Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(3): 7269–7289.
DOI: 10.3934/math.2025333
Received: 19 August 2024
Revised: 29 October 2024
Accepted: 04 November 2024
Published: 28 March 2025

Research article

Estimates of coefficients for bi-univalent Ma-Minda-type functions associated with q-Srivastava-Attiya operator

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Abstract: In this article, we consider new subclasses of analytic and bi-univalent functions associated with the q-Srivastava-Attiya operator in the open unit disk. We obtain coefficient bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of the functions of these new subclasses. Furthermore, we establish the Fekete-Szegö inequality for functions in the classes $\mathcal{T}^{\epsilon}_{\tau,q,\alpha}(\psi)$, $\mathcal{KH}^{\epsilon}_{\tau,q,\alpha}(\delta,\psi)$, and $\mathcal{A}^{\epsilon}_{\tau,q,\alpha}(\delta,\psi)$.

Keywords: subordination; bi-univalent functions; q-Srivastava-Attiya operator; Fekete-Szegö problem
Mathematics Subject Classification: 30C45

1. Introduction

The class of functions f that are analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \ \ and \ \ |z| < 1\}$ and are normalized by the criteria $f(0) = f'(0) - 1 = 0$ is indicated by the symbol $\mathcal{H}$. Equivalently, if $f \in \mathcal{H}$, the Taylor-Maclaurin series representation takes the form:

$$f(z) = z + \sum_{l=2}^{\infty} a_l z^l \ , \quad z \in \mathbb{D}. \tag{1.1}$$

In addition, let us designate $\mathcal{S}$ as the fundamental subclass of $\mathcal{H}$, whose functions are univalent in $\mathbb{D}$. Koebe one-quarter theorem [1], which is widely recognized, guarantees that the image of $\mathbb{D}$ under each function $f \in \mathcal{S}$ contains a disk with a radius of $\frac{1}{4}$. Therefore, it can be concluded that every univalent function f possesses an inverse f^{-1} that satisfies the equation

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{D}$$

and

$$f^{-1}(f(\mu)) = \mu, \left(|\mu| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$\mathcal{G}(\mu) = f^{-1}(\mu) = \mu - a_2\mu^2 + (2a_2^2 - a_3)\mu^3 - (5a_2^3 - 5a_2a_3 + a_4)\mu^4 + \dots\ . \tag{1.2}$$

If f and f^{-1} are both univalent in $\mathbb{D}$, then a function $f \in \mathcal{H}$ is considered bi-univalent in $\mathbb{D}$. The class of bi-univalent functions defined in the unit disk $\mathbb{D}$ is indicated by τ. Because it univalently maps the unit disk $\mathbb{D}$ onto the entire complex plane, minus a slit along the line from $-\frac{1}{4}$ to $-\infty$, the well-known Koebe function is not an element of τ. Therefore, the unit disk $\mathbb{D}$ is absent from the image domain. The well-known Bieberbach conjecture, which asserts that the following coefficient inequality holds for each $f \in \mathcal{S}$ produced by the Taylor-Maclaurin series expansion (1.1), was established in 1985 by Louis de Branges [2],

$$|a_l| \leq l \quad (l \in \mathbb{N}/\{1\}).$$

The class of analytic bi-univalent functions was first introduced and studied by Lewin who proved $|a_2| < 1.51$. Brannan and Clunie subsequently enhanced Lewin's outcome to $|a_2| \leq \sqrt{2}$. Indeed, it is verified for the class of bi-close to convex functions in the article [3]. Researchers Brannan and Taha [4] and Taha [5] looked at different types of bi-univalent functions and found that they are similar to well-known types of univalent functions represented by strongly starlike, starlike and convex functions. We present bi-starlike functions and bi-convex functions, and established non-sharp estimates for the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Next, we recall the definition of subordination between analytic functions. For two functions $f, \mathcal{G} \in \mathcal{H}$, we say that the function f is subordinate to $\mathcal{G}$, if there exists a Schwarz function w, which is analytic in $\mathbb{D}$ with the following property:

$$w(0) = 0,\ |w(z)| < 1, \quad \text{for all } z \in \mathbb{D},$$

such that

$$f(z) = \mathcal{G}(w(z)).$$

This subordination is symbolically written as follows:

$$f \prec \mathcal{G} \ \text{ or } \ f(z) \prec \mathcal{G}(z) \quad (z \in \mathbb{D}).$$

It is well known that if the function $\mathcal{G}$ is univalent in $\mathbb{D}$, then the following equivalence holds (see [6]):

$$f \prec \mathcal{G} \ \ (z \in \mathbb{D}) \Longleftrightarrow f(0) = \mathcal{G}(0) \ \text{ and } \ f(\mathbb{D}) \subseteq \mathcal{G}(\mathbb{D}).$$

The $\mathfrak{q}$-difference operator which was introduced by Jackson is

$$\mathfrak{D}_{\mathfrak{q}} f(z) = \frac{f(z) - f(\mathfrak{q}z)}{z(1-\mathfrak{q})}, \qquad z \in \mathbb{D}/\{0\}. \tag{1.3}$$

The following limit relationship is clear:

$$\lim_{\mathfrak{q}\to 1^-} \mathfrak{D}_{\mathfrak{q}} f(z) = f^{'}(z) \ \text{ and } \ \mathfrak{D}_{\mathfrak{q}} f(0) = f^{'}(0).$$

For a function $f \in \mathbb{D}$ defined by (1.1), we deduce the following result:

$$\mathfrak{D}_{\mathfrak{q}} f(z) = 1 + \sum_{l=2}^{\infty} [l]_{\mathfrak{q}} a_l z^{l-1},$$

where $[l]_{\mathfrak{q}}$ is given by

$$[l]_{\mathfrak{q}} = \frac{1-\mathfrak{q}^l}{1-\mathfrak{q}} \quad (l \in \mathbb{N}\backslash\{1\}).$$

As $\mathfrak{q} \to 1^-$, we have $[l]_{\mathfrak{q}} \to l$ and $[0]_{\mathfrak{q}} = 0$.

The Srivastava-Attiya operator, which has been extensively investigated, was defined by Srivastava and Attiya [7] by using the Hurwitz-Lerch zeta function $\Phi(z, \epsilon, \alpha)$, which is systematically discussed in recent survey papers [8, 9]. To obtain comprehensive information regarding the interconnections between the function $\Phi(z, \epsilon, \alpha)$ and many significant functions within the realm of analytic number theory, readers may consult Chapter I in reference [10]. The following $\mathfrak{q}$-analogue of the Hurwitz-Lerch zeta function $\Phi(z, \epsilon, \alpha)$ was explored by Shah and Noor [11] (see also [12]):

$$\phi_{\mathfrak{q}}(\epsilon, \alpha; z) := \sum_{l=0}^{\infty} \frac{z^l}{[l+\alpha]_{\mathfrak{q}}^{\epsilon}}, \tag{1.4}$$

where $\alpha \in \mathbb{C}\backslash\{\mathbb{Z}_0^-\}$, $\epsilon \in \mathbb{C}$ when $|z| < 1$, $\Re(\epsilon) > 1$, and $|z| = 1$.

$\mathbb{Z}_0^-$ being the set of negative integers. The normalized form of the series (1.4) is defined by

$$\psi_{\mathfrak{q}}(\epsilon, \alpha; z) = [1+\alpha]_{\mathfrak{q}}^{\epsilon} \left(\phi_{\mathfrak{q}}(\epsilon, \alpha; z) - [\alpha]_{\mathfrak{q}}^{-\epsilon}\right) = z + \sum_{l=2}^{\infty} \left(\frac{[1+\alpha]_{\mathfrak{q}}}{[l+\alpha]_{\mathfrak{q}}}\right)^{\epsilon} z^l. \tag{1.5}$$

By using (1.1) and (1.5), Shah and Noor [11] defined the $\mathfrak{q}$-Srivastava-Attiya operator $\mathcal{J}_{\mathfrak{q},\alpha}^{\epsilon} : \mathcal{H} \to \mathcal{H}$ as follows:

Definition 1.1. *(see [11, 12]) The $\mathfrak{q}$-Srivastava-Attiya operator: $\mathcal{J}_{\mathfrak{q},\alpha}^{\epsilon} : \mathcal{H} \to \mathcal{H}$ is defined in terms of the Hadamard product (or convolution) by*

$$\mathcal{J}_{\mathfrak{q},\alpha}^{\epsilon} f(z) = \psi_{\mathfrak{q}}(\epsilon, \alpha; z)(z) * f(z) = z + \sum_{l=2}^{\infty} C_{\mathfrak{q},\alpha}^{\epsilon}(l) a_l z^l, \tag{1.6}$$

where

$$C_{\mathfrak{q},\alpha}^{\epsilon}(l) = \left(\frac{[1+\alpha]_{\mathfrak{q}}}{[l+\alpha]_{\mathfrak{q}}}\right)^{\epsilon}.$$

The mathematical applications of $\mathfrak{q}$-calculus, fractional $\mathfrak{q}$-calculus, and fractional $\mathfrak{q}$-derivative operators in geometric function theory of complex analysis were investigated by Srivastava [13] in his recently published survey-cum-expository review article. Srivastava [13] also exposed the not-yet-widely-understood fact that the so-called $(p, \mathfrak{q})$-variation of classical $\mathfrak{q}$-calculus is a relatively trivial and inconsequential variation of classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [13], p. 340).

