting sizes.*

3. The solvability conditions and the general solution to (1.3) and (1.4)

Let A_1, $\grave{A}_1$, B_1, $\grave{B}_1$, C_1, $\grave{C}_1$, D_1, $\grave{D}_1$, and E_1 be given matrices in (1.3). Set

$$\widehat{A_1} = R_{\grave{A}_1}A_1,\ \widehat{B_1} = B_1L_{\grave{D}_1},\ \widehat{C_1} = R_{\grave{A}_1}C_1,\ \widehat{D_1} = D_1L_{\grave{D}_1},\ \grave{M}_1 = R_{\grave{A}_1}\grave{C}_1, \tag{3.1}$$

$$\grave{S}_1 = \grave{C}_1L_{\grave{M}_1}, \widehat{M_1} = R_{\widehat{A_1}}\widehat{C_1},\ \widehat{N_1} = \widehat{D_1}L_{\widehat{B_1}},\ \widehat{S}_1 = \widehat{C_1}L_{\widehat{M_1}},\ \widehat{E_1} = R_{\grave{A}_1}E_1L_{\grave{D}_1}, \tag{3.2}$$

$$\grave{E}_1 = E_1 - A_1X_1B_1 - C_1X_2D_1. \tag{3.3}$$

Proposition 3.1. *The following statements are equivalent:*
(1) (1.3) *is solvable.*
(2)

$$R_{\grave{M}_1}R_{\grave{A}_1}E_1 = 0,\ E_1L_{\grave{B}_1}L_{\grave{N}_1} = 0,\ R_{\grave{C}_1}E_1L_{\grave{B}_1} = 0,$$
$$R_{\widehat{M_1}}R_{\widehat{A_1}}\widehat{E_1} = 0, \widehat{E}_1L_{\widehat{B_1}}L_{\widehat{N_1}} = 0,\ R_{\widehat{A_1}}\widehat{E}_1L_{\widehat{D_1}} = 0,\ R_{\widehat{C_1}}\widehat{E}_1L_{\widehat{B_1}} = 0.$$

(3)

$$r\begin{pmatrix}\grave{A}_1 & E_1 & \grave{C}_1\end{pmatrix} = r\begin{pmatrix}\grave{A}_1 & \grave{C}_1\end{pmatrix}, \tag{3.4}$$

$$r\begin{pmatrix}\grave{B}_1 \\ E_1 \\ \grave{D}_1\end{pmatrix} = r\begin{pmatrix}\grave{B}_1 \\ \grave{D}_1\end{pmatrix}, \tag{3.5}$$

$$r\begin{pmatrix}\grave{B}_1 & 0 \\ E_1 & \grave{C}_1\end{pmatrix} = r\begin{pmatrix}\grave{B}_1\end{pmatrix} + r\begin{pmatrix}\grave{C}_1\end{pmatrix}, \tag{3.6}$$

$$r\begin{pmatrix}A_1 & E_1 & C_1 & \grave{A}_1 \\ 0 & \grave{D}_1 & 0 & 0\end{pmatrix} = r\begin{pmatrix}A_1 & C_1 & \grave{A}_1\end{pmatrix} + r\begin{pmatrix}\grave{D}_1\end{pmatrix}, \tag{3.7}$$

$$r\begin{pmatrix}B_1 & 0 \\ E_1 & \grave{A}_1 \\ D_1 & 0 \\ \grave{D}_1 & 0\end{pmatrix} = r\begin{pmatrix}B_1 \\ D_1 \\ \grave{D}_1\end{pmatrix} + r\begin{pmatrix}\grave{A}_1\end{pmatrix}, \tag{3.8}$$

$$r\begin{pmatrix}A_1 & E_1 & \grave{A}_1 \\ 0 & D_1 & 0 \\ 0 & \grave{D}_1 & 0\end{pmatrix} = r\begin{pmatrix}A_1 & \grave{A}_1\end{pmatrix} + r\begin{pmatrix}D_1 \\ \grave{D}_1\end{pmatrix}, \tag{3.9}$$

$$r\begin{pmatrix}B_1 & 0 & 0 \\ E_1 & C_1 & \grave{A}_1 \\ \grave{D}_1 & 0 & 0\end{pmatrix} = r\begin{pmatrix}C_1 & \grave{A}_1\end{pmatrix} + r\begin{pmatrix}B_1 \\ \grave{D}_1\end{pmatrix}. \tag{3.10}$$

In this case, the general solution can be expressed as

$$\begin{aligned}
\grave{X}_1 =& \grave{A}_1^{\dagger}\grave{E}_1\grave{B}_1^{\dagger} - \grave{A}_1^{\dagger}\grave{C}_1\grave{M}_1^{\dagger}\grave{E}_1\grave{B}_1^{\dagger} - \grave{A}_1^{\dagger}\grave{S}_1\grave{C}_1^{\dagger}\grave{E}_1\grave{N}_1^{\dagger}\grave{D}_1\grave{B}_1^{\dagger} \\
&- \grave{A}_1^{\dagger}\grave{S}_1\grave{Y}_1R_{\grave{N}_1}\grave{D}_1\grave{B}_1^{\dagger} + L_{\grave{A}_1}\grave{Y}_2 + \grave{Y}_3R_{\grave{B}_1}, \\
\grave{X}_2 =& \grave{M}_1^{\dagger}\grave{E}_1\grave{D}_1^{\dagger} + \grave{S}_1^{\dagger}\grave{S}_1\grave{C}_1^{\dagger}\grave{E}_1\grave{N}_1^{\dagger} + L_{\grave{M}_1}L_{\grave{S}_1}\grave{Y}_4 + L_{\grave{M}_1}\grave{Y}_1R_{\grave{N}_1} + \grave{Y}_5R_{\grave{D}_1}, \\
X_1 =& \widehat{A_1}^{\dagger}\widehat{E_1}\widehat{B_1}^{\dagger} - \widehat{A_1}^{\dagger}\widehat{C_1}\widehat{M_1}^{\dagger}\widehat{E_1}\widehat{B_1}^{\dagger} - \widehat{A_1}^{\dagger}\widehat{S_1}\widehat{C_1}^{\dagger}\widehat{E_1}\widehat{N_1}^{\dagger}\widehat{D_1}\widehat{B_1}^{\dagger} \\
&- \widehat{A_1}^{\dagger}\widehat{S_1}\widehat{Y_1}R_{\widehat{N_1}}\widehat{D_1}\widehat{B_1}^{\dagger} + L_{\widehat{A_1}}\widehat{Y_2} + \widehat{Y_3}R_{\widehat{B_1}}, \\
X_2 =& \widehat{M_1}^{\dagger}\widehat{E_1}\widehat{D_1}^{\dagger} + \widehat{S_1}^{\dagger}\widehat{S_1}\widehat{C_1}^{\dagger}\widehat{E_1}\widehat{N_1}^{\dagger} + L_{\widehat{M_1}}L_{\widehat{S_1}}\widehat{Y_4} + L_{\widehat{M_1}}\widehat{Y_1}R_{\widehat{N_1}} + \widehat{Y_5}R_{\widehat{D_1}}.
\end{aligned} \tag{3.11}$$

Proof. (1) $\Leftrightarrow$ (2) : Utilizing (3.3) to rewrite (1.3), we have that

$$\grave{A}_1\grave{X}_1\grave{B}_1 + \grave{C}_1\grave{X}_2\grave{D}_1 = \grave{E}_1. \tag{3.12}$$

In the view of Lemma 2.4, we have that (3.12) is solvable if and only if

$$R_{\grave{M}_1}R_{\grave{A}_1}\grave{E}_1 = 0,\ \grave{E}_1L_{\grave{B}_1}L_{\grave{N}_1} = 0,\ R_{\grave{C}_1}\grave{E}_1L_{\grave{B}_1} = 0,$$
$$R_{\grave{A}_1}\grave{E}_1L_{\grave{D}_1} = 0.$$

Using straightforward computations, we have their equivalent conditions as

$$R_{\grave{M}_1}R_{\grave{A}_1}E_1 = 0,\ E_1L_{\grave{B}_1}L_{\grave{N}_1} = 0,\ R_{\grave{C}_1}E_1L_{\grave{B}_1} = 0,$$

and

$$\widehat{A_1}X_1\widehat{B_1} + \widehat{C_1}X_2\widehat{D_1} = \widehat{E_1}, \tag{3.13}$$

respectively. In that case, the general solution of (3.12) can be expressed as

$$\begin{aligned}
\grave{X}_1 =& \grave{A}_1^{\dagger}\grave{E}_1\grave{B}_1^{\dagger} - \grave{A}_1^{\dagger}\grave{C}_1\grave{M}_1^{\dagger}\grave{E}_1\grave{B}_1^{\dagger} - \grave{A}_1^{\dagger}\grave{S}_1\grave{C}_1^{\dagger}\grave{E}_1\grave{N}_1^{\dagger}\grave{D}_1\grave{B}_1^{\dagger} \\
&- \grave{A}_1^{\dagger}\grave{S}_1\grave{Y}_1R_{\grave{N}_1}\grave{D}_1\grave{B}_1^{\dagger} + L_{\grave{A}_1}\grave{Y}_2 + \grave{Y}_3R_{\grave{B}_1}, \\
\grave{X}_2 =& \grave{M}_1^{\dagger}\grave{E}_1\grave{D}_1^{\dagger} + \grave{S}_1^{\dagger}\grave{S}_1\grave{C}_1^{\dagger}\grave{E}_1\grave{N}_1^{\dagger} + L_{\grave{M}_1}L_{\grave{S}_1}\grave{Y}_4 + L_{\grave{M}_1}\grave{Y}_1R_{\grave{N}_1} + \grave{Y}_5R_{\grave{D}_1}.
\end{aligned}$$

Finally, the quaternion matrix equation (3.13) has a solution if and only if

$$R_{\widehat{M_1}}R_{\widehat{A_1}}\widehat{E_1} = 0,\ \widehat{E_1}L_{\widehat{B_1}}L_{\widehat{N_1}} = 0,\ R_{\widehat{A_1}}\widehat{E_1}L_{\widehat{D_1}} = 0,\ R_{\widehat{C_1}}\widehat{E_1}L_{\widehat{B_1}} = 0.$$

Consequently, its general solution can be given as

$$\begin{aligned}
X_1 =& \widehat{A_1}^{\dagger}\widehat{E_1}\widehat{B_1}^{\dagger} - \widehat{A_1}^{\dagger}\widehat{C_1}\widehat{M_1}^{\dagger}\widehat{E_1}\widehat{B_1}^{\dagger} - \widehat{A_1}^{\dagger}\widehat{S_1}\widehat{C_1}^{\dagger}\widehat{E_1}\widehat{N_1}^{\dagger}\widehat{D_1}\widehat{B_1}^{\dagger} \\
&- \widehat{A_1}^{\dagger}\widehat{S_1}\widehat{Y_1}R_{\widehat{N_1}}\widehat{D_1}\widehat{B_1}^{\dagger} + L_{\widehat{A_1}}\widehat{Y_2} + \widehat{Y_3}R_{\widehat{B_1}}, \\
X_2 =& \widehat{M_1}^{\dagger}\widehat{E_1}\widehat{D_1}^{\dagger} + \widehat{S_1}^{\dagger}\widehat{S_1}\widehat{C_1}^{\dagger}\widehat{E_1}\widehat{N_1}^{\dagger} + L_{\widehat{M_1}}L_{\widehat{S_1}}\widehat{Y_4} + L_{\widehat{M_1}}\widehat{Y_1}R_{\widehat{N_1}} + \widehat{Y_5}R_{\widehat{D_1}},
\end{aligned}$$

where $\grave{Y}_1, \grave{Y}_2, ..., \grave{Y}_5, \widehat{Y_1}, \widehat{Y_2}, ..., \widehat{Y_5}$ are arbitrary matrices over $\mathbb{H}$ with fit size.

