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Research article

Stability of random attractors for non-autonomous stochastic *p*-Laplacian lattice equations with random viscosity

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Abstract: In this paper, we investigate the stability of pullback random attractors for non-autonomous stochastic p-Laplacian lattice systems, which are influenced by random viscosity and multiplicative white noise. Under appropriate conditions, we first prove the existence and uniqueness of these pullback random attractors and then establish their backward compactness. To ensure their measurability, we demonstrate the equivalence of two different classes of attractors across two distinct universes. Finally, we examine the asymptotic stability of these pullback random attractors by assuming that the time-dependent external forcing term converges to a time-independent external force as time approaches negative infinity.

Keywords: random attractors; stochastic p-Laplacian lattice system; asymptotically autonomous stability; multiplicative white noise
Mathematics Subject Classification: 35B40, 35B41, 37L55

1. Introduction

We consider the following non-autonomous stochastic p-Laplacian lattice system driven by multiplicative white noise:

$$\begin{cases} du_i + \nu(\theta_t\omega)(A_p u)_i dt + \lambda u_i dt = f_i(u_i)dt + g_i(t)dt + u_i \circ dW, \\ u_i(\tau) = u_{i,\tau}, \ \ \tau \in \mathbb{R}, i \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where $\lambda > 0$, and for $i \in \mathbb{Z}$, $u_i \in \mathbb{R}$. The nonlinear function $f_i \in C^1(\mathbb{R}, \mathbb{R})$, and $g(t) = \{g_i(t)\}_{i\in\mathbb{Z}}$ is the time-dependent forcing. W is a scalar Wiener process, and $\nu(\cdot)$ is a random variable on the classical Wiener space $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, which will be special later. The symbol $\circ$ indicates the system is understood in the sense of Stratonovich's integration. For $p \geq 2$, the discrete p-Laplacian operator A_p is defined by

$$(A_p u)_i = |u_i - u_{i-1}|^{p-2}(u_i - u_{i-1}) - |u_{i+1} - u_i|^{p-2}(u_{i+1} - u_i), \ \ i \in \mathbb{Z}. \tag{1.2}$$

The p-Laplacian equation appears in various applications, including those involving nonlinear wave phenomena, fluid mechanics, nonlinear diffusion, non-Newtonian fluids, population dynamics, biological diffusion, chemotaxis, and ecological modeling [12, 16]. The existence and regularity of attractors, whether random, pullback, or global, associated with the p-Laplacian equation have been extensively studied in the literature. For the deterministic case, see [6, 13, 21], and for the stochastic case, refer to [4, 7, 18].

It is well known that the unique solution of a stochastic evolution equation with time-dependent forcing typically generates a non-autonomous random dynamical system (RDS). A central concept in the theory of non-autonomous RDS is the pullback random attractor, which generalizes the foundational work on both autonomous RDS and deterministic non-autonomous systems. Specifically, a typical form of a non-autonomous random attractor is expressed as $\mathcal{A}_\gamma = \{\mathcal{A}_\gamma(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, where γ represents an external parameter arising from various perturbations. It is important to note that existing studies on the stability of $\mathcal{A}_\gamma(\tau, \omega)$ primarily focus on the external parameter γ, rather than the internal time parameter τ. This approach resembles the case of autonomous systems, which leaves the time-dependent features of non-autonomous random attractors inadequately explored. Therefore, the main goal of this paper is to demonstrate the existence of the $\mathcal{D}$-pullback attractor $\mathcal{A}_\mathcal{D} = \{\mathcal{A}_\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ for the cocycle Φ generated by the random Eq (3.2) and to establish its asymptotic autonomous stability. Specifically, we aim to show that

$$dist(\mathcal{A}_\mathcal{D}(\tau, \omega), \mathcal{A}_\infty(\omega)) \to 0 \text{ in probability as } \tau \to -\infty, \tag{1.3}$$

where $\mathcal{A}_\infty(\omega) = \{\mathcal{A}_\infty(\omega) : \omega \in \Omega\}$ is the random attractor for the cocycle Φ_∞ generated by the random Eq (4.2). Note that in this paper, we focus on the random Eq (3.2), which can be viewed as a deterministic equation parameterized by $\omega \in \Omega$, rather than the stochastic Eq (1.1).

Here, we observe that a key step in proving Eq (1.3) is demonstrating the following backward compactness condition:

$$\bigcup_{s\leq\tau} \mathcal{A}_\mathcal{D}(s, \omega) \text{ is precompact in } l^2. \tag{1.4}$$

The backward compactness of non-autonomous attractors has been recently addressed in works such as [3, 14, 15]. In the deterministic setting, the asymptotic robustness of non-autonomous attractors has been explored by Kloeden, Li, and their collaborators [5, 9, 10], while for the stochastic case, we refer to recent studies like [8, 19] and their references.

A major difficulty in studying asymptotic autonomous stability is that the standard pullback asymptotic compactness of solutions to Eq (3.2) is insufficient to prove (1.3) and (1.4). To address this issue, we adopt techniques from [2] and introduce a backward-attracting universe $\mathcal{B}$, which is a subset of the standard-attracting universe $\mathcal{D}$. We then prove the backward pullback asymptotic compactness of solutions to (3.2) in l^2. This development allows us to establish (1.3) and (1.4). Furthermore, we must tackle the measurability of the attractor $\mathcal{A}_\mathcal{B}$. As shown in [11], a union of measurable sets indexed by a countable set is measurable. However, the measurability of the uniformly compact attractor $\mathcal{A}_\mathcal{B} = \{\mathcal{A}_\mathcal{B}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ remains unclear, as the pullback random absorbing set $\mathcal{K}_\mathcal{B}$ is constructed using the supremum of various random sets over the uncountable set $(-\infty, \tau]$. To overcome this challenge, we aim to establish the important relationship $\mathcal{A}_\mathcal{B} = \mathcal{A}_\mathcal{D}$, where $\mathcal{A}_\mathcal{D} = \{\mathcal{A}_\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ represents the usual random attractor of the cocycle Φ generated by the random Eq (3.2), which is measurable.

The structure of the paper is as follows: In the next section, we give some basic results about the non-autonomous random dynamical systems. In Section 3, we provide suitable assumptions to establish the existence of global solutions. In Section 4, we prove the existence and uniqueness of pullback random attractors. The final section is dedicated to studying the asymptotically autonomous stability of the two attractors in l^2 as time $\tau \to -\infty$. Throughout the paper, the letter c will denote a generic constant, whose value may vary from line to line, even within the same line.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system, (X, d) a Polish space, and $\mathcal{D}$ be a collection of some families of nonempty subsets of X.

Definition 2.1. *(See [17]) A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t, s \in \mathbb{R}^+$,*

(i) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;
(iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, \cdot))$;
(iv) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Definition 2.2. *A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is said to be a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathcal{D}$, there exists $T = T(D, \tau, \omega) > 0$ such that*

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega), \ \ \forall t \geq T.$$

If, in addition, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $K(\tau, \omega)$ is a closed nonempty subset of X and K is measurable in ω with respect to $\mathcal{F}$, then we say K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 2.3. *A cocyle Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to +\infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.*

Definition 2.4. *A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback random attractor for Φ if for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

(i) $\mathcal{A}$ is measurable in ω with respect to $\mathcal{F}$ and $\mathcal{A}(\tau, \omega)$ is compact in X;
(ii) $\mathcal{A}$ is invariant; that is, $\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t\omega), \ \forall t \geq 0$;
(iii) $\mathcal{A}$ attracts every set in $\mathcal{D}$; i.e., for every $B \in \mathcal{D}$,

$$\lim_{t\to+\infty} d_X(\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0,$$

where d_X is the Hausdorff semi-distance in X.

Proposition 2.1. *Let Φ is a continuous cocycle that has a closed measurable pullback absorbing set K in $\mathcal{D}$. If Φ is pullback asymptotically compact in X, then Φ has a pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, which is given by*

$$\mathcal{A}(\tau, \omega) = \Omega(K, \tau, \omega) = \bigcup_{D\in\mathcal{D}} \Omega(D, \tau, \omega),$$

where $\Omega(K)$ and $\Omega(D)$ are the omega-limit sets of K and D, respectively.

3. Existence of solutions and random dynamical systems

For $p \geq 2$, the discrete p-Laplacian operator A_p in (1.2) has the following abstract version:

$$A_p(u) = B^*(|Bu|^{p-2}Bu), \ \forall u = (u_i)_{i\in\mathbb{Z}} \in l^2,$$

where $|u|^s = (|u_i|^s)_{i\in\mathbb{Z}}$, $uv = ((u_i v_i))_{i\in\mathbb{Z}}$ and

$$(Bu)_i = u_{i+1} - u_i \text{ and } (B^*u)_i = u_{i-1} - u_i, \ \forall i \in \mathbb{Z}.$$

We consider the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra on Ω induced by the compact-open topology, and $\mathbb{P}$ denotes the Wiener measure. The Wiener shift $\{\theta_t\}_{t\in\mathbb{R}}$ is defined on this probability space as

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t), t \in \mathbb{R}.$$

It is well-established that $\mathbb{P}$ serves as an ergodic invariant measure for the shift family $\{\theta_t\}_{t\in\mathbb{R}}$. Consequently, the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ constitutes a metric dynamical system (see [1]).

