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Research article

Positive periodic stability for a neutral-type host-macroparasite equation

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Abstract: In this work, we study positive periodic solutions of a neutral-type host-macroparasite equation and establish the existence results of positive periodic solutions by using topological degree theory. Furthermore, based on the Lyapunov functional method and differential inequality analysis strategies, the dynamic behaviors of the host-macroparasite model are obtained. Finally, we present a numerical example to verify the effectiveness of the obtained results. It should be pointed out that the properties of neutral operators have significant applications in the proof. Our results have extended existing findings for host-macroparasite equation.

Keywords: positive periodic solution; existence; global exponential stability; topological degree theory
Mathematics Subject Classification: 34K14, 34K20

1. Introduction

In general, a neutral-type functional differential equation (NFDE) is one in which the derivatives of the past history or derivatives of functions of the past history are involved, as well as the present state of the system. In [1], Hale and Verduyn Lunel introduced the definition and basic theories of NFDE. NFDE has wide applications in many areas, including non-destructive transmission line systems, neural networks, population models; see [2–12] and related papers.

In 1995, May and Anderson [13] first studied the host-macroparasite model as follows:

$$x'(t) = -ax(t) + \frac{bx(t)}{[1 + cx(t - \tau)]^{N+1}}, \tag{1.1}$$

where $x(t)$ denotes the number of sexually mature worms in the human community. The means of other parameters can be found in [13]. Elabbasy et al. [14] further considered the oscillation properties of Eq

(1.1). Saker and Alzabut [15] established the existence, global attractivity, and oscillation of a positive periodic solution to an impulsive delay host-macroparasite model by using the continuation theorem of coincidence degree and the Lyapunov functional method. Yao [16] studied the existence and global exponential stability of an almost periodic solution for Eq (1.1) on time scales by using the contraction mapping fixed point theorem, exponential dichotomy, and Gronwall inequality. Yao [17] also studied the existence and exponential stability of almost periodic solutions for a difference host-macroparasite equation. Due to being in a changing environment, the coefficients of a population model should be continuously changing functions. To be more accurate, the coefficients and delays in population models can be periodically time-varying. To the best of the authors' knowledge, there are few results of positive periodic solutions for the neutral-type host-macroparasite model.

Motivated by the above work, in the present paper, we study the following nonautonomous neutral-type host-macroparasite equation with multiply time-varying delays:

$$\big(x(t) - cx(t-\tau)\big)' = -a(t)x(t) + \sum_{i=1}^{n} \frac{b_i(t)x(t-\gamma_i(t))}{[1 + d_i(t)x(t-\gamma_i(t))]^{N_i+1}}, \tag{1.2}$$

where $|c| \neq 1$ and $\tau, N_i > 0$ are constants, $a(t), b_i(t), d_i(t), \gamma_i(t) > 0$ are T-periodic functions.

We list the main innovations of this paper:

(1) We first study positive periodic stability for a neutral-type host-macroparasite equation and generalize the existing results for host-macroparasite model. A neutral-type equation has richer dynamic behaviors.

(2) We develop topological degree theory for studying positive periodic stability. The research method in this article can also be used to study other types of neutral equations.

(3) We innovatively use the properties of neutral-type operators for studying host-macroparasite models.

The remaining framework of this paper is organized as follows: We study the existence of positive periodic solutions of Eq (1.2) in Section 2. Section 3 gives the globally asymptotic and exponential stability of Eq (1.2). Section 4 gives a numerical example for verifying our results. We draw some conclusions in Section 5.

In the whole paper, we use the notations:

$$f^+ = \max_{t\in\mathbb{R}} |f(t)|, \quad f^-(t) = \min_{t\in\mathbb{R}} |f(t)|,$$

where f is a bounded continuous function on $\mathbb{R}$.

2. Existence of positive periodic solutions

The properties of the neutral-type operator are crucial for the proof of our main results. We give the following lemma:

Lemma 2.1. ([18, 19]) Define A on C_T by

$$A : C_T \to C_T, (Ax)(t) = x(t) - cx(t-\tau), \forall t \in \mathbb{R}, \tag{2.1}$$

where $C_T = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t)\}$, c and $\tau > 0$ are constants. If $|c| \neq 1$, then A has a unique continuous bounded inverse A^{-1} satisfying

$$(A^{-1}f)(t) = \begin{cases} \sum\limits_{j\geq 0} c^j f(t - j\tau), & \text{if } |c| < 1, \ \forall f \in C_T, \\ -\sum\limits_{j\geq 1} c^{-j} f(t + j\tau), & \text{if } |c| > 1, \ \forall f \in C_T. \end{cases}$$

Obviously, we have

(1) $\|A^{-1}\| \leq \frac{1}{|1-|c||}$;

(2) $\int_0^T |(A^{-1}f)(t)|dt \leq \frac{1}{|1-|c||} \int_0^T |f(t)|dt, \forall f \in C_T$;

(3) $\int_0^T |(A^{-1}f)(t)|^2 dt \leq \frac{1}{|1-|c||} \int_0^T |f(t)|^2 dt, \forall f \in C_T$.