(2) $\Leftrightarrow$ (3) : By using Lemma 2.3, we have the following seven rank equalities:

$$R_{\grave{M}_1}R_{\grave{A}_1}E_1 = 0 \Leftrightarrow r\begin{pmatrix}\grave{A}_1 & E_1 & \grave{C}_1\end{pmatrix} = r\begin{pmatrix}\grave{A}_1 & \grave{C}_1\end{pmatrix},$$

$$E_1 L_{\grave{B}_1} L_{\grave{N}_1} = 0 \Leftrightarrow r\begin{pmatrix} \grave{B}_1 \\ E_1 \\ \grave{D}_1 \end{pmatrix} = r\begin{pmatrix} \grave{B}_1 \\ \grave{D}_1 \end{pmatrix},$$

$$R_{\grave{C}_1} E_1 L_{\grave{B}_1} = 0 \Leftrightarrow r\begin{pmatrix} \grave{B}_1 & 0 \\ E_1 & \grave{C}_1 \end{pmatrix} = r\left(\grave{A}_1\right) + r\left(\grave{C}_1\right),$$

$$R_{\widehat{M_1}} R_{\widehat{A_1}} \widehat{E_1} = 0 \Leftrightarrow r\begin{pmatrix} A_1 & E_1 & C_1 & \grave{A}_1 \\ 0 & \grave{D}_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & C_1 & \grave{A}_1 \end{pmatrix} + r\left(\grave{D}_1\right),$$

$$\widehat{E_1} L_{\widehat{B_1}} L_{\widehat{N_1}} = 0 \Leftrightarrow r\begin{pmatrix} B_1 & 0 \\ E_1 & \grave{A}_1 \\ D_1 & 0 \\ \grave{D}_1 & 0 \end{pmatrix} = r\begin{pmatrix} B_1 \\ D_1 \\ \grave{D}_1 \end{pmatrix} + r\left(\grave{A}_1\right),$$

$$R_{\widehat{A_1}} \widehat{E_1} L_{\widehat{D_1}} = 0 \Leftrightarrow r\begin{pmatrix} A_1 & E_1 & \grave{A}_1 \\ 0 & D_1 & 0 \\ 0 & \grave{D}_1 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & \grave{A}_1 \end{pmatrix} + r\begin{pmatrix} D_1 \\ \grave{D}_1 \end{pmatrix},$$

$$R_{\widehat{C_1}} \widehat{E_1} L_{\widehat{B_1}} = 0 \Leftrightarrow r\begin{pmatrix} B_1 & 0 & 0 \\ E_1 & C_1 & \grave{A}_1 \\ \grave{D}_1 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} C_1 & \grave{A}_1 \end{pmatrix} + r\begin{pmatrix} B_1 \\ \grave{D}_1 \end{pmatrix}.$$

□

The following corollary investigates the main result in [39].

Corollary 3.2. *Let $A_1, B_1, C_3, D_3, C_4, D_4$, and E_1 be given. Set*

$$A = R_{A_1}C_3,\ B = D_3L_{B_1},\ C = R_{A_1}C_4,\ D = D_4L_{B_1},$$
$$E = R_{A_1}E_1L_{B_1},\ M = R_AC,\ N = DL_B,\ S = CL_M.$$

Then the following are equivalent:
(1)

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1$$

is solvable.
(2)

$$R_MR_AE = 0,\ EL_BL_N = 0,\ R_AEL_D = 0,\ R_CEL_B = 0.$$

In this case, the general solution can be expressed as

$$X_1 = A_1^{\dagger}(E_1 - C_3X_3D_3 - C_4X_4D_4) - T_7B_1 + L_{A_1}T_6,$$
$$X_2 = R_{A_1}(E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^{\dagger} + A_1T_7 + T_8R_{B_1},$$
$$X_3 = A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}ST_2R_NDB^{\dagger} + L_AT_4 + T_5R_B,$$
$$X_4 = M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_ST_1 + L_MT_2R_N + T_3R_D,$$

where $T_1, ..., T_8$ are arbitrary matrices over $\mathbb{H}$ with fitting sizes.

Proof. Consider that $\grave{B}_1$ and $\grave{C}_1$ are identities with suitable sizes in Proposition 3.1. □

Let A_i, B_i, C_i, D_i, F_i, G_i, H_i, J_i, and E_i be given matrices in (1.4). Set

$$\begin{aligned}
&M_i = R_{A_i}C_i,\ N_i = D_iL_{B_i},\ S_i = C_iL_{M_i},\ \widehat{A}_i = R_{A_i}F_i,\ \widehat{B}_i = G_iL_{D_i},\\
&\widehat{C}_i = R_{A_i}H_i,\ \widehat{D}_i = J_iL_{D_i}, \widehat{M}_i = R_{\widehat{A}_i}\widehat{C}_i,\ \widehat{N}_i = \widehat{D}_iL_{\widehat{B}_i},\ \widehat{S}_i = \widehat{C}_iL_{\widehat{M}_i},\ \widehat{E}_i = R_{A_i}E_iL_{D_i},\\
&\widehat{\widehat{A}}_1 = \begin{pmatrix} L_{\widehat{M}_1}L_{\widehat{S}_1} & L_{\widehat{M}_2}L_{\widehat{S}_2}\end{pmatrix},\ \widehat{\widehat{B}}_1 = \begin{pmatrix} R_{\widehat{D}_1}\\ -R_{\widehat{D}_2}\end{pmatrix},\ \widehat{\widehat{C}}_3 = L_{\widehat{M}_1},\ \widehat{\widehat{D}}_3 = R_{\widehat{N}_1},\ \widehat{\widehat{C}}_4 = -L_{\widehat{M}_2},\ \widehat{\widehat{D}}_4 = R_{\widehat{N}_2},\\
&W_1^0 = \widehat{M}_1^{\dagger}\widehat{E}_1\widehat{D}_1^{\dagger} + \widehat{S}_1^{\dagger}\widehat{S}_1\widehat{C}_1^{\dagger}\widehat{E}_1\widehat{N}_1^{\dagger},\ W_2^0 = \widehat{M}_2^{\dagger}\widehat{E}_2\widehat{D}_2^{\dagger} + \widehat{S}_2^{\dagger}\widehat{S}_2\widehat{C}_2^{\dagger}\widehat{E}_2\widehat{N}_2^{\dagger},\ \widehat{\widehat{E}}_1 = W_2^0 - W_1^0,\\
&A = R_{\widehat{\widehat{A}}_1}\widehat{\widehat{C}}_3,\ B = \widehat{\widehat{D}}_3L_{\widehat{\widehat{B}}_1},\ C = R_{\widehat{\widehat{A}}_1}\widehat{\widehat{C}}_4,\ D = \widehat{\widehat{D}}_4L_{\widehat{\widehat{B}}_1},\ E = R_{\widehat{\widehat{A}}_1}\widehat{\widehat{E}}_1L_{\widehat{\widehat{B}}_1},\\
&M = R_AC,\ N = DL_B,\ S = CL_M,\ \grave{E}_i = E_i - F_iZ_iG_i - H_iWJ_i.
\end{aligned} \tag{3.14}$$

Theorem 3.3. *The following statements are equivalent:*
(1) The system (1.4) *is consistent.*
(2)

$$R_{M_i}R_{A_i}E_i = 0,\ E_iL_{B_i}L_{N_i} = 0,\ R_{C_i}E_1L_{B_i} = 0, R_{\widehat{M}_i}R_{\widehat{A}_i}\widehat{E}_i = 0, \widehat{E}_iL_{\widehat{B}_i}L_{\widehat{N}_i} = 0, \tag{3.15}$$

$$R_{\widehat{A}_i}\widehat{E}_iL_{\widehat{D}_i} = 0,\ R_{\widehat{C}_i}\widehat{E}_iL_{\widehat{B}_i} = 0, R_MR_AE = 0,\ EL_BL_N = 0,\ R_AEL_D = 0,\ R_CEL_B = 0. \tag{3.16}$$