In this paper, we utilize the fundamental or quantum (or q−) extension $\phi_{q}(\epsilon,\alpha;z)$. When $q \to 1^{-}$, it produces the well-known Hurwitz-Lerch zeta function $\Phi(z,\epsilon,\alpha)$. Local or non-local symmetries are observed in certain properties of many members of the Hurwitz-Lerch zeta function. family, as previously mentioned. Additional support for our investigation into the practical uses of quantum extensions (or q−) in this research can be located in the chapter titled "Symmetric Quantum Calculus" in reference [14].

Remark 1.1. *The operator $\mathcal{J}^{\epsilon}_{q,\alpha}$ is a generalization of several known operators studied in earlier investigations, which are recalled below:*

1) *The operator $\mathcal{J}^{\epsilon}_{q,\alpha}$ coincides with the Srivastava-Attiya operator in [7], and for $q \to 1^{-}$, the function $\phi_{q}(\epsilon,\alpha;z)$ reduces to the Hurwitz-Lerch zeta function (see [8, 9]). The Srivastava-Attiya operator has several uses, which can be found in [15–17] and the references listed in each of these previous publications.*
2) *The operator $\mathcal{J}^{\epsilon}_{q,\alpha}$ reduces to the q-Bernardi operator for $\epsilon = 1$ as stated in [18].*
3) *The operator $\mathcal{J}^{\epsilon}_{q,\alpha}$ reduces to the q-Libera operator for $\epsilon = \alpha = 1$, as stated in [18].*
4) *The operator $\mathcal{J}^{\epsilon}_{q,\alpha}$ reduces to the Bernardi operator for $q \to 1^{-}$ and $\epsilon = 1$ (see [19]).*
5) *The operator $\mathcal{J}^{\epsilon}_{q,\alpha}$ reduces to the Alexander operator for $q \to 1^{-}, \epsilon = 1$, and $\alpha = 0$ (see [20]).*

We define the subclasses $S^{*;\epsilon}_{q,\alpha}(\delta)$ and $K^{\epsilon}_{q,\alpha}(\delta)$ of the class $\mathcal{H}$ for $0 \le \delta < 1$ using the q-Srivastava-Attiya operator.

Definition 1.2. *A function $f(z)$ of the form (1.1) is in the class $S^{*,\epsilon}_{q,\alpha}(\delta)$ if it satisfies the following condition:*

$$\Re\left\{\frac{z(\mathcal{J}^{\epsilon}_{q,\alpha}f(z))'}{\mathcal{J}^{\epsilon}_{q,\alpha}f(z)}\right\} > \delta, \quad \text{for all } z \in \mathbb{D}.$$

When $\epsilon = 0$, we obtain the result.

Corollary 1.1. *A function $f(z)$ of the form (1.1) is in the class $S^{*,0}_{q,\alpha}(\delta)$ if it satisfies the following condition [21]:*

$$\Re\left\{\frac{z(f(z))'}{f(z)}\right\} > \delta, \quad \text{for all } z \in \mathbb{D}.$$

Definition 1.3. *A function $f(z)$ of the form (1.1) is in the class $K^{\epsilon}_{q,\alpha}(\delta)$ if it satisfies the following condition:*

$$\Re\left\{1 + \frac{z(\mathcal{J}^{\epsilon}_{q,\alpha}f(z))''}{(\mathcal{J}^{\epsilon}_{q,\alpha}f(z))'}\right\} > \delta, \quad \text{for all } z \in \mathbb{D}.$$

Observe that $\mathcal{J}^{\epsilon}_{q,\alpha}f \in K^{\epsilon}_{q,\alpha}(\delta)$ if and only if $z(\mathcal{J}^{\epsilon}_{q,\alpha}f(z))' \in S^{*,\epsilon}_{q,\alpha}(\delta)$. When $\epsilon = 0$, we obtain the result.

Corollary 1.2. *A function $f(z)$ of the form (1.1) is in the class $K^{0}_{q,\alpha}(\delta)$ if it satisfies the following condition [22]:*

$$\Re\left\{1 + \frac{z(f(z))''}{f(z))'}\right\} > \delta, \quad \text{for all } z \in \mathbb{D}.$$

Definition 1.4. *A function $f(z)$ of the form (1.1) is in the class $H^{\epsilon}_{\mathfrak{q},\alpha}(\delta)$ if it satisfies the following condition:*

$$\Re\left\{(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))'\right\} > \delta, \ \ \textit{for all } z \in \mathbb{D}.$$

When $\epsilon = 0$, we obtain the result.

Corollary 1.3. *A function $f(z)$ of the form (1.1) is in the class $H^{0}_{\mathfrak{q},\alpha}(\delta)$ if it satisfies the following condition [23]:*

$$\Re\left\{(f(z)'\right\} > \delta, \ \ \textit{for all } z \in \mathbb{D}.$$

In this study, we derive estimates for the initial coefficients a_2 and a_3 of three novel subclasses of the class τ of bi-univalent functions.

Let ψ be an analytic function with a positive real part in $\mathbb{D}$ such that $\psi(0) = 1$, $\psi(0) > 0$, and $\psi(\mathbb{D})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\psi(z) = 1 + \nu_1 z + \nu_2 z^2 + \nu_3 z^3 + \ldots (\nu_1 > 0).$$

With this brief introduction, we define the following classes of bi-univalent functions and find the coefficient estimates with the help of the $\mathfrak{q}$-Srivastava-Attiya operator.

Definition 1.5. *A function $f \in \tau$ is said to be in the class $\mathcal{T}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\psi)$ if the following subordinations hold*

$$(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))' \prec \psi(z) \ \ \textit{and} \ \ (\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))' \prec \psi(\mu), \ \textit{where} \ \ \mathcal{G}(\mu) = f^{-1}(\mu).$$

Definition 1.6. *A function $f(z) \in \tau$ is said to be in the class $\mathcal{KH}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\delta,\psi), \delta \geq 0$, if the following subordinations hold:*

$$\frac{z(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))'}{f(z)} + \frac{\delta z^2(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))''}{f(z)} \prec \psi(z), \quad (z \in \mathbb{D}),$$

and

$$\frac{\mu(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))'}{\mathcal{G}(\mu)} + \frac{\delta\mu^2(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))''}{\mathcal{G}(\mu)} \prec \psi(\mu), \quad (\mu \in \mathbb{D}),$$

where $\mathcal{G}(\mu) = f^{-1}(\mu)$.

Definition 1.7. *A function $f \in \tau$ is said to be in the class $\mathcal{A}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\psi)$, $\delta > 0$, if the following subordinations hold:*

$$(1-\delta)\frac{z(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))'}{\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z)} + \delta\left(1 + \frac{z(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))''}{(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))'}\right) \prec \psi(z), \qquad (z \in \mathbb{D}),$$

and

$$(1-\delta)\frac{\mu(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))'}{\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu)} + \delta\left(1 + \frac{\mu(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))''}{(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))'}\right) \prec \psi(\mu), \qquad (\mu \in \mathbb{D}),$$

where $\mathcal{G}(\mu) = f^{-1}(\mu)$.

In order to derive our main results, we have to recall the following lemma here.

