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Research article

Finite-time synchronization of fractional-order chaotic systems by applying the maximum-valued method of functions of five variables

Junli You[1] **and Zhengqiu Zhang**[2,*]

[1] School of General Education, Hunan University of Information Technology, Changsha, 410151, China

[2] College of Mathematics, Hunan University, Changsha, 410082, China

* **Correspondence:** Email: zhangzhengqiu@hnu.edu.cn; Tel: +8613367496849.

Abstract: In this discussion, the finite time synchronization (FTSN) of master-slave fractional-order chaotic systems (MSFOCSS) is explored. By adopting the maximum-valued method (MVM) of functions of five variables, two novel criteria on the FTSN are obtained for the MSFOCSS. So far, the studies of the FTSN of the MSFOCSS have been rare. Furthermore, the existing results on the FTSN of MSFOCSS have been achieved often by adopting the LMI method and finite time stability theorems (FTST). Thus, instead of utilizing the past research methods, adopting the MVM to study the FTSN of the MSFOCSS is an innovative study work.

Keywords: finite time synchronization; fractional-order chaotic systems; the maximum-valued method of functions of five variables
Mathematics Subject Classification: 34D06, 93D40

1. Introduction

Chaotic systems (CSS) are quite important systems because they can check the chaotic states of dynamical systems and forecast the future state responses according to the initial conditions. That is, a very small variation in the present state may give rise to disrupt the states with large deviations. So, chaos is one of the most esteemed and intriguing phenomena in control theory. It is a fast that the CSS has widespread applications in practice in many application fields such as secure communications, biological systems, information processing, chemical reactions, economic systems, and so on. So, great advancements in study works have been achieved in synchronizing and stabilization of the CSS by adopting different study methods [7–9,12–16,37–40,43,44]and constructing different controllers [1,2]. In addition, fractional calculus (FCS) is one of the best study approaches for modeling the nonlinear dynamical system of real-world phenomena with more accuracy. Thus, the FCS clearly elucidates the hereditary and memory properties of various materials and processes. In this connection, fractional-order systems (FOSS) are mainly utilized to elucidate the numerous seemingly diverse domains in science and engineering when compared with integer-order dynamical systems [3, 4]. In addition, chaotic phenomena are independent of order because many real-world happenings are controlled in terms of both integers. Hence, it is essential for us to research the synchronization subject of the FOSS. Because of this practical application, the studies on the synchronization (exponential synchronization and asymptotic synchronization and other synchronization) for the FOSS have been a hot research topic and a large number of studies have been realized [5, 6, 10, 11, 17–22].

The exponential/asymptotic synchronization of the system cannot guarantee that the system is bounded in a finite time, or it might be in a large time scale. However, in practical application, in order to better control the dynamics, it is the best thing that the convergence rate of the dynamics should be in a finite time span. But, so far, few works have been achieved on the FTSN for the FOCSS. We only found a few papers [5, 23–30] in which the FTSN subject for FOCSS was studied. In [24], the FTSN for the fractional-order chaotic systems (FOCSS) with different structures under parameter disturbance and external disturbance was studied. By adopting the FTST, some criteria for synchronization of FOCSS were established. In the article [24], employing the sliding-mode control, a finite-time adaptive synchronization method was introduced to realize the generalized projective synchronization of fractional order (FO) memristor or CSS with unknown parameters. The authors designed a new FO integral sliding-mode surface with a faster convergence speed, which can make the error system converge to zero in finite time. The sufficient condition for the sliding-mode surface can be reached by the synchronization error system in finite time. Paper [25] considered the finite-time projective synchronization within a thermal-mechanical FO system in the presence of external disruptions. The study utilized a developed sliding mode surface and used the Lyapunov function method (LFM) to synchronize trajectories. Paper [26] designed a finite-time multiple synchronization controller for FO hyper CSS. The FTSN was explored by employing the FTST for the CSS. This paper [27] researched the synchronization subject of nonlinear delayed FOCSS. The fast synchronization of the considered system was ensured in a finite time by using LMI. In [28], the authors investigated the subject of adaptive sliding mode synchronization control for a class of variable-order FO uncertain coupled systems. By using the graph theory, some novel FTSN criteria were obtained for the systems. Paper [29] introduced a novel fuzzy event-triggered control way for uncertain FOSS in the presence of input delay. A novel practical finite-time chaos synchronization control approach with an event-triggered strategy was proposed. Paper [30] addressed the topic of the FTSN of FO simplest two-component chaotic oscillations operating at high frequency and application to digital cryptography. By constructing a Lyapunov function, an adaptive feedback controller was designed to achieve the FTSN of two oscillators. In [5], by employing LFW, some criteria on asymptotic synchronization for FOCSS were obtained.

Up to now, the FTSN for the MSFOCSS has been explored often by employing the LMI approach [27], LFW [24, 25, 29, 30], FTST [23, 26] and graph theory [28]. Up to now, the results on the FTSN obtained by using LMI for dynamical systems have been very complicated and difficult to verify. In addition, the FTST only can be used to study such a first-order differential inequality: $V^{\eta}(t) \leq a + bV(t)$, where, a, b are constants and η is fractional order cannot be used to study the differential inequalities (3.3) and (3.11) in our paper. Since the results of the FTSN MSFOCSS have been rare, this motivates us to study the FTSN for MSFOCSS by using a new study method. Our MVM

can be used to study the FTSN by studying the complicated differential inequalities, furthermore, our results obtained by MVM are more concise and more easily verified than those obtained by using LMI and FTST. Instead of using the past methods of FTSN for the FOCSS, in this article, by adopting the MVM of functions of the 5 variables and constructing new controllers, two new criteria are established for the MSFOCSS. One of the difficulties we meet is how to construct the new controllers and plan the function of 5 variables can get the possible unique local extreme point of the discussed function. The other difficulty we meet is how to design the assumed conditions can get the setting finite times. Thus, the main contributions of this article include the following three aspects:
(1) The novel controllers are designed to study the FTSN of the discussed MSFOCSS.
(2) The MVM of functions of 5 variables is introduced to study the FTSN of MSFOCSS.
(3) Two new criteria on FTSN of FOCSS are obtained by employing the MVM of functions of 5 variables.