(3)

$$r\begin{pmatrix} A_i & E_i & C_i\end{pmatrix} = r\begin{pmatrix} A_i & C_i\end{pmatrix},\ r\begin{pmatrix} B_i\\ E_i\\ D_i\end{pmatrix} = r\begin{pmatrix} B_i\\ D_i\end{pmatrix},\ r\begin{pmatrix} B_i & 0\\ E_i & C_i\end{pmatrix} = r\begin{pmatrix} B_i\end{pmatrix} + r\begin{pmatrix} C_i\end{pmatrix}, \tag{3.17}$$

$$r\begin{pmatrix} F_i & E_i & H_i & A_i\\ 0 & D_i & 0 & 0\end{pmatrix} = r\begin{pmatrix} F_i & H_i & A_i\end{pmatrix} + r\begin{pmatrix} D_i\end{pmatrix}, \tag{3.18}$$

$$r\begin{pmatrix} G_i & 0\\ E_i & A_i\\ J_i & 0\\ D_i & 0\end{pmatrix} = r\begin{pmatrix} G_i\\ J_i\\ D_i\end{pmatrix} + r\begin{pmatrix} A_i\end{pmatrix}, \tag{3.19}$$

$$r\begin{pmatrix} E_i & F_i & A_i\\ J_i & 0 & 0\\ D_i & 0 & 0\end{pmatrix} = r\begin{pmatrix} F_i & A_i\end{pmatrix} + r\begin{pmatrix} J_i\\ D_i\end{pmatrix}, \tag{3.20}$$

$$r\begin{pmatrix} E_i & H_i & A_i\\ G_i & 0 & 0\\ D_i & 0 & 0\end{pmatrix} = r\begin{pmatrix} H_i & A_i\end{pmatrix} + r\begin{pmatrix} G_i\\ D_i\end{pmatrix}, \tag{3.21}$$

$$r\begin{pmatrix} 0 & J_1 & J_2 & 0 & 0 & 0 & 0\\ H_1 & -E_1 & 0 & A_1 & F_1 & 0 & 0\\ H_2 & 0 & E_2 & 0 & 0 & A_2 & F_2\\ 0 & D_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & D_2 & 0 & 0 & 0 & 0\end{pmatrix} = r\begin{pmatrix} J_1 & J_2\\ D_1 & 0\\ 0 & D_2\end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & F_1 & 0 & 0\\ H_2 & 0 & 0 & A_2 & F_2\end{pmatrix}, \tag{3.22}$$

$$r\begin{pmatrix} 0 & J_1 & J_2 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & 0 \\ H_2 & 0 & E_2 & 0 & A_2 \\ 0 & G_1 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 \\ 0 & 0 & D_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} J_1 & J_2 \\ G_1 & 0 \\ D_1 & 0 \\ 0 & G_2 \\ 0 & D_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & 0 \\ H_2 & 0 & A_2 \end{pmatrix}, \tag{3.23}$$

$$r\begin{pmatrix} 0 & J_1 & J_2 & 0 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & F_1 & 0 \\ H_2 & 0 & E_2 & 0 & 0 & A_2 \\ 0 & D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} J_1 & J_2 \\ D_1 & 0 \\ 0 & D_2 \\ 0 & G_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & F_1 & 0 \\ H_2 & 0 & 0 & A_2 \end{pmatrix}, \tag{3.24}$$

$$r\begin{pmatrix} 0 & J_1 & J_2 & 0 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & 0 & 0 \\ H_2 & 0 & E_2 & 0 & A_2 & F_2 \\ 0 & G_1 & 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} J_1 & J_2 \\ G_1 & 0 \\ D_1 & 0 \\ 0 & D_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & 0 & 0 \\ H_2 & 0 & A_2 & F_2 \end{pmatrix}. \tag{3.25}$$

In this case, the general solution can be expressed as

$$\begin{aligned}
X_i &= A_i^{\dagger}\grave{E}_iB_i^{\dagger} - A_i^{\dagger}C_iM_i^{\dagger}\grave{E}_iB_i^{\dagger} - A_i^{\dagger}S_iC_i^{\dagger}\grave{E}_iN_i^{\dagger}D_iB_i^{\dagger} - A_i^{\dagger}S_iT_{i1}R_{N_i}D_iB_i^{\dagger} + L_{A_i}T_{i2} + T_{i3}R_{B_i},\\
Y_i &= M_i^{\dagger}\grave{E}_iD_i^{\dagger} + S_i^{\dagger}S_iC_i^{\dagger}\grave{E}_iN_i^{\dagger} + L_{M_i}L_{S_i}T_{i4} + L_{M_i}T_{i1}R_{N_i} + T_{i5}R_{D_i},\\
Z_i &= \widehat{A_i}^{\dagger}\widehat{E_i}\widehat{B_i}^{\dagger} - \widehat{A_i}^{\dagger}\widehat{C_i}\widehat{M_i}^{\dagger}\widehat{E_i}\widehat{B_i}^{\dagger} - \widehat{A_i}^{\dagger}\widehat{S_i}\widehat{C_i}^{\dagger}\widehat{E_i}\widehat{N_i}^{\dagger}\widehat{D_i}\widehat{B_i}^{\dagger} - \widehat{A_i}^{\dagger}\widehat{S_i}\widehat{T_{i1}}R_{\widehat{N_i}}\widehat{D_i}\widehat{B_i}^{\dagger} + L_{\widehat{A_i}}\widehat{T_{i2}} + \widehat{T_{i3}}R_{\widehat{B_i}},\\
W &= W_1 := \widehat{M_1}^{\dagger}\widehat{E_1}\widehat{D_1}^{\dagger} + \widehat{S_1}^{\dagger}\widehat{S_1}\widehat{C_1}^{\dagger}\widehat{E_1}\widehat{N_1}^{\dagger} + L_{\widehat{M_1}}L_{\widehat{S_1}}\widehat{T_{14}} + L_{\widehat{M_1}}\widehat{T_{11}}R_{\widehat{N_1}} + \widehat{T_{15}}R_{\widehat{D_1}},\\
W &= W_2 := \widehat{M_2}^{\dagger}\widehat{E_2}\widehat{D_2}^{\dagger} + \widehat{S_2}^{\dagger}\widehat{S_2}\widehat{C_2}^{\dagger}\widehat{E_2}\widehat{N_2}^{\dagger} + L_{\widehat{M_2}}L_{\widehat{S_2}}\widehat{T_{24}} + L_{\widehat{M_2}}\widehat{T_{21}}R_{\widehat{N_2}} + \widehat{T_{25}}R_{\widehat{D_2}},
\end{aligned}$$

where $T_{i1}, T_{i2}, ..., T_{i5}, \widehat{T_{i1}}, \widehat{T_{i2}}, ..., \widehat{T_{i5}}$ *are arbitrary matrices over* $\mathbb{H}$ *with fit size, and*

$$\begin{aligned}
\widehat{T_{14}} &= \begin{pmatrix} I_{m_1} \\ 0 \end{pmatrix}(\widehat{\widehat{A_1}}^{\dagger}(\widehat{\widehat{E_1}} - \widehat{\widehat{C_3}}\widehat{T_{11}}\widehat{\widehat{D_3}} - \widehat{\widehat{C_4}}\widehat{T_{21}}\widehat{\widehat{D_4}}) - T_7\widehat{\widehat{B_1}} + L_{\widehat{\widehat{A_1}}}T_6),\\
\widehat{T_{24}} &= \begin{pmatrix} 0 \\ I_{m_1} \end{pmatrix}(\widehat{\widehat{A_1}}^{\dagger}(\widehat{\widehat{E_1}} - \widehat{\widehat{C_3}}\widehat{T_{11}}\widehat{\widehat{D_3}} - \widehat{\widehat{C_4}}\widehat{T_{21}}\widehat{\widehat{D_4}}) - T_7\widehat{\widehat{B_1}} + L_{\widehat{\widehat{A_1}}}T_6),\\
\widehat{T_{15}} &= (R_{\widehat{\widehat{A_1}}}(\widehat{\widehat{E_1}} - \widehat{\widehat{C_3}}\widehat{T_{11}}\widehat{\widehat{D_3}} - \widehat{\widehat{C_4}}\widehat{T_{21}}\widehat{\widehat{D_4}})\widehat{\widehat{B_1}}^{\dagger} + \widehat{\widehat{A_1}}T_7 + T_8R_{\widehat{\widehat{B_1}}})\begin{pmatrix} I_{q_1} & 0 \end{pmatrix},\\
\widehat{T_{25}} &= (R_{\widehat{\widehat{A_1}}}(\widehat{\widehat{E_1}} - \widehat{\widehat{C_3}}\widehat{T_{11}}\widehat{\widehat{D_3}} - \widehat{\widehat{C_4}}\widehat{T_{21}}\widehat{\widehat{D_4}})\widehat{\widehat{B_1}}^{\dagger} + \widehat{\widehat{A_1}}T_7 + T_8R_{\widehat{\widehat{B_1}}})\begin{pmatrix} 0 & I_{q_1} \end{pmatrix},\\
\widehat{T_{11}} &= A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}ST_2R_NDB^{\dagger} + L_AT_4 + T_5R_B,\\
\widehat{T_{21}} &= M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_ST_1 + L_MT_2R_N + T_3R_D,
\end{aligned}$$

where $T_1, ..., T_8$ *are arbitrary matrices over* $\mathbb{H}$ *with fitting sizes.*

Proof. (1) ⇔ (2) : Split (1.4) to the two equations

$$A_1X_1B_1 + C_1Y_1D_1 + F_1Z_1G_1 + H_1W_1J_1 = E_1, \tag{3.26}$$

$$A_2X_2B_2 + C_2Y_2D_2 + F_2Z_2G_2 + H_2W_2J_2 = E_2. \tag{3.27}$$

It is clear that (1.4) is consistent if and only if (3.26) and (3.27) are consistent, respectively, and $W_1 = W_2$. As a consequence of Proposition 3.1, we have that (3.26) and (3.27) are solvable, respectively, if and only if

$$R_{M_i}R_{A_i}E_i = 0,\ E_iL_{B_i}L_{N_i} = 0,\ R_{C_i}E_1L_{B_i} = 0,$$
$$R_{\widehat{M_i}}R_{\widehat{A_i}}\widehat{E_i} = 0, \widehat{E_i}L_{\widehat{B_i}}L_{\widehat{N_i}} = 0, R_{\widehat{A_i}}\widehat{E_i}L_{\widehat{D_i}} = 0,\ R_{\widehat{C_i}}\widehat{E_i}L_{\widehat{B_i}} = 0,\ (i = 1, 2).$$

In this case, the general solution can be expressed as

$$X_i = A_i^\dagger \grave{E}_iB_i^\dagger - A_i^\dagger C_iM_i^\dagger \grave{E}_iB_i^\dagger - A_i^\dagger S_iC_i^\dagger \grave{E}_iN_i^\dagger D_iB_i^\dagger - A_i^\dagger S_iT_{i1}R_{N_i}D_iB_i^\dagger + L_{A_i}T_{i2} + T_{i3}R_{B_i},$$
$$Y_i = M_i^\dagger \grave{E}_1D_i^\dagger + S_i^\dagger S_iC_i^\dagger \grave{E}_1N_i^\dagger + L_{M_i}L_{S_i}T_{i4} + L_{M_i}T_{i1}R_{N_i} + T_{i5}R_{D_i},$$
$$Z_i = \widehat{A_i}^\dagger\widehat{E_i}\widehat{B_i}^\dagger - \widehat{A_i}^\dagger\widehat{C_i}\widehat{M_i}^\dagger\widehat{E_i}\widehat{B_i}^\dagger - \widehat{A_i}^\dagger\widehat{S_i}\widehat{C_i}^\dagger\widehat{E_i}\widehat{N_i}^\dagger\widehat{D_i}\widehat{B_i}^\dagger - \widehat{A_i}^\dagger\widehat{S_i}\widehat{T_{i1}}R_{\widehat{N_i}}\widehat{D_i}\widehat{B_i}^\dagger + L_{\widehat{A_i}}\widehat{T_{i2}} + \widehat{T_{i3}}R_{\widehat{B_i}},$$

$$\begin{aligned} W_1 &= \widehat{M_1}^\dagger\widehat{E_1}\widehat{D_1}^\dagger + \widehat{S_1}^\dagger\widehat{S_1}\widehat{C_1}^\dagger\widehat{E_1}\widehat{N_1}^\dagger + L_{\widehat{M_1}}L_{\widehat{S_1}}\widehat{T_{14}} + L_{\widehat{M_1}}\widehat{T_{11}}R_{\widehat{N_1}} + \widehat{T_{15}}R_{\widehat{D_1}},\\ W_2 &= \widehat{M_2}^\dagger\widehat{E_2}\widehat{D_2}^\dagger + \widehat{S_2}^\dagger\widehat{S_2}\widehat{C_2}^\dagger\widehat{E_2}\widehat{N_2}^\dagger + L_{\widehat{M_2}}L_{\widehat{S_2}}\widehat{T_{24}} + L_{\widehat{M_2}}\widehat{T_{21}}R_{\widehat{N_2}} + \widehat{T_{25}}R_{\widehat{D_2}}. \end{aligned} \tag{3.28}$$