Lemma 1.1. *[24] If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ..., \tag{1.7}$$

where $\mathcal{P}$ is the family of all functions $p(z)$ analytic in $\mathbb{D}$ and satisfying $\Re\{p(z)\} > 0$. Then the following sharp estimate holds:

$$|c_l| \leq 2 \quad (l = 1, 2, ...).$$

Also, $\gamma \in \mathbb{R}$ for all, we obtain

$$|c_2 - \gamma c_1^2| \leqq \max\{1, |\gamma|\}.$$

2. Main results

Theorem 2.1. *Let $f(z) \in \mathcal{T}^{\epsilon}_{\tau,\mathrm{q},\alpha}(\psi)$ and be given by (1.1). Then*

$$|a_2| \leq \frac{\nu_1\sqrt{\nu_1}}{\sqrt{|3C^{\epsilon}_{\mathrm{q},\alpha}(3)\nu_1^2 - 4(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2\nu_2 + 4(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2\nu_1|}} \text{ and } |a_3| \leq \frac{\nu_1}{3|C^{\epsilon}_{\mathrm{q},\alpha}(3)|} + \frac{|\nu_1^2}{2|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2}. \tag{2.1}$$

Proof. Let $f \in \mathcal{T}^{\epsilon}_{\tau,\mathrm{q},\alpha}(\psi)$ and $\mathcal{G} = f^{-1}$. Then there are analytic functions $r, \mathcal{F} : \mathbb{D} \to \mathbb{D}$, with $r(0) = \mathcal{F}(0) = 0$, satisfying

$$(\mathcal{J}^{\epsilon}_{\mathrm{q},\alpha} f(z))' = \psi(r(z)) \text{ and } (\mathcal{J}^{\epsilon}_{\mathrm{q},\alpha}\mathcal{G}(z))' = \psi(\mathcal{F}(\mu)). \tag{2.2}$$

The functions p_1 and p_2 are defined as follows:

$$p_1(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \text{ and } p_2(z) = \frac{1 + \mathcal{F}(z)}{1 - \mathcal{F}(z)} = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ...,$$

or, equivalently,

$$r(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + (c_2 - \frac{c_1^2}{2})z^2 + ...\right) \tag{2.3}$$

and

$$\mathcal{F}(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2}\left(b_1 z + (b_2 - \frac{b_1^2}{2})z^2 + ...\right). \tag{2.4}$$

It is clear that p_1 and p_2 are analytic in $\mathbb{D}$ and $p_1(0) = p_2(0) = 1$. Also p_1 and p_2 have a positive real part in $\mathbb{D}$ and hence $|b_i| \leq 2$ and $|c_i| \leq 2$ $(i \in \mathbb{N}\backslash\{1\})$.

Substituting (2.3) and (2.4) into (2.2) and using (1), we can obtain

$$\psi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}\nu_1 c_1 z + \left(\frac{1}{2}\nu_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}\nu_2 c_1^2\right)z^2 + ... \tag{2.5}$$

and

$$\psi\left(\frac{p_2(z) - 1}{p_2(z) + 1}\right) = 1 + \frac{1}{2}\nu_1 b_1 \mu + \left(\frac{1}{2}\nu_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}\nu_2 b_1^2\right)\mu^2 + ... \tag{2.6}$$

Since $f \in \tau$ has the Maclaurin series given by (1.1), a computation shows that its inverse $\mathcal{G} = f^{-1}$ has the expansion $\mathcal{G}(\mu) = f^{-1}(\mu) = \mu - a_2\mu^2 + (2a_2^2 - a_3)\mu^3 + ...$

Since

$$(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))^{'} = 1 + 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)a_2z + 3C^{\epsilon}_{\mathfrak{q},\alpha}(3)a_3z^2 + ...$$

and

$$(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))^{'} = 1 - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)a_2\mu + 3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(2a_2^2 - a_3)\mu^2 + ...,$$

it follows from (2.5), (2.6), and (2.2) that

$$2a_2 = \frac{\nu_1 c_1}{2C^{\epsilon}_{\mathfrak{q},\alpha}(2)}, \tag{2.7}$$

$$3C^{\epsilon}_{\mathfrak{q},\alpha}(3)a_3 = \frac{1}{2}\nu_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}\nu_2 c_1^2, \tag{2.8}$$

$$2a_2 = \frac{\nu_1 b_1}{-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)}, \tag{2.9}$$

$$3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(2a_2^2 - a_3) = \frac{1}{2}\nu_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}\nu_2 b_1^2. \tag{2.10}$$

From (2.7) and (2.9), we obtain

$$c_1 = -b_1 \tag{2.11}$$

and

$$2a_2^2 = \frac{\nu_1^2(c_1^2 + b_1^2)}{16(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2}. \tag{2.12}$$

Adding Eqs (2.8) and (2.10) and using (2.12), we now obtain

$$a_2^2 = \frac{\nu_1^3(c_2 + b_2)}{4[3C^{\epsilon}_{\mathfrak{q},\alpha}(3)\nu_1^2 - 4(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2\nu_2 + 4(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2\nu_1]}.$$

Applying Lemma 1.1 for the coefficients b_2 and c_2, we immediately have

$$|a_2| \leq \frac{\nu_1\sqrt{\nu_1}}{\sqrt{|3C^{\epsilon}_{\mathfrak{q},\alpha}(3)\nu_1^2 - 4(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2\nu_2 + 4(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2\nu_1|}}.$$

As stated in (2.7), this provides us with the bound on $|a_2|$. Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8) and also from (2.11), we get $c_1^2 = b_1^2$, hence

$$a_3 = \frac{1}{12C^{\epsilon}_{\mathfrak{q},\alpha}(3)}\nu_1(c_2 - b_2) + \frac{1}{16(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2}(\nu_1^2 c_1^2).$$

Using (2.12) and applying Lemma 1.1 once again for the coefficients b_2 and c_2, we have

$$|a_3| \leq \frac{\nu_1}{3|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|} + \frac{\nu_1^2}{2|C^{\epsilon}_{\mathfrak{q},\alpha}(2)|^2}.$$

This completes the proof of Theorem 2.1. □

When $\epsilon = 0$, we obtain the result presented by Ali et al. [25].

Corollary 2.1. *Let $f(z) \in \mathcal{T}_\tau(\psi)$ and be given by (1.1). Then*

$$|a_2| \leq \frac{\nu_1 \sqrt{\nu_1}}{\sqrt{\left|3\nu_1^2 - 4\nu_2 + 4\nu_1\right|}} \text{ and } |a_3| \leq \frac{\nu_1}{3} + \frac{\nu_1^2}{4}. \tag{2.13}$$

Theorem 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{KH}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\delta, \psi)$. Then*

$$|a_2^2| \leq \frac{\nu_1^3}{\left|\left[\left(-1 + 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+2\delta)\right)^2 (\nu_1 - \nu_2) + \left(3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right)\nu_1^2\right]\right|}, \tag{2.14}$$

and

$$|a_3| \leq \frac{\nu_1 + |\nu_2 - \nu_1|}{3|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|(1+2\delta) + 2|C^{\epsilon}_{\mathfrak{q},\alpha}(2)|(1+\delta)}. \tag{2.15}$$

Proof. Let $f(z) \in \mathcal{KH}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\delta, \psi)$. Then there are analytic functions $r, \mathcal{F} : \mathbb{D} \to \mathbb{D}$, with $r(0) = \mathcal{F}(0) = 0$, satisfying

$$\frac{z(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha} f(z))'}{f(z)} + \frac{\delta z^2 (\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha} f(z))''}{f(z)} = \psi(r(z)), \quad (z \in \mathbb{D}), \tag{2.16}$$

and

$$\frac{\mu(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha} \mathcal{G}(\mu))'}{\mathcal{G}(\mu)} + \frac{\delta \mu^2 (\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha} \mathcal{G}(\mu))''}{\mathcal{G}(\mu)} = \psi(\mathcal{F}(\mu)), \quad (\mu \in \mathbb{D}), \tag{2.17}$$

where $\mathcal{G}(\mu) = f^{-1}(\mu)$. Since $\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha} f(z) = z + C^{\epsilon}_{\mathfrak{q},\alpha}(2) a_2 z^2 + C^{\epsilon}_{\mathfrak{q},\alpha}(3) a_3 z^3 + ...$ and $\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha} \mathcal{G}(\mu) = \mu - C^{\epsilon}_{\mathfrak{q},\alpha}(2) a_2 \mu^2 + C^{\epsilon}_{\mathfrak{q},\alpha}(3)(2a_2^2 - a_3)\mu^3 + ...$, we have

$$1 + \left[-1 + 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right] a_2 z + \left(\left[-1 + 3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right] a_3 + \left[1 - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right] a_2^2\right) z^2 ... ,$$

and

$$\begin{aligned} 1 &+ \left[1 - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right] a_2 \mu + \left(-\left[-1 + 3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right] a_3 \right. \\ &+ \left. \left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta) - 1\right] a_2^2\right) \mu^2 .... \end{aligned}$$

Equating the coefficients (2.16), (2.17), (2.5), and (2.6) on both sides, we have

$$\left[-1 + 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right] a_2 = \frac{\nu_1 c_1}{2}, \tag{2.18}$$

$$\left[-1 + 3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right] a_3 + \left[1 - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right] a_2^2 = \frac{1}{2}\nu_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}\nu_2 c_1^2, \tag{2.19}$$

$$\left[1 - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right] a_2 = \frac{\nu_1 b_1}{2}, \tag{2.20}$$

$$-\left[-1 + 3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right] a_3 + \left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta) - 1\right] a_2^2 \tag{2.21}$$

$$= \frac{1}{2}\nu_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}\nu_2 b_1^2. \tag{2.22}$$

