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Research article

The existence of a nontrivial solution to an elliptic equation with critical Sobolev exponent and a general potential well

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Abstract: The purpose of this paper is to examine a class of elliptic problems that involve negative potentials $a \in L^{\frac{N}{2}}(\Omega)$ and critical nonlinearities. To discuss this, the well-known eigenvalue problem $-\Delta - a$ is considered. Under some mild assumptions, an existence result is obtained, which extends the existing results to the critical case.

Keywords: elliptic equation; existence; general potential; critical Sobolev exponent
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1. Introduction

1.1. Elliptic problems with critical nonlinearities

Over the years, with the aid of variational methods, for varying conditions of the potential and nonlinearity, the existence and multiplicity of solutions for elliptic equations have been extensively discussed in the literature, among which we highlight [1, 4, 5]. One interesting characteristic is that the potential function can be negative or indefinite, as shown in [7, 8, 11]. On the other hand, equations with critical growth raise interest; see [2, 6, 12].

Li and Wang [11] established the existence result for the following equation:

$$\begin{cases} -\Delta u - a(x)u = f(x, u) \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $a \in L^{\frac{N}{2}}(\Omega)$ and $f(x, u)$ is superlinear at $u = 0$ and subcritical at $u = \infty$. They established the existence result for the equation above without assuming that the Ambrosetti-Rabinowitz condition holds.

Ke and Tang [8] studied (1.1) where $a \in L^{\frac{N}{2}}(\Omega)$ and g has super-linear but sub-critical growth. By introducing a new super-linear condition, they proved the existence and multiplicity of solutions. In [7], they gave the existence and multiplicity results for Eq (1.1) with $a(x) = \lambda_k - V(x)$, where $V \in L^{\frac{N}{2}}(\Omega)$, g is sublinear, and λ_k denotes the kth eigenvalue for the elliptic linear operator $-\Delta + V(x)$ with zero Dirichlet boundary condition.

The works [13, 15, 16] were devoted to studying the critical equation

$$\begin{cases} -\Delta u - \lambda u = f(x,u) + |u|^{2^*-2}u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.2}$$

where $f = 0$ in [13, 16] and f is a lower order perturbation of the critical power $|u|^{2^*-2}u$ in [13, 16]. In [10], they were concerned with the existence and bifurcation of nontrivial solutions for Eq (1.2) with $\lambda = 0$ and $f(x,u) = \mu g(x,u)$ with $\mu > 0$. The single solution results obtained by [10] extend the main results of [15].

Inspired by the research mentioned above, this paper focuses on the existence of nontrivial solutions for the critical problem

$$\begin{cases} -\Delta u - a(x)u = f(x,u) + |u|^{2^*-2}u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain, $N \geq 3$, and $0 < a \in L^{\frac{N}{2}}(\Omega)$. We believe that it is an intriguing question to ask whether there exists a nontrivial solution for Eq (1.3) with $a \in L^{\frac{N}{2}}(\Omega)$. Although there have been some works on this type of potential (see [3, 8, 11]), to the best of our knowledge, no attempt has been made to answer this question for the critical term. In this paper, we will provide an affirmative answer to this question. The result we obtained extends the results of [8] and [11] to the critical case.

The main objective of this paper is to construct nontrivial solutions of (1.3) using variational techniques. Our strategy will depend on whether $\lambda_1 \leq 0$ or $\lambda_1 > 0$, where λ_1 denotes the eigenvalue for the elliptic linear operator $-\Delta - a$ with zero Dirichlet boundary condition. If $\lambda_1 \leq 0$, we will use the Linking theorem to obtain a nontrivial solution. On the other hand, if $\lambda_1 > 0$, the Mountain Pass theorem will be effective. In order to achieve this, we will consider the well-known eigenvalue problem $-\Delta - a$.

1.2. Notations

In order to study the problem mentioned above, we shall consider its weak formulation, given by

$$\begin{aligned} &\int_\Omega \nabla u(x)\nabla\phi(x)dx - \int_\Omega a(x)u(x)\phi(x)dx \\ &= \int_\Omega f(x,u(x))\phi(x)dx + \int_\Omega |u(x)|^{2^*-2}u(x)\phi(x)dx, \quad \forall\phi,\ u \in H_0^1(\Omega), \end{aligned}$$

where the Hilbert space $H_0^1(\Omega)$ is defined as the closure $\mathcal{D}(\Omega)$ in $H^1(\mathbb{R}^N)$ with the scalar product

$$\langle u, v\rangle = \int_\Omega \nabla u(x)\nabla v(x)dx,$$

for any $u, v \in H_0^1(\Omega)$ and the norm $||u||^2 = \langle u, u\rangle$.

We can observe that Eq (1.3) is an Euler-Lagrange equation of the functional $\mathcal{J} : H_0^1(\Omega) \to \mathbb{R}$ defined as follows

$$\mathcal{J}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega a(x)u^2(x)dx - \int_\Omega F(x,u(x))dx - \frac{1}{2^*}|u|_{2^*}^{2^*}.$$

Moreover, $\mathcal{J} \in C^1(H_0^1(\Omega), \mathbb{R})$ and for any $u,\ \phi \in H_0^1(\Omega)$, we have

$$\begin{aligned}\langle \mathcal{J}'(u), \phi\rangle = &\int_\Omega \nabla u(x)\nabla\phi(x)dx - \int_\Omega a(x)u(x)\phi(x)dx \\ &- \int_\Omega f(x,u(x))\phi(x)dx - \int_\Omega |u(x)|^{2^*-2}u(x)\phi(x)dx.\end{aligned}$$