We now introduce an Ornstein-Uhlenbeck process on the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, generated by the Wiener process. Specifically, we define

$$z(\theta_t\omega) := -\gamma \int_{-\infty}^{0} e^{\gamma r}\theta_t\omega(r)dr.$$

This integral is well-defined for any path ω exhibiting sub-exponential growth. It is well known that z satisfies the stochastic differential equation

$$dz + \gamma z dt = dW(t),$$

and $t \to z(\theta_t\omega)$ is a θ_t invariant stationary solution, commonly referred to as the Ornstein-Uhlenbeck process. Furthermore, there exists a θ-invariant, full-measure subspace (still denoted by Ω below) of Ω such that, for $\omega \in \Omega$, $|z(\omega)|$ is tempered, the mapping $t \to z(\theta_t\omega)$ is continuous in t, and

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t |z(\theta_s\omega)|ds = \mathbb{E}[|z(\omega)|] = \frac{1}{\sqrt{\pi\gamma}}. \tag{3.1}$$

By considering the change of variables $v(t) = e^{-\gamma z(\theta_t\omega)}u(t)$, problem (1.1) is transformed into:

$$\begin{cases} \frac{dv}{dt} + \nu(\theta_t\omega)e^{(p-2)\gamma z(\theta_t\omega)}(A_p v) + (\lambda - \gamma z(\theta_t\omega))v = e^{-\gamma z(\theta_t\omega)} f(e^{\gamma z(\theta_t\omega)}v) + e^{-\gamma z(\theta_t\omega)} g(t), \\ v(\tau) = v_\tau = e^{-\gamma z(\theta_\tau\omega)}u_\tau, \ \tau \in \mathbb{R}, \end{cases} \tag{3.2}$$

where $v = (v_i)_{i\in\mathbb{Z}} \in l^2$, $v_\tau = (v_{i,\tau})_{i\in\mathbb{Z}} \in l^2$, $f(v) = (f_i(v_i))_{i\in\mathbb{Z}}$, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$, and $\lambda, \gamma > 0$, we then impose the following conditions:

(H1) The viscosity $\nu(\cdot)$ is a positive random variable satisfying:

$$t \to \nu(\theta_t\omega) \text{ is continuous and } \lim_{t\to\pm\infty}\frac{\nu(\theta_t\omega)}{t} = 0, \ \omega \in \Omega.$$

(H2) For each $i \in \mathbb{Z}$, $f_i \in C^1(\mathbb{R}, \mathbb{R})$, and there exist constants $q \geq 2$, $\alpha_{1,2,3} > 0$, and $\beta = (\beta_i)_{i\in\mathbb{Z}} \in l^2$ such that

$$f_i(s)s \leq -\alpha_1(|s|^p + |s|^q) + \beta_i^2, \ |f_i(s)| \leq \alpha_2|s|^{q-1} + |\beta_i|, \ \frac{df_i}{ds}(s) \leq \alpha_3,$$

for all $s \in \mathbb{R}$, $i \in \mathbb{Z}$, where p is the same as the p in the p-Laplacian operator.

(H3) The time-dependent forcing $g \in C(\mathbb{R}, l^2)$, and there exists $g_\infty := (g_{\infty,i})_{i\in\mathbb{Z}} \in l^2$ such that

$$\lim_{\tau\to-\infty}\int_{-\infty}^{\tau}\|g(s) - g_\infty\|^2 ds = 0. \tag{3.3}$$

From [20], one can see that the hypothesis (3.3) leads to the following conditions:
(i) Backward temperedness of g:

$$\sup_{r\leq\tau}\int_{-\infty}^{r} e^{a(s-r)}\|g(s)\|^2 ds < \infty, \ \forall a > 0, \ \tau \in \mathbb{R}; \tag{3.4}$$

(ii) Backward asymptotic smallness of g:

$$\lim_{k\to+\infty}\sup_{r\leq\tau}\int_{-\infty}^{\tau} e^{a(s-r)}\sum_{|i|\geq k}\xi_i|g_i(r)|^2 dr = 0, \ \forall a > 0, \ \tau \in \mathbb{R}. \tag{3.5}$$

Lemma 3.1. *Assume H1–H3 hold. Then, for any $\tau \in \mathbb{R}$ and $v_\tau \in l^2$, there exists $T_{max} > 0$ such that Eq (3.2) admits a unique local solution*

$$v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \tau + T_{max}], l^2)$$

with the initial condition $v(\tau, \tau, \omega, v_\tau) = v_\tau$. Furthermore, the solution $v(t, \tau, v_\tau)$ depends continuously on the initial data (τ, v_τ).

Proof. Let

$$G(t, v, \omega) = -\nu(\theta_t\omega)e^{(p-2)\gamma z(\theta_t\omega)}(A_p v) - (\lambda - \gamma z(\theta_t\omega))v + e^{-\gamma z(\theta_t\omega)}f(e^{\gamma z(\theta_t\omega)}v) + e^{-\gamma z(\theta_t\omega)}g(t).$$

By Theorem 1 in [5], the discrete Laplace operator A_p with $p \geq 2$ is locally Lipschitz continuous in l^2. Fixed $\omega \in \Omega$, by assumptions H2 and H3, along with the continuity of $\nu(\theta.\omega)$ and $z(\theta.\omega)$, it follows that $G(t, v, \omega)$ is continuous in l^2.

Therefore, using standard arguments, Eq (3.2) possesses a local solution $v(\cdot, \omega, v_\tau) \in C([\tau, \tau + T_{max}), l^2)$, where $[\tau, \tau + T_{max})$ is the maximal interval of existence for the solution.

To prove uniqueness, assume v_1 and v_2 are two solutions of (3.2) with initial data $v_{1,\tau}$ and $v_{2,\tau}$, respectively. Define $\widehat{v} = v_1 - v_2$. Subtracting the equations for v_1 and v_2, we find that $\widehat{v}$ satisfies:

$$\frac{d\widehat{v}}{dt} + \nu(\theta_t\omega)e^{(p-2)\gamma z(\theta_t\omega)}(A_p v_1 - A_p v_2) + (\lambda - \gamma z(\theta_t\omega))\widehat{v} = e^{-\gamma z(\theta_t\omega)}(f(e^{\gamma z(\theta_t\omega)}v_1) - f(e^{\gamma z(\theta_t\omega)}v_2)). \tag{3.6}$$

Using conditions (H1), (H2), and the monotonicity of A_p, it can be shown that for $t \in [\tau, \tau + T_{max}]$,

$$\frac{1}{2}\frac{d}{dt}\|\widehat{v}\|^2 \leq (\alpha_3 - (\lambda - \gamma z(\theta_t\omega)))\|\widehat{v}\|^2. \tag{3.7}$$

Therefore, for all $t \in [\tau, \tau + T_{max}]$, we have

$$\|\widehat{v}(t)\|^2 \leq e^{ct}\|v_{1,\tau} - v_{2,\tau}\|^2, \tag{3.8}$$

where $c > 0$ is a constant depending on τ, T_{max} and ω, implying that the solution is unique and depends continuously on the initial data in l^2. □

Lemma 3.2. *Assume H1–H3 hold and*

$$\delta(\omega) := \lambda - 2\alpha_3 - 2\gamma|z(\omega)|. \tag{3.9}$$

Then, for any $\tau \in \mathbb{R}$ and $v_\tau \in l^2$, Eq (3.2) admits a unique global solution $v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \tau + T], l^2)$ for every $T > 0$ with $v(\tau, \tau, \omega, v_\tau) = v_\tau$.