From (2.18) and (2.20), we obtain

$$c_1 = -b_1 \tag{2.23}$$

and

$$2a_2^2 = \frac{\nu_1^2(c_1^2 + b_1^2)}{4[-1 + 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+2\delta)]^2}. \tag{2.24}$$

Adding Eqs (2.19) and (2.21) and using (2.24), we now obtain

$$a_2^2 = \frac{\nu_1^3(c_2 + b_2)}{4\left[\left(-1 + 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+2\delta)\right)^2(\nu_1 - \nu_2) + \left(3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right)\nu_1^2\right]}.$$

Applying Lemma 1.1 for the coefficients b_2 and c_2, we immediately get

$$|a_2^2| \le \frac{\nu_1^3}{\left|\left[\left(-1 + 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+2\delta)\right)^2(\nu_1 - \nu_2) + \left(3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right)\nu_1^2\right]\right|}.$$

Since $\nu_1 > 0$, the last inequality gives the desired estimate on $|a_2|$ given in (2.14). Next, in order to find the bound on $|a_3|$, by subtracting (2.21) from (2.19) and also from (2.23), we get $c_1^2 = b_1^2$, hence

$$a_3 = \frac{\nu_1\left(\left[-1 + 6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]c_2 + \left[-1 + 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]b_2\right)}{\left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 4C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]\left[-2 + 6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}$$
$$+ \frac{b_1^2(\nu_2 - \nu_1)\left[-1 + 3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}{2\left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 4C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]\left[-2 + 6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}.$$

Applying Lemma 1.1 once again for the coefficients b_2 and c_2, we obtain

$$|a_3| \le \frac{\nu_1 + |\nu_2 - \nu_1|}{3|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|(1+2\delta) + 2|C^{\epsilon}_{\mathfrak{q},\alpha}(2)|(1+\delta)}.$$

This estimate is exactly that found in (2.15).

When $\epsilon = 0$, we obtain the result presented by Ali et al. [25]. □

Corollary 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{KH}_\tau(\delta, \psi)$. Then*

$$|a_2| \le \frac{\nu_1\sqrt{\nu_1}}{\sqrt{\left|(1+2\delta)^2(\nu_1 - \nu_2) + (1+4\delta)\nu_1^2\right|}},$$

and

$$|a_3| \le \frac{\nu_1 + |\nu_2 - \nu_1|}{(1+4\delta)}.$$

When $\epsilon = 0$ and $\delta = 0$, the coefficient estimates for Ma-Minda bi-starlike functions are obtained [26].

Corollary 2.3. *Let $f(z)$ given by (1.1) be in the class $\mathcal{KH}_{\tau}(0,\psi)$. Then*

$$|a_2| \leq \frac{\nu_1\sqrt{\nu_1}}{\sqrt{\left|\nu_1-\nu_2+\nu_1^2\right|}},$$

and

$$|a_3| \leq \nu_1 + |\nu_2-\nu_1|.$$

Theorem 2.3. *Let f given by (1.1) be in the class $\mathcal{A}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\psi)$. Then*

$$|a_2^2| \leq \frac{\nu_1^3}{\left[|C^{\epsilon}_{\mathfrak{q},\alpha}(2)|^2(1+\delta)^2(\nu_1-\nu_2)+\left(2|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|(1+2\delta)-|C^{\epsilon}_{\mathfrak{q},\alpha}(2)|^2(1+3\delta)\right)\nu_1^2\right]}, \tag{2.25}$$

and

$$|a_3| \leq \frac{\nu_1+|\nu_2-\nu_1|}{2|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|(1+2\delta)-|C^{\epsilon}_{\mathfrak{q},\alpha}(2)|^2(1+3\delta)}. \tag{2.26}$$

Proof. Let $f(z) \in \mathcal{A}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\delta,\psi)$. Then there are analytic functions $r,\mathcal{F}:\mathbb{D}\to\mathbb{D}$, with $r(0)=\mathcal{F}(0)=0$, satisfying

$$(1-\delta)\frac{z(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))'}{\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z)}+\delta\left(1+\frac{z(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))''}{(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}f(z))'}\right)=\psi(r(z)), \qquad (z\in\mathbb{D}), \tag{2.27}$$

and

$$(1-\delta)\frac{\mu(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))'}{\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu)}+\delta\left(1+\frac{\mu(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))''}{(\mathcal{J}^{\epsilon}_{\mathfrak{q},\alpha}\mathcal{G}(\mu))'}\right)=\psi(r(\mu)), \qquad (\mu\in\mathbb{D}), \tag{2.28}$$

where $\mathcal{G}(\mu)=f^{-1}(\mu)$. By (2.27) and (2.28), we have

$$1+(1+\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(2)a_2z+\left(2(1+2\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(3)a_3-(1+3\delta)(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2a_2^2\right)z^2+...$$

and

$$1-(1+\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(2)a_2\mu+\left(\left[4(1+2\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(3)-(1+3\delta)\left(C^{\epsilon}_{\mathfrak{q},\alpha}(2)\right)^2\right]a_2^2+\left[-2(1+2\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(3)\right]a_3\right)\mu^2... .$$

Equating the coefficients (2.27), (2.28), (2.5), and (2.6) on both sides, we have

$$(1+\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(2)a_2=\frac{\nu_1c_1}{2}, \tag{2.29}$$

$$2(1+2\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(3)a_3-(1+3\delta)(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2a_2^2=\frac{1}{2}\nu_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}\nu_2c_1^2, \tag{2.30}$$

$$-(1+\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(2)a_2=\frac{\nu_1b_1}{2}, \tag{2.31}$$

$$-2(1+2\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(3)a_3+\left[4(1+2\delta)C^{\epsilon}_{\mathfrak{q},\alpha}(3)-(1+3\delta)\left(C^{\epsilon}_{\mathfrak{q},\alpha}(2)\right)^2\right]a_2^2=\frac{1}{2}\nu_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}\nu_2b_1^2. \tag{2.32}$$

From (2.29) and (2.31), we obtain

$$c_1=-b_1, \tag{2.33}$$

and

$$2a_2^2 = \frac{\nu_1^2(c_1^2 + b_1^2)}{4(1+\delta)^2\left(C^{\epsilon}_{\mathrm{q},\alpha}(2)\right)^2}. \tag{2.34}$$

Adding Eqs (2.30) and (2.32) and using (2.34), we now obtain

$$a_2^2 = \frac{\nu_1^3(c_2 + b_2)}{4\left[(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2(1+\delta)^2(\nu_1 - \nu_2) + \left(2C^{\epsilon}_{\mathrm{q},\alpha}(3)(1+2\delta) - (C^{\epsilon}_{\mathrm{q},\alpha}(2))^2(1+3\delta)\right)\nu_1^2\right]}.$$

Applying Lemma 1.1 for the coefficients b_2 and c_2, we immediately get

$$|a_2^2| \leq \frac{\nu_1^3}{\left[|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2(1+\delta)^2(\nu_1 - \nu_2) + \left(2|C^{\epsilon}_{\mathrm{q},\alpha}(3)|(1+2\delta) - |C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2(1+3\delta)\right)\nu_1^2\right]},$$

which yields the desired estimate on $|a_2|$ as described in (2.25). Next, in order to find the bound on $|a_3|$, by subtracting (2.32) from (2.30) and also from (2.33), we get $c_1^2 = b_1^2$, hence

$$a_3 = \frac{(\nu_1/2)\left[\left(4C^{\epsilon}_{\mathrm{q},\alpha}(3)(1+2\delta) - (C^{\epsilon}_{\mathrm{q},\alpha}(2))^2(1+3\delta)\right)c_2 + (C^{\epsilon}_{\mathrm{q},\alpha}(2))^2(1+3\delta)b_2\right] + b_1^2(\nu_2 - \nu_1)\left[C^{\epsilon}_{\mathrm{q},\alpha}(3)(1+2\delta)\right]}{4C^{\epsilon}_{\mathrm{q},\alpha}(3)(1+2\delta)\left(2C^{\epsilon}_{\mathrm{q},\alpha}(3)(1+2\delta) - (C^{\epsilon}_{\mathrm{q},\alpha}(2))^2(1+3\delta)\right)}.$$

Applying Lemma 1.1 once again for the coefficients b_2 and c_2, we obtain

$$|a_3| \leq \frac{\nu_1 + |\nu_2 - \nu_1|}{2|C^{\epsilon}_{\mathrm{q},\alpha}(3)|(1+2\delta) - |C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2(1+3\delta)}.$$

This estimate is exactly that found in (2.26). □

When $\epsilon = 0$, we obtain the result presented by Ali et al. [25].