Now we recall an eigenvalue problem (see [11, 17]) related to the problem mentioned earlier. Let $\{\lambda_k\}$ be the positive and increasing sequence of the eigenvalues of the following problem

$$\begin{cases} -\Delta u - a(x)u = \lambda u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \backslash \Omega \end{cases} \tag{1.4}$$

and the sequence $\{e_k\}$ of the eigenfunctions corresponding to $\{\lambda_k\}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^1(\Omega)$. Moreover,

$$-\infty < \lambda_1 \le \lambda_2 \le ... \le \lambda_k \le \lambda_{k+1} \le ...,$$

where

$$\lambda_{k+1} = \min_{u\in P_{k+1}\backslash\{0\}} \frac{\|u\|^2 - \int_\Omega a(x)u^2(x)dx}{\int_\Omega u^2(x)dx}, k \in \mathbb{N},$$

$P_{k+1} = \{u \in H_0^1(\Omega) | \left\langle u, e_j\right\rangle = 0,\ j = 1, ..., k\}$. If $\lambda_1 \le 0$, for convenience, we set

$$-\infty < \lambda_1 \le \lambda_2 \le ... \le \lambda_k \le 0 < \lambda_{k+1} \le ....$$

Corresponding, let

$$Y = \{e_1, ..., e_k\} \text{ and } Z = \{u \in H_0^1(\Omega), \langle u, v\rangle_{L^2} = 0, v \in Y\}, \tag{1.5}$$

thereby, $H_0^1(\Omega) = Y \oplus Z$. We need the following constants:

$$S = \inf_{u\in H_0^1(\Omega)\backslash\{0\}} \frac{\|u\|^2}{(\int_\Omega |u(x)|^{2^*}dx)^{\frac{2}{2^*}}}, \tag{1.6}$$

$$S_a = \inf_{u\in H_0^1(\Omega)\backslash\{0\}} S_a(u),$$

where

$$S_a(u) = \frac{\|u\|^2 - \int_\Omega a(x)|u(x)|^2dx}{(\int_\Omega |u(x)|^{2^*}dx)^{\frac{2}{2^*}}}. \tag{1.7}$$

1.3. *Main results*

In this subsection, we present the main result of the paper. We consider the nonlinear partial differential Eq (1.3) with a Caratheodory function $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying conditions (f_1)–(f_4) (or (f_4')):

(f_1) $\sup\{|f(x,t)|\text{: a.e. } x \in \Omega, |t| \leq M\} < +\infty$ for any $M > 0$;
(f_2) $\lim\limits_{t\to 0} \frac{f(x,t)}{t} = 0$ and $\lim\limits_{|t|\to+\infty} \frac{f(x,t)}{|t|^{2^*-1}} = 0$ uniformly in $x \in \Omega$;
(f_3) there exists $\alpha > 2$ such that

$$0 < \alpha F(x,t) \leq f(x,t)t, \text{ for } t \neq 0,$$

where $F(x,t) = \int_0^t f(x,\tau)d\tau$;
(f_4) $\lim\limits_{t\to+\infty} \frac{F(x,t)}{t^4} = +\infty,$ when $N = 3$, uniformly in $x \in \Omega$;
$\lim\limits_{t\to+\infty} \frac{F(x,t)}{t^2|\ln t|} = +\infty,$ when $N = 4$, uniformly in $x \in \Omega$;
$\lim\limits_{t\to+\infty} \frac{F(x,t)}{t^2} = +\infty,$ when $N \geq 5$, uniformly in $x \in \Omega$;
(f_4') there exists $u_0 \in H_0^1(\Omega)\backslash\{0\}$ with $u_0 \geq 0$ a.e. in $\mathbb{R}^N$ such that $S_a(u) < S$ for any $u \in \mathbb{U}$, where $\mathbb{U} = \text{span}\{e_1, e_2, ..., e_k, u_0\},\ k \in \mathbb{N}$.

Our main result is as follows:

Theorem 1. *Suppose that conditions (f_1)–(f_3) and (f_4) (or (f_4')) hold. Then, problem (1.3) admits a nontrivial solution $u \in H_0^1(\Omega)\backslash\{0\}$.*

Remark 1. *We note that the function $a(x)$ and the critical exponent term pose natural difficulties in this problem. One difficulty is that the boundedness of the Palais-Smale sequence fails, and we need to apply certain inequalities to recover it. Another difficulty is to prove that the required level c is below the threshold. We use various techniques to overcome these difficulties. Our result extends the results of [8] and [11] to the critical case. Furthermore, if we set $s = 1$ in the fractional problem, our result generalizes the single-solution results of [10] and [15].*

The structure of the remaining part of this paper is as follows. In Sections 2 and 3, we present several lemmas and estimates that are crucial for the proof of the main theorem. These results, based on the relevant lemmas from Sections 2 and 3, are then used in Section 4 to complete the proof of the main theorem.

2. Preliminaries

In this section, we will first present some relevant information that will be useful. Some of the lemmas provided are standard, and readers familiar with them may proceed directly to the estimation part.

Lemma 1. *Assume conditions (f_1) and (f_2) hold; then for any $\varepsilon > 0$, there exists $M = M(\varepsilon) > 0$ such that for a.e. $x \in \Omega$ and any $t \in \mathbb{R}^N$,*

$$|f(x,t)| \leq 2^*\varepsilon|t|^{2^*-1} + M(\varepsilon)$$

and

$$|F(x,t)| \le \varepsilon |t|^{2^*} + M(\varepsilon)|t|,$$

where $F(x,t)$ is defined as in (f_3).

Lemma 2. *[17] Let $1 < r < \infty$. If $\{u_n\}$ is bounded in $L^r(\Omega)$ and $u_n \to u$ a.e. in Ω, then $u_n \rightharpoonup u$ in $L^r(\Omega)$.*

Lemma 3. *[17] The following assertions are true:*
(a) The embedding $H_0^1(\Omega) \hookrightarrow L^\nu(\Omega)$ is compact for any $\nu \in [1, 2^*)$.
(b) The embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous.