Proof. Take the inner product of (3.2) with $v(t)$; it follows that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu(\theta_t\omega)e^{(p-2)\gamma z(\theta_t\omega)}(A_p v, v) + (\lambda - \gamma z(\theta_t\omega)\|v\|^2 = e^{-\gamma z(\theta_t\omega)}(f(u), v) + e^{-\gamma z(\theta_t\omega)}(g(t), v). \tag{3.10}$$

Also by Theorem 1 in [5], $(A_p v, v) \geq 0$. By (H1), we have

$$e^{-\gamma z(\theta_t\omega)}(f(u), v) \leq -\alpha_1 e^{-2\gamma z(\theta_t\omega)}(\|u\|_p^p + \|u\|_q^q) + e^{-2\gamma z(\theta_t\omega)}\|\beta\|^2. \tag{3.11}$$

By the Young inequality,

$$e^{-\gamma z(\theta_t\omega)}(g(t), v) \leq \frac{1}{\lambda}e^{-2\gamma z(\theta_t\omega)}\|g(t)\|^2 + \frac{\lambda}{4}\|v\|^2. \tag{3.12}$$

Therefor, we have

$$\frac{d}{dt}\|v\|^2 + \frac{\lambda}{2}\|v\|^2 + 2\alpha_1 e^{-2\gamma z(\theta_t\omega)}(\|u\|_p^p + \|u\|_q^q) \leq (2\gamma z(\theta_t\omega) - \lambda)\|v\|^2 + 2e^{-2\gamma z(\theta_t\omega)}\|\beta\|^2 + \frac{2}{\lambda}e^{-2\gamma z(\theta_t\omega)}\|g(t)\|^2. \tag{3.13}$$

Multiplying (3.13) by $e^{\int_\tau^t \delta(\theta_s\omega)ds}$, where $\delta(\omega)$ is given by (3.9), we have

$$\begin{aligned}
&\frac{d}{dt}e^{\int_\tau^t \delta(\theta_s\omega)ds}\|v\|^2 + \frac{\lambda}{2}e^{\int_\tau^t \delta(\theta_s\omega)ds}\|v\|^2 + 2\alpha_1 e^{\int_\tau^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}(\|u\|_p^p + \|u\|_q^q) \\
\leq &(2\gamma z(\theta_t\omega) - \lambda + \delta(\theta_t\omega))e^{\int_\tau^t \delta(\theta_s\omega)ds}\|v\|^2 + 2e^{\int_\tau^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}\|\beta\|^2 + \frac{2}{\lambda}e^{\int_\tau^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}\|g(t)\|^2.
\end{aligned} \tag{3.14}$$

Integrating (3.14) on $r \geq \tau$, we have

$$\begin{aligned}
&\|v(r)\|^2 + \frac{\lambda}{2}\int_\tau^r e^{\int_r^t \delta(\theta_s\omega)ds}\|v(t)\|^2 dt + 2\alpha_1 \int_\tau^r e^{\int_r^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}(\|u(t)\|_p^p + \|u(t)\|_q^q)dt \\
\leq &e^{\int_r^\tau \delta(\theta_s\omega)ds}\|v(\tau)\|^2 + \int_\tau^r (2\gamma z(\theta_t\omega) - \lambda + \delta(\theta_t\omega))e^{\int_r^t \delta(\theta_s\omega)ds}\|v(t)\|^2 dt \\
&+ 2\int_\tau^r e^{\int_r^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}dt\|\beta\|^2 + \frac{2}{\lambda}\int_\tau^r e^{\int_r^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}\|g(t)\|^2 dt \\
\leq &e^{\int_r^\tau \delta(\theta_s\omega)ds}\|v(\tau)\|^2 + 2\int_\tau^r e^{\int_r^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}dt\|\beta\|^2 + \frac{2}{\lambda}\int_\tau^r e^{\int_r^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}\|g(t)\|^2 dt.
\end{aligned} \tag{3.15}$$

The right-hand side of this inequality is bounded for $r \in [\tau, \tau + T](\forall T > 0)$, due to the continuity of $\delta(\theta.\omega)$, $z(\theta.\omega)$, and $g(\cdot)$. Thus, the solution extends globally, implying $T_{max} = +\infty$. □

Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times l^2 \to l^2$ be a mapping given by, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in l^2$,

$$\Phi(t, \tau, \omega, v_\tau) = v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau). \tag{3.16}$$

Then, by Lemma 3.2, Φ is a continuous cocycle on l^2 over the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

4. Existence of pullback random attractors

In this section, we establish the existence of pullback random attractors for the cocycle Φ associated with the random Eq (3.2). To achieve this, we first construct attractors within two distinct universes and then demonstrate the equivalence between the two types of attractors defined in these universes.

From (3.1) and (3.9), the expectation of $\delta(\cdot)$ is given by

$$\mathbb{E}\delta = \lambda - 2\alpha_3 - 2\gamma\mathbb{E}|z| = \lambda - 2\alpha_3 - \frac{2\gamma}{\sqrt{\pi\gamma}}. \tag{4.1}$$

We consider two types of attracting universes:

(1) The universe $\mathcal{D}$:

This consists of families $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, where each $D(\tau, \omega)$ is a nonempty bounded subset of l^2, and

$$\lim_{t\to+\infty} e^{-\mathbb{E}\delta t}\|D(\tau - t, \theta_{-t}\omega)\|^2 = 0. \tag{4.2}$$

(2) The universe $\mathcal{B}$:

This consists of families $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, where each $B(\tau, \omega)$ is a nonempty bounded subset of l^2, and

$$\lim_{t\to+\infty} e^{-\mathbb{E}\delta t}\sup_{s\le\tau}\|B(s - t, \theta_{-t}\omega)\|^2 = 0. \tag{4.3}$$

Note $\mathcal{B}$ is a subset of $\mathcal{D}$.

4.1. Backward-pullback absorption

Lemma 4.1. *Let (H1)–(H3) and (3.9) be satisfied and*

$$\mathbb{E}\delta > 0. \tag{4.4}$$

Then, for any $(\tau, \omega, B, D) \in \mathbb{R} \times \Omega \times \mathcal{B} \times \mathcal{D}$, there exist constants $T_{\mathcal{B}} = T_{\mathcal{B}}(\tau, \omega) > 0$ and $T_{\mathcal{D}} = T_{\mathcal{D}}(\tau, \omega) > 0$ such that,

$$\sup_{l\le\tau}\|v(l, l - t, \theta_{-l}\omega, v_{l-t})\|^2 \le 2M\sup_{l\le\tau}R(l, \omega), \tag{4.5}$$

and

$$\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \le 2MR(\tau, \omega), \tag{4.6}$$

where $M > 0$ is a constant independent of τ, ω, $\mathcal{B}$, and $\mathcal{D}$, $v_{l-t} \in B(l - t, \theta_{-t}\omega)$ for $l \le \tau$, $v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$, and $R(l, \omega)$ is given by

$$R(l, \omega) = \int_{-\infty}^{0} e^{\int_0^r \delta(\theta_s\omega)ds - 2\gamma z(\theta_r\omega)}(1 + \|g(r + l)\|^2)dr. \tag{4.7}$$

Proof. Using $l-t$ instead of τ and $\theta_{-l}\omega$ instead of ω in (3.15), we have

$$\begin{aligned} &\|v(l)\|^2 + \frac{\lambda}{2}\int_{l-t}^{l} e^{\int_l^r \delta(\theta_{s-l}\omega)ds}\|v(r)\|^2 dr + 2\alpha_1 \int_{l-t}^{l} e^{\int_l^r \delta(\theta_{s-l}\omega)ds - 2\gamma z(\theta_{r-l}\omega)}(\|u(r)\|_p^p + \|u(r)\|_q^q)dr \\ \leq & e^{\int_l^{l-t} \delta(\theta_{s-l}\omega)ds}\|v(l-t)\|^2 + 2\int_{l-t}^{l} e^{\int_l^r \delta(\theta_{s-l}\omega)ds - 2\gamma z(\theta_{r-l}\omega)}dr\|\beta\|^2 + \frac{2}{\lambda}\int_{l-t}^{l} e^{\int_l^r \delta(\theta_{s-l}\omega)ds - 2\gamma z(\theta_{r-l}\omega)}\|g(r)\|^2 dr \\ \leq & e^{\int_0^{-t} \delta(\theta_s\omega)ds}\|v(l-t)\|^2 + M\int_{-\infty}^{0} e^{\int_0^r \delta(\theta_s\omega)ds - 2\gamma z(\theta_r\omega)}(1 + \|g(r+l)\|^2)dr. \end{aligned} \tag{4.8}$$

In particular, let $l=\tau$, we obtain

$$\|v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \leq e^{\int_0^{-t} \delta(\theta_s\omega)ds}\|v(\tau-t)\|^2 + M\int_{-\infty}^{0} e^{\int_0^r \delta(\theta_s\omega)ds - 2\gamma z(\theta_r\omega)}(1 + \|g(r+\tau)\|^2)dr. \tag{4.9}$$

By the property of $|z(\theta_t\omega)|$, there exists a bounded random variable $c(\omega) > 0$ such that, for all $t \geq 0$,

$$-(\lambda - 2\alpha_3)t - 2\gamma\int_0^{-t} |z(\theta_s\omega)|ds \leq -(\lambda - 2\alpha_3 - \frac{2\gamma}{\sqrt{\pi\gamma}})t + c(\omega) = -\mathbb{E}\delta t + c(\omega). \tag{4.10}$$