Based on Lemma 2.1, Eq (1.2) can be written by the following equation:

$$(Ax)'(t) = -a(t)(Ax)(t) - a(t)cx(t-\tau) + \sum_{i=1}^{n} \frac{b_i(t)x(t-\gamma_i(t))}{[1 + d_i(t)x(t-\gamma_i(t))]^{N_i+1}}, \tag{2.2}$$

Obviously, Eq (1.2) is equivalent to Eq (2.2). The existence of periodic solutions of Eq (2.2) will be proved as a consequence of a generalized continuation theorem; see Theorem 6.3 in [20]. In our case, we consider

Lemma 2.2. Suppose that there exists an open bounded set $\Omega \subset C_T$ such that

(i) The equation

$$(Ax)'(t) = \lambda\Big[- a(t)(Ax)(t) - a(t)cx(t-\tau) + \sum_{i=1}^{n} \frac{b_i(t)x(t-\gamma_i(t))}{[1 + d_i(t)x(t-\gamma_i(t))]^{N_i+1}} \Big] \tag{2.3}$$

has no solutions on $\partial\Omega$, where $\lambda \in (0, 1)$;

(ii) For $Ax \in \partial\Omega \cap \mathbb{R}$,

$$g(Ax) = \frac{1}{T} \int_0^T \Big[- a(t)Ax - a(t)cx + \sum_{i=1}^{n} \frac{b_i(t)x}{[1 + d_i(t)x]^{N_i+1}} \Big] dt \neq 0;$$

(iii) $deg_B(g, \Omega \cap \mathbb{R}, 0) \neq 0$.

Then, Eq (2.2) has at least one periodic solution Ax on Ω.

Remark 2.1. If Ax is a periodic solution of Eq (2.2), let $Ax = y$; by Lemma 2.1, then $x = A^{-1}y$ is a periodic solution of Eq (2.2).

For obtaining the existence of a positive periodic solution of Eq (2.2) by using the topological degree theory, we need to find an priori bound for any periodic solution of Eq (2.2).

We need the following assumption:

(H_1) $\frac{1}{a^+} \sum_{i=1}^{n} \frac{b_i^- \frac{1}{N_i d_i^+}}{[1+\frac{1}{N_i}]^{N_i+1}} - \frac{cR_0}{1-c} > 0$, where R_0 is defined by Lemma 2.3.

Lemma 2.3. If $c \in [0, 1)$, then every non-negative T-periodic solution $(Ax)(t)$ of Eq (2.3) is bounded above for each $\lambda \in (0, 1)$.

Proof: Let $(Ax)(t)$ be an non-negative T-periodic solution of Eq (2.3) and $(Ax)^+ = (Ax)(\xi) = R$, where $\xi \in [0, T]$. If $(Ax)(t) \geq 0$, by Lemma 2.1 and $c \in [0, 1)$, then

$$x(t) = A^{-1}Ax(t) = \sum_{j\geq 0} c^j Ax(t - j\tau) \geq 0. \tag{2.4}$$

It follows by $(Ax)'(\xi) = 0$ that

$$0 = -a(\xi)(Ax)(\xi) - a(\xi)cx(\xi - \tau) + \sum_{i=1}^{n} \frac{b_i(\xi)x(\xi - \gamma_i(\xi))}{[1 + d_i(\xi)x(\xi - \gamma_i(\xi))]^{N_i+1}}.$$

In view of (2.4), then

$$\begin{aligned} 0 &\le -a(\xi)R + \sum_{i=1}^{n} \frac{b_i(\xi)x(\xi - \gamma_i(\xi))}{[1 + d_i(\xi)x(\xi - \gamma_i(\xi))]^{N_i+1}} \\ &< -a^- R + \sum_{i=1}^{n} \frac{b_i^+ x(\xi - \gamma_i(\xi))}{[1 + d_i^- x(\xi - \gamma_i(\xi))]^{N_i+1}}. \end{aligned} \tag{2.5}$$

Consider the function

$$f(x) = \frac{x}{(1 + d_i^- x)^{N_i+1}}, \ , i = 1, 2, \cdots, n, \ x \ge 0.$$

Since $f'(x) = \frac{1 - d_i^- N_i x}{(1 + d_i^- x)^{N_i+2}}$, $f(x)$ is increasing on $[0, \frac{1}{N_i d_i^-}]$ and decreasing on $[\frac{1}{N_i d_i^-}, +\infty)$. Hence,

$$\frac{b_i^+ x(\xi - \gamma_i(\xi))}{[1 + d_i^- x(\xi - \gamma_i(\xi))]^{N_i+1}} \le \frac{b_i^+ \frac{1}{N_i d_i^-}}{[1 + d_i^- \frac{1}{N_i d_i^-}]^{N_i+1}} := M_i, \ i = 1, 2, \cdots, n. \tag{2.6}$$

From (2.5) and (2.6), we have

$$R < \frac{\sum_{i=1}^{n} M_i}{a^-} := R_0.$$

Remark 2.2. Using $(Ax)(t) < R_0$ for $t \in \mathbb{R}$ and Lemma 2.1, we have

$$x(t) = A^{-1}Ax(t) < \frac{R_0}{1 - c} \text{ for all } t \in \mathbb{R}.$$

Lemma 2.4. If assumption (H_1) holds and $c \in [0, 1)$, then every non-negative T-periodic solution $(Ax)(t)$ of Eq. (2.3) is bounded below for each $\lambda \in (0, 1)$.