By equating W_1 and W_2 in (3.28), we have the following matrix equation:

$$\widehat{\widehat{A_1}}\widehat{\widehat{X_1}} + \widehat{\widehat{X_2}}\widehat{\widehat{B_1}} + \widehat{\widehat{C_3}}\widehat{\widehat{X_3}}\widehat{\widehat{D_3}} + \widehat{\widehat{C_4}}\widehat{\widehat{X_4}}\widehat{\widehat{D_4}} = \widehat{\widehat{E_1}}, \tag{3.29}$$

where

$$\widehat{\widehat{X_1}} = \begin{pmatrix}\widehat{T_{14}}\\ \widehat{T_{24}}\end{pmatrix},\ \widehat{\widehat{X_2}} = \begin{pmatrix}\widehat{T_{15}}\\ \widehat{T_{25}}\end{pmatrix},\ \widehat{\widehat{X_3}} = \widehat{T_{11}},\ \widehat{\widehat{X_4}} = \widehat{T_{21}}.$$

Utilizing Corollary 3.2, we have that (3.29) is solvable if and only if

$$R_MR_AE = 0,\ EL_BL_N = 0,\ R_AEL_D = 0,\ R_CEL_B = 0.$$

In this case, the general solution can be expressed as

$$\widehat{\widehat{X_1}} = \widehat{\widehat{A_1}}^\dagger(\widehat{\widehat{E_1}} - \widehat{\widehat{C_3}}\widehat{\widehat{X_3}}\widehat{\widehat{D_3}} - \widehat{\widehat{C_4}}\widehat{\widehat{X_4}}\widehat{\widehat{D_4}}) - T_7\widehat{\widehat{B_1}} + L_{\widehat{\widehat{A_1}}}T_6,$$
$$\widehat{\widehat{X_2}} = R_{\widehat{\widehat{A_1}}}(\widehat{\widehat{E_1}} - \widehat{\widehat{C_3}}\widehat{\widehat{X_3}}\widehat{\widehat{D_3}} - \widehat{\widehat{C_4}}\widehat{\widehat{X_4}}\widehat{\widehat{D_4}})\widehat{\widehat{B_1}}^\dagger + \widehat{\widehat{A_1}}T_7 + T_8R_{\widehat{\widehat{B_1}}},$$
$$\widehat{\widehat{X_3}} = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger ST_2R_NDB^\dagger + L_AT_4 + T_5R_B,$$
$$\widehat{\widehat{X_4}} = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_ML_ST_1 + L_MT_2R_N + T_3R_D,$$

respectively, where $T_1, ..., T_8$ are facultative matrices over $\mathbb{H}$ with fitting sizes.

(2) ⇔ (3) : Utilizing Lemma 2.3, it is easy to check that

$$R_{M_i}R_{A_i}E_i = 0 \Leftrightarrow r\begin{pmatrix}A_i & E_i & C_i\end{pmatrix} = r\begin{pmatrix}A_i & C_i\end{pmatrix},$$

$$E_iL_{B_i}L_{N_i}=0\Leftrightarrow r\begin{pmatrix}B_i\\E_i\\D_i\end{pmatrix}=r\begin{pmatrix}B_i\\D_i\end{pmatrix},$$

$$R_{C_i}E_1L_{B_i}=0\Leftrightarrow r\begin{pmatrix}B_i&0\\E_i&C_i\end{pmatrix}=r\begin{pmatrix}B_i\end{pmatrix}+r\begin{pmatrix}C_i\end{pmatrix},$$

$$R_{\widehat{M}_i}R_{\widehat{A}_i}\widehat{E}_i=0\Leftrightarrow r\begin{pmatrix}F_i&E_i&H_i&A_1\\0&D_i&0&0\end{pmatrix}=r\begin{pmatrix}F_i&H_i&A_i\end{pmatrix}+r\begin{pmatrix}D_i\end{pmatrix},$$

$$\widehat{E}_iL_{\widehat{B}_i}L_{\widehat{N}_i}=0\Leftrightarrow r\begin{pmatrix}G_i&0\\E_i&A_i\\J_i&0\\D_i&0\end{pmatrix}=r\begin{pmatrix}G_i\\J_i\\D_i\end{pmatrix}+r\begin{pmatrix}A_i\end{pmatrix},$$

$$R_{\widehat{A}_i}\widehat{E}_iL_{\widehat{D}_i}=0\Leftrightarrow r\begin{pmatrix}E_i&F_i&A_i\\J_i&0&0\\D_i&0&0\end{pmatrix}=r\begin{pmatrix}F_i&A_i\end{pmatrix}+r\begin{pmatrix}J_i\\D_i\end{pmatrix},$$

$$R_{\widehat{C}_i}\widehat{E}_iL_{\widehat{B}_i}=0\Leftrightarrow r\begin{pmatrix}E_i&H_i&A_i\\G_i&0&0\\D_i&0&0\end{pmatrix}=r\begin{pmatrix}H_i&A_i\end{pmatrix}+r\begin{pmatrix}G_i\\D_i\end{pmatrix}.$$

Under the conditions (3.15), the system (1.4) is solvable and hence there is a special solution $(X_i^0, Y_i^0, Z_i^0, W^0)$.

$$\begin{aligned}R_MR_AE=0\quad&\Leftrightarrow r\begin{pmatrix}\widehat{\widehat{E}}_1&\widehat{\widehat{C}}_4&\widehat{\widehat{C}}_3&\widehat{\widehat{A}}_1\\\widehat{\widehat{B}}_1&0&0&0\end{pmatrix}=r(\widehat{\widehat{C}}_4\ \widehat{\widehat{C}}_3\ \widehat{\widehat{A}}_1)+r(\widehat{\widehat{B}}_1)\\
&\Leftrightarrow r\begin{pmatrix}W_2^0-W_1^0&-I&I&0&0\\I&0&0&\widehat{D_1}&0\\-I&0&0&0&\widehat{D_2}\\0&\widehat{M_2}&0&0&0\\0&0&\widehat{M_1}&0&0\end{pmatrix}=r\begin{pmatrix}-I&I\\\widehat{M_2}&0\\0&\widehat{M_1}\end{pmatrix}+r\begin{pmatrix}I&\widehat{D_1}&0\\-I&0&\widehat{D_2}\end{pmatrix}\\
&\Leftrightarrow r\begin{pmatrix}W_2^0-W_1^0&-I&I&0&0&0&0&0&0\\I&0&0&J_1&0&0&0&0&0\\-I&0&0&0&J_2&0&0&0&0\\0&H_2&0&0&0&F_2&0&A_2&0\\0&0&H_1&0&0&0&F_1&0&A_1\\0&0&0&D_1&0&0&0&0&0\\0&0&0&0&D_2&0&0&0&0\end{pmatrix}\\
&=r\begin{pmatrix}-I&I&0&0&0&0\\H_2&0&F_2&0&A_2&0\\0&H_1&0&F_1&0&A_1\end{pmatrix}+r\begin{pmatrix}I&J_1&0\\-I&0&J_2\\0&D_1&0\\0&0&D_2\end{pmatrix}\ \Leftrightarrow (3.22).\end{aligned}$$

$$EL_BL_N = 0 \iff r\begin{pmatrix} \widehat{\widehat{E_1}} & \widehat{\widehat{A_1}} \\ \widehat{\widehat{D_4}} & 0 \\ \widehat{\widehat{D_3}} & 0 \\ \widehat{\widehat{B_1}} & 0 \end{pmatrix} = r\begin{pmatrix} \widehat{\widehat{D_4}} \\ \widehat{\widehat{D_3}} \\ \widehat{\widehat{B_1}} \end{pmatrix} + r(\widehat{\widehat{A_1}})$$

$$\iff r\begin{pmatrix} W_2^0 - W_1^0 & I & -I & 0 & 0 \\ I & 0 & 0 & \widehat{N_1} & 0 \\ -I & 0 & 0 & 0 & \widehat{N_2} \\ 0 & \widehat{C_1} & 0 & 0 & 0 \\ 0 & 0 & \widehat{C_2} & 0 & 0 \\ 0 & \widehat{M_1} & 0 & 0 & 0 \\ 0 & 0 & \widehat{M_2} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} I & -I \\ \widehat{C_1} & 0 \\ 0 & \widehat{C_2} \\ \widehat{M_1} & 0 \\ 0 & \widehat{M_2} \end{pmatrix} + r\begin{pmatrix} I & \widehat{N_1} & 0 \\ -I & 0 & \widehat{N_2} \end{pmatrix}$$

$$\iff r\begin{pmatrix} W_2^0 - W_1^0 & I & -I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & J_1 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & J_2 & 0 & 0 \\ 0 & H_1 & 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & H_2 & 0 & 0 & 0 & A_2 \\ 0 & 0 & 0 & G_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_2 & 0 & 0 \\ 0 & 0 & 0 & D_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_2 & 0 & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} I & -I & 0 & 0 \\ H_1 & 0 & A_1 & 0 \\ 0 & H_2 & 0 & A_2 \end{pmatrix} + r\begin{pmatrix} I & J_1 & 0 \\ -I & 0 & J_2 \\ 0 & G_1 & 0 \\ 0 & 0 & G_2 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{pmatrix} \iff (3.23).$$

Similarly, it can carry out that $R_AEL_D = 0 \iff$ (3.24) and $R_CEL_B = 0 \iff$ (3.25). □