Lemma 4. *Suppose that $|\Omega| < \infty$, and conditions (f_1) and (f_2) hold. If $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then*

$$\int_\Omega F(x, u_n(x))dx \to \int_\Omega F(x, u(x))dx.$$

Proof. The proof is standard and omitted here. □

Lemma 5. *[7, Lemma 3.1] If $a \in L^{\frac{N}{2}}(\Omega)$, $|\Omega| < \infty$ and $u_n \rightharpoonup u \in H_0^1(\Omega)$, then*

$$\int_\Omega a(x)|u_n(x)|^2 dx \to \int_\Omega a(x)|u(x)|^2 dx.$$

Lemma 6. *[17, Lemma 2.14] If $a \in L^{\frac{N}{2}}(\Omega)$ and $|\Omega| < \infty$, then*

$$\lambda_1 = \inf_{u \in H_0^1(\Omega), |u|_2 = 1} \int_\Omega (|\nabla u|^2 - a(x)u^2(x))dx > -\infty. \tag{2.1}$$

Lemma 7. *[11, Lemma 3.1] If $a \in L^{\frac{N}{2}}(\Omega)$ and $|\Omega| < \infty$, then*

$$\hat{\delta} = \inf_{u \in Z, \|u\| = 1} \int_\Omega (|\nabla u|^2 - a(x)u^2(x))dx > 0, \tag{2.2}$$

where Z is defined in (1.5).

3. Estimates

Next, we will provide some estimates. Firstly, let us recall that the limiting problem

$$-\triangle u = |u|^{2^*-2}u \text{ in } \mathbb{R}^N$$

admits a solution u^* in $H^1(\mathbb{R}^N)$ (see, for instance, [2, 14]). Now, for any $\varepsilon > 0$, let us consider the following functions: U_ε, u_ε, and v_ε defined as

$$U_\varepsilon = \varepsilon^{-\frac{(N-2)}{2}} u^*\Big(\frac{x}{\varepsilon}\Big), \ x \in \mathbb{R}^N,$$

$$u_\varepsilon(x) = \eta(x) U_\varepsilon(x), \ x \in \mathbb{R}^N$$

and

$$v_\varepsilon(x) = \frac{u_\varepsilon}{|u_\varepsilon|_{2^*}},\ x \in \mathbb{R}^N,$$

where $\eta \in C^\infty(\mathbb{R}^N)$ is such that $0 \le \eta \le 1$ in $\mathbb{R}^N$, $\eta \equiv 1$ in B_δ and $\eta \equiv 0$ in $\mathbb{R}^N \backslash B_{2\delta}$, with $B_\delta = B(0,\delta)$ and $\delta > 0$ such that $B_{4\delta} \subset \Omega$. Note that $u_\varepsilon \in H_0^1(\Omega)$ for any $\varepsilon > 0$. What's more, according to [9], one has

$$|v_\varepsilon|_{2^*} = 1,\ \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx = S + O(\varepsilon^{\frac{N-2}{2}}),$$

$$\Upsilon(\varepsilon) := \int_{\mathbb{R}^N} |v_\varepsilon|^2 dx = \begin{cases} O(\varepsilon^{\frac{1}{2}}),\ N = 3, \\ O(\varepsilon|\ln\varepsilon|),\ N = 4, \\ O(\varepsilon),\ N \ge 5, \end{cases} \tag{3.1}$$

and

$$|u_\varepsilon|_{2^*} \le 2L \text{ for } \varepsilon \text{ small enough,}$$

where L is some positive constant.

Known to all, for problem (1.3), the compactness condition holds true only within a suitable threshold related to the best critical Sobolev constant. Now we will deal with the problem.

Lemma 8. *Suppose that (f_1)–(f_4) hold. There is a $v \in H_0^1(\Omega)\backslash\{0\}$ such that*

$$\max_{t\ge 0} \mathcal{J}(tv) < \frac{1}{N} S^{\frac{N}{2}}.$$

Proof. Obviously, $\mathcal{J}(u) \le \mathcal{J}_0(u)$, where

$$\mathcal{J}_0(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u(x))dx - \frac{1}{2^*}|u|_{2^*}^{2^*}.$$

Hence, it is sufficient to prove that $\max\limits_{t\ge 0} \mathcal{J}_0(tv) < \frac{1}{N} S^{\frac{N}{2}}$. By ($f_1$)–($f_3$), there exists $t_\varepsilon > 0$ such that $\mathcal{J}_0(t_\varepsilon v_\varepsilon) = \max\limits_{t\ge 0} \mathcal{J}_0(tv_\varepsilon)$. By $\frac{d\mathcal{J}_0(tv_\varepsilon)}{dt}|_{t=t_\varepsilon} = 0$ and noticing

$$|v_\varepsilon|_{2^*} = 1,\ \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx = S + O(\varepsilon^{\frac{N-2}{2}}),$$

we derive that

$$0 = t_\varepsilon \|v_\varepsilon\|^2 - \int_\Omega f(x, t_\varepsilon v_\varepsilon) v_\varepsilon dx - t_\varepsilon{}^{2^*-1} \le t_\varepsilon \|v_\varepsilon\|^2 - t_\varepsilon{}^{2^*-1},$$

and it follows that

$$t_\varepsilon{}^{2^*-2} \le \|v_\varepsilon\|^2 \le S + O(\varepsilon^{\frac{N-2}{2}}).$$

This implies that $t_\varepsilon \le C$, where C is independent of $\varepsilon(>0)$ with ε small.

We claim that $t_\varepsilon \geq C > 0$ for sufficiently small ε. If not, there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that $t_{\varepsilon_n} \to 0$ as $n \to \infty$. It follows that $t_{\varepsilon_n} v_{\varepsilon_n} \to 0$ in $H_0^1(\Omega)$ as $n \to \infty$. Hence,

$$0 < c_0 \leq \max_{t\geq 0} \mathcal{J}_0(tv_{\varepsilon_n}) = \mathcal{J}_0(t_{\varepsilon_n} v_{\varepsilon_n}) \leq \frac{1}{2}\|t_{\varepsilon_n} v_{\varepsilon_n}\|^2 \to 0,$$

where c_0 denotes the Mountain Pass level of $\mathcal{J}_0$. This is a contradiction.