Proof: Let $(Ax)(t)$ be a non-negative T-periodic solution of Eq (2.3) and $(Ax)^- = (Ax)(\eta) = r$, where $\eta \in [0, T]$. It follows by $(Ax)'(\eta) = 0$ that

$$0 = -a(\eta)(Ax)(\eta) - a(\eta)cx(\eta - \tau) + \sum_{i=1}^{n} \frac{b_i(\eta)x(\eta - \gamma_i(\eta))}{[1 + d_i(\eta)x(\eta - \gamma_i(\eta))]^{N_i+1}}.$$

In view of (2.4) and Remark 2.2, then

$$0 > -a^+ r - a^+ \frac{cR_0}{1 - c} + \sum_{i=1}^{n} \frac{b_i^- x(\eta - \gamma_i(\eta))}{[1 + d_i^+ x(\eta - \gamma_i(\eta))]^{N_i+1}},$$

i.e.,

$$a^+ r > -a^+ \frac{cR_0}{1 - c} + \sum_{i=1}^{n} \frac{b_i^- x(\eta - \gamma_i(\eta))}{[1 + d_i^+ x(\eta - \gamma_i(\eta))]^{N_i+1}}.$$

Take the maximum value on both sides of the above inequality; by assumption (H_1), then,

$$a^+ r > -a^+ \frac{cR_0}{1-c} + \sum_{i=1}^{n} \frac{b_i^- \frac{1}{N_i d_i^+}}{[1 + d_i^+ \frac{1}{N_i d_i^+}]^{N_i+1}},$$

thus,

$$r > \frac{1}{a^+} \sum_{i=1}^{n} \frac{b_i^- \frac{1}{N_i d_i^+}}{[1 + d_i^+ \frac{1}{N_i d_i^+}]^{N_i+1}} - \frac{cR_0}{1-c} := r_0.$$

Remark 2.3. Since $(Ax)(t) > r_0$ for all $t \in \mathbb{R}$, by Lemma 2.1, we have

$$x(t) = A^{-1}Ax(t) > \frac{r_0}{1-c} \text{ for all } t \in \mathbb{R}.$$

Now, we show the existence of at least one positive periodic solution of Eq (1.2).

Theorem 2.1. Assume that assumption (H_1) holds and $c \in [0, 1)$. Then, Eq (1.2) has at least one positive T-periodic solution.

Proof: The proof of this result is based on Lemma 2.2. Since assumption (H_1) holds, from Lemmas 2.3 and 2.4, the periodic solutions of Eq (2.3) exists lower and upper bounds for all $\lambda \in (0, 1)$. Define the set $\Xi \subset C_T$ by

$$\Xi = \{Ax \in C_T : \ r_0 < (Ax)(t) < R_0, \ t \in [0, T]\},$$

where positive constants r_0 and R_0 are defined by Lemmas 2.3 and 2.4, respectively. From Lemmas 2.3 and 2.4, condition (i) of Lemma 2.2 holds. Next, we prove that condition (ii) of Lemma 2.2 holds. For $Ax \in \partial\Xi$, if $Ax = r_0$ with $r > r_0$, we have

$$\begin{aligned} g(r) &= \frac{1}{T} \int_0^T \Big[-a(t)r - a(t)cA^{-1}r + \sum_{i=1}^{n} \frac{b_i(t)A^{-1}r}{[1 + d_i(t)A^{-1}r]^{N_i+1}} \Big] dt \\ &> \frac{1}{T} \int_0^T \Big[-a^+ r - a^+ \frac{cR_0}{1-c} + \sum_{i=1}^{n} \frac{b_i^- \frac{1}{N_i+1-d_i^+}}{[1 + d_i^+ \frac{1}{N_i+1-d_i^+}]^{N_i+1}} \Big] dt. \end{aligned}$$

Let $r \to r_0$ in the above inequality, we have $g(r_0) > 0$. On the other hand, if $Ax = R_0$ with $R < R_0$, we have

$$\begin{aligned} g(R) &= \frac{1}{T} \int_0^T \Big[-a(t)R - a(t)cA^{-1}R + \sum_{i=1}^{n} \frac{b_i(t)A^{-1}R}{[1 + d_i(t)A^{-1}R]^{N_i+1}} \Big] dt \\ &< \frac{1}{T} \int_0^T \Big[-a^+ R + \sum_{i=1}^{n} \frac{b_i^+ \frac{1}{N_i+1-d_i^-}}{[1 + d_i^- \frac{1}{N_i+1-d_i^-}]^{N_i+1}} \Big] dt. \end{aligned}$$