2. Preliminaries

We consider the following drive fractional-order Lorenz system with Caputo derivative:

$$\begin{cases} \frac{d^\eta N_1(t)}{dt^\eta} = b_1[N_3(t) - N_1(t)], \\ \frac{d^\eta N_2(t)}{dt^\eta} = b_1[N_4(t) - N_2(t)], \\ \frac{d^\eta N_3(t)}{dt^\eta} = b_2 N_1(t) - N_3(t) - N_1(t)N_5(t), \\ \frac{d^\eta N_4(t)}{dt^\eta} = b_2 N_2(t) - N_4(t) - N_2(t)N_5(t), \\ \frac{d^\eta N_5(t)}{dt^\eta} = N_1(t)N_3(t) + N_2(t)N_4(t) - b_3 N_5(t), \end{cases} \tag{2.1}$$

which is discussed in [5]. We also consider the corresponding slave fractional-order Lorenz systems with Caputo derivative discussed in [5] as follows:

$$\begin{cases} \frac{d^\eta K_1(t)}{dt^\eta} = b_1[K_3(t) - K_1(t)] + Q_1(t), \\ \frac{d^\eta K_2(t)}{dt^\eta} = b_1[K_4(t) - K_2(t)] + Q_2(t), \\ \frac{d^\eta K_3(t)}{dt^\eta} = b_2 K_1(t) - K_3(t) - K_1(t)K_5(t) + Q_3(t), \\ \frac{d^\eta K_4(t)}{dt^\eta} = b_2 K_2(t) - K_4(t) - K_2(t)K_5(t) + Q_4(t), \\ \frac{d^\eta K_5(t)}{dt^\eta} = K_1(t)K_3(t) + K_2(t)K_4(t) - b_3 K_5(t) + Q_5(t), \end{cases} \tag{2.2}$$

where $b_i (i = 1, 2, 3)$ are constants, $Q_m(t), m = 1, 2, 3, 4, 5$ are the control functions that will be designed later, $0 < \eta < 1$ is the fractional order, and n is a positive integer. Defining the error functions as $\alpha_m(t) = K_m(t) - N_m(t), m = 1, 2, 3, 4, 5$, the fractional-order error system with the Caputo derivative can be expressed as

$$\begin{cases} \frac{d^\eta \alpha_1(t)}{dt^\eta} = b_1[\alpha_3(t) - \alpha_1(t)] + Q_1(t), \\ \frac{d^\eta \alpha_2(t)}{dt^\eta} = b_1[\alpha_4(t) - \alpha_2(t)] + Q_2(t), \\ \frac{d^\eta \alpha_3(t)}{dt^\eta} = b_2\alpha_1(t) - \alpha_3(t) - \alpha_1(t)\alpha_5(t) - \alpha_1(t)N_5(t) - \alpha_5(t)N_1(t) + Q_3(t), \\ \frac{d^\eta \alpha_4(t)}{dt^\eta} = b_2\alpha_2(t) - \alpha_4(t) - \alpha_2(t)\alpha_5(t) - \alpha_2(t)N_5(t) - \alpha_5(t)N_2(t) + Q_4(t), \\ \frac{d^\eta \alpha_5(t)}{dt^\eta} = \alpha_1(t)\alpha_3(t) + \alpha_2(t)\alpha_4(t) - b_3\alpha_5(t) + \alpha_1(t)N_3(t) + \alpha_2(t)N_4(t), \\ \quad +\alpha_3(t)N_1(t) + \alpha_4(t)N_2(t) + Q_5(t). \end{cases} \tag{2.3}$$

Now, by designing the control functions $Q_m(t), m = 1, 2, 3, 4, 5$, employing the control skills, we will study the FTSN for system (2.1) and system (2.2).

Definition 2.1. The system (2.1) and the system (2.2) are said to realize FTSN. If for each solution of system (2.1) and system (2.2) expressed by $[N_1(t), N_2(t), \cdots, N_5(t)]^T$ and $[K_1(t), K_2(t), \cdots, K_5(t)]^T$, there is a positive constant T that depends on the initial values of the error system (2.3) such that

$$\lim_{t\to T} |N_m(t) - K_m(t)| = 0, \quad |N_m(t) - K_m(t)| = 0, t \geq T.$$

Definition 2.2. [32] The fractional integral of function $H_1(t)$ is defined as

$$I^\eta H_1(t) = \frac{1}{G(\eta)} \int_{t_0}^{t} (t-u)^{\eta-1} H_1(u) du, \quad \eta > 0,$$

where $G(.)$ is the Gamma function defined by

$$G(\eta) = \int_0^\infty t^{\eta-1} e^{-t} dt.$$

Definition 2.3. [31] Caputo fractional derivative (CFD) of order η of function $H_1(t)$ is defined by

$$D^\eta H_1(t) = \frac{1}{G(l-\eta)} \int_{t_0}^{t} \frac{H_1^{(n)}(u)}{(t-u)^{\eta-n+1}} du,$$

where $0 \leq l-1 < \eta < l, G(.)$ is the Gamma function, and l is an integer that is more than 1. Especially, when $0 < \eta < 1$,

$$D^\eta H_1(t) = \frac{1}{G(1-\eta)} \int_{t_0}^{t} \frac{H_1'(u)}{(t-u)^\eta} du.$$