Corollary 2.4. *Let $f(z)$ given by (1.1) be in the class $\mathcal{A}_\tau(\delta, \psi)$. Then*

$$|a_2| \leq \frac{\nu_1\sqrt{\nu_1}}{\sqrt{\left|(1+\delta)^2(\nu_1 - \nu_2) + (1+\delta)\nu_1^2\right|}},$$

and

$$|a_3| \leq \frac{\nu_1 + |\nu_2 - \nu_1|}{(1+\delta)}.$$

The coefficient estimates for Ma-Minda bi-starlike functions are obtained when $\epsilon = 0$ and $\delta = 0$. Conversely, for Ma-Minda bi-convex functions, the coefficient estimates are obtained when $\delta = 1$ [26].

Corollary 2.5. *Let $f(z)$ given by (1.1) be in the class $\mathcal{CN}_\tau(\psi)$. Then*

$$|a_2| \leq \frac{\nu_1\sqrt{\nu_1}}{\sqrt{2\left|2\nu_1 - 2\nu_2 + \nu_1^2\right|}},$$

and

$$|a_3| \leq \frac{1}{2}(\nu_1 + |\nu_2 - \nu_1|).$$

3. Fekete-Szegö inequality for the functions classes

Fekete and Szegö published their findings in 1933 [27], setting a precise limit for the functional $a_3 - \gamma a_2^2$. This limit, known as the conventional Fekete-Szegö inequality, was determined using real values of γ $(0 \le \gamma \le 1)$. Establishing accurate bounds for a function within a compact family of functions ($f \in \mathcal{H}$) for a real parameter γ is a tough problem. The Fekete-Szegö coefficient bounds for different analytic subclasses have been established by other authors [28–30]. In this context, the Fekete-Szegö inequality for functions belonging to the classes $f(z) \in \mathcal{T}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\psi)$, $\mathcal{KH}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\delta,\psi)$, and $\mathcal{A}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\delta,\psi)$, is studied, using the findings of Zaprawa [31].

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{T}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\psi)$. Then for some $\gamma \in \mathbb{R}$,*

$$|a_3 - \gamma a_2^2| \leqq \begin{cases} \dfrac{\nu_1}{6|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|}, & (\gamma \in [\gamma_1, \gamma_2]), \\ \dfrac{\nu_1}{6|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|}\left|\dfrac{1}{2} - \dfrac{\nu_2}{2\nu_1} + \dfrac{3\nu_1 C^{\epsilon}_{\mathfrak{q},\alpha}(3)}{4(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2} - \dfrac{3\gamma C^{\epsilon}_{\mathfrak{q},\alpha}(3)}{8(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2}\right|, & (\gamma \notin [\gamma_1, \gamma_2]), \end{cases} \tag{3.1}$$

where

$$\gamma_1 = \frac{2}{3|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|}\left(\frac{-2(\nu_1+\nu_2)|C^{\epsilon}_{\mathfrak{q},\alpha}(2)|^2 + 3\nu_1^2|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|}{\nu_1^2}\right)$$

and

$$\gamma_2 = \frac{2}{3|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|}\left(\frac{2(3\nu_1-\nu_2)|C^{\epsilon}_{\mathfrak{q},\alpha}(2)|^2 + 3\nu_1^2|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|}{\nu_1^2}\right).$$

Proof. Using for those in the Eqs (2.9) and (2.10), we get

$$2a_2 = \frac{\nu_1 b_1}{-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)}$$

$$\implies a_2^2 = \frac{\nu_1^2 b_1^2}{16(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2}, \tag{3.2}$$

and

$$3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(2a_2^2 - a_3) = \frac{1}{2}\nu_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}\nu_2 b_1^2$$

$$\implies -3C^{\epsilon}_{\mathfrak{q},\alpha}(3)a_3 = \frac{1}{2}\nu_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}\nu_2 b_1^2 - 6C^{\epsilon}_{\mathfrak{q},\alpha}(3)a_2^2$$

$$-3C^{\epsilon}_{\mathfrak{q},\alpha}(3)a_3 = \frac{1}{2}\nu_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}\nu_2 b_1^2 - 6C^{\epsilon}_{\mathfrak{q},\alpha}(3)\left(\frac{\nu_1^2 b_1^2}{16(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2}\right)$$

$$\implies a_3 = -\frac{\nu_1}{6C^{\epsilon}_{\mathfrak{q},\alpha}(3)}\left(b_2 - \frac{b_1^2}{2}\right) - \frac{\nu_2 b_1^2}{12C^{\epsilon}_{\mathfrak{q},\alpha}(3)} + \frac{\nu_1^2 b_1^2}{8(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2}. \tag{3.3}$$

Now, from (3.2) and (3.3), we can easily see that

$$a_3 - \gamma a_2^2 = -\frac{\nu_1}{6C^{\epsilon}_{\mathfrak{q},\alpha}(3)}\left(b_2 - \frac{b_1^2}{2}\right) - \frac{\nu_2 b_1^2}{12C^{\epsilon}_{\mathfrak{q},\alpha}(3)} + \frac{\nu_1^2 b_1^2}{8(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2} - \gamma\frac{\nu_1^2 b_1^2}{16(C^{\epsilon}_{\mathfrak{q},\alpha}(2))^2}$$

$$\Longrightarrow a_3 - \gamma a_2^2 = -\frac{\nu_1 b_2}{6C^{\epsilon}_{\mathrm{q},\alpha}(3)} + \left(\frac{\nu_1}{12C^{\epsilon}_{\mathrm{q},\alpha}(3)} - \frac{\nu_2}{12C^{\epsilon}_{\mathrm{q},\alpha}(3)} + \frac{\nu_1^2}{8(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2} - \gamma\frac{\nu_1^2}{16(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2}\right)b_1^2$$

$$a_3 - \gamma a_2^2 = \frac{\nu_1}{6C^{\epsilon}_{\mathrm{q},\alpha}(3)}\left\{-b_2 + \left(\frac{1}{2} - \frac{\nu_2}{2\nu_1} + \frac{3\nu_1 C^{\epsilon}_{\mathrm{q},\alpha}(3)}{4(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2} - \frac{3\gamma\nu_1 C^{\epsilon}_{\mathrm{q},\alpha}(3)}{8(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2}\right)b_1^2\right\},$$

and

$$|a_3 - \gamma a_2^2| = \frac{\nu_1}{6|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}\left|b_2 - \left(\frac{1}{2} - \frac{\nu_2}{2\nu_1} + \frac{3\nu_1 C^{\epsilon}_{\mathrm{q},\alpha}(3)}{4(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2} - \frac{3\gamma\nu_1 C^{\epsilon}_{\mathrm{q},\alpha}(3)}{8(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2}\right)b_1^2\right|.$$

Therefore, in view of Lemma 1.1, we conclude that

$$|a_3 - \gamma a_2^2| \leqq \frac{\nu_1}{6|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}\max\left\{1, \left|\frac{1}{2} - \frac{\nu_2}{2\nu_1} + \frac{3\nu_1 C^{\epsilon}_{\mathrm{q},\alpha}(3)}{4(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2} - \frac{3\gamma\nu_1 C^{\epsilon}_{\mathrm{q},\alpha}(3)}{8(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2}\right|\right\}.$$

Moreover, we have

$$\begin{aligned}
&\left|\frac{1}{2} - \frac{\nu_2}{2\nu_1} + \frac{3\nu_1 C^{\epsilon}_{\mathrm{q},\alpha}(3)}{4(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2} - \frac{3\gamma\nu_1 C^{\epsilon}_{\mathrm{q},\alpha}(3)}{8(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2}\right| \leqq 1\\
\Longleftrightarrow & -1 - \frac{1}{2} + \frac{\nu_2}{2\nu_1} - \frac{3\nu_1|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{4|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2}\\
&\leqq -\frac{3\gamma\nu_1|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{8|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2}\\
&\leqq 1 - \frac{1}{2} + \frac{\nu_2}{2\nu_1} - \frac{3\nu_1|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{4|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2}\\
\Longleftrightarrow & -4 - 2 + \frac{2\nu_2}{\nu_1} - \frac{3\nu_1|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2}\\
&\leqq -\frac{3\gamma\nu_1|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{2|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2}\\
&\leqq 4 - 2 + \frac{2\nu_2}{\nu_1} - \frac{3\nu_1|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2}\\
\Longleftrightarrow & \frac{-6|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2\nu_1 + 2\nu_2|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2 - 3\nu_1^2|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{\nu_1^2}\\
&\leqq -\frac{3\gamma|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{2}\\
&\leqq \frac{2|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2 + 2\nu_2|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2 - 3\nu_1^2|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{\nu_1^2}\\
\Longleftrightarrow & \frac{2}{3|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}\left(\frac{-2(\nu_1+\nu_2)|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2 + 3\nu_1^2|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{\nu_1^2}\right)\\
&\leqq \gamma\\
&\leqq \frac{2}{3|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}\left(\frac{2(3\nu_1-\nu_2)|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2 + 3\nu_1^2|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}{\nu_1^2}\right)\\
\Longleftrightarrow & \ \gamma_1 \leqq \gamma \leqq \gamma_2.
\end{aligned}$$