According to condition (f_3), for any $M > 0$, there exists $T_M > 0$ such that for $t > T_M$, we obtain that

$$F(x,t) \geq \begin{cases} Mt^4, & N = 3, \\ Mt^2|\ln t|, & N = 4, \\ Mt^2, & N \geq 5. \end{cases} \tag{3.2}$$

Hence, for sufficiently small ε, we can conclude that

$$\int_{|x|<\varepsilon^{\frac{1}{2}}} F(x, v_\varepsilon) \geq \begin{cases} CM \int_{|x|<\varepsilon^{\frac{1}{2}}} \varepsilon^{-1} dx = CM\varepsilon^{\frac{1}{2}}, \quad N = 3, \\ C^2 M \int_{|x|<\varepsilon^{\frac{1}{2}}} \varepsilon^{-1} \ln(C\varepsilon^{-\frac{1}{2}}) dx = CM\varepsilon \ln(C\varepsilon^{-\frac{1}{2}}), \ N = 4, \\ CM \int_{|x|<\varepsilon^{\frac{1}{2}}} \varepsilon^{-\frac{N-2}{2}} dx = CM\varepsilon, \quad N \geq 5. \end{cases} \tag{3.3}$$

More details about the estimate can be found in [9, Lemma 3.4]. By the arbitrariness of M and (3.1), we obtain

$$\lim_{\varepsilon\to 0^+} \frac{\int_{|x|<\varepsilon^{\frac{1}{2}}} F(x, v_\varepsilon) dx}{\Upsilon(\varepsilon)} = +\infty, \tag{3.4}$$

where $\Upsilon(\varepsilon)$ is defined in (3.1). Thus, by (3.4) and the fact that $F(x,u) \geq 0$, for sufficiently small ε, one has

$$\begin{aligned} \mathcal{J}_0(t_\varepsilon v_\varepsilon) \leq & \frac{1}{2}\|t_\varepsilon v_\varepsilon\|^2 + \frac{1}{2}\int_\Omega |t_\varepsilon v_\varepsilon|^2 dx - \int_\Omega F(x, t_\varepsilon v_\varepsilon) dx - \frac{1}{2^*}\int_\Omega |t_\varepsilon v_\varepsilon|^{2^*} dx \\ \leq & \max_{t\geq 0}\Big(\frac{1}{2}\|t_\varepsilon v_\varepsilon\|^2 - \frac{1}{2^*}\int_\Omega |t_\varepsilon v_\varepsilon|^{2^*} dx\Big) + \frac{1}{2}\int_\Omega |t_\varepsilon v_\varepsilon|^2 dx - \int_\Omega F(x, t_\varepsilon v_\varepsilon) dx \\ \leq & \frac{S^{\frac{2}{N}}}{N} + O(\varepsilon^{\frac{N-2}{2}}) - \int_{|x|<\varepsilon^{\frac{1}{2}}} F(x, t_\varepsilon v_\varepsilon) - \int_{|x|\geq\varepsilon^{\frac{1}{2}}} F(x, t_\varepsilon v_\varepsilon) dx + C\int_\Omega |v_\varepsilon|^2 dx \\ \leq & \frac{1}{N}S^{\frac{2}{N}} + O(\varepsilon^{\frac{N-2}{2}}) - \int_{|x|<\varepsilon^{\frac{1}{2}}} F(x, t_\varepsilon v_\varepsilon) dx + C\int_\Omega |v_\varepsilon|^2 dx \\ < & \frac{1}{N}S^{\frac{2}{N}}. \end{aligned}$$

This completes the proof. □

Lemma 9. *For $u \in H_0^1(\Omega)\backslash\{0\}$, we have*

$$\sup_{\xi\geq 0}\Big(\frac{\xi^2}{2}\|u\|^2 - \frac{\xi^2}{2}\int_\Omega a(x)u^2(x)dx - \frac{\xi^{2^*}}{2^*}|u|_{2^*}^{2^*}\Big) = \frac{1}{N}S_a^{\frac{N}{2}}(u), \tag{3.5}$$

where S_a is defined in (1.7).

Proof. Let $\mathcal{M} : [0, +\infty) \to \mathbb{R}$ be the following function

$$\mathcal{M}(\xi) = \frac{\xi^2}{2} \|u\|^2 - \frac{\xi^2}{2} \int_\Omega a(x)u^2(x)dx - \frac{\xi^{2^*}}{2^*}|u|_{2^*}^{2^*}.$$

Note that

$$\mathcal{M}'(\xi) = \xi \|u\|^2 - \xi \int_\Omega a(x)u^2(x)dx - \xi^{2^*-1}|u|_{2^*}^{2^*},$$

and $\mathcal{M}'(\xi) \geq 0$ if and only if

$$\xi \leq \bar{\xi} := \Big(\frac{\|u\|^2 - \int_\Omega a(x)u^2(x)dx}{|u|_{2^*}^{2^*}}\Big)^{\frac{1}{2^*-2}}.$$

It follows that $\mathcal{M}_{\max}(\xi) = \mathcal{M}(\bar{\xi})$. By accurate calculation,

$$\sup_{\xi\geq 0} \mathcal{M}(\xi) = \max_{\xi\geq 0} \mathcal{M}(\xi) = \mathcal{M}(\bar{\xi}) = \frac{1}{N} S_a^{\frac{N}{2}}(u).$$

This concludes the proof. □

4. Proof of the main result

Firstly, we are going to prove that there exists a bounded Palais-Smale sequence for $\mathcal{J}$ in $H_0^1(\Omega)$.