Let $R \to R_0$ in the above inequality, we have $g(R_0) < 0$. Hence, condition (ii) of Lemma 2.2 holds. It remains to show that condition (iii) of Lemma 2.2 holds. Define the map $\mathcal{H}(x, \mu) : \mathbb{R} \times [0, 1]$ by

$$\mathcal{H}(x, \mu) = \mu x + \frac{1-\mu}{T} \int_0^T \Big[-a(t)Ax - a(t)cx + \sum_{i=1}^{n} \frac{b_i(t)x}{[1 + d_i(t)x]^{N_i+1}} \Big] dt \neq 0.$$

Obviously, $\mathcal{H}(x,\mu)$ does not vanish on $\partial\Xi$ for any $\mu \in [0,1]$. So, we have

$$\begin{aligned} deg_B\{\mathcal{H}(\cdot,0),\Omega\cap\mathbb{R},0\} &= deg_B\{\mathcal{H}(\cdot,1),\Omega\cap\mathbb{R},0\} \\ &= deg_B\{x,\Omega\cap KerL,0\} \neq 0. \end{aligned}$$

Applying Lemma 2.2, Eq (2.2) has at least one positive T-periodic solution $(Ax)(t)$. Let $(Ax)(t) = y(t)$, then Eq (1.2) has at least one positive T-periodic solution $(A^{-1}y)(t)$.

Remark 2.4. Mawhin's continuation and its generalizations are often used to study the existence for periodic solutions for functional differential equations, see [21–25]. However, when studying the existence of positive periodic solutions using the above theorem, it is very difficult to estimate the prior bound of the solution. When proving Theorem 2.1, we used mathematical analysis methods to estimate the range of positive periodic solutions, thereby obtaining the existence of positive periodic solutions.

Remark 2.5. The proof of Lemma 2.2 can be obtained by Mawhin's continuation theorem.

Mawhin's continuation theorem [26]: Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$, is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$. if all the following conditions hold:

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \forall \lambda \in (0,1)$,
(2) $Nx \notin ImL, \forall x \in \partial\Omega \cap KerL$,
(3) $deg_B\{QN, \Omega \cap KerL, 0\} \neq 0$.

Then, equation $Lx = Nx$ has a solution on $\bar{\Omega} \cap D(L)$.
Let

$$L : C_T \to C_T,\ Lx = (Ax)'(t),$$

$$N : C_T \to C_T,\ Nx = -a(t)(Ax)(t) - a(t)cx(t-\tau) + \sum_{i=1}^{n} \frac{b_i(t)x(t-\gamma_i(t))}{[1 + d_i(t)x(t-\gamma_i(t))]^{N_i+1}}.$$

Then, Lemma 2.2 is similar to Mawhin's continuation theorem.

3. Stability of positive periodic solution

In this section, we will deal with the globally asymptotic and exponential stability of Eq (1.2). As is general in the literature on population models, our asymptotic results are obtained by constructing appropriate Lyapunov functionals. Particularly, we define the region of stability of the solutions of Eq (1.2) as the following set:

$$\Gamma = \{(Ax)(t) \in C(\mathbb{R},\mathbb{R}) : 0 < (Ax)(t) < L\}.$$

To reach our stability results, we make the following assumptions:

(H_2) The delays involved in Eq (1.2) are continuously differentiable and satisfy:

$$\gamma_i'(t) \leq \gamma_i^* < 1,\ i = 1,2,\cdots,n;$$

(H_3) $\frac{a^-(1-2c)}{1-c} - \frac{1}{1-c}\sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} > 0$.

We first state and prove the globally asymptotic theorem.

Theorem 3.1. Assume that assumptions (H_1)–(H_3) hold and $c \in [0,1)$. Then, Eq (1.2) has a unique

asymptotically stable T-periodic solution.
Proof: Let $(Ax)(t), (Ay)(t) \in \Gamma$ be two solutions of Eq (2.2). Consider the following functional:

$$V(t) = |(Ax)(t) - (Ay)(t)| + a^- c \int_{t-\tau}^{t} |x(s) - y(s)| ds + \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} \int_{t-\gamma_i(t)}^{t} |x(s) - y(s)| ds.$$

Calculating the upper right Dini derivative of $V(t)$ along the solutions of Eq (2.2), we have

$$\begin{aligned} D^+V(t) \le & -a(t)|(Ax)(t) - (Ay)(t)| - a(t)c|x(t-\tau) - y(t-\tau)| \\ & + \sum_{i=1}^{n} b_i(t) \left| \frac{x(t-\gamma_i(t))}{[1 + d_i(t)x(t-\gamma_i(t))]^{N_i+1}} - \frac{y(t-\gamma_i(t))}{[1 + d_i(t)y(t-\gamma_i(t))]^{N_i+1}} \right| \\ & + a^- c|x(t) - y(t)| - a^- c|x(t-\tau) - y(t-\tau)| \\ & + \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} |x(t) - y(t)| - \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} |x(t-\gamma_i(t)) - y(t-\gamma_i(t))|(1 - \gamma_i'(t)). \end{aligned} \tag{3.1}$$