We take the supremum of (4.8) with respect to $l \in (-\infty, \tau]$, then we infer from (4.10) that

$$\begin{aligned} \sup_{l\leq\tau}\|v(l, l-t, \theta_{-l}\omega, v_{l-t})\|^2 &\leq e^{-(\lambda-2\alpha_3)t - 2\gamma\int_0^{-t}|z(\theta_s\omega)|ds}\sup_{l\leq\tau}\|v(l-t)\|^2 + M\sup_{l\leq\tau}R(l,\omega) \\ &\leq e^{-\mathbb{E}\delta t + c(\omega)}\sup_{l\leq\tau}\|v(l-t)\|^2 + M\sup_{l\leq\tau}R(l,\omega). \end{aligned} \tag{4.11}$$

Since $v_{l-t} \in B(l-t, \theta_{-t}\omega)$ for all $l \leq \tau$ is tempered. we have

$$e^{-\mathbb{E}\delta t + c(\omega)}\sup_{l\leq\tau}\|v(l-t)\|^2 \leq e^{-\mathbb{E}\delta t + c(\omega)}\sup_{l\leq\tau}\|B(l-t, \theta_{-t}\omega)\|^2 \to 0 \ (t \to +\infty). \tag{4.12}$$

Therefore, there exists $T_{\mathcal{B}} = T_{\mathcal{B}}(\tau, \omega) > 0$ such that for all $t \geq T_{\mathcal{B}}$,

$$e^{-\mathbb{E}\delta t + c(\omega)}\sup_{l\leq\tau}\|v(l-t)\|^2 \leq M\int_{-\infty}^{0} e^{\int_0^r \delta(\theta_s\omega)ds - 2\gamma z(\theta_r\omega)}(1 + \|g(r+l)\|^2)dr, \tag{4.13}$$

thus (4.5) is proved. On the other hand, if $v_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$, we obtain

$$\begin{aligned} \|v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 &\leq e^{-(\lambda-2\alpha_3)t - 2\gamma\int_0^{-t}|z(\theta_s\omega)|ds}\|D(\tau-t, \theta_{-t}\omega)\|^2 + MR(\tau,\omega) \\ &\leq e^{-\mathbb{E}\delta t + C(\omega)}\|D(\tau-t, \theta_{-t}\omega)\|^2 + MR(\tau,\omega) \\ &\to MR(\tau,\omega) \text{ as } t \to +\infty. \end{aligned} \tag{4.14}$$

This completes the proof. □

Corollary 4.1. *Assume (H1)–(H3), (3.9), and (4.4) hold. Then, the cocycle* Φ *generated by (3.2) possesses two types of pullback random absorbing sets:*

(i) Φ has a $\mathcal{B}$-pullback random absorbing set $\mathcal{K}_{\mathcal{B}} = \{\mathcal{K}_{\mathcal{B}}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{B}$, which is given by, for any $(\tau, \omega) \in \mathbb{R} \times \Omega$,

$$\mathcal{K}_{\mathcal{B}}(\tau, \omega) = \{v \in l^2 : \|v\|^2 \leq 2M \sup_{l \leq \tau} R(l, \omega)\}. \tag{4.15}$$

(ii) Φ has a $\mathcal{D}$-pullback random absorbing set, $\mathcal{K}_{\mathcal{D}} = \{\mathcal{K}_{\mathcal{D}}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, which is given by, for any $(\tau, \omega) \in \mathbb{R} \times \Omega$,

$$\mathcal{K}_{\mathcal{D}}(\tau, \omega) = \{v \in l^2 : \|v\|^2 \leq 2MR(\tau, \omega)\}. \tag{4.16}$$

Here, the constant $M > 0$ and the function R are the same as those defined in Lemma 4.1.

Proof. (i) By Lemma 4.1, for each $(\tau, \omega) \in \mathbb{R} \times \Omega$, there exists a time $T_{\mathcal{B}} = T_{\mathcal{B}}(\tau, \omega) > 0$ such that

$$\cup_{t \geq T_{\mathcal{B}}} \cup_{s \leq \tau} \Phi(t, s - t, \theta_{-t}\omega)\mathcal{B}(s - t, \theta_{-t}\omega) \subset \mathcal{K}_{\mathcal{B}}(\tau, \omega).$$

Next, it remains to verify that $\mathcal{K}_{\mathcal{B}} \in \mathcal{B}$.

By (3.9), there exists $T(\omega) > 0$ such that

$$\left| \int_0^r (\delta(\theta_s\omega) - \mathbb{E}\delta) ds \right| \leq \frac{\mathbb{E}\delta}{2}|r| \text{ and } |z(\theta_r\omega)| \leq \frac{\mathbb{E}\delta}{8\gamma}|r|, \ \forall |r| \geq T(\omega). \tag{4.17}$$

This implies that for all $r \leq -t \leq -T(\omega)$, we have

$$\begin{aligned}
&\int_{-t}^r \delta(\theta_s\omega) ds - 2\gamma z(\theta_r\omega) \\
=& \int_{-t}^0 (\delta(\theta_s\omega) - \mathbb{E}\delta) ds + \int_0^r (\delta(\theta_s\omega) - \mathbb{E}\delta) ds + (r - t)\mathbb{E}\delta - 2\gamma z(\theta_r\omega) \\
\leq& \frac{\mathbb{E}\delta}{2} t + \frac{\mathbb{E}\delta}{2}|r| + (r - t)\mathbb{E}\delta + \frac{\mathbb{E}\delta}{4}|r| \\
=& -\frac{\mathbb{E}\delta}{2} t + \frac{\mathbb{E}\delta}{4} r.
\end{aligned} \tag{4.18}$$

Thus, as $t \to +\infty$, we have

$$\begin{aligned}
&2Me^{-\mathbb{E}\delta t} \sup_{l \leq \tau} R(l - t, \theta_{-t}\omega) \\
=&2Me^{-\mathbb{E}\delta t} \sup_{l \leq \tau} \int_{-\infty}^0 e^{\int_0^r \delta(\theta_{s-t}\omega) ds - 2\gamma z(\theta_{r-t}\omega)} (1 + \|g(r + l - t)\|^2) dr \\
=&2Me^{-\mathbb{E}\delta t} \sup_{l \leq \tau} \int_{-\infty}^{-t} e^{\int_{-t}^r \delta(\theta_s\omega) ds - 2\gamma z(\theta_r\omega)} (1 + \|g(r + l)\|^2) dr \\
\leq&2Me^{-\frac{\mathbb{E}\delta}{2} t} \sup_{l \leq \tau} \int_{-\infty}^{-t} e^{\frac{\mathbb{E}\delta}{4} r} (1 + \|g(r + l)\|^2) dr \\
\leq&2Me^{-\frac{\mathbb{E}\delta}{2} t} \sup_{l \leq \tau} \int_{-\infty}^0 e^{\frac{\mathbb{E}\delta}{4} r} (1 + \|g(r + l)\|^2) dr \to 0.
\end{aligned} \tag{4.19}$$

Therefore, $\mathcal{K}_{\mathcal{B}} \in \mathcal{B}$.

(ii) Since $R(\tau, \omega) \leq \sup_{l \leq \tau} R(l, \omega)$, it follows that $\mathcal{K}_{\mathcal{D}} \subseteq \mathcal{K}_{\mathcal{B}} \in \mathcal{B} \subseteq \mathcal{D}$. Hence, we find that $\mathcal{K}_{\mathcal{D}}$ serves as a $\mathcal{D}$-pullback random absorbing set for Φ. □

4.2. Backward tail-estimates

Lemma 4.2. *Suppose that (H1)–(H3), (3.9), and (4.4) hold true. Then, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have that, for all $v_{l-t} \in B(l-t, \theta_{-t}\omega)$,*

$$\lim_{t,k\to+\infty} \sup_{l\leq\tau} \sum_{|i|\geq k} \xi_i |v_i(l, l-t, \theta_{-l}\omega, v_{i,l-t})|^2 = 0, \tag{4.20}$$

and for all $v_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$,

$$\lim_{t,k\to+\infty} \sum_{|i|\geq k} \xi_i |v_i(\tau, \tau-t, \theta_{-\tau}\omega, v_{i,\tau-t})|^2 = 0. \tag{4.21}$$

Proof. For a smooth function $\xi : [0,\infty) \to [0,1]$ with $\xi(s) = 0$ for $0 \leq s \leq 1$ and $\xi(s) = 1$ for $s \geq 2$, define

$$\xi_k = (\xi_{i,k})_{i\in\mathbb{Z}} = (\xi(\frac{|i|}{k}))_{i\in\mathbb{Z}}, \ \forall k \in \mathbb{N}.$$