Taking $\epsilon = 0$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let $f(z)$ given by (1.1) be in the class $\mathcal{T}_\tau(\psi)$. Then*

$$|a_3 - \gamma a_2^2| \leqq \begin{cases} \dfrac{\nu_1}{6}, & (\gamma \in [\gamma_1, \gamma_2]), \\ \dfrac{\nu_1}{6}\left|\dfrac{1}{2} - \dfrac{\nu_2}{2\nu_1} + \dfrac{3\nu_1}{4} - \dfrac{3\gamma}{8}\right|, & (\gamma \notin [\gamma_1, \gamma_2]), \end{cases}$$

where

$$\gamma_1 = \frac{2}{3}\left(\frac{-2(\nu_1+\nu_2) + 3\nu_1^2}{\nu_1^2}\right)$$

and

$$\gamma_2 = \frac{2}{3}\left(\frac{2(3\nu_1-\nu_2) + 3\nu_1^2}{\nu_1^2}\right).$$

Theorem 3.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{KH}^{\epsilon}_{\tau,\mathrm{q},\alpha}(\delta,\psi)$. Then for some $\gamma \in \mathbb{R}$,*

$$|a_3 - \gamma a_2^2| \leqq \begin{cases} \dfrac{\nu_1}{2[-1+3\eta]}, & (\gamma \in [\gamma_1, \gamma_2]), \\ \dfrac{\nu_1}{2[-1+3\eta]}\left|\dfrac{\nu_1\left[6\eta - 2C^{\epsilon}_{\mathrm{q},\alpha}(2)(1+\delta) - 1\right]}{2\left[1-2C^{\epsilon}_{\mathrm{q},\alpha}(2)(1+\delta)\right]^2} + \dfrac{(\nu_1+\nu_2)}{2\nu_1} - \dfrac{\gamma\nu_1[-1+3\eta]}{2\left[1-2C^{\epsilon}_{\mathrm{q},\alpha}(2)(1+\delta)\right]^2}\right|, & (\gamma \notin [\gamma_1, \gamma_2]), \end{cases} \tag{3.4}$$

where

$$\eta = |C^{\epsilon}_{\mathrm{q},\alpha}(3)|(1+2\delta)$$

and

$$\gamma_1 = \frac{(3\nu_1+\nu_2)\left[1-2|C^{\epsilon}_{\mathrm{q},\alpha}(2)|(1+\delta)\right]^2}{\nu_1^2\left[-1+3|C^{\epsilon}_{\mathrm{q},\alpha}(3)|(1+2\delta)\right]} + \frac{\left[6|C^{\epsilon}_{\mathrm{q},\alpha}(3)|(1+2\delta) - 2|C^{\epsilon}_{\mathrm{q},\alpha}(2)|(1+\delta) - 1\right]}{\left[-1+3|C^{\epsilon}_{\mathrm{q},\alpha}(3)|(1+2\delta)\right]},$$

$$\gamma_2 = -\frac{(\nu_1-\nu_2)\left[1-2|C^{\epsilon}_{\mathrm{q},\alpha}(2)|(1+\delta)\right]^2}{\nu_1^2\left[-1+3|C^{\epsilon}_{\mathrm{q},\alpha}(3)|(1+2\delta)\right]} + \frac{\nu_1\left[6|C^{\epsilon}_{\mathrm{q},\alpha}(3)|(1+2\delta) - 2|C^{\epsilon}_{\mathrm{q},\alpha}(2)|(1+\delta) - 1\right]}{\nu_1\left[-1+3|C^{\epsilon}_{\mathrm{q},\alpha}(3)|(1+2\delta)\right]}.$$

Proof. Using the relations in Eqs (2.20) and (2.21), we get

$$\begin{aligned} &\left[1-2C^{\epsilon}_{\mathrm{q},\alpha}(2)(1+\delta)\right]a_2 = \frac{\nu_1 b_1}{2} \\ \Longleftrightarrow\ & a_2^2 = \frac{\nu_1^2 b_1^2}{4\left[1-2C^{\epsilon}_{\mathrm{q},\alpha}(2)(1+\delta)\right]^2}. \end{aligned} \tag{3.5}$$

$$\begin{aligned} &-\left[-1+3C^{\epsilon}_{\mathrm{q},\alpha}(3)(1+2\delta)\right]a_3 + \left[6C^{\epsilon}_{\mathrm{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathrm{q},\alpha}(2)(1+\delta) - 1\right]a_2^2 \\ &= \frac{1}{2}\nu_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}\nu_2 b_1^2, \end{aligned}$$

$$\Longleftrightarrow a_3 = \left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta) - 1\right] \left(\frac{\nu_1^2 b_1^2}{4\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}\right) - \frac{\nu_1\left(2b_2-b_1^2\right)-\nu_2 b_1^2}{4\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}. \tag{3.6}$$

Now, from (3.5) and (3.6), we can see that

$$a_3 - \gamma a_2^2 = \left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta) - 1\right]\left(\frac{\nu_1^2 b_1^2}{4\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}\right) - \frac{\nu_1\left(2b_2-b_1^2\right)-\nu_2 b_1^2}{4\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]} - \gamma\frac{\nu_1^2 b_1^2}{4\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2}$$

$$\Longleftrightarrow a_3 - \gamma a_2^2 = \frac{\nu_1 b_2}{2\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]} + \left(\frac{\nu_1^2\left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta) - 1\right]}{4\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]} + \frac{(\nu_1+\nu_2)}{4\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]} - \frac{\gamma\nu_1^2}{4\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2}\right) b_1^2$$

$$\Longleftrightarrow a_3 - \gamma a_2^2 = \frac{\nu_1}{2\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}\left\{b_2 + \left(\frac{\nu_1\left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta) - 1\right]}{2\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2} + \frac{(\nu_1+\nu_2)}{2\nu_1} - \frac{\gamma\nu_1\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}{2\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2}\right) b_1^2\right\}$$

$$\Longleftrightarrow |a_3 - \gamma a_2^2| = \frac{\nu_1}{2\left[-1+3|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|(1+2\delta)\right]}\left|b_2 + \left(\frac{\nu_1\left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta) - 1\right]}{2\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2} + \frac{(\nu_1+\nu_2)}{2\nu_1} - \frac{\gamma\nu_1\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}{2\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2}\right) b_1^2\right|.$$

Therefore, in view of Lemma 1.1, we conclude that

$$|a_3 - \gamma a_2^2| \leqq \frac{\nu_1}{2\left[-1+3|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|(1+2\delta)\right]}\max\left\{1, \left|\frac{\nu_1\left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta) - 1\right]}{2\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2} + \frac{(\nu_1+\nu_2)}{2\nu_1} - \frac{\gamma\nu_1\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}{2\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2}\right|\right\}.$$

Moreover, we have

$$\left|\frac{\nu_1\left[6C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta) - 2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta) - 1\right]}{2\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2} + \frac{(\nu_1+\nu_2)}{2\nu_1} - \frac{\gamma\nu_1\left[-1+3C^{\epsilon}_{\mathfrak{q},\alpha}(3)(1+2\delta)\right]}{2\left[1-2C^{\epsilon}_{\mathfrak{q},\alpha}(2)(1+\delta)\right]^2}\right| \leqq 1$$

$$
\begin{aligned}
&\Longleftrightarrow -1-\frac{\nu_1\left[6|C^{\epsilon}_{q,\alpha}(3)|(1+2\delta)-2|C^{\epsilon}_{q,\alpha}(2)|(1+\delta)-1\right]}{2\left[1-2|C^{\epsilon}_{q,\alpha}(2)|\,(1+\delta)\right]^2}-\frac{(\nu_1+\nu_2)}{2\nu_1}\\
&\leqq -\frac{\gamma\nu_1\left[-1+3|C^{\epsilon}_{q,\alpha}(3)|(1+2\delta)\right]}{2\left[1-2|C^{\epsilon}_{q,\alpha}(2)|(1+\delta)\right]^2}\\
&\leqq 1-\frac{\nu_1\left[6|C^{\epsilon}_{q,\alpha}(3)|(1+2\delta)-2|C^{\epsilon}_{q,\alpha}(2)|(1+\delta)-1\right]}{2\left[1-2|C^{\epsilon}_{q,\alpha}(2)|\,(1+\delta)\right]^2}-\frac{(\nu_1+\nu_2)}{2\nu_1}\\
&\Longleftrightarrow -\frac{(\nu_1-\nu_2)\left[1-2|C^{\epsilon}_{q,\alpha}(2)|\,(1+\delta)\right]^2}{\nu_1^2\left[-1+3|C^{\epsilon}_{q,\alpha}(3)|(1+2\delta)\right]}+\frac{\nu_1\left[6|C^{\epsilon}_{q,\alpha}(3)|(1+2\delta)-2|C^{\epsilon}_{q,\alpha}(2)|(1+\delta)-1\right]}{\nu_1\left[-1+3|C^{\epsilon}_{q,\alpha}(3)|(1+2\delta)\right]}\\
&\leqq \gamma\\
&\leqq \frac{(3\nu_1+\nu_2)\left[1-2|C^{\epsilon}_{q,\alpha}(2)|\,(1+\delta)\right]^2}{\nu_1^2\left[-1+3|C^{\epsilon}_{q,\alpha}(3)|(1+2\delta)\right]}+\frac{\left[6|C^{\epsilon}_{q,\alpha}(3)|(1+2\delta)-2|C^{\epsilon}_{q,\alpha}(2)|(1+\delta)-1\right]}{\left[-1+3|C^{\epsilon}_{q,\alpha}(3)|(1+2\delta)\right]}\\
&\Longleftrightarrow \gamma_1\leqq\gamma\leqq\gamma_2.
\end{aligned}
$$