Lemma 2.1. [32] If the CFD $D^\eta H_1(t)$ is integrable, then the η integration of $D^\eta H_1(t)$ is defined as

$$I^\eta [D^\eta H_1(t)] = H_1(t) - \sum_{k=0}^{n-1} \frac{H_1^{(k)}(t_0)}{k!} (t-t_0)^k.$$

Especially, for $0 < \eta \leq 1$, one has

$$I^\eta [D^\eta H_1(t] = H_1(t) - H_1(t_0).$$

Lemma 2.2. [33] Suppose that $H_1(t) \in R$ is a continuous and derivable function. Then for $t \geq 0$,

$$\frac{1}{2} D^\eta H_1^2(t) \leq H_1(t) D^\eta H_1(t), \eta \in (0,1).$$

Lemma 2.3. [35,42] If function $f(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)), x_i \in (a_i, b_i), i = 1,2,3,4,5$ has a unique possible local extremum point $(\alpha_1^0, \alpha_2^0, \alpha_3^0, \alpha_4^0, \alpha_5^0)$, and $\lim_{(x_1,x_2,x_3,x_4,x_5)\to(y_1,y_2,y_3,y_4,y_5)} f(x_1,x_2,x_3,x_4,x_5) = z_1$ exists, where $y_i = a_i$ or $b_i (i = 1,2,3,4,5), a_i$ may be $-\infty, b_i$ may be $+\infty$, then

$$\begin{aligned} & f(x_1, x_2, \cdots, x_5) \\ \leq & \max_{(x_1,x_2,\cdots,x_5)\in R^5} f(x_1, x_2, \cdots, x_5) \\ = & \max\left\{ f(\alpha_1^0, \alpha_2^0, \alpha_3^0, \alpha_4^0, \alpha_5^0), z_1 \right\}. \end{aligned}$$

3. Main results

The control functions in system (2.3) are designed as follows:

$$\begin{aligned} Q_1(t) &= -\ \alpha_5(t)N_3(t),\\ Q_2(t) &= \alpha_4(t)N_5(t),\\ Q_3(t) &= \alpha_1(t)N_5(t),\\ Q_4(t) &= \alpha_4^{-1}(t)(k_5+k_6),\\ Q_5(t) &= -\alpha_2(t)N_4(t)+k_4,\\ Q_4(t) &= 0, \alpha_4(t)=0; \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} Q_1(t) &= k_2,\\ Q_2(t) &= k_3+\alpha_2^{-1}(t)\beta,\\ Q_3(t) &= \alpha_1(t)\alpha_5(t),\\ Q_4(t) &= \alpha_2(t)N_5(t),\\ Q_5(t) &= -\alpha_1(t)N_3(t)-\alpha_2(t)N_4(t)+k_1\alpha_1(t),\\ Q_2(t) &= 0, \alpha_2(t)=0, \end{aligned} \tag{3.2}$$

where $k_5 < 0, k_6 < 0, \beta < 0, k_1, k_2, k_3, k_4$ are constants.

Theorem 3.1. *Set $\eta \in (0,1)$. Then the system (2.1) and the system (2.2) can achieve the FTSN by utilizing the designed controllers (3.1) if the following conditions are met:*

(h_1)

$$b_2 < b_1, b_3 > 0, 0.5(b_1+b_2) < 1, (b_1+b_2)^2 \neq 4b_1;$$

(h_2)

$$k_5 + \frac{k_4^2}{4b_3} < 0.$$

Proof. A Lyapunov function is considered as follows:

$$n(t) = \frac{1}{2}\Big[\alpha_1^2(t)+\alpha_2^2(t)+\alpha_3^2(t)+\alpha_4^2(t)+\alpha_5^2(t)\Big].$$

From system (2.3), by Lemma 2.2, it follows that

$$\begin{aligned} & D_t^{\eta} n(t)\\ \le\ & \alpha_1(t)\Big[b_1(\alpha_3(t)-\alpha_1(t))+Q_1(t)\Big]+\alpha_2(t)[b_1(\alpha_4(t)-\alpha_2(t))+Q_2(t)\Big]+\alpha_3(t)\Big[b_2\alpha_1(t)\\ & -\alpha_3(t)-\alpha_1(t)\alpha_5(t)-\alpha_1(t)N_5(t)-\alpha_5(t)N_2(t)+Q_3(t)\Big]+\alpha_4(t)\Big[b_2\alpha_2(t)-\alpha_4(t)-\\ & \alpha_2(t)\alpha_5(t)-\alpha_2(t)N_5(t)-\alpha_5(t)N_2(t)+Q_4(t)\Big]+\alpha_5(t)\Big[\alpha_1(t)\alpha_3(t)+\alpha_2(t)\alpha_4(t)-b_3\\ & \times\alpha_5(t)+\alpha_1(t)N_3(t)+\alpha_2(t)N_4(t)+\alpha_3(t)N_1(t)+\alpha_4(t)N_2(t)+Q_5(t)\Big]\\ =\ & (b_1+b_2)\alpha_1(t)\alpha_3(t)-b_1\alpha_1^2(t)+(b_1+b_2)\alpha_2(t)\alpha_4(t)-b_1\alpha_2^2(t)-\alpha_3^2(t)-\alpha_4^2(t)-b_3\alpha_5^2(t) \end{aligned}$$