Taking the inner product of (3.2) with $\xi_k v$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}} \xi_{i,k}|v_i|^2 + \nu(\theta_t\omega)e^{(p-2)\gamma z(\theta_t\omega)}(A_p v, \xi_k v) + (\lambda - \gamma z(\theta_t\omega))(v, \xi_k v) \\ =&e^{-\gamma z(\theta_t\omega)}(f(u), \xi_k v) + e^{-\gamma z(\theta_t\omega)}(g(t), \xi_k v). \end{aligned} \tag{4.22}$$

By Lemma 2 in [5], we have

$$(A_p v, \xi_k v) = (|Bv|^{p-2}(Bv), B(\xi_k v)) \geq -\frac{c}{k}\|v\|_p^p. \tag{4.23}$$

For the first term on the right side of (4.22), we have

$$e^{-\gamma z(\theta_t\omega)}(f(u), \xi_k v) = e^{-2\gamma z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \xi_{i,k}(f_i(u_i), u_i) \leq e^{-2\gamma z(\theta_t\omega)}(-\alpha_1 \sum_{i\in\mathbb{Z}} \xi_{i,k}(|u_i|^p + |u_i|^q) + \sum_{i\in\mathbb{Z}} \xi_{i,k}|\beta_i|^2). \tag{4.24}$$

Using Youngs inequality, we obtain

$$e^{-\gamma z(\theta_t\omega)}(g(t), \xi_k v) \leq \frac{\lambda}{2}\sum_{i\in\mathbb{Z}} \xi_{i,k}|v_i|^2 + \frac{1}{2\lambda}e^{-2\gamma z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \xi_{i,k}|g_i(t)|^2. \tag{4.25}$$

Therefore, we have

$$\begin{aligned} &\frac{d}{dt}\sum_{i\in\mathbb{Z}} \xi_{i,k}|v_i|^2 + (\lambda - 2\gamma z(\theta_t\omega))\sum_{i\in\mathbb{Z}} \xi_{i,k}|v_i|^2 + 2\alpha_1 e^{-2\gamma z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \xi_{i,k}(|u_i|^p + |u_i|^q) \\ \leq&\frac{c}{k}\nu(\theta_t\omega)e^{-2\gamma z(\theta_t\omega)}\|u\|_p^p + 2e^{-2\gamma z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \xi_{i,k}|\beta_i|^2 + \frac{1}{\lambda}e^{-2\gamma z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \xi_{i,k}|g_i(t)|^2. \end{aligned} \tag{4.26}$$

Multiplying (4.26) by $e^{\int_{l-h}^{t}\delta(\theta_r\omega)dr}$ and then integrating on $[l-h, l](h > 0)$, replacing ω by $\theta_{-l}\omega$, we get

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}\xi_{i,k}|v_i(l)|^2+\int_{l-h}^{l}(\lambda-2\gamma z(\theta_{t-l}\omega)-\delta(\theta_{t-l}\omega))e^{\int_l^t\delta(\theta_{r-l}\omega)dr}\sum_{i\in\mathbb{Z}}\xi_{i,k}|v_i(t)|^2dt\\
&+2\alpha_1\int_{l-h}^{l}e^{\int_l^t\delta(\theta_{r-l}\omega)dr-2\gamma z(\theta_{t-l}\omega)}\sum_{i\in\mathbb{Z}}\xi_{i,k}(|u_i(t)|^p+|u_i(t)|^q)dt\\
\le& e^{\int_l^{l-h}\delta(\theta_{r-l}\omega)dr}\sum_{i\in\mathbb{Z}}\xi_{i,k}|v_i(l-h)|^2+\frac{c}{k}\int_{l-h}^{l}\nu(\theta_{t-l}\omega)e^{\int_l^t\delta(\theta_{r-l}\omega)dr-2\gamma z(\theta_{t-l}\omega)}\|u(t)\|_p^pdt\\
&+c\int_{l-h}^{l}e^{\int_l^t\delta(\theta_{r-l}\omega)dr-2\gamma z(\theta_{t-l}\omega)}\sum_{i\in\mathbb{Z}}\xi_{i,k}(|\beta_i|^2+|g_i(t)|^2)dt.
\end{aligned}\tag{4.27}$$

By taking the supremum of (4.27) over $l\in(-\infty,\tau)$, we have

$$\begin{aligned}
&\sup_{l\le\tau}\sum_{|i|>k}\xi_i|v_i(l)|^2\\
\le&\sup_{l\le\tau}e^{\int_l^{l-h}\delta(\theta_{r-l}\omega)dr}\sum_{|i|>k}\xi_i|v_i(l-h)|^2+\frac{c}{k}\sup_{l\le\tau}\int_{l-h}^{l}\nu(\theta_{t-l}\omega)e^{\int_l^t\delta(\theta_{r-l}\omega)dr-2\gamma z(\theta_{t-l}\omega)}\|u(t)\|_p^pdt\\
&+c\sup_{l\le\tau}\int_{l-h}^{l}e^{\int_l^t\delta(\theta_{r-l}\omega)dr-2\gamma z(\theta_{t-l}\omega)}\sum_{|i|>k}\xi_i(|\beta_i|^2+|g_i(t)|^2)dt=:\sum_{j=1}^{3}J_j.
\end{aligned}\tag{4.28}$$

Since $v_{l-h}\in B(l-h,\theta_{-h}\omega)$, it follows from (4.10) that

$$\begin{aligned}
J_1&\le e^{\int_0^{-h}\delta(\theta_r\omega)dr}\sup_{l\le\tau}\sum_{i\in\mathbb{Z}}|v_i(l-h)|^2\\
&=e^{\int_0^{-h}\delta(\theta_r\omega)dr}\sup_{l\le\tau}\|v(l-h)\|^2\\
&\le e^{-\mathbb{E}\delta h+c(\omega)}\sup_{l\le\tau}\|B(l-h,\theta_{-h}\omega)\|^2\to0\text{ as }h\to+\infty.
\end{aligned}\tag{4.29}$$

By (4.8), we obtain

$$\begin{aligned}
J_2&=\frac{c}{k}\sup_{l\le\tau}\int_{l-h}^{l}\nu(\theta_{t-l}\omega)e^{\int_l^t\delta(\theta_{r-l}\omega)dr-2\gamma z(\theta_{t-l}\omega)}\|u(t)\|_p^pdt\\
&=\frac{c}{k}\sup_{l\le\tau}\int_{-h}^{0}\nu(\theta_t\omega)e^{\int_0^t\delta(\theta_r\omega)dr-2\gamma z(\theta_t\omega)}\|u(t+l)\|_p^pdt\\
&\le cR(l,\omega)\to0\text{ as }h\to+\infty.
\end{aligned}\tag{4.30}$$

By (4.17), we have

$$\begin{aligned}
&\int_0^t\delta(\theta_r\omega)dr-2\gamma z(\theta_t\omega)\\
=&\int_0^t(\delta(\theta_r\omega)-\mathbb{E}\delta)dr+\mathbb{E}\delta t-2\gamma z(\theta_t\omega)\\
\le&\frac{\mathbb{E}\delta}{2}|t|+\mathbb{E}\delta t+2\gamma|z(\theta_t\omega)|\\
=&\frac{\mathbb{E}\delta}{4}t.
\end{aligned}\tag{4.31}$$

which, along with (3.5), we obtain

$$\begin{aligned} J_3 &= c\sup_{l\le\tau}\int_{l-h}^{l} e^{\int_l^t \delta(\theta_{r-l}\omega)dr-2\gamma z(\theta_{t-l}\omega)}\sum_{|i|>k}\xi_i(|\beta_i|^2+|g_i(t)|^2)dt \\ &= c\sup_{l\le\tau}\int_{-h}^{0} e^{\int_0^t \delta(\theta_r\omega)dr-2\gamma z(\theta_t\omega)}\sum_{|i|>k}\xi_i(|\beta_i|^2+|g_i(t+l)|^2)dt \\ &\le c\sup_{l\le\tau}\int_{-h}^{0} e^{\frac{\mathbb{E}\delta}{4}t}\sum_{|i|>k}\xi_i(|\beta_i|^2+|g_i(t+l)|^2)dt \to 0 \text{ as } k,h\to+\infty. \end{aligned} \tag{4.32}$$

This result, along with (4.29) and (4.30), implies (4.20). Similar to the steps above, we can prove (4.21). □

4.3. Existence of pullback random attractors

Theorem 4.1. *Assume (H1)–(H3), (3.9), and (4.4) hold. Let* Φ *be the RDS generated by (3.2), then*

(i) Φ *has a* $\mathcal{B}$*-pullback random attractor* $\mathcal{A}_{\mathcal{B}}\in\mathcal{B}$*, given by*

$$\mathcal{A}_{\mathcal{B}}(\tau,\omega)=\bigcap_{s>0}\overline{\bigcup_{t\ge s}\Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{K}_{\mathcal{B}}(\tau-t,\theta_{-t}\omega)}. \tag{4.33}$$