4. Main results

Let A_i, C_i, F_i, H_i, and $E_i = E_i^{\eta^*}$ be given matrices of feasible dimensions over $\mathbb{H}$ in (1.1). Set

$$M_i = R_{A_i}C_i,\ S_i = C_iL_{M_i},\ \widehat{A_i} = R_{A_i}F_i,\ \widehat{B_i} = (R_{C_i}F_i)^{\eta^*},\ \widehat{C_i} = R_{A_i}H_i,$$
$$\widehat{D_i} = (R_{C_i}H_i)^{\eta^*}, \widehat{M_i} = R_{\widehat{A_i}}\widehat{C_i},\ \widehat{N_i} = \widehat{D_i}L_{\widehat{B_i}},\ \widehat{S_i} = \widehat{C_i}L_{\widehat{M_i}},\ \widehat{E_i} = R_{A_i}E_i(R_{C_i})^{\eta^*},$$
$$\widehat{\widehat{A_1}} = \begin{pmatrix} L_{\widehat{M_1}}L_{\widehat{S_1}} & L_{\widehat{M_2}}L_{\widehat{S_2}} \end{pmatrix},\ \widehat{\widehat{B_1}} = \begin{pmatrix} R_{\widehat{D_1}} \\ -R_{\widehat{D_2}} \end{pmatrix},\ \widehat{\widehat{C_3}} = L_{\widehat{M_1}},\ \widehat{\widehat{D_3}} = R_{\widehat{N_1}},\ \widehat{\widehat{C_4}} = -L_{\widehat{M_2}},\ \widehat{\widehat{D_4}} = R_{\widehat{N_2}},$$
$$W_1^0 = \widehat{M_1}^{\dagger}\widehat{E_1}\widehat{D_1}^{\dagger} + \widehat{S_1}^{\dagger}\widehat{S_1}\widehat{C_1}^{\dagger}\widehat{E_1}\widehat{N_1}^{\dagger},\ W_2^0 = \widehat{M_2}^{\dagger}\widehat{E_2}\widehat{D_2}^{\dagger} + \widehat{S_2}^{\dagger}\widehat{S_2}\widehat{C_2}^{\dagger}\widehat{E_2}\widehat{N_2}^{\dagger},\ \widehat{\widehat{E_1}} = W_2^0 - W_1^0,$$
$$A = R_{\widehat{\widehat{A_1}}}\widehat{\widehat{C_3}},\ B = \widehat{\widehat{D_3}}L_{\widehat{\widehat{B_1}}},\ C = R_{\widehat{\widehat{A_1}}}\widehat{\widehat{C_4}},\ D = \widehat{\widehat{D_4}}L_{\widehat{\widehat{B_1}}},\ E = R_{\widehat{\widehat{A_1}}}\widehat{\widehat{E_1}}L_{\widehat{\widehat{B_1}}},$$
$$M = R_AC,\ N = DL_B,\ S = CL_M,\ \grave{E}_i = E_i - F_iZ_iF_i^{\eta^*} - H_iWH_i^{\eta^*}\ (i = 1, 2).$$

Theorem 4.1. *The following statements are equivalent:*
(1) (1.1) *is consistent.*
(2) The following Moore-Penrose inverse conditions are satisfied:

$$R_{M_i}R_{A_i}E_i=0,\ R_{C_i}E_1(R_{A_i})^{\eta^*}=0, R_{\widehat{M_i}}R_{\widehat{A_i}}\widehat{E_i}=0, \widehat{E_i}L_{\widehat{B_i}}L_{\widehat{N_i}}=0,\ R_{\widehat{A_i}}\widehat{E_i}L_{\widehat{D_i}}=0,$$
$$R_{\widehat{C_i}}\widehat{E_i}L_{\widehat{B_i}}=0, R_MR_AE=0,\ EL_BL_N=0,\ R_AEL_D=0,\ R_CEL_B=0.$$

(3)

$$r\begin{pmatrix}A_i & E_i & C_i\end{pmatrix}=r\begin{pmatrix}A_i & C_i\end{pmatrix},\ r\begin{pmatrix}A_i^{\eta^*} & 0\\ E_i & C_i\end{pmatrix}=r\begin{pmatrix}A_i\end{pmatrix}+r\begin{pmatrix}C_i\end{pmatrix},$$
$$r\begin{pmatrix}F_i & E_i & H_i & A_i\\ 0 & C_i^{\eta^*} & 0 & 0\end{pmatrix}=r\begin{pmatrix}F_i & H_i & A_i\end{pmatrix}+r\begin{pmatrix}C_i\end{pmatrix},$$
$$r\begin{pmatrix}F_i & E_i & H_i & C_i\\ 0 & A_i^{\eta^*} & 0 & 0\end{pmatrix}=r\begin{pmatrix}F_i & H_i & C_i\end{pmatrix}+r\begin{pmatrix}A_i\end{pmatrix},$$
$$r\begin{pmatrix}E_i & F_i & A_i\\ H_i^{\eta^*} & 0 & 0\\ C_i^{\eta^*} & 0 & 0\end{pmatrix}=r\begin{pmatrix}F_i & A_i\end{pmatrix}+r\begin{pmatrix}H_i & C_i\end{pmatrix},$$
$$r\begin{pmatrix}E_i & H_i & A_i\\ F_i^{\eta^*} & 0 & 0\\ C_i^{\eta^*} & 0 & 0\end{pmatrix}=r\begin{pmatrix}H_i & A_i\end{pmatrix}+r\begin{pmatrix}F_i & C_i\end{pmatrix},$$
$$r\begin{pmatrix}0 & H_1^{\eta^*} & H_2^{\eta^*} & 0 & 0 & 0 & 0\\ H_1 & -E_1 & 0 & A_1 & F_1 & 0 & 0\\ H_2 & 0 & E_2 & 0 & 0 & A_2 & F_2\\ 0 & C_1^{\eta^*} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & C_2^{\eta^*} & 0 & 0 & 0 & 0\end{pmatrix}=r\begin{pmatrix}H_1 & C_1 & 0\\ H_2 & 0 & C_2\end{pmatrix}+r\begin{pmatrix}H_1 & A_1 & F_1 & 0 & 0\\ H_2 & 0 & 0 & A_2 & F_2\end{pmatrix},$$
$$r\begin{pmatrix}0 & H_1^{\eta^*} & H_2^{\eta^*} & 0 & 0\\ H_1 & -E_1 & 0 & A_1 & 0\\ H_2 & 0 & E_2 & 0 & A_2\\ 0 & F_1^{\eta^*} & 0 & 0 & 0\\ 0 & C_1^{\eta^*} & 0 & 0 & 0\\ 0 & 0 & F_2^{\eta^*} & 0 & 0\\ 0 & 0 & C_2^{\eta^*} & 0 & 0\end{pmatrix}=r\begin{pmatrix}H_1 & F_1 & C_1 & 0 & 0\\ H_2 & 0 & 0 & F_2 & C_2\end{pmatrix}+r\begin{pmatrix}H_1 & A_1 & 0\\ H_2 & 0 & A_2\end{pmatrix},$$
$$r\begin{pmatrix}0 & H_1^{\eta^*} & H_2^{\eta^*} & 0 & 0 & 0\\ H_1 & -E_1 & 0 & A_1 & F_1 & 0\\ H_2 & 0 & E_2 & 0 & 0 & A_2\\ 0 & C_1^{\eta^*} & 0 & 0 & 0 & 0\\ 0 & 0 & C_2^{\eta^*} & 0 & 0 & 0\\ 0 & 0 & F_2^{\eta^*} & 0 & 0 & 0\end{pmatrix}=r\begin{pmatrix}H_1 & C_1 & 0 & 0\\ H_2 & 0 & C_2 & F_2\end{pmatrix}+r\begin{pmatrix}H_1 & A_1 & F_1 & 0\\ H_2 & 0 & 0 & A_2\end{pmatrix},$$

$$r\begin{pmatrix} 0 & H_1^{\eta^*} & H_2^{\eta^*} & 0 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & 0 & 0 \\ H_2 & 0 & E_2 & 0 & A_2 & F_2 \\ 0 & F_1^{\eta^*} & 0 & 0 & 0 & 0 \\ 0 & C_1^{\eta^*} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2^{\eta^*} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} H_1 & F_1 & C_1 & 0 \\ H_2 & 0 & 0 & C_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & 0 & 0 \\ H_2 & 0 & A_2 & F_2 \end{pmatrix}.$$

In this case, the general solution can be expressed as

$$X_i = \frac{1}{2}[\grave{X}_i + (\grave{X}_i)^{\eta^*}],\ Y_i = \frac{1}{2}[\grave{Y}_i + (\grave{Y}_i)^{\eta^*}],$$
$$Z_i = \frac{1}{2}[\grave{Z}_i + (\grave{Z}_i)^{\eta^*}],\ W = \frac{1}{2}[\grave{W} + (\grave{W})^{\eta^*}],$$

where

$$\begin{aligned}
\grave{X}_i =& A_i^\dagger \grave{E}_i (A_i^{\eta^*})^\dagger - A_i^\dagger C_i M_i^\dagger \grave{E}_i (A_i^{\eta^*})^\dagger - A_i^\dagger S_i C_i^\dagger \grave{E}_i N_i^\dagger D_i (A_i^{\eta^*})^\dagger \\
& - A_i^\dagger S_i T_{i1} R_{N_i} D_i (A_i^{\eta^*})^\dagger + L_{A_i} T_{i2} + T_{i3} R_{A_i^{\eta^*}}, \\
\grave{Y}_i =& M_i^\dagger \grave{E}_1 (C_i^{\eta^*})^\dagger + S_i^\dagger S_i C_i^\dagger \grave{E}_1 (M_i^{\eta^*})^\dagger + L_{M_i} L_{S_i} T_{i4} + L_{M_i} T_{i1} (L_{M_i})^{\eta^*} + T_{i5} R_{D_i}, \\
\grave{Z}_i =& \widehat{A_i}^\dagger \widehat{E_i} \widehat{B_i}^\dagger - \widehat{A_i}^\dagger \widehat{C_i} \widehat{M_i}^\dagger \widehat{E_i} \widehat{B_i}^\dagger - \widehat{A_i}^\dagger \widehat{S}_i \widehat{C_i}^\dagger \widehat{E_i} \widehat{N_i}^\dagger \widehat{D_i} \widehat{B_i}^\dagger \\
& - \widehat{A_i}^\dagger \widehat{S}_i \widehat{T_{i1}} R_{\widehat{N_i}} \widehat{D_i} \widehat{B_i}^\dagger + L_{\widehat{A_i}} \widehat{T_{i2}} + \widehat{T_{i3}} R_{\widehat{B_i}}, \\
\grave{W} =& \grave{W}_1 := \widehat{M_1}^\dagger \widehat{E_1} \widehat{D_1}^\dagger + \widehat{S}_1^\dagger \widehat{S}_1 \widehat{C_1}^\dagger \widehat{E_1} \widehat{N_1}^\dagger + L_{\widehat{M_1}} L_{\widehat{S_1}} \widehat{T_{14}} + L_{\widehat{M_1}} \widehat{T_{11}} R_{\widehat{N_1}} + \widehat{T_{15}} R_{\widehat{D_1}}, \\
\text{or } \grave{W} =& \grave{W}_2 := \widehat{M_2}^\dagger \widehat{E_2} \widehat{D_2}^\dagger + \widehat{S}_2^\dagger \widehat{S}_2 \widehat{C_2}^\dagger \widehat{E_2} \widehat{N_2}^\dagger + L_{\widehat{M_2}} L_{\widehat{S_2}} \widehat{T_{24}} + L_{\widehat{M_2}} \widehat{T_{21}} R_{\widehat{N_2}} + \widehat{T_{25}} R_{\widehat{D_2}},
\end{aligned}$$

where $T_{i1}, T_{i2}, ..., T_{i5}, \widehat{T_{i1}}, \widehat{T_{i2}}, ..., \widehat{T_{i5}}$, *are arbitrary matrices over* $\mathbb{H}$ *with fit size.*