$$
\begin{aligned}
& +\alpha_1(t)Q_1(t)+\alpha_1(t)\alpha_5(t)N_3(t)+\alpha_2(t)Q_2(t)-\alpha_2(t)\alpha_4(t)N_5(t)+\alpha_3(t)Q_3(t)-\alpha_1(t) \\
& \times\alpha_3(t)N_5(t)+\alpha_4(t)Q_4(t)+\alpha_5(t)Q_5(t)+\alpha_2(t)\alpha_5(t)N_4(t)+k_4\alpha_5(t)+k_5+k_6 \\
= \; & (b_1+b_2)\alpha_1(t)\alpha_3(t)-b_1\alpha_1^2(t)+(b_1+b_2)\alpha_2(t)\alpha_4(t)-b_1\alpha_2^2(t)-\alpha_3^2(t)-\alpha_4^2(t)-b_3\alpha_5^2(t) \\
& +k_4\alpha_5(t)+k_5+k_6. \qquad (3.3)
\end{aligned}
$$

Let

$$
\begin{aligned}
& g(\alpha_1(t),\alpha_2(t),\alpha_3(t),\alpha_4(t),\alpha_5(t)) \\
= \; & (b_1+b_2)\alpha_1(t)\alpha_3(t)-b_1\alpha_1^2(t)+(b_1+b_2)\alpha_2(t)\alpha_4(t)-b_1\alpha_2^2(t)-\alpha_3^2(t)-\alpha_4^2(t)-b_3\alpha_5^2(t) \\
& +k_4\alpha_5(t)+k_5,(\alpha_1(t),\alpha_2(t),\alpha_3(t),\alpha_4(t),\alpha_5(t))\in R^5. \qquad (3.4)
\end{aligned}
$$

From (3.4), we get

$$
\begin{cases}
\dfrac{\partial g}{\partial\alpha_1}=(b_1+b_2)\alpha_3(t)-2b_1\alpha_1(t),\\
\dfrac{\partial g}{\partial\alpha_2}=(b_1+b_2)\alpha_4(t)-2b_1\alpha_2(t),\\
\dfrac{\partial g}{\partial\alpha_3}=(b_1+b_2)\alpha_1(t)-2\alpha_3(t),\\
\dfrac{\partial g}{\partial\alpha_4}=(b_1+b_2)\alpha_2(t)-2\alpha_4(t),\\
\dfrac{\partial g}{\partial\alpha_5}=k_4-2b_3\alpha_5(t).
\end{cases}
$$

Letting

$$
\begin{cases}
\dfrac{\partial g}{\partial\alpha_1}=(b_1+b_2)\alpha_3(t)-2b_1\alpha_1(t)=0,\\
\dfrac{\partial g}{\partial\alpha_2}=(b_1+b_2)\alpha_4(t)-2b_1\alpha_2(t)=0,\\
\dfrac{\partial g}{\partial\alpha_3}=(b_1+b_2)\alpha_1(t)-2\alpha_3(t)=0,\\
\dfrac{\partial g}{\partial\alpha_4}=(b_1+b_2)\alpha_2(t)-2\alpha_4(t)=0,\\
\dfrac{\partial g}{\partial\alpha_5}=k_4-2b_3\alpha_5(t)=0,
\end{cases}
$$

by using (h_1), we obtain the unique possible local extremum point $(\alpha_1^0,\alpha_2^0,\alpha_3^0,\alpha_4^0,\alpha_5^0)$ of $g(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$, where

$$
\alpha_1^0=0,\alpha_2^0=0,\alpha_3^0=0,\alpha_4^0=0,\alpha_5^0=\frac{k_4}{2b_3}.
$$

From (3.4) and using (h_2), it follows that

$$
\begin{aligned}
& g(\alpha_1^0,\alpha_2^0,\alpha_3^0,\alpha_4^0,\alpha_5^0) \\
= \; & -b_3\Big(\frac{k_4}{2b_3}\Big)^2+k_4\frac{k_4}{2b_3}+k_5
\end{aligned}
$$

$$= \quad k_5 + \frac{k_4^2}{4b_3} < 0. \tag{3.5}$$

From (3.4), using inequality $2ab \le a^2 + b^2$, we obtain

$$\begin{aligned}
& g(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t)) \\
\le\ & 0.5(b_1 + b_2)[\alpha_1^2(t) + \alpha_3^2(t)] - b_1\alpha_1^2(t) + 0.5(b_1 + b_2)[\alpha_2^2(t) + \alpha_4^2(t)] - b_1\alpha_2^2(t) - \alpha_3^2(t) \\
& -\alpha_4^2(t) - b_3\alpha_5^2(t) + k_4\alpha_5(t) + k_5 \\
=\ & 0.5(b_2 - b_1)\alpha_1^2(t) + [0.5(b_1 + b_2) - 1]\alpha_3^2(t) + [0.5(b_1 + b_2) - 1]\alpha_4^2(t) + 0.5(b_2 - b_1)\alpha_2^2(t) \\
& -b_3\alpha_5^2(t) + k_4\alpha_5(t) + k_5.
\end{aligned} \tag{3.6}$$

From (3.6), we get

$$\begin{aligned}
& g(\infty, \infty, \infty, \infty, \infty) \\
\le\ & \lim_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\to(\infty,\infty,\infty,\infty,\infty)} \Big\{0.5(b_2 - b_1)\alpha_1^2(t) + [0.5(b_1 + b_2) - 1]\alpha_3^2(t) + [0.5(b_1 + b_2) \\
& -1]\alpha_4^2(t) + 0.5(b_2 - b_1)\alpha_2^2(t) - b_3\alpha_5^2(t) + k_4\alpha_5(t) + k_5\Big\} \\
\le\ & k_5 \\
<\ & 0.
\end{aligned} \tag{3.7}$$