(ii) $\mathcal{A}_{\mathcal{B}}$ *is backward compact in* l^2*, meaning that the union* $\bigcup_{l\le\tau}\mathcal{A}_{\mathcal{B}}(l,\omega)$ *is pre-compact in* l^2 *for each* $\tau\in\mathbb{R}$*,* $\omega\in\Omega$*;*

(iii) Φ *has a* $\mathcal{D}$*-pullback random attractor* $\mathcal{A}_{\mathcal{D}}\in\mathcal{D}$*, given by*

$$\mathcal{A}_{\mathcal{D}}(\tau,\omega)=\bigcap_{s>0}\overline{\bigcup_{t\ge s}\Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{K}_{\mathcal{D}}(\tau-t,\theta_{-t}\omega)}. \tag{4.34}$$

(iv) $\mathcal{A}_{\mathcal{B}}=\mathcal{A}_{\mathcal{D}}\in\mathcal{D}$*, that is,* $\mathcal{A}_{\mathcal{B}}$ *is a unique* $\mathcal{D}$*-pullback random attractor of the cocycle* Φ *and is backward compact in* l^2*.*

Proof. (i) We first show that Φ is $\mathcal{B}$-pullback asymptotically compact in l^2. Let $B\in\mathcal{B}$, $\tau\in\mathbb{R}$, and $\omega\in\Omega$, and consider a sequence $v_{l_n-t_n}\in B(l_n-t_n,\theta_{-t_n}\omega)$ where $l_n\le\tau$ and $t_n\to+\infty$. Define $v^n=\Phi(t_n,l_n-t_n,\theta_{-t_n}\omega)v_{l_n-t_n}$. We need to show $\{v^n\}$ is pre-compact in l^2. By Lemma 4.1, we know that there exists $N_1=N_1(\tau,\omega,B)$ such that for all $n\ge N_1$,

$$\|v^n\|^2\le 2M\sup_{l\le\tau}R(l,\omega). \tag{4.35}$$

By Lemma 4.2, we obtain that there exist $I>k$ and $N_2=N_2(\varepsilon)>N_1$ such that for all $n\ge N_2$,

$$\sum_{|i|>I}\xi_i|v_i^n|^2\le\varepsilon. \tag{4.36}$$

By (4.35), $\{(|v_i^n|)_{|i|\le I}: n\ge N_2\}$ is bounded in $\mathbb{R}^{2I+1}$ and thus pre-compact in $\mathbb{R}^{2I+1}$. Hence, $\{(|v_i^n|)_{|i|\le I}: n\ge N_2\}$ has a finite ε-net in $\mathbb{R}^{2I+1}$, which together with (4.36) implies that $\{(|v_i^n|)_{i\in\mathbb{Z}}: n\ge N_2\}$ has a finite 2ε-net in l^2. On the other hand, the finite set $\{(|v_i^n|)_{i\in\mathbb{Z}}: n\le N_2\}$ has a 2ε-net, thus $\{v^n: n\in\mathbb{N}\}$ is pre-compact in l^2.

(ii) Let $\{v^n : n \in \mathbb{N}\}$ be any sequence in $\bigcup_{l\le\tau} \mathcal{A}_{\mathcal{B}}(l,\omega)$; then, there exists $l_n \le \tau$ such that $v^n \in \mathcal{A}_{\mathcal{B}}(l_n,\omega)$ for each $n \in \mathbb{N}$. Let $t_n \to +\infty$, by the invariance of $\mathcal{A}_{\mathcal{B}}$, we obtain $v^n = \Phi(t_n, l_n - t_n, \theta_{-t_n}\omega)\mathcal{A}_{\mathcal{B}}(l_n - t_n, \theta_{-t_n}\omega)$, which implies that there exists $v_{l_n-t_n} \in \mathcal{A}_{\mathcal{B}}(l_n - t_n, \theta_{-t_n}\omega)$ such that

$$v^n = \Phi(t_n, l_n - t_n, \theta_{-t_n}\omega, v_{l_n-t_n}). \tag{4.37}$$

Since $v_{l_n-t_n} \in \mathcal{A}_{\mathcal{B}}(l_n - t_n, \theta_{-t_n}\omega)$ when $l_n \le \tau$, which along with the pre-compact of the sequence $\{v^n : n \in \mathbb{N}\}$ implies that $\bigcup_{l\le\tau} \mathcal{A}_{\mathcal{B}}(l,\omega)$ is pre-compact in l^2.

(iii) By Corollary 4.1(ii), $\mathcal{K}_{\mathcal{D}} = \{\mathcal{K}_{\mathcal{D}}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is a $\mathcal{D}$-pullback random absorbing set for Φ. Following a similar proof (i), Φ is $\mathcal{D}$-pullback asymptotically compact in l^2. Therefore, Φ admits a $\mathcal{D}$-pullback random attractor $\mathcal{A}_{\mathcal{D}}$.

(iv) By the definition of $\mathcal{K}_{\mathcal{D}}$ and $\mathcal{K}_{\mathcal{B}}$, we obtain that $\mathcal{K}_{\mathcal{D}} \subseteq \mathcal{K}_{\mathcal{B}}$; together with the definition of $\mathcal{A}_{\mathcal{D}}$ and $\mathcal{A}_{\mathcal{B}}$, we obtain that $\mathcal{A}_{\mathcal{D}} \subseteq \mathcal{A}_{\mathcal{B}}$. On the other hand, since $\mathcal{A}_{\mathcal{B}} \in \mathcal{B} \subseteq \mathcal{D}$, $\mathcal{A}_{\mathcal{B}}$ is attracted by $\mathcal{A}_{\mathcal{D}}$. Using invariance of $\mathcal{A}_{\mathcal{B}}$, we obtain

$$dist_{l^2}(\mathcal{A}_{\mathcal{B}}(\tau,\omega), \mathcal{A}_{\mathcal{D}}(\tau,\omega)) = dist_{l^2}(\Phi(t, \tau - t, \theta_{-t}\omega)\mathcal{A}_{\mathcal{B}}(\tau - t, \theta_{-t}\omega), \mathcal{A}_{\mathcal{D}}(\tau,\omega)) \to 0.$$

Thus, $\mathcal{A}_{\mathcal{B}}(\tau,\omega) = \mathcal{A}_{\mathcal{D}}(\tau,\omega)$, and Φ has the unique $\mathcal{D}$-pullback random attractor $\mathcal{A}_{\mathcal{B}}$, which is backward compact in l^2. □

5. Asymptotically autonomous stability of pullback random attractors

In this section, we explore the asymptotic autonomous stability of the time section $\mathcal{A}_{\mathcal{B}}(\tau,\omega)$ of the pullback random attractor $\mathcal{A}_{\mathcal{B}} = \{\mathcal{A}_{\mathcal{B}}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in l^2 as τ approaches negative infinity.

Consider the autonomous version of (1.1):

$$\begin{cases} \frac{du_i}{dt} + \nu(\theta_t\omega)(A_p u)_i + \lambda u_i = f_i(u_i) + g_{\infty,i} + u_i \frac{dW}{dt}, \\ u_i(\tau) = u_{i,\tau}, \ \tau \in \mathbb{R}, i \in \mathbb{Z}. \end{cases} \tag{5.1}$$

Using the variables $v(t) = e^{-\gamma z(\theta_t\omega)}u(t)$, we obtain the following random equation:

$$\begin{cases} \frac{dv}{dt} + \nu(\theta_t\omega)e^{(p-2)\gamma z(\theta_t\omega)}(A_p v) + (\lambda - \gamma z(\theta_t\omega))v = e^{-\gamma z(\theta_t\omega)} f(e^{\gamma z(\theta_t\omega)}v) + e^{-\gamma z(\theta_t\omega)} g_\infty, \\ v(\tau) = v_\tau = e^{-\gamma z(\theta_\tau\omega)}u_\tau, \ \tau \in \mathbb{R}, \end{cases} \tag{5.2}$$

where $v = (v_i)_{i\in\mathbb{Z}} \in l^2$, $v_\tau = (v_{i,\tau})_{i\in\mathbb{Z}} \in l^2$, $f(v) = (f_i(v_i))_{i\in\mathbb{Z}}$, $g_\infty = \{g_{\infty,i}\}_{i\in\mathbb{Z}} \in l^2$ and $\lambda, \gamma > 0$.