Proof. First, we prove that the system (1.1) is consistent if and only if the system

$$\begin{aligned} A_1\grave{X}_1A_1^{\eta^*} + C_1\grave{Y}_1C_1^{\eta^*} + F_1\grave{Z}_1F_1^{\eta^*} + H_1\grave{W}H_1^{\eta^*} = E_1, \\ A_2\grave{X}_2A_2^{\eta^*} + C_2\grave{Y}_2C_2^{\eta^*} + F_2\grave{Z}_2F_2^{\eta^*} + H_2\grave{W}H_2^{\eta^*} = E_2 \end{aligned} \tag{4.1}$$

is consistent. If the system (1.1) has a solution, then it is clear that the system (4.1) is solvable.

On the contrary, suppose that the system (4.1) is consistent with the solution ($\grave{X}_i,\ \grave{Y}_i,\ \grave{Z}_i,\ \grave{W}$). Claim that $X_i := \frac{1}{2}[\grave{X}_i + (\grave{X}_i)^{\eta^*}],\ Y_i := \frac{1}{2}[\grave{Y}_i + (\grave{Y}_i)^{\eta^*}],\ Z_i := \frac{1}{2}[\grave{Z}_i + (\grave{Z}_i)^{\eta^*}],\ W := \frac{1}{2}[\grave{W} + (\grave{W})^{\eta^*}]$ are solutions of (1.1). It is clear that $X_i,\ Y_i,\ Z_i,\ W$ are η^*-Hermitian matrices. Second, we apply Theorem 3.3 to obtain the solvability conditions and the general solution to (1.1). □

Let A_1, C_1, F_1, H_1, and $E_1 = E_1^{\eta^*}$ be given matrices over $\mathbb{H}$ in (1.2). Set

$$M_1 = R_{A_1}C_1,\ S_1 = C_1L_{M_1},\ \widehat{A_1} = R_{A_1}F_1,\ \widehat{B_1} = (R_{C_1}F_1)^{\eta^*},\ \widehat{C_1} = R_{A_1}H_1,$$
$$\widehat{D_1} = (R_{C_1}H_1)^{\eta^*}, \widehat{M_1} = R_{\widehat{A_1}}\widehat{C_1},\ \widehat{N_1} = \widehat{D_1}L_{\widehat{B_1}},\ \widehat{S}_1 = \widehat{C_1}L_{\widehat{M_1}},\ \widehat{E_1} = R_{A_1}E_1(R_{C_1})^{\eta^*}.$$

Corollary 4.2. *The following statements are equivalent:*
(1) (1.2) *is solvable.*
(2)

$$R_{M_1}R_{A_1}E_1 = 0,\ R_{C_1}E_1(R_{A_1})^{\eta^*} = 0, R_{\widehat{M_1}}R_{\widehat{A_i}}\widehat{E_1} = 0,$$
$$R_{\widehat{A_1}}\widehat{E_1}L_{\widehat{D_1}} = 0, R_{\widehat{C_1}}\widehat{E_1}L_{\widehat{B_1}} = 0, \widehat{E_1}L_{\widehat{B_1}}L_{\widehat{N_1}} = 0.$$

(3)

$$r\begin{pmatrix} A_1 & E_1 & C_1 \end{pmatrix} = r\begin{pmatrix} A_1 & C_1 \end{pmatrix},\ r\begin{pmatrix} A_1^{\eta^*} & 0 \\ E_1 & C_1 \end{pmatrix} = r\begin{pmatrix} A_1 \end{pmatrix} + r\begin{pmatrix} C_1 \end{pmatrix},$$
$$r\begin{pmatrix} F_1 & E_1 & H_1 & A_1 \\ 0 & C_1^{\eta^*} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_1 & H_1 & A_1 \end{pmatrix} + r\begin{pmatrix} C_1 \end{pmatrix},$$
$$r\begin{pmatrix} F_1 & E_1 & H_1 & C_1 \\ 0 & A_1^{\eta^*} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_1 & H_1 & C_1 \end{pmatrix} + r\begin{pmatrix} A_1 \end{pmatrix},$$
$$r\begin{pmatrix} E_1 & F_1 & A_1 \\ H_1^{\eta^*} & 0 & 0 \\ C_1^{\eta^*} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_1 & A_1 \end{pmatrix} + r\begin{pmatrix} H_1 & C_1 \end{pmatrix},$$
$$r\begin{pmatrix} E_1 & H_1 & A_1 \\ F_1^{\eta^*} & 0 & 0 \\ C_1^{\eta^*} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} H_1 & A_1 \end{pmatrix} + r\begin{pmatrix} F_1 & C_1 \end{pmatrix}.$$

In this case, the general solution to the system can be expressed as

$$X_1 = \frac{1}{2}[\grave{X}_1 + (\grave{X}_1)^{\eta^*}],\ Y_1 = \frac{1}{2}[\grave{Y}_1 + (\grave{Y}_1)^{\eta^*}],\ Z_1 = \frac{1}{2}[\grave{Z}_1 + (\grave{Z}_1)^{\eta^*}],\ W = \frac{1}{2}[\grave{W} + (\grave{W})^{\eta^*}],$$

where

$$\begin{aligned}
\grave{X}_1 =& A_1^\dagger \grave{E}_1(A_1^{\eta^*})^\dagger - A_1^\dagger C_1 M_1^\dagger \grave{E}_1(A_1^{\eta^*})^\dagger - A_1^\dagger S_1 C_1^\dagger \grave{E}_1 N_1^\dagger D_1(A_1^{\eta^*})^\dagger \\
&- A_1^\dagger S_1 T_{11} R_{N_1} D_1(A_1^{\eta^*})^\dagger + L_{A_1}T_{12} + T_{13}R_{A_1^{\eta^*}},\\
\grave{Y}_1 =& M_1^\dagger \grave{E}_1(C_1^{\eta^*})^\dagger + S_1^\dagger S_1 C_1^\dagger \grave{E}_1(M_1^{\eta^*})^\dagger + L_{M_1}L_{S_1}T_{14} + L_{M_1}T_{11}(L_{M_1})^{\eta^*} + T_{15}R_{D_1},\\
\grave{Z}_1 =& \widehat{A_1}^\dagger \widehat{E_1}\widehat{B_1}^\dagger - \widehat{A_1}^\dagger \widehat{C_1}\widehat{M_1}^\dagger \widehat{E_1}\widehat{B_1}^\dagger - \widehat{A_1}^\dagger \widehat{S_1}\widehat{C_1}^\dagger \widehat{E_1}\widehat{N_1}^\dagger \widehat{D_1}\widehat{B_1}^\dagger \\
&- \widehat{A_1}^\dagger \widehat{S_1}\widehat{T_{11}}R_{\widehat{N_1}}\widehat{D_1}\widehat{B_1}^\dagger + L_{\widehat{A_1}}\widehat{T_{12}} + \widehat{T_{13}}R_{\widehat{B_1}},\\
\grave{W} =& \grave{W}_1 := \widehat{M_1}^\dagger \widehat{E_1}\widehat{D_1}^\dagger + \widehat{S_1}^\dagger \widehat{S_1}\widehat{C_1}^\dagger \widehat{E_1}\widehat{N_1}^\dagger + L_{\widehat{M_1}}L_{\widehat{S_1}}\widehat{T_{14}} + L_{\widehat{M_1}}\widehat{T_{11}}R_{\widehat{N_1}} + \widehat{T_{15}}R_{\widehat{D_1}}.
\end{aligned}$$

Proof. Apply Theorem 4.1 whenever $A_2 = 0, C_2 = 0, F_2 = 0, H_2 = 0, E_2 = 0$. □

5. Algorithms and numerical examples

Algorithm 5.1. *Calculate the general solution to the system* (1.4).

1) ***Input*** *the system* (1.4) *with coefficients* A_k, B_k, C_k, D_k, F_k, G_k, H_k, J_k ($k = 1,2$) *with viable dimensions over* $\mathbb{H}$.
2) *Compute all matrices, which appeared in (3.1)–(3.3).*
3) *Check whether the generalized inverses conditions in Theorem 3.3 and the ranks conditions* (3.17)–(3.25) *are satisfied or not. If not, return to "The system* (1.4) *is inconsistent".*
4) *Else compute* X_k, Y_k, Z_k, W *by Theorem 3.3.*
5) ***Output*** *the general solution of the system* (1.4) *is* X_k, Y_k, Z_k, W.