From (3.5) and (3.7), by using Lemma 2.3, we have

$$\begin{aligned}
& g(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t)) \\
\le\ & \max_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\in R^5} \Big\{g(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t))\Big\} \\
\le\ & \max\Big\{g(\alpha_1^0, \alpha_2^0, \alpha_3^0, \alpha_4^0, \alpha_5^0), g(\infty, \infty, \infty, \infty, \infty)\Big\} \\
\le\ & 0.
\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.3) gives

$$D_t^\eta n(t) \le k_6. \tag{3.9}$$

q-Integrating (3.9) over $[0, t]$ gives

$$\begin{aligned}
0 \le n(t) \le\ & n(0) + \frac{1}{G(\eta)} \int_0^t (t - u)^{\eta-1} k_6 du \\
\le\ & n(0) + \frac{k_6 t^\eta}{\eta G(p)}.
\end{aligned} \tag{3.10}$$

When $t \ge \hat{t}_1 = \left[\frac{-n(0)\eta G(\eta)}{k_6}\right]^{\frac{1}{\eta}}$, from (3.10), we have

$$0 \le n(t) \le 0, t \ge \hat{t}_1.$$

Namely

$$\lim_{t\to\hat{t}_1} n(t) = 0, n(t) = 0, t \ge \hat{t}_1.$$

Namely

$$\lim_{t\to\hat{t}_1}|K_m(t)-N_m(t)|=0, |K_m(t)-N_m(t)|=0, t\geq\hat{t}_1.$$

This ends the argument of Theorem 3.1.

Theorem 3.2. *Let $\eta\in(0,1)$. The system (2.1) and system (2.2) can achieve the FTS by utilizing the constructed controllers (3.2) if the following conditions are satisfied:*

(l_1)

$$0.5|b_1+b_2|<b_1+0.5|k_1|, |b_1+b_2|<2b_3, |b_1+b_2|<2, |k_1|<2b_3;$$

(l_2)

$$b_3>0, (b_1+b_2)^2>4b_1.$$

Proof. A Lyapunov function is introduced as follows:

$$n(t)=\frac{1}{2}\Big[\alpha_1^2(t)+\alpha_2^2(t)+\alpha_3^2(t)+\alpha_4^2(t)+\alpha_5^2(t)\Big].$$

From system (2.3), by using Lemma 2.2, it follows that

$$\begin{aligned}
& D_t^{\eta}n(t) \\
\leq\ & \alpha_1(t)\Big[b_1(\alpha_3(t)-\alpha_1(t))+Q_1(t)\Big]+\alpha_2(t)[b_1(\alpha_4(t)-\alpha_2(t))+Q_2(t)\Big]+\alpha_3(t)\Big[b_2\alpha_1(t) \\
& -\alpha_3(t)-\alpha_1(t)\alpha_5(t)-\alpha_1(t)N_5(t)-\alpha_5(t)N_2(t)+Q_3(t)\Big]+\alpha_4(t)\Big[b_2\alpha_2(t)-\alpha_4(t)- \\
& \alpha_2(t)\alpha_5(t)-\alpha_2(t)N_5(t)-\alpha_5(t)N_2(t)+Q_4(t)\Big]+\alpha_5(t)\Big[\alpha_1(t)\alpha_3(t)+\alpha_2(t)\alpha_4(t)-b_3 \\
& \times\alpha_5(t)+\alpha_1(t)N_3(t)+\alpha_2(t)N_4(t)+\alpha_3(t)N_1(t)+\alpha_4(t)N_2(t)+Q_5(t)\Big] \\
=\ & (b_1+b_2)\alpha_1(t)\alpha_3(t)-b_1\alpha_1^2(t)+(b_1+b_2)\alpha_2(t)\alpha_4(t)-b_1\alpha_2^2(t)-\alpha_3^2(t)-\alpha_4^2(t)-b_3\alpha_5^2(t) \\
& +\alpha_1(t)Q_1(t)+\alpha_1(t)\alpha_5(t)N_3(t)+\alpha_2(t)Q_2(t)-\alpha_2(t)\alpha_4(t)N_5(t)+\alpha_3(t)Q_3(t)-\alpha_1(t) \\
& \times\alpha_3(t)N_5(t)+\alpha_4(t)Q_(t)+\alpha_5(t)Q_5(t)+\alpha_2(t)\alpha_5(t)N_4(t) \\
=\ & (b_1+b_2)\alpha_1\alpha_3-b_1\alpha_1^2+(b_1+b_2)\alpha_2\alpha_4-b_1\alpha_2^2-\alpha_3^2-\alpha_4^2-b_3\alpha_5^2+k_1\alpha_1\alpha_5+k_2\alpha_1 \\
& +k_3\alpha_2+\beta.
\end{aligned}\tag{3.11}$$

Set

$$\begin{aligned}
& \hat{f}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \\
=\ & (b_1+b_2)\alpha_1\alpha_3-b_1\alpha_1^2+(b_1+b_2)\alpha_2\alpha_4-b_1\alpha_2^2-\alpha_3^2-\alpha_4^2-b_3\alpha_5^2+k_1\alpha_1\alpha_5+k_2\alpha_1+k_3\alpha_2, \\
& (\alpha_1,\alpha_2,\cdots,\alpha_5)\in R^5.
\end{aligned}\tag{3.12}$$