Consider the function

$$\tilde{f}(x) = \frac{x}{[1 + d_i(t)x]^{N_i+1}}, \ i = 1, 2, \cdots, n, \ x \ge 0.$$

Since $\tilde{f}'(x) = \frac{1-d_i(t)N_i x}{(1+d_i(t)x)^{N_i+2}}$, then $|\tilde{f}'(x)| \le 1$. From mean value theorem, we have

$$\left| \frac{x(t-\gamma_i(t))}{[1 + d_i(t)x(t-\gamma_i(t))]^{N_i+1}} - \frac{y(t-\gamma_i(t))}{[1 + d_i(t)y(t-\gamma_i(t))]^{N_i+1}} \right| \le |x(t-\gamma_i(t)) - y(t-\gamma_i(t))|. \tag{3.2}$$

By assumption (H_2), we have

$$\frac{1-\gamma_i(t)}{1-\gamma_i^*} > 1. \tag{3.3}$$

In view of (3.1)–(3.3) and Lemma 2.1, we have

$$\begin{aligned} D^+V(t) \le & -a^-|(Ax)(t) - (Ay)(t)| - a^- c|x(t-\tau) - y(t-\tau)| \\ & + \sum_{i=1}^{n} b_i^+ \Big| x(t-\gamma_i(t)) - y(t-\gamma_i(t)) \Big| \\ & + a^- c|x(t) - y(t)| - a^- c|x(t-\tau) - y(t-\tau)| \\ & + \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} |x(t) - y(t)| - \sum_{i=1}^{n} b_i^+ \Big| x(t-\gamma_i(t)) - y(t-\gamma_i(t)) \Big| \\ = & -a^-|(Ax)(t) - (Ay)(t)| + \Big(a^- c + \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} \Big) |x(t) - y(t)| \\ \le & -a^-|(Ax)(t) - (Ay)(t)| + \frac{1}{1-c} \Big(a^- c + \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} \Big) |(Ax)(t) - (Ay)(t)| \\ = & -\Big(\frac{a^-(1-2c)}{1-c} - \frac{1}{1-c} \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} \Big) |(Ax)(t) - (Ay)(t)|. \end{aligned} \tag{3.4}$$

It follows by assumption (H_3) and (3.4) that there exists a positive constant α such that

$$D^+V(t) \leq -\alpha|(Ax)(t) - (Ay)(t)| \text{ for } t \geq 0,$$

then,

$$V(t) + \alpha \int_0^t |(Ax)(s) - (Ay)(s)|ds \leq V(0) < +\infty \text{ for } t \geq 0,$$

and

$$\int_0^t |(Ax)(s) - (Ay)(s)|ds \leq \frac{V(0)}{\alpha} < +\infty \text{ for } t \geq 0.$$

Since $|(Ax)(s) - (Ay)(s)| \in L^1([0, \infty))$, by Barbalat's Lemma [27], we have

$$\lim_{t\to+\infty} |(Ax)(t) - (Ay)(t)| = 0.$$

Using $|x(t) - y(t)| \leq \frac{1}{1-c}|(Ax)(t) - (Ay)(t)|$, we also have

$$\lim_{t\to+\infty} |x(t) - y(t)| = 0.$$

Hence, all solutions of the Eq (1.2) in Γ converge to a T-periodic solution and there exists a unique periodic solution of Eq (1.2) in Γ.

Now, we show the globally exponential stability of Eq (1.2). Let

$$F(\varepsilon) = \frac{a^-(1-2c)}{1-c} - \varepsilon - \frac{1}{1-c}\sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} e^{\varepsilon\gamma_i^+},\ i = 1, 2, \cdots, n. \tag{3.5}$$

By assumption (H_3), we have $F(0) > 0$. From the continuity of F, there exists a constant λ_0 such that

$$F(\varepsilon) > 0 \quad \text{for } 0 \leq \varepsilon \leq \lambda_0. \tag{3.6}$$

Theorem 3.2. Assume that assumptions (H_1)–(H_3) hold and $c \in [0, 1)$. Then, each T-periodic solution of Eq (1.2) is globally exponentially stable.

Proof: Let $(Ax)(t), (Ay)(t) \in \Gamma$ be two solutions of Eq (2.2). Consider the following Lyapunov functional:

$$\Phi(t) = |(Ax)(t) - (Ay)(t)|e^{\lambda t} + a^- c \int_{t-\tau}^{t} |x(s) - y(s)|e^{\lambda(s+\tau)}ds + \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*} \int_{t-\gamma_i(t)}^{t} |x(s) - y(s)|e^{\lambda(s+\gamma_i^+)}ds.$$

Calculating the upper right Dini derivative of $\Phi(t)$ along the solutions of Eq (2.2), we have

$$\begin{aligned} D^+\Phi(t) = {} & |(Ax)(t) - (Ay)(t)|\lambda e^{\lambda t} + [(Ax)'(t) - (Ay)'(t)]sgn\{(Ax)(t) - (Ay)(t)\}e^{\lambda t} \\ & + a^- c|x(t) - y(t)|e^{\lambda(t+\tau)} - a^- c|x(t-\tau) - y(t-\tau)|e^{\lambda t} \\ & + \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*}|x(t) - y(t)|e^{\lambda(t+\gamma_i^+)} - \sum_{i=1}^{n} \frac{b_i^+}{1-\gamma_i^*}|x(t-\gamma_i(t)) - y(t-\gamma_i(t))|(1-\gamma_i'(t))e^{\lambda(t-\gamma_i(t)+\gamma_i^+)}. \end{aligned} \tag{3.7}$$