Using a method similar to that outlined in Section 3, it can be demonstrated that if assumptions (H1)–(H3) and (3.9) are satisfied, then for every $v_0 \in l^2$ and $\omega \in \Omega$, the autonomous random Eq (5.2) has a unique solution $v_\infty(\cdot,\omega,v_0) : [0,\infty) \to l^2$. Consequently, we can define an autonomous RDS as follows:

$$\Phi_\infty : \mathbb{R}^+ \times \Omega \times l^2 \to l^2,$$

given by, for every $t \ge 0$ and $\omega \in \Omega$,

$$\Phi_\infty(t,\omega,v_0) = v_\infty(t,\omega,v_0). \tag{5.3}$$

Let $D_\infty = \{D_\infty(\omega) : \omega \in \Omega\}$ be a family of bounded, nonempty subsets of l^2. This family is termed tempered if, for every $\omega \in \Omega$,

$$\lim_{t\to-\infty} e^{-\mathbb{E}\delta t}\|D_\infty(\theta_{-t}\omega)\| = 0. \tag{5.4}$$

Let $\mathcal{D}_\infty$ be the universe of all families of bounded, nonempty random subsets of l^2 satisfying (5.4). It is straightforward to prove that Φ_∞ has a unique $\mathcal{D}_\infty$-pullback random attractor $\mathcal{A}_\infty = \{\mathcal{A}_\infty(\omega) : \omega \in \Omega\} \in \mathcal{D}_\infty$. The primary objective of this section is to establish that the time section $\mathcal{A}_{\mathcal{B}}(\tau, \omega)$ of $\mathcal{A}_{\mathcal{B}}$ is upper semi-continuous with respect to $\mathcal{A}_\infty(\omega)$ as $\tau \to -\infty$ in the sense of the Hausdorff semi-distance of l^2.

Next, we analyze the asymptotically autonomous convergence of solutions to Eqs (3.2) and (5.2).

Lemma 5.1. *Suppose that the assumptions (H1)–(H3) and (3.9) hold. Then the solution of problem (3.2) convergences to the solution of (5.2), that is,*

$$\lim_{\tau\to-\infty} \|\Phi(t, \tau, \omega, v_\tau) - \Phi_\infty(t, \omega, v_0)\| = 0, \ \ \forall t \geq 0, \ \ \omega \in \Omega,$$

whenever $\|v_\tau - v_0\| \to 0$ *as* $\tau \to -\infty$.

Proof. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $T > 0$. For any $t \in [0, T]$, denote

$$\widetilde{v}(t) = v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau) - v_\infty(t, \omega, v_0). \tag{5.5}$$

By (3.2) and (5.2), we have

$$\frac{d\widetilde{v}(t)}{dt} + \nu(\theta_t\omega)e^{(p-2)\gamma z(\theta_t\omega)}A_p\widetilde{v} + (\lambda - \gamma z(\theta_t\omega))\widetilde{v} = e^{-\gamma z(\theta_t\omega)}(f(u(t+\tau)) - f(u_\infty(t))) + e^{-\gamma z(\theta_t\omega)}(g(t+\tau) - g_\infty). \tag{5.6}$$

Taking the inner product of (5.6) with $\widetilde{v}(t)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\widetilde{v}(t)\|^2 + \nu(\theta_t\omega)e^{(p-2)\gamma z(\theta_t\omega)}(A_p\widetilde{v}, \widetilde{v}) + (\lambda - \gamma z(\theta_t\omega))\|\widetilde{v}\|^2 \\ =&e^{-\gamma z(\theta_t\omega)}(f(u(t + \tau)) - f(u_\infty(t)), \widetilde{v}) + e^{-\gamma z(\theta_t\omega)}(g(t + \tau) - g_\infty, \widetilde{v}). \end{aligned} \tag{5.7}$$

We now estimate all the terms on the right-hand side of (5.7). For the first term, we obtain from (H1) that

$$e^{-\gamma z(\theta_t\omega)}(f(u(t + \tau)) - f(u_\infty(t), \widetilde{v}) \leq \alpha_3\|\widetilde{v}\|^2. \tag{5.8}$$

For the second term, we obtain from Young's inequality that

$$e^{-\gamma z(\theta_t\omega)}(g(t + \tau) - g_\infty, \widetilde{v}) \leq \frac{1}{\lambda}e^{-2\gamma z(\theta_t\omega)}\|g(t + \tau) - g_\infty\|^2 + \frac{\lambda}{4}\|\widetilde{v}\|^2. \tag{5.9}$$

Therefore, we have

$$\begin{aligned} &\frac{d}{dt}\|\widetilde{v}(t)\|^2 + 2\nu(\theta_t\omega)e^{(p-2)\gamma z(\theta_t\omega)}(A_p\widetilde{v}, \widetilde{v}) + \frac{\lambda}{2}\|\widetilde{v}\|^2 \\ \leq&(2\gamma z(\theta_t\omega) - \lambda + 2\alpha_3)\|\widetilde{v}\|^2 + \frac{2}{\lambda}e^{-2\gamma z(\theta_t\omega)}\|g(t + \tau) - g_\infty\|^2. \end{aligned} \tag{5.10}$$

Multiplying (5.10) by $e^{\int_0^t \delta(\theta_s\omega)ds}$, and then integrating on $[0,r]$, $r \in [0,T]$, we have

$$\begin{aligned}\|\widetilde{v}(r)\|^2 \le & e^{\int_r^0 \delta(\theta_s\omega)ds}\|\widetilde{v}(0)\|^2 + \int_0^r (2\gamma z(\theta_t\omega) - \lambda + 2\alpha_3 + \delta(\theta_t\omega))e^{\int_r^t \delta(\theta_s\omega)ds}\|\widetilde{v}(t)\|^2 dt \\ & + \frac{2}{\lambda}\int_0^r e^{\int_r^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}\|g(t+\tau) - g_\infty\|^2 dt \\ \le & e^{\int_r^0 \delta(\theta_s\omega)ds}\|\widetilde{v}(0)\|^2 + \frac{2}{\lambda}\int_0^r e^{\int_r^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}\|g(t+\tau) - g_\infty\|^2 dt.\end{aligned} \tag{5.11}$$

By the property of $\delta(\theta_s\omega)$ and $z(\theta_t\omega)$, we have

$$\begin{aligned}&\int_0^r e^{\int_r^t \delta(\theta_s\omega)ds - 2\gamma z(\theta_t\omega)}\|g(t+\tau) - g_\infty\|^2 dt \\ \le & c_T(\omega)\int_0^T \|g(t+\tau) - g_\infty\|^2 dt \\ \le & c_T(\omega)\int_{-\infty}^{\tau+T} \|g(t) - g_\infty\|^2 dt \to 0 \text{ as } \tau \to -\infty.\end{aligned} \tag{5.12}$$

Note that $\|\widetilde{v}(0)\|^2 = \|v_\tau - v_0\|^2 \to 0$ as $\tau \to -\infty$, so we obtain $\|\widetilde{v}(t)\|^2 \to 0$ as $\tau \to -\infty$. Hence, the proof is completed. □

Next, we establish the asymptotically autonomous stability of the time section of the pullback random attractor $\mathcal{A}_{\mathcal{B}} = \{\mathcal{A}_{\mathcal{B}}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in l^2 as time $\tau \to -\infty$.

Theorem 5.1. *Assume (H1)–(H3), (3.9), and (4.4) hold. Let $\mathcal{A}_{\mathcal{B}}$ be the $\mathcal{B}$-pullback random attractor of non-autonomous RDS Φ, and let $\mathcal{A}_\infty$ be the $\mathcal{D}_\infty$-pullback random attractor of autonomous RDS Φ_∞. Then $\mathcal{A}_{\mathcal{B}}$ is asymptotically autonomous to $\mathcal{A}_\infty$ in probability, that is, $\forall \varepsilon > 0$,*

$$\lim_{\tau\to-\infty} \mathbb{P}\{\omega \in \Omega : dist(\mathcal{A}_{\mathcal{B}}(\tau,\omega), \mathcal{A}_\infty(\omega)) > \varepsilon\} = 0. \tag{5.13}$$

Proof. Define the set

$$\Omega_1 = \{\omega \in \Omega : \lim_{\tau\to-\infty} dist(\mathcal{A}_{\mathcal{B}}(\tau,\omega), \mathcal{A}_\infty(\omega)) = 0\}. \tag{5.14}$$

We aim to show that $\mathbb{P}(\Omega_1) = 1$. Suppose instead that $\mathbb{P}(\Omega_1) < 1$, which implies $\Omega_2 = \Omega\backslash\Omega_1 \neq \emptyset$. Then, there exists $\omega_0 \in \Omega_2$, meaning $\omega_0 \notin \Omega_1$. Consequently, there exist $\varepsilon_0 > 0$ and and a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n \to -\infty$ as $n \to \infty$ such that

$$dist(\mathcal{A}_{\mathcal{B}}(\tau_n,\omega_0), \mathcal{A}_\infty(\omega_0)) \ge 3\varepsilon_0, \ \forall n \in \mathbb{N}. \tag{5.15}$$