From (3.12), we have

$$\begin{cases} \dfrac{\partial \hat{f}}{\partial \alpha_1} = -2b_1\alpha_1 + (b_1+b_2)\alpha_3 + k_1\alpha_5 + k_2, \\ \dfrac{\partial \hat{f}}{\partial \alpha_2} = -2b_1\alpha_2 + (b_1+b_2)\alpha_4 + k_3, \\ \dfrac{\partial \hat{f}}{\partial \alpha_3} = -2\alpha_3 + (b_1+b_2)\alpha_1, \\ \dfrac{\partial \hat{f}}{\partial \alpha_4} = -2\alpha_4 + (b_1+b_2)\alpha_2, \\ \dfrac{\partial \hat{f}}{\partial \alpha_5} = -2b_3\alpha_5 + k_1\alpha_1. \end{cases}$$

Setting

$$\begin{cases} \dfrac{\partial \hat{f}}{\partial \alpha_1} = -2b_1\alpha_1 + (b_1+b_2)\alpha_3 + k_1\alpha_5 + k_2 = 0, \\ \dfrac{\partial \hat{f}}{\partial \alpha_2} = -2b_1\alpha_2 + (b_1+b_2)\alpha_4 + k_3 = 0, \\ \dfrac{\partial \hat{f}}{\partial \alpha_3} = -2\alpha_3 + (b_1+b_2)\alpha_1 = 0, \\ \dfrac{\partial \hat{f}}{\partial \alpha_4} = -2\alpha_4 + (b_1+b_2)\alpha_2 = 0, \\ \dfrac{\partial \hat{f}}{\partial \alpha_5} = -2b_3\alpha_5 + k_1\alpha_1 = 0, \end{cases}$$

we obtain the unique local extremum point $(\alpha_1^0, \alpha_2^0, \cdots, \alpha_5^0)$ of $\hat{f}(\alpha_1, \alpha_2, \cdots, \alpha_5)$, where

$$\alpha_1^0 = \frac{2b_3k_2}{4b_1b_3 - k_1^2 - b_3(b_1+b_2)^2};$$

$$\alpha_2^0 = \frac{2k_3}{4b_1 - (b_1+b_2)^2}, \quad \alpha_3^0 = \frac{k_2b_3(b_1+b_2)}{4b_1b_3 - k_1^2 - b_3(b_1+b_2)^2};$$

$$\alpha_4^0 = \frac{k_3(b_1+b_2)}{4b_1 - (b_1+b_2)^2}, \quad \alpha_5^0 = \frac{k_1k_2}{4b_1b_3 - k_1^2 - b_3(b_1+b_2)^2}.$$

By applying inequality $2ab \le a^2 + b^2$, we have

$$\begin{aligned} & \hat{f}(\alpha_1, \alpha_2, \cdots, \alpha_5) \\ \le\ & 0.5|b_1+b_2|(\alpha_1^2+\alpha_3^2) - b_1\alpha_1^2 + 0.5|b_1+b_2|(\alpha_2^2+\alpha_4^2) - b_1\alpha_2^2 - \alpha_3^2 - \alpha_4^2 - b_3\alpha_5^2 + \\ & 0.5|k_1|(\alpha_1^2+\alpha_5^2) + k_2\alpha_1 + k_3\alpha_2 \\ =\ & \Big[0.5|b_1+b_2| - b_1 - 0.5|k_1|\Big]\alpha_1^2 + (0.5|b_1+b_2| - b_1)\alpha_2^2 - [0.5\times \\ & \times|b_1+b_2| - 1]\alpha_3^2 + [0.5|b_1+b_2| - 1]\alpha_4^2 + (0.5|k_1| - b_3)\alpha_5^2 + k_2\alpha_1 + k_3\alpha_2 \\ \le\ & k_2\alpha_1 + k_3\alpha_2. \end{aligned} \tag{3.13}$$

From (3.13), we obtain by using (l_2),

$$\begin{aligned} & \hat{f}(\alpha_1^0, \alpha_2^0, \cdots, \alpha_5^0) \\ \leq\ & \frac{2b_3k_2^2}{4b_1b_3 - k_1^2 - b_3(b_1+b_2)^2} + \frac{2k_3^2}{4b_1 - (b_1+b_2)^2} \\ \leq\ & 0. \end{aligned} \tag{3.14}$$

At the same time, by using (l_1), we have

$$\begin{aligned} & \hat{f}(\infty, \infty, \cdots, \infty) \\ \leq\ & \lim_{(\alpha_1,\alpha_2,\cdots,\alpha_5)\to(\infty,\infty,\infty,\infty,\infty)} \Big\{\Big[0.5|b_1+b_2| - b_1 - 0.5|k_1|\Big]\alpha_1^2 + (|b_1+b_2| - b_3)\alpha_2^2 \\ & -\alpha_3^2 + [0.5|b_1+b_2| - 1]\alpha_4^2 + (0.5|k_1| - b_3)\alpha_5^2 + k_2\alpha_1 + k_3\alpha_2\Big\} \\ =\ & -\infty < 0. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), by using Lemma 2.3, it follows that

$$\begin{aligned} & \hat{f}(\alpha_1, \alpha_2, \cdots, \alpha_5) \\ \leq\ & \max_{(\alpha_1,\alpha_2,\cdots,\alpha_4,\alpha_5)\in R^5} \{\hat{f}(\alpha_1, \alpha_2, \cdots, \alpha_5)\} \\ \leq\ & \max\Big\{\hat{f}(\alpha_1^0, \alpha_2^0, \alpha_3^0, \alpha_4^0, \alpha_5^0), \hat{f}(\infty, \infty, \cdots, \infty)\Big\} \\ \leq\ & 0. \end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.11) gives

$$D_t^\eta n(t) \leq \beta. \tag{3.17}$$

q-Integrating (3.17) over $[0, t]$ gives

$$\begin{aligned} 0 \leq n(t) \ &\leq\ n(0) + \frac{1}{G(\eta)}\int_0^t (t-u)^{\eta-1}\beta du \\ &\leq\ n(0) + \frac{\beta t^\eta}{\eta G(\eta)}. \end{aligned} \tag{3.18}$$