From (3.5), (3.7), assumptions (H_1)–(H_3), and Lemma 2.1, we have

$$\begin{aligned}
D^+\Phi(t) \le\ & e^{\lambda t}\Big[|(Ax)(t)-(Ay)(t)|\lambda \\
& - a^-|(Ax)(t)-(Ay)(t)| - a^-c|x(t-\tau)-y(t-\tau)| \\
& + \sum_{i=1}^{n} b_i^+\Big|x(t-\gamma_i(t))-y(t-\gamma_i(t))\Big| \\
& + a^-c|x(t)-y(t)|e^{\lambda\tau} - a^-c|x(t-\tau)-y(t-\tau)| \\
& + \sum_{i=1}^{n}\frac{b_i^+}{1-\gamma_i^*}|x(t)-y(t)|e^{\lambda\gamma_i^+} - \sum_{i=1}^{n} b_i^+|x(t-\gamma_i(t))-y(t-\gamma_i(t))|\Big] \\
=\ & e^{\lambda t}\Big[|(Ax)(t)-(Ay)(t)|(\lambda-a^-) \\
& + \Big(a^-c + \sum_{i=1}^{n}\frac{b_i^+}{1-\gamma_i^*}e^{\lambda\gamma_i^+}\Big)|x(t)-y(t)|\Big] \\
\le\ & -e^{\lambda t}\Big[-\lambda + a^- - \frac{1}{1-c}\Big(a^-c + \sum_{i=1}^{n}\frac{b_i^+}{1-\gamma_i^*}e^{\lambda\gamma_i^+}\Big)\Big]|(Ax)(t)-(Ay)(t)| \\
=\ & -e^{\lambda t}\Big[-\lambda + \frac{a^-(1-2c)}{1-c} - \frac{1}{1-c}\sum_{i=1}^{n}\frac{b_i^+}{1-\gamma_i^*}e^{\lambda\gamma_i^+}\Big]|(Ax)(t)-(Ay)(t)|.
\end{aligned}$$

Thus,

$$D^+\Phi(t) \le -e^{\lambda t}F(\lambda)|(Ax)(t)-(Ay)(t)|.$$

Let $\lambda = \lambda_0$, in view of (3.6), we have

$$D^+\Phi(t) \le -e^{\lambda_0 t}F(\lambda_0)|(Ax)(t)-(Ay)(t)| < 0 \text{ for } t \ge 0.$$

Hence, $\Phi(t)$ is decreasing for all $t \ge 0$ along the solutions of Eq (2.2); consequently, we get

$$|(Ax)(t)-(Ay)(t)|e^{\lambda_0 t} \le \Phi(t) \le \Phi(0),$$

and

$$|(Ax)(t)-(Ay)(t)| \le \Phi(0)e^{-\lambda_0 t}.$$

Using Lemma 2.1, we have

$$|x(t)-y(t)| \le \frac{\Phi(0)}{1-c}e^{-\lambda_0 t}.$$

Hence, each T-periodic solution of Eq (1.2) is globally exponentially stable.

Remark 3.1. The proofs of Theorems 3.1 and 3.2 are based on proper Lyapunov functions and the properties of neutral-type operators. It should be pointed out that the Lyapunov functions of this paper are different from those of the related papers; see [7, 15–17].

4. An example

For Eq (1.2), let

$$c = 10^{-3},\ n = 2,\ \tau = 0.2,\ \gamma_1(t) = \frac{1}{3}\sin t,\ \gamma_2(t) = \frac{2}{3}\cos t,$$

$$a(t) = 6 - 2\sin t,\ b_1(t) = 3 - 2\sin t,\ b_2(t) = 3 - 2\cos t,$$

$$d_1(t) = 2 - \sin t,\ d_2(t) = 2 - \cos t,\ N_1 = N_2 = 3.$$

Regard the scalar neutral-type host-macroparasite equation as follows:

$$\begin{aligned}(x(t) - 10^{-3}x(t-0.2))' = -(6 - 2\sin t)x(t) + \frac{(3-2\sin t)x(t-\frac{1}{3}\sin t)}{[1+(2-\sin t)x(t-\frac{1}{3}\sin t)]^2} \\ + \frac{(3-2\cos t)x(t-\frac{2}{3}\cos t)}{[1+(2-\cos t)x(t-\frac{2}{3}\cos t)]^2}.\end{aligned} \tag{4.1}$$