We further take a sequence $\{x_n\}_{n=1}^\infty$ from $\mathcal{A}_{\mathcal{B}}(\tau_n,\omega_0)$ such that

$$dist(x_n, \mathcal{A}_\infty(\omega_0)) \ge 3\varepsilon_0. \tag{5.16}$$

Define

$$\mathcal{A}_{\mathcal{B}}(\tau_0,\omega) = \bigcup_{\tau\le\tau_0} \mathcal{A}_{\mathcal{B}}(\tau,\omega) \subset \bigcup_{\tau\le\tau_0} \mathcal{K}_{\mathcal{B}}(\tau,\omega), \tag{5.17}$$

where, by the monotonicity of $\mathcal{K}_{\mathcal{B}}(\tau, \omega)$, we have

$$\cup_{\tau \le \tau_0} \mathcal{K}_{\mathcal{B}}(\tau, \omega) = \mathcal{K}_{\mathcal{B}}(\tau_0, \omega).$$

From Corollary 4.1, it follows that $\mathcal{A}_{\mathcal{B}}(\tau_0, \omega) \in \mathcal{D}_\infty$, meaning $\mathcal{A}_{\mathcal{B}}(\tau_0, \omega)$ can be attracted by the attractor $\mathcal{A}_\infty$. Thus, there exists $n_0 \in \mathbb{N}$ such that

$$dist(\Phi_\infty(|\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0) \mathcal{A}_{\mathcal{B}}(\tau_0, \omega_0), \mathcal{A}_\infty(\omega_0)) \le \varepsilon_0.$$

By the continuity of Φ_∞, we obtain

$$dist(\Phi_\infty(|\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0) \overline{\mathcal{A}_{\mathcal{B}}(\tau_0, \omega_0)}, \mathcal{A}_\infty(\omega_0)) \le \varepsilon_0. \tag{5.18}$$

On the other hand, by the invariance of $\mathcal{A}_{\mathcal{B}}$, we have

$$\mathcal{A}_{\mathcal{B}}(\tau_n, \omega_0) = \Phi(|\tau_{n0}|, \tau_n - |\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0) \mathcal{A}_{\mathcal{B}}(\tau_n - |\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0).$$

Therefore, $x_n \in \mathcal{A}_{\mathcal{B}}(\tau_n, \omega_0)$ can be written as

$$x_n = \Phi(|\tau_{n0}|, \tau_n - |\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0) y_n, \tag{5.19}$$

where $y_n \in \mathcal{A}_{\mathcal{B}}(\tau_n - |\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0)$. Since $\{y_n : n \ge n_0\} \subset \mathcal{A}_{\mathcal{B}}(\tau_0, \theta_{\tau_{n_0}} \omega_0)$, the backward compactness of $\mathcal{A}_{\mathcal{B}}$ ensures that $\mathcal{A}_{\mathcal{B}}(\tau_0, \theta_{\tau_{n_0}} \omega_0)$ is precompact in l^2. Hence, $\{y_n\}_{n=1}^\infty$ has a convergence subsequence such that, for some $y_0 \in \overline{\mathcal{A}_{\mathcal{B}}(\tau_0, \theta_{\tau_{n_0}} \omega_0)}$,

$$y_{n_k} \to y_0 \text{ as } k \to \infty.$$

Applying the asymptotically autonomous convergence for Φ, we find, for sufficiently large k,

$$||x_{n_k} - \Phi_\infty(|\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0) y_0|| = ||\Phi(|\tau_{n0}|, \tau_{n_k} - |\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0) y_{n_k} - \Phi_\infty(|\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0) y_0|| \le \varepsilon_0.$$

This, along with (5.18) implies

$$dist(x_{n_k}, \mathcal{A}_\infty(\omega_0)) \le ||x_{n_k} - \Phi_\infty(|\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0) y_0|| + dist(\Phi_\infty(|\tau_{n0}|, \theta_{\tau_{n_0}} \omega_0) y_0, \mathcal{A}_\infty(\omega_0)) \le 2\varepsilon_0,$$

which indeed is a contradiction to (5.15), and thus (5.13) is proved. □

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

References

1. L. Arnold, *Random dynamical systems*, Berlin: Springer-Verlag, 1998. https://doi.org/10.1007/978-3-662-12878-7
2. T. Caraballo, B. Guo, N. Tuan, R. Wang, Asymptotically autonomous robustness of random attractors for a class of weakly dissipative stochastic wave equations on unbounded domains, *Proc. Roy. Soc. Edinb. A*, **151** (2021), 1700–1730. https://doi.org/10.1017/prm.2020.77
3. P. Chen, R. Wang, X. Zhang, Asymptotically autonomous robustness of random attractors for 3D BBM equations driven by nonlinear colored noise, *SIAM J. Math. Anal.*, **56** (2024), 254–274. https://doi.org/10.1137/22M1529129
4. P. Chen, X. Zhang, Upper semi-continuity of attractors for non-autonomous fractional stochastic parabolic equations with delay, *Discrete Cont. Dyn.-B*, **26** (2021), 4325–4357. https://doi.org/10.3934/dcdsb.2020290
5. A. Gu, P. E. Kloeden, Asymptotic behavior of a nonautonomous p-Laplacian lattice system, *Int. J. Bifurcat. Chaos*, **26** (2016), 1650174. https://doi.org/10.1142/S0218127416501741
6. P. G. Geredeli, On the existence of regular global atractor for p-Laplacian evolution equations, *Appl. Math. Optim.*, **71** (2015), 517–532. https://doi.org/10.1007/s00245-014-9268-y
7. B. Gess, Random attractors for singular stochastic evolution equations, *J. Differ. Equations*, **255** (2013), 524–559. https://doi.org/10.1016/j.jde.2013.04.023
8. K. Kinra, M. T. Mohan, R. Wang, Asymptotically autonomous robustness in probability of non-autonomous random attractors for stochastic convective Brinkman-Forchheimer equations on $\mathbb{R}^3$, *Int. Math. Res. Notices*, **7** (2024), 5850–5893. https://doi.org/10.1093/imrn/rnad279
9. P. E. Kloeden, T. Lorenz, Construction of nonautonomous forward attractors, *Proc. Amer. Math. Soc.*, **144** (2016), 259–268. http://dx.doi.org/10.1090/proc/12735
10. P. E. Kloeden, J. Simsen, M. S. Simsen, Asymptotically autonomous multivalued Cauchy problems with spatially variable exponebnts, *J. Math. Anal. Appl.*, **445** (2017), 513–531. https://doi.org/10.1016/j.jmaa.2016.08.004
11. P. E. Kloeden, M. Rasmussen, *Nonautonomous dynamical systems*, Providence: American Mathematical Society, 2011.
12. J. L. Lions, *Quelques methodes de resolution des problemes aux limites non lineaires (French)*, Paris: Dunod, 1969.
13. Y. Li, J. Yin, Existence, regularity and approximation of global attractors for weaky dissipative p-Laplacian quations, *Discrete Cont. Dyn.-S*, **9** (2016), 1939–1957. https://doi.org/10.3934/dcdss.2016079
14. Y. Li, R. Wang, J. Yin, Backward compact attractors for non-autonomous Benjamin Bona-Mahony equations on unbounded channels, *Discrete Cont. Dyn.-B*, **22** (2017), 2569–2586. https://doi.org/10.3934/dcdsb.2017092
15. Y. Li, L. She, J. Yin, Longtime robustness and semi-uniform conpactness of a pullback attractor via nonautonomous PDE, *Discrete Cont. Dyn.-B*, **23** (2018), 1535–1557. https://doi.org/10.3934/dcdsb.2018058

16. J. L. Vázquez, *The porous medium equation: mathematical theory*, Oxford: Oxford University Press, 2007. https://doi.org/10.1093/acprof:oso/9780198569039.001.0001

17. B. Wang, Sufficient and necessary criteria for existence of pullback attractors for noncompact random dynamical systems, *J. Differ. Equations*, **253** (2012), 1544–1583. https://doi.org/10.1016/j.jde.2012.05.015

18. B. Wang, B. Guo, Asymptotic behavior of non-autonomous stochastic parabolic equations with nonlinear Laplacian principal part, *Electron. J. Differ. Eq.*, **2013** (2013), 1–25.

19. R. Wang, K. Kinra, M. T. Mohan, Asymptotically autonomous robustness in probability of random attractors for stochastic Navier-Stokes equations on unbounded Poincaré domains, *SIAM J. Math. Anal.*, **55** (2023), 2644–2676. https://doi.org/10.1137/22M1517111

20. R. Wang, Y. Li, Asymptotic autonomy of kernel section for Newton-Boussinesq equations on unbounded zonary domains, *Dynam. Part. Differ. Eq.*, **16** (2019), 295–316. https://doi.org/10.4310/dpde.2019.v16.n3.a4

21. M. Yang, C. Sun, C. Zhong, Existence of a global attractor for a p-Laplacian equation in $\mathbb{R}^N$, *Nonlinear Anal.-Theor.*, **66** (2007), 1–13. https://doi.org/10.1016/j.na.2005.11.004

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