Example 5.1. *Let*

$$A_1 = \begin{pmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & -\mathbf{2k} \end{pmatrix},\ A_2 = \begin{pmatrix} \mathbf{0} & \mathbf{3-i} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ B_1 = \begin{pmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ B_2 = \begin{pmatrix} \mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$
$$C_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{k} \end{pmatrix},\ C_2 = \begin{pmatrix} \mathbf{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ D_1 = \begin{pmatrix} \mathbf{j} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},\ D_2 = \begin{pmatrix} \mathbf{1} & \mathbf{3k} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix},$$
$$F_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{j} & \mathbf{1} \end{pmatrix},\ F_2 = \begin{pmatrix} \mathbf{i} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ G_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{j} & \mathbf{0} \end{pmatrix},\ G_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1+i} & \mathbf{0} \end{pmatrix},$$
$$H_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{j} \end{pmatrix},\ H_2 = \begin{pmatrix} \mathbf{1} & \mathbf{j} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ J_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{k} & \mathbf{0} \end{pmatrix},\ J_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{1+j} & \mathbf{0} \end{pmatrix},$$
$$E_1 = \begin{pmatrix} -\mathbf{i} & \mathbf{0} \\ \mathbf{i+2k} & \mathbf{0} \end{pmatrix},\ E_2 = \begin{pmatrix} \mathbf{2+2i+j+k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

By straight calculations using the quaternion package on MATLAB software, we can find

$$r\begin{pmatrix} A_i & E_i & C_i \end{pmatrix} = r\begin{pmatrix} A_i & C_i \end{pmatrix} = \begin{cases} 2, & if\ i = 1, \\ 1, & if\ i = 2, \end{cases}$$
$$r\begin{pmatrix} B_i \\ E_i \\ D_i \end{pmatrix} = r\begin{pmatrix} B_i \\ D_i \end{pmatrix} = 2,\ r\begin{pmatrix} B_i & 0 \\ E_i & C_i \end{pmatrix} = r\begin{pmatrix} B_i \end{pmatrix} + r\begin{pmatrix} C_i \end{pmatrix} = 2,$$
$$r\begin{pmatrix} F_i & E_i & H_i & A_1 \\ 0 & D_i & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_i & H_i & A_i \end{pmatrix} + r\begin{pmatrix} D_i \end{pmatrix} = \begin{cases} 4, & if\ i = 1, \\ 3, & if\ i = 2, \end{cases}$$
$$r\begin{pmatrix} G_i & 0 \\ E_i & A_i \\ J_i & 0 \\ D_i & 0 \end{pmatrix} = r\begin{pmatrix} G_i \\ J_i \\ D_i \end{pmatrix} + r\begin{pmatrix} A_i \end{pmatrix} = \begin{cases} 4, & if\ i = 1, \\ 3, & if\ i = 2, \end{cases}$$
$$r\begin{pmatrix} E_i & F_i & A_i \\ J_i & 0 & 0 \\ D_i & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_i & A_i \end{pmatrix} + r\begin{pmatrix} J_i \\ D_i \end{pmatrix} = \begin{cases} 4, & if\ i = 1, \\ 3, & if\ i = 2, \end{cases}$$
$$r\begin{pmatrix} E_i & H_i & A_i \\ G_i & 0 & 0 \\ D_i & 0 & 0 \end{pmatrix} = r\begin{pmatrix} H_i & A_i \end{pmatrix} + r\begin{pmatrix} G_i \\ D_i \end{pmatrix} = \begin{cases} 4, & if\ i = 1, \\ 3, & if\ i = 2, \end{cases}$$

$$r\begin{pmatrix} 0 & J_1 & J_2 & 0 & 0 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & F_1 & 0 & 0 \\ H_2 & 0 & E_2 & 0 & 0 & A_2 & F_2 \\ 0 & D_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} J_1 & J_2 \\ D_1 & 0 \\ 0 & D_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & F_1 & 0 & 0 \\ H_2 & 0 & 0 & A_2 & F_2 \end{pmatrix} = 7,$$

$$r\begin{pmatrix} 0 & J_1 & J_2 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & 0 \\ H_2 & 0 & E_2 & 0 & A_2 \\ 0 & G_1 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 \\ 0 & 0 & D_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} J_1 & J_2 \\ G_1 & 0 \\ D_1 & 0 \\ 0 & G_2 \\ 0 & D_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & 0 \\ H_2 & 0 & A_2 \end{pmatrix} = 7,$$

$$r\begin{pmatrix} 0 & J_1 & J_2 & 0 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & F_1 & 0 \\ H_2 & 0 & E_2 & 0 & 0 & A_2 \\ 0 & D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} J_1 & J_2 \\ D_1 & 0 \\ 0 & D_2 \\ 0 & G_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & F_1 & 0 \\ H_2 & 0 & 0 & A_2 \end{pmatrix} = 7,$$

$$r\begin{pmatrix} 0 & J_1 & J_2 & 0 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & 0 & 0 \\ H_2 & 0 & E_2 & 0 & A_2 & F_2 \\ 0 & G_1 & 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} J_1 & J_2 \\ G_1 & 0 \\ D_1 & 0 \\ 0 & D_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & F_1 & 0 \\ H_2 & 0 & 0 & A_2 \end{pmatrix} = 7.$$

Consequently, the system (1.4) *is consistent. Moreover, it is easy to check that*

$$X_1 = \begin{pmatrix} \mathbf{i} & \mathbf{1} \\ \mathbf{0} & \mathbf{j} \end{pmatrix},\ X_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{k} \end{pmatrix},\ Y_1 = \begin{pmatrix} \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},\ Y_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{2-i} & \mathbf{0} \end{pmatrix},$$
$$Z_1 = \begin{pmatrix} \mathbf{-i} & \mathbf{2k} \\ \mathbf{j} & \mathbf{0} \end{pmatrix},\ Z_2 = \begin{pmatrix} \mathbf{0} & \mathbf{j} \\ \mathbf{0} & \mathbf{k} \end{pmatrix},\ W = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{i} \end{pmatrix}.$$

Algorithm 5.2. *Calculate the general solution to the system* (1.1).

1) ***Input*** *the system* (1.1) *with coefficients* A_k, C_k, F_k, H_k, E_k $(k = 1, 2)$ *with viable dimensions over* $\mathbb{H}$.
2) *Compute all matrices coefficients in* (1.1) *by Theorem 4.1.*
3) *Check whether the generalized inverses conditions and the ranks conditions in Theorem 4.1 are satisfied or not. If not, return to "The system* (1.1) *is inconsistent".*
4) *Else compute* X_k, Y_k, Z_k, W *by Theorem 4.1 whenever* $\eta = \mathbf{k}$.
5) ***Output*** *the general solution of the system* (1.4), *which is* $X_k = X_k^{\eta^*}$, $Y_k = Y_k^{\eta^*}$, $Z_k = Z_k^{\eta^*}$, $W = W^{\eta^*}$.

Example 5.2. *Consider the system* (1.1), *whenever* $\eta = \mathbf{k}$, *where*

$$A_1 = C_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{j} \end{pmatrix},\ A_2 = C_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{pmatrix},\ F_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{k} & \mathbf{-2} \end{pmatrix},\ F_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{-i} \end{pmatrix},$$

$$H_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{j-1} & \mathbf{2} \end{pmatrix},\ H_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{k-1} & \mathbf{-1} \end{pmatrix},\ E_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2+9j} \end{pmatrix},\ E_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{pmatrix}.$$

By straight calculations using the quaternion package on MATLAB software, we can find

$$r\begin{pmatrix} A_i & E_i & C_i \end{pmatrix} = r\begin{pmatrix} A_i & C_i \end{pmatrix} = 1,\ r\begin{pmatrix} A_i^{\eta^*} & 0 \\ E_i & C_i \end{pmatrix} = r\begin{pmatrix} A_i \end{pmatrix} + r\begin{pmatrix} C_i \end{pmatrix} = 2,$$

$$r\begin{pmatrix} F_i & E_i & H_i & A_i \\ 0 & C_i^{\eta^*} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_i & H_i & A_i \end{pmatrix} + r\begin{pmatrix} C_i \end{pmatrix} = 2,$$

$$r\begin{pmatrix} F_i & E_i & H_i & C_i \\ 0 & A_i^{\eta^*} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_i & H_i & C_i \end{pmatrix} + r\begin{pmatrix} A_i \end{pmatrix} = 2,$$

$$r\begin{pmatrix} E_i & F_i & A_i \\ H_i^{\eta^*} & 0 & 0 \\ C_i^{\eta^*} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} F_i & A_i \end{pmatrix} + r\begin{pmatrix} H_i & C_i \end{pmatrix} = 2,$$

$$r\begin{pmatrix} E_i & H_i & A_i \\ F_i^{\eta^*} & 0 & 0 \\ C_i^{\eta^*} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} H_i & A_i \end{pmatrix} + r\begin{pmatrix} F_i & C_i \end{pmatrix} = 2,$$

$$r\begin{pmatrix} 0 & H_1^{\eta^*} & H_2^{\eta^*} & 0 & 0 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & F_1 & 0 & 0 \\ H_2 & 0 & E_2 & 0 & 0 & A_2 & F_2 \\ 0 & C_1^{\eta^*} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2^{\eta^*} & 0 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} H_1 & C_1 & 0 \\ H_2 & 0 & C_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & F_1 & 0 & 0 \\ H_2 & 0 & 0 & A_2 & F_2 \end{pmatrix} = 4,$$

$$r\begin{pmatrix} 0 & H_1^{\eta^*} & H_2^{\eta^*} & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & 0 \\ H_2 & 0 & E_2 & 0 & A_2 \\ 0 & F_1^{\eta^*} & 0 & 0 & 0 \\ 0 & C_1^{\eta^*} & 0 & 0 & 0 \\ 0 & 0 & F_2^{\eta^*} & 0 & 0 \\ 0 & 0 & C_2^{\eta^*} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} H_1 & F_1 & C_1 & 0 & 0 \\ H_2 & 0 & 0 & F_2 & C_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & 0 \\ H_2 & 0 & A_2 \end{pmatrix} = 4,$$

$$r\begin{pmatrix} 0 & H_1^{\eta^*} & H_2^{\eta^*} & 0 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & F_1 & 0 \\ H_2 & 0 & E_2 & 0 & 0 & A_2 \\ 0 & C_1^{\eta^*} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2^{\eta^*} & 0 & 0 & 0 \\ 0 & 0 & F_2^{\eta^*} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} H_1 & C_1 & 0 & 0 \\ H_2 & 0 & C_2 & F_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & F_1 & 0 \\ H_2 & 0 & 0 & A_2 \end{pmatrix} = 4,$$

$$r\begin{pmatrix} 0 & H_1^{\eta^*} & H_2^{\eta^*} & 0 & 0 & 0 \\ H_1 & -E_1 & 0 & A_1 & 0 & 0 \\ H_2 & 0 & E_2 & 0 & A_2 & F_2 \\ 0 & F_1^{\eta^*} & 0 & 0 & 0 & 0 \\ 0 & C_1^{\eta^*} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2^{\eta^*} & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} H_1 & F_1 & C_1 & 0 \\ H_2 & 0 & 0 & C_2 \end{pmatrix} + r\begin{pmatrix} H_1 & A_1 & 0 & 0 \\ H_2 & 0 & A_2 & F_2 \end{pmatrix} = 4.$$