After simper computations, we obtain

$$a^+ = 8,\ a^- = 4,\ b_1^+ = 5,\ b_1^- = 1,\ b_2^+ = 5,\ b_2^- = 1,$$

$$d_1^+ = 3,\ d_1^- = 1,\ d_2^+ = 3,\ d_2^- = 1,\ \gamma_1^* = \frac{1}{3},\ \gamma_2^* = \frac{2}{3},$$

$$M_1 = \frac{b_1^+\frac{1}{N_1 d_1^-}}{[1+\frac{1}{N_1}]^{N_1+1}} = \frac{135}{256},\ M_2 = \frac{b_2^+\frac{1}{N_2 d_2^-}}{[1+\frac{1}{N_2}]^{N_2+1}} = \frac{135}{256},$$

$$R_0 = \frac{\sum_{i=1}^{2} M_i}{a^-} = \frac{135}{512}.$$

Then, we have

$$\frac{1}{a^+}\sum_{i=1}^{2}\frac{b_i^-\frac{1}{N_i d_i^+}}{[1+\frac{1}{N_i}]^{N_i+1}} - \frac{cR_0}{1-c} \approx 8.52\times 10^{-3} > 0.$$

Hence, Eq (4.1) satisfies all assumptions in Theorem 2.1, and then it follows that Eq (4.1) has at least one positive periodic solution. In Figure 1, we give the numerical simulations of positive periodic solutions to Eq (4.1) with different initial conditions $x(0) = 1.615$, $x(0) = 1.421$, and $x(0) = 1.171$.

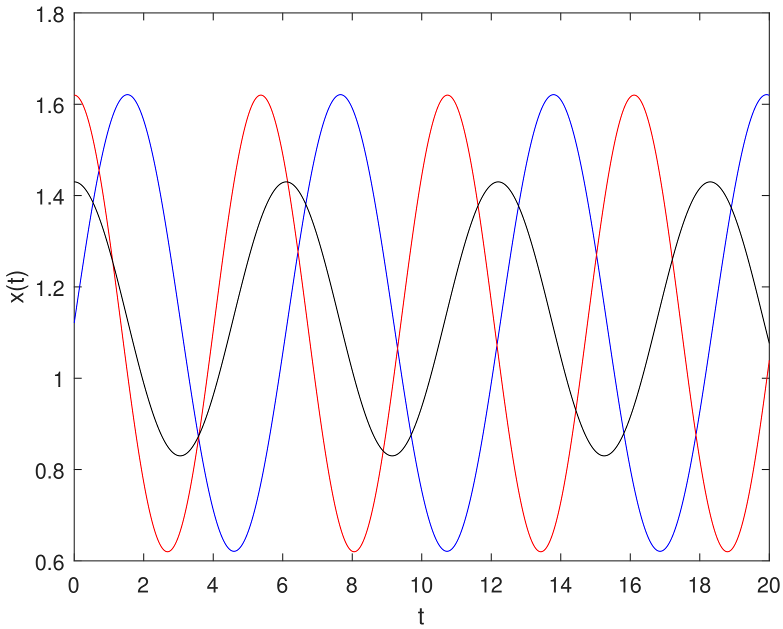

Figure 1. Positive periodic solutions of Eq (4.1).

5. Conclusions

Neutral-type functional differential equations are more complex equations compared to functional differential equations, which contain rich dynamical behaviors. In this paper, we deal with positive periodic solutions of a neutral-type host-macroparasite equation. First, based on a generalized continuation theorem, some sufficient conditions have been proposed to ensure the existence of a positive periodic solution. Then, by the Lyapunov-Krasovskii functional method and some inequality techniques, the asymptotic properties of the positive periodic solution are addressed. Finally, we give a numerical example for verifying the correctness of the results obtained.

Among the projections of this work, we will concentrate on the possible extension of the present study to a neutral-type host-macroparasite equation on time scales, which can unify discrete and continuous equations.

Author contributions

Axiu Shu: Methodology, writing-review and editing; Xiaoliang Li: Supervision, methodology; Bo Du: Writing-original draft; Tao Wang: Formal analysis. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. J. K. Hale, S. M. V. Lunel, *Introduction to functional differential equations*, New York: Springer, 1993. https://doi.org/10.1007/978-1-4612-4342-7
2. Q. Wang, B. X. Dai, Existence of positive periodic solutions for neutral population model with delays and impulse, *Nonlinear Anal.-Theor.*, **69** (2008), 3919–3930. https://doi.org/10.1016/j.na.2007.10.033
3. X. Ma, X. B. Shu, J. Z. Mao, Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay, *Stoch. Dynam.*, **20** (2020), 2050003. https://doi.org/10.1142/S0219493720500033
4. X. B. Shu, F. Xu, Y. J. Shi, S-Asymptotically ω-positive periodic solutions for a class of neutral fractional differential equations, *Appl. Math. Comput.*, **270** (2015), 768–776. https://doi.org/10.1016/j.amc.2015.08.080
5. S. P. Lu, W. G. Ge, Existence of positive periodic solutions for neutral logarithmic population model with multiple delays, *J. Comput. Appl. Math.*, **166** (2004), 371–383. https://doi.org/10.1016/j.cam.2003.08.033
6. N. H. Chen, Existence of periodic solutions for shunting inhibitory cellular neural networks with neutral delays, *Discrete Dyn. Nat. Soc.*, **2013** (2013), 143706. https://doi.org/10.1155/2013/143706
7. M. L. Tang, X. H. Tang, Positive periodic solutions for neutral multi-delay logarithmic population model, *J. Inequal. Appl.*, **2012** (2012), 10. https://doi.org/10.1186/1029-242X-2012-10
8. Q. X. Zhu, Event-triggered sampling problem for exponential stability of stochastic nonlinear delay systems driven by Levy processes, *IEEE T. Automat. Contr.*, **70** (2025), 1176–1183. https://doi.org/10.1109/TAC.2024.3448128
9. Z. Zhao, Q. X. Zhu, Stabilization of stochastic highly nonlinear delay systems with neutral-term, *IEEE T. Automat. Contr.*, **68** (2023), 2544–2551. https://doi.org/10.1109/TAC.2022.3186827