□

Taking $\epsilon = 0$ and $\delta = 0$ in Theorem 3.2, we obtain the following corollary.

Corollary 3.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{KH}_\tau(0,\psi)$. Then*

$$
|a_3-\gamma a_2^2| \leqq \begin{cases} \dfrac{\nu_1}{4}, & (\gamma\in[\gamma_1,\gamma_2]),\\[2ex] \dfrac{\nu_1}{4}\left|\dfrac{3\nu_1}{2}+\dfrac{(\nu_1+\nu_2)}{2\nu_1}-\dfrac{2\gamma\nu_1}{2}\right|, & (\gamma\notin[\gamma_1,\gamma_2]),\end{cases}
$$

$$
\gamma_1=\frac{3\nu_1+\nu_2}{2\nu_1^2}+\frac{3}{2},
$$

and

$$
\gamma_2=-\frac{\nu_1-\nu_2}{2\nu_1^2}+\frac{3\nu_1}{2\nu_1}.
$$

Theorem 3.3. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{A}^{\epsilon}_{\tau,q,\alpha}(\delta,\psi)$. Then for some $\gamma\in\mathbb{R}$,*

$$
|a_3-\gamma a_2^2| \leqq \begin{cases} \dfrac{\nu_1}{4(1+2\delta)|C^{\epsilon}_{q,\alpha}(3)|}, & (\gamma\in[\gamma_1,\gamma_2]),\\[2ex] \dfrac{\nu_1}{4(1+2\delta)|C^{\epsilon}_{q,\alpha}(3)|}\left|\dfrac{2(1+3\delta)\nu_1^2}{(1+\delta)^2}+\dfrac{(\nu_2-\nu_1)}{2}-\dfrac{4\gamma\nu_1^2(1+2\delta)C^{\epsilon}_{q,\alpha}(3)}{\left[(1+\delta)C^{\epsilon}_{q,\alpha}(2)\right]^2}\right|, & (\gamma\notin[\gamma_1,\gamma_2]),\end{cases} \tag{3.7}
$$

where

$$
\gamma_1=-\frac{(2-\nu_2+\nu_1)\left[(1+\delta)|C^{\epsilon}_{q,\alpha}(2)|\right]^2}{8\nu_1^2(1+2\delta)|C^{\epsilon}_{q,\alpha}(3)|}+\frac{(1+3\delta)|C^{\epsilon}_{q,\alpha}(2)|^2}{2(1+2\delta)|C^{\epsilon}_{q,\alpha}(3)|}
$$

and

$$\gamma_2 = \frac{(2+\nu_2-\nu_1)\left[(1+\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(2)|\right]^2}{8\nu_1^2(1+2\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(3)|} + \frac{(1+3\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2}{2(1+2\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}.$$

Proof. Using for those in Eqs (2.29) and (2.30), we get

$$(1+\delta)C^{\epsilon}_{\mathrm{q},\alpha}(2)a_2 = \frac{\nu_1 c_1}{2}$$

$$\Longleftrightarrow a_2^2 = \frac{\nu_1^2 c_1^2}{\left[(1+\delta)C^{\epsilon}_{\mathrm{q},\alpha}(2)\right]^2}, \tag{3.8}$$

$$2(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)a_3 - (1+3\delta)(C^{\epsilon}_{\mathrm{q},\alpha}(2))^2 a_2^2 = \frac{1}{2}\nu_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}\nu_2 c_1^2$$

$$\Longleftrightarrow a_3 = \frac{(1+3\delta)\nu_1^2 c_1^2}{2(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)(1+\delta)^2} + \frac{\nu_1 c_2}{4(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)} + \frac{(\nu_2-\nu_1)c_1^2}{8(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)}. \tag{3.9}$$

Now, from (3.8) and (3.9), we can easily see that

$$a_3 - \gamma a_2^2 = \frac{(1+3\delta)\nu_1^2 c_1^2}{2(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)(1+\delta)^2} + \frac{\nu_1 c_2}{4(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)} + \frac{(\nu_2-\nu_1)c_1^2}{8(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)} - \gamma\frac{\nu_1^2 c_1^2}{\left[(1+\delta)C^{\epsilon}_{\mathrm{q},\alpha}(2)\right]^2},$$

$$a_3 - \gamma a_2^2 = \frac{\nu_1 c_2}{4(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)} + \left(\frac{(1+3\delta)\nu_1^2}{2(1+2\delta)(1+\delta)^2 C^{\epsilon}_{\mathrm{q},\alpha}(3)} + \frac{(\nu_2-\nu_1)}{8(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)} - \frac{\gamma\nu_1^2}{\left[(1+\delta)C^{\epsilon}_{\mathrm{q},\alpha}(2)\right]^2}\right)c_1^2,$$

$$a_3 - \gamma a_2^2 = \frac{\nu_1}{4(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)}\left\{c_2 + \left(\frac{2(1+3\delta)\nu_1^2}{(1+\delta)^2} + \frac{(\nu_2-\nu_1)}{2} - \frac{4\gamma\nu_1^2(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)}{\left[(1+\delta)C^{\epsilon}_{\mathrm{q},\alpha}(2)\right]^2}\right)c_1^2\right\},$$

$$|a_3 - \gamma a_2^2| = \frac{\nu_1}{4(1+2\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}\left|c_2 + \left(\frac{2(1+3\delta)\nu_1^2}{(1+\delta)^2} + \frac{(\nu_2-\nu_1)}{2} - \frac{4\gamma\nu_1^2(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)}{\left[(1+\delta)C^{\epsilon}_{\mathrm{q},\alpha}(2)\right]^2}\right)c_1^2\right|.$$

Therefore, in view of Lemma 1.1, we conclude that

$$|a_3 - \gamma a_2^2| \leqq \frac{\nu_1}{4(1+2\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}\max\left\{1, \left|\frac{2(1+3\delta)\nu_1^2}{(1+\delta)^2} + \frac{(\nu_2-\nu_1)}{2} - \frac{4\gamma\nu_1^2(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)}{\left[(1+\delta)C^{\epsilon}_{\mathrm{q},\alpha}(2)\right]^2}\right|\right\}.$$

Moreover, we have

$$\left|\frac{2(1+3\delta)\nu_1^2}{(1+\delta)^2} + \frac{(\nu_2-\nu_1)}{2} - \frac{4\gamma\nu_1^2(1+2\delta)C^{\epsilon}_{\mathrm{q},\alpha}(3)}{\left[(1+\delta)C^{\epsilon}_{\mathrm{q},\alpha}(2)\right]^2}\right| \leqq 1$$

$$\Longleftrightarrow -\frac{(2-\nu_2+\nu_1)\left[(1+\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(2)|\right]^2}{8\nu_1^2(1+2\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(3)|} + \frac{(1+3\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(2)|^2}{2(1+2\delta)|C^{\epsilon}_{\mathrm{q},\alpha}(3)|}$$

$$\leqq \gamma$$

$$\leqq \frac{(2+\nu_2-\nu_1)\left[(1+\delta)|C^{\epsilon}_{\mathfrak{q},\alpha}(2)|\right]^2}{8\nu_1^2(1+2\delta)|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|} + \frac{(1+3\delta)\left|C^{\epsilon}_{\mathfrak{q},\alpha}(2)\right|^2}{2(1+2\delta)|C^{\epsilon}_{\mathfrak{q},\alpha}(3)|}$$

$$\Longleftrightarrow \gamma_1 \leqq \gamma \leqq \gamma_2.$$

□

Taking $\epsilon = 0$ and $\delta = 0$ from Theorem 3.3, we obtain the following corollary.