Lemma 10. *Suppose that (f_1)–(f_3) hold, $a \in L^{\frac{N}{2}}(\Omega)$ and $|\Omega| < \infty$. If $c \in (-\infty, \frac{1}{N}S^{\frac{N}{2}})$ and $\{u_n\}$ is sequence in $H_0^1(\Omega)$ such that $\mathcal{J}'(u_n) \to 0$ and $\mathcal{J}(u_n) \to c$, as $n \to +\infty$, then there exists $\hat{u} \in H_0^1(\Omega)$ such that, up to a subsequence,*

$$\|u_n - \hat{u}\| \to 0, \ as\ n \to +\infty. \tag{4.1}$$

Proof. By Eq (1.5), we can write $u_n = y_n + z_n$, where $y_n \in Y$ and $z_n \in Z$. Let's set $\max\left\{\frac{1}{\alpha}, \frac{1}{2^*}\right\} < \beta < \frac{1}{2}$, where α is defined in (f_3). By $\alpha\beta > 1$ and $\beta \geq \frac{1}{2^*}$, as well as applying (f_3), Lemmas 3, 6 and 7, we derive that

$$
\begin{aligned}
&\mathcal{J}(u_n) - \beta\langle \mathcal{J}'(u_n), u_n\rangle \\
=&\Big(\frac{1}{2}-\beta\Big)\int_\Omega (|\nabla u_n|^2 - a(x)u_n^2(x))dx + \Big(\beta - \frac{1}{2^*}\Big)|u_n|_{2^*}^{2^*} \\
&- \int_\Omega F(x,u_n(x))dx + \int_\Omega \beta f(x,u_n(x))u_n(x)dx \\
\geq&\Big(\frac{1}{2}-\beta\Big)\int_\Omega (|\nabla u_n|^2 - a(x)u_n^2)dx + \Big(\beta - \frac{1}{2^*}\Big)|u_n|_{2^*}^{2^*} + (\alpha\beta - 1)\int_\Omega F(x,u_n)dx \\
=&\Big(\frac{1}{2}-\beta\Big)\int_\Omega (|\nabla y_n|^2 - a(x)y_n^2(x) + |\nabla z_n|^2 - a(x)z_n^2(x))dx \\
&+ (\alpha\beta - 1)\int_\Omega F(x,u_n(x))dx + \Big(\beta - \frac{1}{2^*}\Big)|u_n|_{2^*}^{2^*} \\
\geq&\Big(\frac{1}{2}-\beta\Big)(\lambda_1|y_n|_2^2 + \hat{\delta}\,\|z_n\|^2) + \Big(\beta - \frac{1}{2^*}\Big)|u_n|_{2^*}^{2^*} + C_1(\alpha\beta-1)|u_n|_\alpha^\alpha - C_3 \\
\geq&\Big(\frac{1}{2}-\beta\Big)(\lambda_1|y_n|_2^2 + \hat{\delta}\,\|z_n\|^2) + \Big(\beta - \frac{1}{2^*}\Big)|u_n|_{2^*}^{2^*} + C_1(\alpha\beta-1)|y_n|_\alpha^\alpha - C_3 \\
\geq&\Big(\frac{1}{2}-\beta\Big)(\lambda_1|y_n|_2^2 + \hat{\delta}\,\|z_n\|^2) + \Big(\beta - \frac{1}{2^*}\Big)|u_n|_{2^*}^{2^*} + C_1C(\alpha\beta-1)|y_n|_2^\alpha - \bar{C} - C_3 \\
\geq&\Big(\frac{1}{2}-\beta\Big)\lambda_1|y_n|_2^2 + \Big(\frac{1}{2}-\beta\Big)\hat{\delta}\,\|z_n\|^2 + C_4|y_n|_2^2 - C_5 \\
=&\Big[C_4 + \Big(\frac{1}{2}-\beta\Big)\lambda_1\Big]|y_n|_2^2 + \Big[\Big(\frac{1}{2}-\beta\Big)\hat{\delta}\Big]\,\|z_n\|^2 - C_5.
\end{aligned}
$$

And then, it is easy to know

$$
\begin{aligned}
C + |y_n|_2 + \|z_n\| \geq& \mathcal{J}(u_n) - \beta\langle \mathcal{J}'(u_n), u_n\rangle \\
\geq& C_6|y_n|_2^2 + \Big[\Big(\frac{1}{2}-\beta\Big)\hat{\delta}\Big]\,\|z_n\|^2 - C_5.
\end{aligned}
$$

Hence, $\{u_n\}$ is bounded in $H_0^1(\Omega)$ because of the boundedness of $\{y_n\}$ and $\{z_n\}$, using the fact that $\dim Y$ is finite.

Now, in order to verify the PS-condition, we need to establish several results (Steps 1–5) whose proofs are standard. Now we give a brief statement. Let $\{u_n\}$ be a sequence in $H_0^1(\Omega)$ such that

$$
\mathcal{J}'(u_n) \to 0 \text{ and } \mathcal{J}(u_n) \to c, \text{ as } n \to +\infty.
$$

Step 1. We aim to show that $\hat{u}$ is a solution of (1.3). We assume that as $n \to +\infty$, there exists $\hat{u}$ in $H_0^1(\Omega)$ such that $u_n \rightharpoonup \hat{u}$ in $H_0^1(\Omega)$, because of the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$. This convergence can be summarized as follows

$$
\begin{cases}
u_n \rightharpoonup \hat{u} \text{ in } H_0^1(\Omega), \\
u_n \to \hat{u} \text{ in } L^2(\Omega), \\
u_n(x) \to \hat{u}(x) \text{ a.e. in } \Omega.
\end{cases}
\tag{4.2}
$$

In addition, we have the following convergence:

$$
\langle u_n, v\rangle \to \langle \hat{u}, v\rangle,\ \forall v \in H_0^1(\Omega)
$$

and by Lemma 5, we have

$$\int_\Omega a(x)u_n^2dx \to \int_\Omega a(x)\hat{u}^2dx, \text{ as } n \to +\infty.$$