10. M. Roohi, S. Mirzajani, A. R. Haghighi, A. Basse-O'Connor, Robust design of two-level non-integer SMC based on deep soft actor-critic for synchronization of chaotic fractional order memristive neural networks, *Fractal Fract.*, **8** (2024), 548. https://doi.org/10.3390/fractalfract8090548

11. M. Roohi, S. Mirzajani, A. R. Haghighi, A. Basse-O'Connor, Robust stabilization of fractional-order hybrid optical system using a single-input TS-fuzzy sliding mode control strategy with input nonlinearities, *AIMS Mathematics*, **9** (2024), 25879–25907. https://doi.org/10.3934/math.20241264

12. M. Roohi, S. Mirzajani, A. Basse-O'Connor, A no-chatter single-input finite-time PID sliding mode control technique for stabilization of a class of 4D chaotic fractional-order laser systems, *Mathematics*, **11** (2023), 4463. https://doi.org/10.3390/math11214463

13. R. M. Anderson, R. M. May, *Infectious diseases of humans: dynamics and control*, Oxford: Oxford University Press, 1991. https://doi.org/10.1093/oso/9780198545996.001.0001

14. M. Elabbasy, S. H. Saker, K. Saif, Oscillation in host macroparasite model with delay time, *Far East Journal of Applied Mathematics*, **4** (2000), 119–142.

15. S. H. Sakera, J. O. Alzabut, Periodic solutions, global attractivity and oscillation of an impulsive delay host-macroparasite model, *Math. Comput. Model.*, **45** (2007), 531–543. https://doi.org/10.1016/j.mcm.2006.07.001

16. Z. J. Yao, Existence and global exponential stability of an almost periodic solution for a host-macroparasite equation on time scales, *Adv. Differ. Equ.*, **2015** (2015), 41.

17. Z. J. Yao, Existence and exponential stability of almost periodic positive solution for host-macroparasite difference model, *Int. J. Biomath.*, **9** (2016), 1650028. https://doi.org/10.1142/S1793524516500285

18. M. Zhang, Periodic solutions of linear and quasilinear neutral functional differential equations, *J. Math. Anal. Appl.*, **189** (1995), 378–392. https://doi.org/10.1006/jmaa.1995.1025

19. S. P. Lu, W. G. Ge, Z. X. Zheng, Periodic solutions to neutral differential equation with deviating arguments, *Appl. Math. Comupt.*, **152** (2004), 17–27. https://doi.org/10.1016/S0096-3003(03)00530-7

20. P. Amster, *Topological methods in the study of boundary value problems*, New York: Springer, 2014. https://doi.org/10.1007/978-1-4614-8893-4

21. H. Meng, F. Long, Periodic solutions for a Liénard type p-Laplacian differential equation, *J. Comput. Appl. Math.*, **224** (2009), 696–701. https://doi.org/10.1016/j.cam.2008.06.001

22. F. X. Zhang, Y. Li, Existence and uniqueness of periodic solutions for a kind of duffing type p-Laplacian equation, *Nonlinear Analy.-Real*, **9** (2008), 985–989. https://doi.org/10.1016/j.nonrwa.2007.01.013

23. H. J. Yang, X. L. Han, Existence and uniqueness of periodic solutions for a class of higher order differential equations, *Mediterr. J. Math.*, **20** (2023), 282. https://doi.org/10.1007/s00009-023-02477-0

24. X. M. Wu, J. W. Li, Z. C. Wang, Existence of positive periodic solutions for a generalized prey-predator model with harvesting term, *Comput. Math. Appl.*, **55** (2008), 1895–1905. https://doi.org/10.1016/j.camwa.2007.11.020
25. Z. Q. Zhang, Periodic solutions of a predator-prey system with stage-structures for predator and prey, *J. Math. Anal. Appl.*, **302** (2005), 291–305. https://doi.org/10.1016/j.jmaa.2003.11.033
26. R. E. Gains, J. L. Mawhin, *Coincide degree and nonlinear differential equation*, Berlin: Springer, 1977. https://doi.org/10.1007/BFb0089537
27. H. K. Khalil, *Nonlinear systems*, 3 Eds., New Jersey: Prentice Hall, 2002.