Corollary 3.3. *Let $f(z)$ given by (1.1) be in the class $\mathcal{A}_\tau(0,\psi)$. Then*

$$|a_3 - \gamma a_2^2| \leqq \begin{cases} \dfrac{\nu_1}{4}, & (\gamma \in [\gamma_1, \gamma_2]), \\ \dfrac{\nu_1}{4}\left|2\nu_1^2(1-2\gamma) + \dfrac{(\nu_2-\nu_1)}{2}\right|, & (\gamma \notin [\gamma_1, \gamma_2]), \end{cases} \tag{3.10}$$

where

$$\gamma_1 = -\frac{2-\nu_2+\nu_1}{8\nu_1^2} + \frac{1}{2}$$

and

$$\gamma_2 = \frac{2+\nu_2-\nu_1}{8\nu_1^2} + \frac{1}{2}.$$

In these Theorems 3.1–3.3, we use the technique of [32].

4. Conclusions

In our present investigation, we have introduced and studied the coefficient problems associated with each of the following three new subclasses: $\mathcal{T}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\psi)$, $\mathcal{KH}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\delta,\psi)$, $\mathcal{A}^{\epsilon}_{\tau,\mathfrak{q},\alpha}(\psi)$ of the class of bi-univalent Ma-Minda-type functions associated with the $\mathfrak{q}$-Srivastava-Attiya operator in the open unit disk $\mathbb{D}$. These bi-univalent Ma-Minda-type functions associated with $\mathfrak{q}$-Srivastava-Attiya operator classes are given by Definitions 1.2 to 1.7, respectively. For functions in each of these three bi-univalent Ma-Minda-type functions associated with $\mathfrak{q}$-Srivastava-Attiya operator classes, we have derived the estimates of the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these new subclasses, along with estimates for the Fekete-Szegö functional problem, for functions belonging to each of these bi-univalent function classes. Our results also further generalize the results of Theorems 2.1–2.3 of R. M. Ali et al. and some results of W. Ma and D. Minda. The results presented in this paper are a beneficial supplement for the research of geometric function theory of complex analysis.

Author contributions

Norah Saud Almutairi: Investigation, supervision, writing–original draft, writing–review, and editing; Adarey Saud Almutairi: Supervision; Awatef Shahen: Supervision; Hanan Darwish: Supervision, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

The first author would like to thank her father Saud Dhaifallah Almutairi for supporting this work.

Funding

The author(s) express their sincere gratitude and appreciation to [Onaizah Colleges , Saudi Arabia] for providing APC funding for this research.

Conflicts of interest

The authors declare that they have no conflict of interest.

References

1. P. L. Duren, *Univalent functions*, Springer Science & Business Media, 2001.
2. L. de-Branges, A proof of the Bieberbach conjecture, *Acta Math.*, **154** (1985), 137–152.
3. S. Sivasubramanian, R. Sivakumar, S. Kanas, S. Kim, Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions, *Ann. Pol. Math.*, **113** (2015), 295–304. https://doi.org/10.4064/ap113-3-6
4. D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, *Math. Anal. Appl.*, 1988, 53–60. https://doi.org/10.1016/B978-0-08-031636-9.50012-7
5. T. S. Taha, *Topics in univalent function theory*, Ph.D. Thesis, University of London, 1981.
6. S. S. Miller, P. T. Mocanu, *Differential subordinations: Theory and applications*, 1st Eds., Boca Raton: CRC Press, 2000. https://doi.org/10.1201/9781482289817
7. H. M. Srivastava, A. A. Attiya, An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, *Integral Transforms Spec. Funct.*, **18** (2007), 207–216. https://doi.org/10.1080/10652460701208577
8. H. M. Srivastava, The Zeta and related functions: Recent developments, *J. Adv. Eng. Comput.*, **3** (2019), 329–354. http://dx.doi.org/10.25073/jaec.201931.229
9. F. Ghanim, B. Batiha, A. H. Ali, M. Darus, Geometric properties of a linear complex operator on a subclass of meromorphic functions: An analysis of Hurwitz-Lerch-Zeta functions, *Appl. Math. Nonlinear Sci.*, **8** (2023), 2229–2240. https://doi.org/10.2478/amns.2023.1.00407
10. H. Bateman, *Higher transcendental functions*, New York: McGraw-Hill Book Company, **I-III** (1953).
11. S. A. Shah, K. I. Noor, Study on the q-analogue of a certain family of linear operators, *Turkish J. Math.*, **43** (2019), 2707–2714. https://doi.org/10.3906/mat-1907-41
12. E. Deniz, M. Kamali, S. Korkmaz, A certain subclass of bi-univalent functions associated with Bell numbers and q-Srivastava-Attiya operator, *AIMS Mathematics*, **5** (2020), 7259–7271. https://doi.org/10.3934/math.2020464

13. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. Sci.*, **44** (2020), 327–344. https://doi.org/10.1007/s40995-019-00815-0

14. V. Kac, P. Cheung, Symmetric quantum calculus, In: *Quantum calculus*, New York: Springer, 2002, 99–104. https://doi.org/10.1007/978-1-4613-0071-7_26

15. H. M. Srivastava, A. R. S. Juma, H. M. Zayed, Univalence conditions for an integral operator defined by a generalization of the Srivastava-Attiya operator, *Filomat*, **32** (2018), 2101–2114. https://doi.org/10.2298/FIL1806101S

16. Y. J. Sim, O. S. Kwon, N. E. Cho, H. M. Srivastava, Bounds for the real parts and arguments of normalized analytic functions defined by the Srivastava-Attiya operator, *J. Comput. Anal. Appl.*, **28** (2020), 628–645.

17. H. M. Srivastava, A. Prajapati, P. Gochhayat, Third-order differential subordination and differential superordination results for analytic functions involving the Srivastava-Attiya operator, *Appl. Math. Inform. Sci.*, **12** (2018), 469–481. http://dx.doi.org/10.18576/amis/120301

18. K. I. Noor, S. Riaz, M. A. Noor, On q-Bernardi integral operator, *TWMS J. Pure Appl. Math.*, **8** (2017), 3–11.

19. S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, **135** (1969), 429–446. https://doi.org/10.2307/1995025

20. J. W. Alexander, Functions which map the interior of the unit circle upon simple region, *Ann. Math.*, **17** (1915), 12–22. https://doi.org/10.2307/2007212

21. R. H. Nevanlinna, Über die konforme abbildung von sterngebieten, *Översikt av Finska Vetens. Soc. Förh., Avd.*, **63** (1921), 1–21.

22. E. Study, *Vorlesungen über ausgewählte gegenstände der geometrie, zweites heft; konforme abbildung einfach-zusammenhängender bereiche*, 1913.

23. W. Kaplan, Close-to-convex schlicht functions, *Michigan Math. J.*, **1** (1952), 169–185. https://doi.org/10.1307/mmj/1028988895

24. C. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, 1975.

25. R. M. Ali, S. K. Lee, V. Ravichandran, S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.*, **25** (2012), 344–351. https://doi.org/10.1016/j.aml.2011.09.012

26. W. C. Ma, D. Minda, A unified treatment of some special classes of univalent functions, In: *Proceedings of the conference on complex analysis*, 1992, 157–169.

27. M. Fekete, G. Szegö, Eine Bemerkung über ungerade schlichte Funktionen, *J. Lond. Math. Soc.*, **s1-8** (1933), 85–89. https://doi.org/10.1112/jlms/s1-8.2.85

28. A. Cătaş, On the Fekete-Szegö problem for certain classes of meromorphic functions using a p,q-derivative operator and a p,q-wright type hypergeometric function, *Symmetry*, **13** (2021), 2143.

29. N. S. Almutairi, A. Shahen, A. Cătaş, H. Darwish, On the Fekete–Szegö problem for certain classes of (γ, δ)-starlike and (γ, δ)-convex functions related to quasi-subordinations, *Symmetry*, **16** (2024), 1043. https://doi.org/10.3390/sym16081043

30. G. Murugusundaramoorthy, L. Cotîrla, Bi-univalent functions of complex order defined by Hohlov operator associated with legendrae polynomial, *AIMS Mathematics*, **7** (2022), 8733–8750. https://doi.org/10.3934/math.2022488

31. P. Zaprawa, On the Fekete-Szegö problem for classes of bi-univalent functions, *Bull. Belg. Math. Soc. Simon Stevin*, **21** (2014), 169–178. https://doi.org/10.36045/bbms/1394544302

32. H. M. Srivastava, M. Kamali, A. Urdaletova, A study of the Fekete-Szegö functional and coefficient estimates for subclasses of analytic functions satisfying a certain subordination condition and associated with the Gegenbauer polynomials, *AIMS Mathematics*, **7** (2022), 2568–2584. https://doi.org/10.3934/math.2022144