We know $\{u_n\}$ is bounded in $L^{2^*}(\Omega)$ because of $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, and $|u_n|^{2^*-2}u_n$ is bounded in $L^{\frac{2^*}{2^*-1}}(\Omega)$. We obtain that

$$|u_n|^{2^*-2}u_n \rightharpoonup |\hat{u}|^{2^*-2}\hat{u} \text{ in } L^{\frac{2^*}{2^*-1}}(\Omega), \text{ as } n \to +\infty. \tag{4.3}$$

Thereby,

$$\int_\Omega |u_n(x)|^{2^*-2}u_n(x)\phi(x)dx \to \int_\Omega |\hat{u}(x)|^{2^*-2}\hat{u}(x)\phi(x)dx,$$

as $n \to +\infty$, $\forall \phi \in L^{2^*}(\Omega)$, and so,

$$\int_\Omega |u_n(x)|^{2^*-2}u_n(x)\phi(x)dx \to \int_\Omega |\hat{u}(x)|^{2^*-2}\hat{u}(x)\phi(x)dx,$$

as $n \to +\infty$, $\forall \phi \in H_0^1(\Omega)$. By Lemma 1, we obtain that $f(x, u_n)$ is bounded in $L^{\frac{2^*}{2^*-1}}(\Omega)$. It follows from (4.2) that $f(x, u_n(x)) \to f(x, \hat{u}(x))$ a.e. in Ω, as $n \to \infty$. Thus, by Lemma 2, we obtain that $f(x, u_n(x)) \rightharpoonup f(x, \hat{u}(x))$ in $L^{\frac{2^*}{2^*-1}}(\Omega)$, as $n \to \infty$. Then,

$$\int_\Omega f(x, u_n(x))\phi(x) \to \int_\Omega f(x, \hat{u}(x))\phi(x), \text{ as } n \to +\infty, \ \forall \phi \in L^{2^*}(\Omega),$$

so that

$$\int_\Omega f(x, u_n(x))\phi(x) \to \int_\Omega f(x, \hat{u}(x))\phi(x), \text{ as } n \to +\infty, \ \forall \phi \in H_0^1(\Omega).$$

By assumption, for any $\phi \in H_0^1(\Omega)$, $\langle \mathcal{J}'(u_n), \phi \rangle \to 0$, as $n \to \infty$. Therefore,

$$\langle \hat{u}, \phi \rangle - \int_\Omega a(x)\hat{u}\phi dx - \int_\Omega \left(f(x, \hat{u})\phi - |\hat{u}|^{2^*-2}\hat{u}\phi \right) dx = 0, \forall \phi \in H_0^1(\Omega).$$

Hence, $\hat{u}$ is a solution of (1.3).

Step 2. We claim that the following equality holds true:

$$\mathcal{J}(\hat{u}) = \frac{1}{N}|\hat{u}|_{2^*}^{2^*} + \frac{1}{2}\int_\Omega f(x, \hat{u}(x))\hat{u}(x)dx - \int_\Omega F(x, \hat{u}(x))dx \geq 0.$$

Noticing that

$$\langle \mathcal{J}'(\hat{u}), \hat{u} \rangle = \|\hat{u}\|^2 - \int_\Omega a(x)\hat{u}^2(x)dx - \int_\Omega f(x, \hat{u}(x))\hat{u}(x)dx - |\hat{u}|_{2^*}^{2^*} = 0$$

and

$$\mathcal{J}(\hat{u}) = \frac{1}{2}\|\hat{u}\|^2 - \frac{1}{2}\int_\Omega a(x)\hat{u}^2(x)dx - \int_\Omega F(x, \hat{u}(x))dx - \frac{1}{2^*}|\hat{u}|_{2^*}^{2^*},$$

by (f_3), we obtain

$$\mathcal{J}(\hat{u}) = \frac{1}{N}\,|\hat{u}|_{2^*}^{2^*} + \frac{1}{2}\int_\Omega \Big(f(x,\hat{u}(x))\hat{u}(x) - 2F(x,\hat{u}(x))\Big)dx \geq 0.$$

Step 3. We claim that

$$\mathcal{J}(u_n) = \mathcal{J}(\hat{u}) + \frac{1}{2}\,\|u_n - \hat{u}\|^2 - \frac{1}{2^*}|u_n - \hat{u}|_{2^*}^{2^*} + o(1),\ \text{as } n \to +\infty.$$

Recall $\mathcal{J}(u_n) = \frac{1}{2}\,\|u_n\|^2 - \frac{1}{2}\int_\Omega a(x)u_n^2(x)dx - \int_\Omega F(x,u_n(x))dx - \frac{1}{2^*}\,|u_n|_{2^*}^{2^*}$. By Brezis-Lieb lemma, we also get

$$\begin{aligned} \|u_n - \hat{u}\|^2 &= \|u_n\|^2 - \|\hat{u}\|^2 + o(1), \\ |u_n|_{2^*}^{2^*} &= |u_n - \hat{u}|_{2^*}^{2^*} + |\hat{u}|_{2^*}^{2^*} + o(1). \end{aligned} \tag{4.4}$$

As $u_n \rightharpoonup \hat{u}$ in $H_0^1(\Omega)$ and by Lemma 5, we obtain

$$\int_\Omega a(x)\hat{u}_n^2(x)dx \to \int_\Omega a(x)\hat{u}^2(x)dx,\ \text{as } n \to +\infty.$$

By Lemma 4, it follows that

$$\int_\Omega F(x,u_n(x))dx \to \int_\Omega F(x,u(x))dx,\ \text{as } n \to +\infty.$$

Hence,

$$\begin{aligned} \mathcal{J}(u_n) =& \frac{1}{2}\,\|u_n - \hat{u}\|^2 + \frac{1}{2}\,\|\hat{u}\|^2 - \frac{1}{2}\int_\Omega a(x)\hat{u}(x)^2dx - \int_\Omega F(x,u(x))dx \\ &- \frac{1}{2^*}\,\|u_n - \hat{u}\|_{2^*}^{2^*} - \frac{1}{2^*}\,\|\hat{u}\|_{2^*}^{2^*} + o(1) \\ =& \mathcal{J}(\hat{u}) + \frac{1}{2}\,\|u_n - \hat{u}\|^2 - \frac{1}{2^*}\,\|u_n - \hat{u}\|_{2^*}^{2^*} + o(1),\ \text{as } n \to +\infty. \end{aligned}$$