When $t \geq \hat{t}_2 = \left[\frac{-n(0)\eta G(\eta)}{\beta}\right]^{\frac{1}{\eta}}$, by (3.18), we have

$$0 \leq n(t) \leq 0, t \geq \hat{t}_2.$$

Then

$$\lim_{t\to\hat{t}_2} n(t) = 0, n(t) = 0, t \geq \hat{t}_2.$$

Namely

$$\lim_{t\to\hat{t}_2} |K_m(t) - N_m(t)| = 0, |K_m(t) - N_m(t)| = 0, t \geq \hat{t}_2. \tag{3.19}$$

From (3.19), the argument of Theorem 3.2 is ended.
Remark 1. So far, the results on FTSN of FOCSS are obtained often by employing the LMI way [27], FTST [23,26], and LFW [24,25,29,30], and graph theory [28]. In our discussion, the maximum-valued method of functions of five variables is cited to study the FTSN of MSFOCSS. Namely, by using the MVM of functions of 5 variables, two innovative results on FTSN of the discussed Lorenz chaotic systems are gotten.
Remark 2. In [5], the exponential synchronization for the MSFOCSS was studied by LFW. In our article, by using the MVM,innovative results are established for the same MSFOCSS. Thus, our study develops the study fields of synchronization for the MSFOCSS.
Remark 3. The FTSN method in our article cannot be extended to study the preassigned-time intermittent control of memristive chaotic systems and Fixed-time synchronization of dynamical networks [34, 43, 44] since the setting time in our results is related to the initial values of the error system.
Remark 4. The result in Theorem 3.1 and the result in Theorem 3.2 are different since their controllers are different and the assumed conditions are different.
Remark 5. The maximum-valued method and techniques used in our study are quite different from the integral inequality method and techniques used in [42] and techniques used in [43, 44].

4. Numerical examples

In this section, two examples are given to show the validity of our results.
Example 4.1. We are concerned about the drive system (2.1) and response system (2.2) with control functions (3.1) for $i = 5$, where $b_1 = 1, b_2 = 0.5, b_3 = 4, k_4 = 1.4, k_5 = -3, k_6 = -1$. It is easy to test if (h_1) and (h_2) are met. Thus, by Theorem 3.1, the system (2.1) and the system (2.2) can realize the FTSN utilizing the (3.1).

The curves of the drive fractional-order Lorenz system are shown in the Figure 1, and the curves of the corresponding slave fractional-order Lorenz systems are shown in Figure 2; the fractional-order error system's curves that achieve the FTSN are shown in Figure 3. Figures 1 and 2 show that the system fluctuates regularly with time, and the kinetic behavior shown is oscillating. Figure 3 shows that the system error oscillates with time and converges to 0 at finite time t. In this case, dynamic behavior is in a synchronized state.

In our example, without utilizing the past study methods in [5, 24–30], the controllers in our paper are essentially different from those in [5, 24–30], so our result on FTS cannot be dealt with using their methods in [5, 24–30].

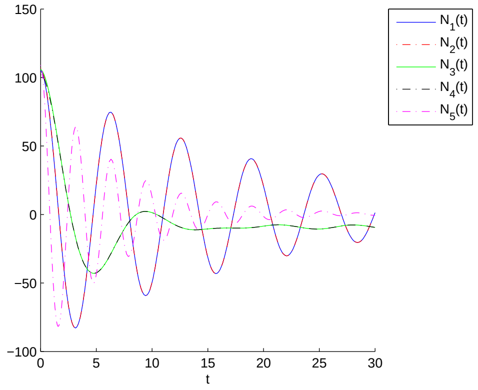

Figure 1. Curves of the variables $N_i(t)$ of drive system (2.1) when $k_4 = 1.4, k_5 = -3, k_6 = -1$.

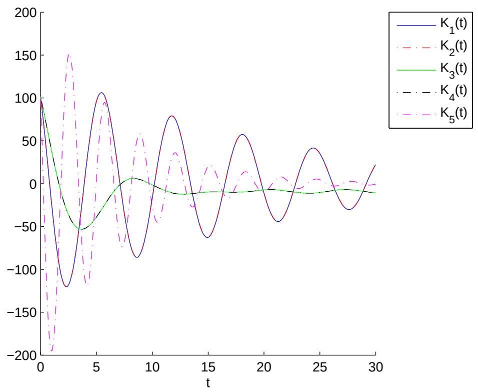

Figure 2. Curves of the variables $K_i(t)$ of response system (2.2) when $k_4 = 1.4, k_5 = -3,\ k_6 = -1$.