Consequently, the system (1.1) *is consistent. Moreover, we have the general solution as*

$$X_1 = X_1^{\mathbf{k}^*} = \begin{pmatrix} \mathbf{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{pmatrix}, \; X_2 = X_2^{\mathbf{k}^*} = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{j} & -\mathbf{i} \end{pmatrix},$$
$$Y_1 = Y_1^{\mathbf{k}^*} = \begin{pmatrix} -4\mathbf{j} & -6\mathbf{j} \\ -6\mathbf{j} & -9\mathbf{j} \end{pmatrix}, \; Y_2 = Y_2^{\mathbf{k}^*} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{j} \end{pmatrix},$$
$$Z_1 = Z_1^{\mathbf{k}^*} = \begin{pmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \; Z_2 = Z_2^{\mathbf{k}^*} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{pmatrix}, \; W = W^{\mathbf{k}^*} = \begin{pmatrix} \mathbf{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

6. Conclusions

The motivation for this work stems from established literature on the consistency conditions and general solutions of linear matrix equations, particularly the Sylvester-type matrix equation (1.5). We have established the solvability conditions for the quaternion matrix equations (1.3), and hence the necessary and sufficient condition of (1.4) to be solvable. The general solution form of the two-sided Sylvester-type quaternion system of matrix equation (1.1) of seven η-Hermitian variables has been investigated, when the solvability conditions are met. Finally, algorithms and numerical examples have been used to check the validity of the main results.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Mahmoud S. Mehany and Faizah D. Alanazi: Conceptualization, formal analysis, investigation, methodology, software, validation, writing the original draft, review, and editing. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at Ain Shams University, Faculty of Education, Mathematics Department, Roxy, Cairo, Egypt. Moreover, the authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, Kingdom of Saudi Arabia, for funding this research work through the project number "NBU-FPEJ-2025−1102−01".

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. T. Jiang, Z. Guo, D. Zhang, V. I. Vasilev, A fast algorithm for the Schrödinger equation in quaternionic quantum mechanics, *Appl. Math. Lett.*, **150** (2024), 108975. https://doi.org/10.1016/j.aml.2023.108975
2. Z. Guo, T. Jiang, G. Wang, V. I. Vasilev, An efficient algorithm for the eigenvalue problem of a Hermitian quaternion matrix in quantum chemistry, *J. Comput. Appl. Math.*, **463** (2025), 116516. https://doi.org/10.1016/j.cam.2025.116516
3. T. Jiang, Z. Jiang, Z. Zhang, Two novel algebraic techniques for quaternion least squares problems in quaternionic quantum mechanics, *Adv. Appl. Clifford Algebr.*, **26** (2016), 169–182. https://doi.org/10.1007/s00006-015-0581-6
4. L. Rodman, *Topics in quaternion linear algebra*, Princeton University Press, 2014.
5. C. C. Took, D. P. Mandic, F. Zhang, On the unitary diagonalisation of a special class of quaternion matrices, *Appl. Math. Lett.*, **24** (2011), 1806–1809. https://doi.org/10.1016/j.aml.2011.04.038
6. C. C. Took, D. P. Mandic, Quaternion-valued stochastic gradient-based adaptive IIR filtering, *IEEE Trans. Signal Process.*, **58** (2010), 3895–3901. https://doi.org/10.1109/TSP.2010.2047719
7. C. C. Took, D. P. Mandic, Augmented second-order statistics of quaternion random signals, *Signal Process.*, **91** (2011), 214–224. https://doi.org/10.1016/j.sigpro.2010.06.024
8. C. C. Took, D. P. Mandic, The quaternion LMS algorithm for adaptive filtering of hypercomplex processes, *IEEE Trans. Signal Process.*, **57** (2009), 1316–1327. https://doi.org/10.1109/TSP.2008.2010600
9. W. R. Hamilton, II. On quaternions; or on a new system of imaginaries in algebra, *Lond Edinb. Dublin Philos. Mag. J. Sci.*, **25** (1844), 10–13. https://doi.org/10.1080/14786444408644923
10. J. H. Conway, D. A. Smith, *On quaternions and octonions: Their geometry, arithmetic, and symmetry*, 1st Eds., New York: A K Peters/CRC Press, 2003. https://doi.org/10.1201/9781439864180
11. G. Kamberov, P. Norman, F. Pedit, U. Pinkall, *Quaternions, spinors, and surfaces*, American Mathematical Society, **299** (2002).
12. G. Nebe, Finite quaternionic matrix groups, *Represent. Theory*, **2** (1998), 106–223.
13. J. P. Ward, *Quaternions and Cayley numbers: Algebra and applications*, Dordrecht: Kluwer Academic Publishers, 1997.
14. A. Dmytryshyn, B. Kågström, Coupled Sylvester-type matrix equations and block diagonalization, *SIAM J. Matrix Anal. Appl.*, **36** (2015), 580–593. https://doi.org/10.1137/151005907
15. V. L. Syrmos, F. L. Lewis, Output feedback eigenstructure assignment using two Sylvester equations, *IEEE Trans. Automat. Control*, **38** (1993), 495–499. https://doi.org/10.1109/9.210155
16. Y. Zhang, D. Jiang, J. Wang, A recurrent neural network for solving Sylvester equation with time-varying coefficients, *IEEE Trans. Neural Netw.*, **13** (2002), 1053–1063. https://doi.org/10.1109/TNN.2002.1031938

17. A. Varga, Robust pole assignment via Sylvester equation based state feedback parametrization, In: *IEEE International symposium on computer-aided control system design (Cat. No.00TH8537)*, Anchorage: IEEE, 2002. https://doi.org/10.1109/CACSD.2000.900179

18. J. K. Baksalary, R. Kala, The matrix equation $AXB + CYD = E$, *Linear Algebra Appl.*, **30** (1980), 141–147. https://doi.org/10.1016/0024-3795(80)90189-5

19. A. B. Özgüler, The equation $AXB + CYD = E$ over a principal ideal Domain, *SIAM J. Matrix Anal. Appl.*, **12** (1991), 581–591. https://doi.org/10.1137/0612044

20. Y. Tian, The solvability of two linear matrix equations, *Linear Multilinear Algebra*, **48** (2000), 123–147. https://doi.org/10.1080/03081080008818664

21. Q. W. Wang, A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity, *Linear Algebra Appl.*, **384** (2004), 43–54. https://doi.org/10.1016/j.laa.2003.12.039

22. Z. H. He, A system of Coupled quaternion matrix equations with seven unknowns and its applications, *Adv. Appl. Clifford Algebras*, **29** (2019), 38. https://doi.org/10.1007/s00006-019-0955-2

23. J. J. Sylvester, Sur léquation en matrices px= xq, *CR Acad. Sci. Paris*, **99** (1884), 67–71.

24. M. S. Mehany, An analysis to some systems of matrix equations with ϕ-Hermitian solutions for some nonstandard involution ϕ over the real quaternion algebra, *Iran. J. Sci.*, 2025. https://doi.org/10.1007/s40995-024-01758-x

25. M. S. Mehany, Q. W. Wang, L. S. Liu, A system of Sylvester-like quaternion tensor equations with an application, *Front. Math.*, **19** (2024), 749–768. https://doi.org/10.1007/s11464-021-0389-8

26. M. S. Mehany, Q. W. Wang, Three symmetrical systems of coupled Sylvester-like quaternion matrix equations, *Symmetry*, **14** (2022), 550. https://doi.org/10.3390/sym14030550

27. M. Y. Xie, Q. W. Wang, Z. H. He, M. M. Saad, A system of Sylvester-type quaternion matrix equations with ten variables, *Acta. Math. Sin. English Ser.*, **38** (2022), 1399–1420. https://doi.org/10.1007/s10114-022-9040-1

28. L. S. Liu, Q. W. Wang, M. S. Mehany, A Sylvester-type matrix equation over the Hamilton quaternions with an application, *Mathematics*, **10** (2022), 1758. https://doi.org/10.3390/math10101758

29. L. S. Liu, S. Zhang, H. X. Chang, Four symmetric systems of the matrix equations with an application over the Hamilton quaternions, *AIMS Mathematics*, **9** (2024), 33662–33691. https://doi.org/10.3934/math.20241607

30. X. L. Xu, Q. W. Wang, The consistency and the general common solution to some quaternion matrix equations, *Ann. Funct. Anal.*, **14** (2023), 53. https://doi.org/10.1007/s43034-023-00276-y

31. Z. H. He, J. Liu, T. Y. Tam, The general ϕ-Hermitian solution to mixed pairs of quaternion matrix Sylvester equations, *Electron. J. Linear Algebra*, **32** (2017), 475–499. https://doi.org/10.13001/1081-3810.3606

32. Z. H. He, Some new results on a system of Sylvester-type quaternion matrix equations, *Linear Multilinear Algebra*, **69** (2021), 3069–3091. https://doi.org/10.1080/03081087.2019.1704213

33. I. I. Kyrchei, Determinantal representations of general and (Skew-)Hermitian solutions to the generalized Sylvester-type quaternion matrix equation, *Abstr. Appl. Anal.*, **2019** (2019), 5926832. https://doi.org/10.1155/2019/5926832

34. I. Kyrchei, Cramer's Rules of η-(skew-)Hermitian solutions to the quaternion Sylvester-type matrix equations, *Adv. Appl. Clifford Algebras*, **29** (2019), 56. https://doi.org/10.1007/s00006-019-0972-1

35. A. Rehman, I. Kyrchei, M. Z. U. Rahman, V. Leiva, C. Castro, Solvability and algorithm for Sylvester-type quaternion matrix equations with potential applications, *AIMS Mathematics*, **9** (2024), 19967–19996. http://dx.doi.org/10.3934/math.2024974

36. A. M. E. Bayoumi, A shift-splitting Jacobi-gradient iterative algorithm for solving the matrix equation $AV - \overline{V}B = C$, *Opti. Contr. Appl. Met.*, **45** (2024), 1593–1602. https://doi.org/10.1002/oca.3112

37. S. K. Mitra, A pair of simultaneous linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ and a matrix programming problem, *Linear Algebra Appl.*, **131** (1990), 107–123. https://doi.org/10.1016/0024-3795(90)90377-O

38. G. Matsaglia, G. P. H. Styan, Equalities and inequalities for ranks of matrices, *Linear Multilinear Algebra*, **2** (1974), 269–292. https://doi.org/10.1080/03081087408817070

39. Q. W. Wang, Z. H. He, Some matrix equations with applications, *Linear Multilinear Algebra*, **60** (2012), 1327–1353. https://doi.org/10.1080/03081087.2011.648635

AIMS Press