Step 4. We claim that $\|u_n - \hat{u}\|^2 = |u_n - \hat{u}|_{2^*}^{2^*} + o(1),$ as $n \to +\infty$ holds. By (4.2)–(4.4), we infer that

$$\begin{aligned} &\int_\Omega (|u_n(x)|^{2^*-2}\,u_n(x) - |\hat{u}(x)|^{2^*-2}\,\hat{u}(x))(u_n(x) - \hat{u}(x))dx \\ =& \int_\Omega |u_n(x)|^{2^*}dx - \int_\Omega |\hat{u}(x)|^{2^*}dx + o(1) \\ =& \int_\Omega |u_n(x) - \hat{u}(x)|^{2^*}dx + o(1),\ \text{as } n \to \infty. \end{aligned} \tag{4.5}$$

Moreover,

$$\int_\Omega (f(x,u_n(x)) - f(x,\hat{u}(x)))(u_n(x) - \hat{u}(x))dx \to 0,\ as\ n \to +\infty. \tag{4.6}$$

Easily to know that

$$
\begin{aligned}
&\langle \mathcal{J}'(u_n) - \mathcal{J}'(\hat{u}), u_n - \hat{u}\rangle \\
&= \|u_n - \hat{u}\|^2 - \int_\Omega (|u_n(x)|^{2^*-2} u_n(x) - |\hat{u}(x)|^{2^*-2} \hat{u}(x))(u_n(x) - \hat{u}(x))dx \\
&- \int_\Omega (f(x, u_n(x)) - f(x, \hat{u}(x)))(u_n(x) - \hat{u}(x))dx - \int_\Omega a(x)(u_n(x) - \hat{u}(x))^2 dx \\
&= \|u_n - \hat{u}\|^2 - \int_\Omega |u_n(x) - \hat{u}(x)|^{2^*} dx + o(1),
\end{aligned} \tag{4.7}
$$

as $n \to +\infty$. On the other hand, by the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$ and Step 1, it follows that

$$
o(1) = \langle \mathcal{J}'(u_n), u_n - \hat{u}\rangle = \langle \mathcal{J}'(u_n) - \mathcal{J}'(\hat{u}), u_n - \hat{u}\rangle,\ n \to \infty. \tag{4.8}
$$

Hence, from (4.7) and (4.8), we get the assertion of Step 4.

Step 5. We conclude the proof of Lemma 10. As $n \to +\infty$, by Steps 2–4, we derive that

$$
\begin{aligned}
\mathcal{J}(u_n) =& \mathcal{J}(\hat{u}) + \frac{1}{2}\|u_n - \hat{u}(x)\|^2 - \frac{1}{2^*}|u_n - \hat{u}|_{2^*}^{2^*} + o(1) \\
\geq& \frac{1}{2}\|u_n - \hat{u}\|^2 - \frac{1}{2^*}|u_n - \hat{u}|_{2^*}^{2^*} + o(1) \\
\geq& \frac{1}{N}\|u_n - \hat{u}\|^2 + o(1),\ \text{as } n \to \infty.
\end{aligned} \tag{4.9}
$$

Let $\|u_n - \hat{u}\|^2 \to L,$ as $n \to \infty$. By Step 4, $|u_n - \hat{u}|_{2^*}^{2^*} \to L,$ as $n \to \infty$. In view of $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and the definition of S in (1.6), so $L^{\frac{2}{2^*}} S \leq L$, then $L = 0$ or $L \geq S^{\frac{N}{2}}$. If $L \geq S^{\frac{N}{2}}$, by (4.9) there will be

$$
c \geq \frac{1}{N} L \geq \frac{1}{N} S^{\frac{N}{2}},
$$

which against the assumption. Thus, $L = 0$, which implies $u_n \to \hat{u}$ in $H_0^1(\Omega)$, as $n \to +\infty$. This completes the proof. □

If $\lambda_1 \leq 0$, now we will show that $\mathcal{J}$ fits with the geometric assumptions of Rabinowitz's linking theorem, e.g., see Theorem 2.12 in [17].

Lemma 11. *Suppose that (f_1)-(f_3) hold. Then, for Y and Z defined in (1.5), we have*
(i) *there are* $\rho, \kappa > 0$ *such that for any* $u \in Z$ *with* $\|u\| = \rho$, *we have* $\mathcal{J}(u) \geq \kappa$;
(ii) $\mathcal{J}(u) \leq 0$ *for any* $u \in Y$;
(iii) *there exists* $R_0 > \rho$ *such that* $\mathcal{J}(u) \leq 0$ *for any* $u \in F$ *with* $\|u\| \geq R_0$, *where* $F = \{u + tz | u \in Y, t > 0, z \in Z\backslash\{0\}\}$.