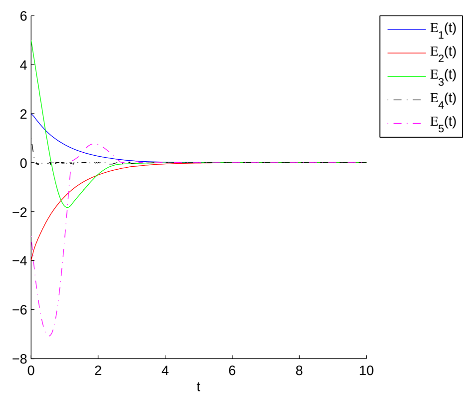

Figure 3. Curves of the variables $\alpha_i = E_i(t)$ of error system (2.3) when $k_4 = 1.4, k_5 = -3, k_6 = -1$.

Example 4.2. We are concerned about the drive system (2.1) and response system (2.2) with control functions (3.2). For $i = 5$, the parameters of both systems are taken as $b_1 = 0.2, b_2 = -1.5, b_3 = 3$, and controllers (4.2) with $\beta = -2, k_1 = 2, k_2 = 2.5$, and $k_3 = 0.1$. It is easy to test if the (l_1) and (l_2) are met. Thus, by Theorem 3.2, the system (2.1) and the system (2.2) can attain the FTSN by employing the controllers (3.2).

With different constraints and controllers, the curves of synchronization for drive and response system under the controller (3.2) are shown in Figures 4–6. Figures 4 and 5 show the dynamic behavior of the system is very similar, with slight fluctuations in the master-slave system. The figure shows synchronization errors between systems (2.1) and (2.2), whose values converge to 0 at positive time t.

In our discussion, without utilizing the past study methods in [5, 24–30], the controllers in our paper are very different from those in [5]. In [5], the asymptotic synchronization was explored, while in our paper, the FTSN is considered. Our method, the controllers designed, and the results on FTSN are different from those in the existing articles. So, our result on FTSN cannot be dealt with their methods in [5, 24–30].

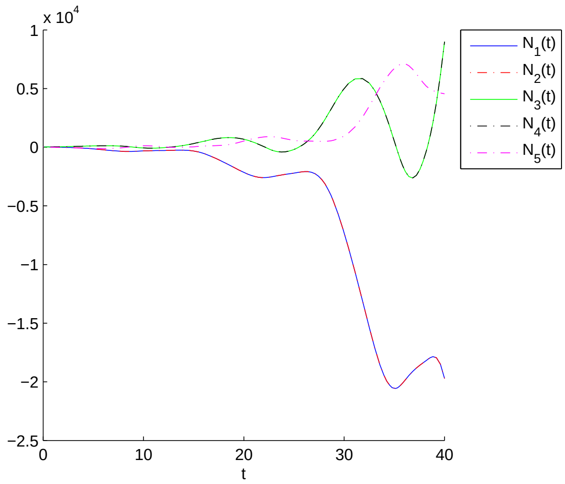

Figure 4. Curves of the variables $N_i(t)$ of drive system (2.1) when $\beta = -2, k_1 = 2, k_2 = 2.5, k_3 = 0.1$.

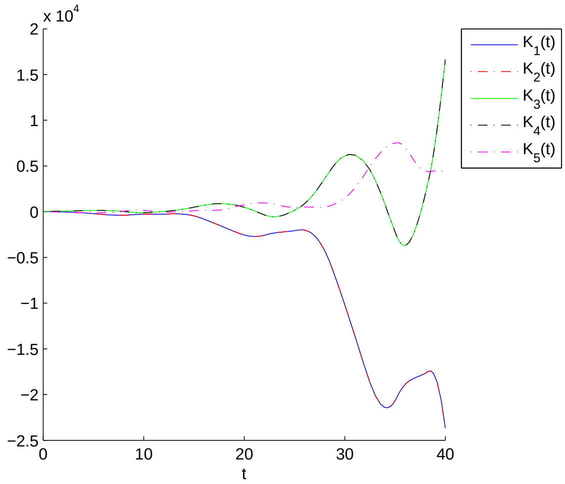

Figure 5. Curves of the variables $K_i(t)$ of response system (2.2) when $\beta = -2, k_1 = 2, k_2 = 2.5, k_3 = 0.1$.

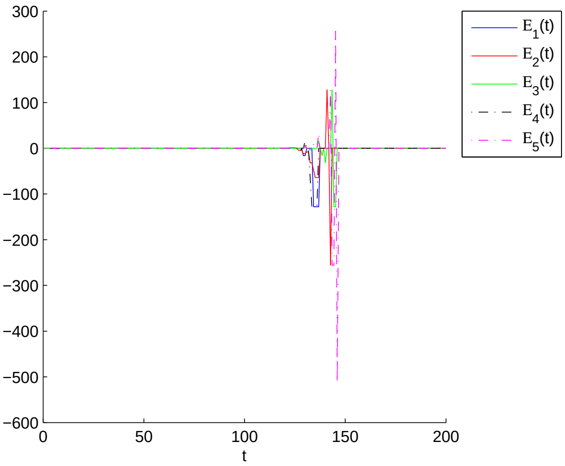

Figure 6. Curves of the variables $\alpha_i = E_i(t)$ of error system (2.3) when $\beta = -2, k_1 = 2, k_2 = 2.5, k_3 = 0.1$.

5. Conclusions

In this article, without adopting previous methods, such as the Fo integral inequality way, the LMI approach, and FTST, by using the maximum-valued method of functions of five variables, two criteria assuring the FTSN for the discussed MSFOCSS have been established. Our study method and results on the FTSN have been quite novel for MSFOCSS. In the future, we will transform our study direction to study the synchronization for the discrete-time fractional-order dynamical systems.

Author contributions

Junli You: Conceptualization, methodology, software, validation, formal analysis, writing-review and editing, and project administration; Zhengqiu Zhang: Conceptualization, methodology, validation, formal analysis, investigation, and writing-original draft preparation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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