Proof. Let $u \in Z$, by (f_2) and (f_3), we have

$$
\begin{aligned}
\mathcal{J}(u) =& \frac{1}{2}\int_\Omega (|\nabla u|^2 - a(x)u^2(x))dx - \int_\Omega F(x, u(x))dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*} dx \\
\geq& \frac{\hat{\delta}}{2}\|u\|^2 - \frac{\varepsilon}{2}|u|_2^2 - M(\varepsilon)|u|_\varsigma^\varsigma - \frac{1+\varepsilon}{2^*}|u|_{2^*}^{2^*} \\
=& \frac{1}{2}\left(\hat{\delta} - \varepsilon k_2\right)\|u\|^2 - k_\varsigma M(\varepsilon)\|u\|^\varsigma - \frac{(1+\varepsilon)S^{\frac{2^*}{2}}}{2^*}\|u\|^{2^*},
\end{aligned}
$$

where $\varsigma \in (2, 2^*)$. Fixed ε such that $\hat{\delta} - \varepsilon k_2 > 0$. For $u \in Z$, $\|u\| = \rho$ sufficiently small,

$$\kappa = \inf_{u \in H_0^1(\Omega), |u|_2 = \rho} \mathcal{J}(u) \geq \frac{1}{2}\Big(\hat{\delta} - \varepsilon k_2\Big)\rho^2 - k_\varsigma M(\varepsilon)\rho^\varsigma - \frac{(1+\varepsilon)S^{\frac{2^*}{2}}}{2^*}\rho^{2^*} > 0.$$

Now we will give the proof of (*ii*). Thanks to $\lambda_k \leq 0$, for any $u \in Y = \text{span}\{e_1, ..., e_k\}$,

$$\begin{aligned} \mathcal{J}(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega a(x)u^2(x)dx - \int_\Omega F(x, u(x))dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx \\ \leq& \frac{1}{2}\lambda_k |u|_2^2 - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx \leq 0. \end{aligned} \tag{4.10}$$

For any $u \in F$, we also have

$$\begin{aligned} \mathcal{J}(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega a(x)u(x)^2dx - \int_\Omega F(x, u(x))dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx \\ \leq& \frac{1}{2}\|u\|^2 - \frac{C}{2^*}\|u\|_{2^*}^{2^*}. \end{aligned}$$

Hence, by $2^* > 2$, $\mathcal{J}(u) \to -\infty$, as $\|u\| \to +\infty$, thanks to the fact that in any finite-dimensional space all norms are equivalent. □

Proof of Theorem 1. We give the proof as follows for two cases. For the one case where $\lambda_1 > 0$, we can apply the Mountain pass theorem to show the existence of a nontrivial solution for Eq (1.3). The proof follows the standard procedure, so we omit it here.

Let us define the Mountain Pass level of $\mathcal{J}$ as

$$c_m = \inf_{\gamma_m \in \Gamma_m} \max_{t \in [0,1]} \mathcal{J}(\gamma_m(t)),$$

where

$$\Gamma_m = \{\gamma_m \in C([0, 1], H_0^1(\Omega)) : \gamma_m(0) = 0, \gamma_m(1) = e\}.$$

By combining Lemmas 8 and 10, we can conclude that there exists a nontrivial solution for Eq (1.3).

For the case where $\lambda_1 \leq 0$, we construct the functions $z, \bar{z} \in Z$ as follows:

$$z = u_0 - \Sigma_{i=1}^k \Big(\int_\Omega u_0(x)e_i(x)dx\Big)e_i, \ \bar{z} = \frac{z}{\|z\|},$$

where u_0 is defined in (f_4'). Let c be the linking critical level of $\mathcal{J}$, that is,

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in M} \mathcal{J}(\gamma(u))$$

is the linking critical level of $\mathcal{J}$, where

$$\Gamma = \{\gamma \in C(\overline{M}, H_0^1(\Omega)) : \gamma = id \text{ on } \partial M\}$$

and

$$M = \{u = y + t\bar{z} | \|u\| \leq R_0, t \geq 0, y \in Y\}.$$

Our goal is to show that $c < \frac{1}{N}S^{\frac{N}{2}}$. Since $F = Y \oplus \text{span}\{\bar{z}\}$ is a linear space, we have

$$\sup_{u\in F}\mathcal{J}(u) = \sup_{u\in F,\xi\neq 0}\mathcal{J}(|\xi|\frac{u}{|\xi|}) = \sup_{u\in F,\xi>0}\mathcal{J}(\xi u) \leq \sup_{u\in F,\xi\geq 0}\mathcal{J}(\xi u), \tag{4.11}$$

so, in view of $M \subset F$, we obtain

$$c \leq \sup_{u\in M}\mathcal{J}(u) \leq \sup_{u\in F}\mathcal{J}(u) \leq \sup_{u\in F,\xi\geq 0}\mathcal{J}(\xi u). \tag{4.12}$$

By (f_3), $\mathcal{J}(u) \leq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega a(x)u^2(x)dx - \frac{1}{2^*}|u|_{2^*}^{2^*}$. By Lemma 9, for any $u \in H_0^1\backslash\{0\}$,

$$\sup_{\xi\geq 0}\mathcal{J}(\xi u) \leq \frac{1}{N}S_a^{\frac{N}{2}}(u). \tag{4.13}$$

Therefore, combining (4.12), (4.13), and (f_4'), we derive that

$$c \leq \sup_{u\in F,\xi\geq 0}\mathcal{J}(\xi u) \leq \frac{1}{N}\sup_{u\in F}S_a^{\frac{N}{2}}(u) \leq \frac{1}{N}S^{\frac{N}{2}},$$

thanks to $F = Y \oplus \text{span}\{\bar{z}\} = \{e_1, e_2, ..., e_k, u_0\}$. Then, combining Lemma 10 with Lemma 11, we deduce that there is a critical point u of $\mathcal{J}$ such that $\mathcal{J}(0) = 0 < \kappa < \mathcal{J}(u) < \frac{1}{N}S^{\frac{N}{2}}$. This completes the proof. □

5. Conclusions

Despite much literature concerning the existence of nontrivial solutions for the critical problem, to our knowledge, there are no available results involving the potential $a \in \frac{N}{2}(\Omega)$. Under some mild assumptions, we obtain an existence result. For the potential $a \in \frac{N}{2}(\Omega)$, we would like to go further in this direction, not just in bounded Ω.

Author contributions

Ye Xue: Conceptualization, Investigation, Methodology, Supervision, Validation, Writing-original draft, Writing-review and editing; Yongzhen Ge: Conceptualization, Investigation, Methodology, Supervision, Validation, Writing-original draft, Writing-review and editing; Yunlan Wei: Validation, Writing-original draft, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

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Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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