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Research article

Second-order nonlinear neutral differential equations with delay term: Novel oscillation theorems

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Abstract: This work aims to propose novel criteria to guarantee the oscillation of solutions for second-order differential equations. To analyze the oscillatory characteristics of the studied equation, new necessary conditions are introduced. We used a variety of analysis techniques to support these findings, forming fresh connections to tackle some issues that have impeded earlier studies. As a result, by using the Riccati transformation and the principles of comparison, we were able to acquire results that both expand upon and enhance those found in previous research. Several examples are presented to illustrate the significance of our findings.

Keywords: oscillation theorems; second-order; delay terms; differential equation
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1. Introduction

A quantity in the modeled phenomenon is represented by each dependent variable in differential equations (DEs), which are mathematical models used to explore natural phenomena. Differential equations are essential to many engineering applications and have enabled us to comprehend a wide range of intricate events in our daily lives. They are now crucial instruments in technology and applied sciences, utilized to investigate media, conversations, phone signals, and internet buying data. More conventionally, astronomers utilized them to explain how stars moved and how planets orbited. Additionally, they have numerous uses in the fields of biology and medicine; see [1–3].

This study focuses on developing oscillation criteria for second-order neutral differential equations

$$\left(a(\eta)\left((y(\eta)+\varpi(\eta)y(\zeta(\eta)))'\right)^{\beta}\right)'+\varsigma(\eta)y^{\beta}(\xi(\eta))=0, \tag{1.1}$$

where $\eta \geq \eta_0$,. In this study, we make the following assumptions:

(H_1) $a \in C([\eta_0,\infty),(0,\infty))$, $\varpi \in C([\eta_0,\infty),[0,\infty))$, $\varsigma \in C([\eta_0,\infty),[0,\infty))$, $\varsigma(\eta)$ does not vanish identically, $\varpi(\eta) < 1$ and satisfies the canonical case given by

$$\int_{\eta_0}^{\infty} a^{-1/\beta}(s)\,\mathrm{d}s = \infty; \tag{1.2}$$

(H_2) $\zeta, \xi \in C([\eta_0,\infty),\mathbb{R})$, $\zeta(\eta) \le \eta$, $\xi(\eta) \le \eta$ and $\lim_{\eta\to\infty}\zeta(\eta) = \lim_{\eta\to\infty}\xi(\eta) = \infty$ and β is a quotient of adding positive integers;

By a solution of (1.1), we mean a function $y \in C^1\left([\eta_y,\infty),\mathbb{R}\right)$, $\eta_y \ge \eta_0$, which has the property $a(\eta)(y'(\eta))^{\beta} \in C^1([\eta_0,\infty),\mathbb{R})$, and satisfies (1.1) on $[\eta_y,\infty)$. We consider only those solutions y of (1.1) which satisfy $\sup\{|y(\eta)| : \eta \ge \eta_y\} > 0$, for all $\eta > \eta_y$. y is referred to as oscillatory if it is neither finally positive nor eventually negative; if it is, it is referred to as non-oscillatory. If every solution to the equation oscillates, the equation is said to be oscillatory.

A particular subset of functional differential equations (FDEs) known as neutral differential equations (NDEs) have derivatives that are affected by the function's present values as well as its derivatives from previous times. This special feature creates a different analytical framework and sets NDEs apart from conventional FDEs. Since NDEs and FDEs frequently occur in systems where both previous values and rates of change have an impact on future states, their link is crucial. Since NDEs depict systems with memory effects, their importance is especially clear in domains like control theory and signal processing. For example, acceleration may be influenced by both velocity and current position in mechanical systems with inertia. The significance of NDEs in precisely modeling and simulating dynamic systems is highlighted by this relationship. Furthermore, since knowledge of one typically yields important insights into the other, the study of NDEs complements that of FDEs [4–7]. Problems involving masses linked to a shaky, flexible rod are among the many domains in which these equations find application [8–10].

One essential tool for comprehending and simulating a variety of natural and technical systems is the ordinary differential equation (ODE). Even though ODEs are widely used, the complexity and diversity of real-world occurrences frequently require the addition of sophisticated argumentation in order to get more thorough and accurate solutions; see [11, 12]. Many nonlinear systems exhibit behavior that traditional linear differential equations are unable to adequately describe, which has highlighted the significance of integrating sophisticated arguments into ODEs. These systems may be represented more realistically thanks to advanced nonlinear dynamics, which improves predictions and insights. Additionally, systems subject to minor perturbations can be analyzed using perturbation methods, which offers a way to comprehend how complex systems behave in various scenarios. Additionally, stability analysis is critical for figuring out how ODE solutions behave over the long run, which is critical in disciplines like epidemiology and control theory; see [13, 14].

The investigation of higher-order equation oscillation conditions, especially second-order differential equations with delays, has advanced significantly in recent years. This explains the enormous interest in the qualitative features of these kinds of equations. Numerous real-world models exhibit oscillation phenomena; for example, the works [15, 16] discuss mathematical biology models in which cross-diffusion terms may be used to design oscillation and/or delay actions. The oscillation theory of this kind of equation has been thoroughly developed as part of this methodology.

See Sun et al. [17] for some similar works, and Dzurina et al. [18] acquired a few oscillation conditions for

$$(a(\eta)|y'(\eta)|^{\beta-1}y'(\eta))' + \varsigma(\eta)|y[\xi(\eta)]|^{\beta-1}y[\xi(\eta)] = 0. \tag{1.3}$$

Sahiner and Wang [19, 20] established some oscillation results for equations

$$(a(\eta)(y(\eta) + \varpi(\eta)y(\eta-\zeta_0))')' + \varsigma(\eta)f(y(\xi(\eta))) = 0.$$

Studies by Xu and Weng [21] and Zhao and Meng [22], which concentrate on oscillation criteria and asymptotic behavior, are examples of subsequent contributions. Recent works by Baculikova and Dzurina [23] and Grace [24] are noteworthy as they offer important new information on oscillation conditions for second-order delay differential equations of type

$$\left(a(\eta)((y(\eta) + \varpi(\eta)y(\zeta(\eta)))')^{\beta}\right)' + \varsigma(\eta)y^{\beta}(\xi(\eta)) = 0. \tag{1.4}$$

Last, further useful standards for evaluating the asymptotic and oscillatory behavior of solutions are presented in recent works by Al-Jaser et al. [25], Batiha et al. [26], and Hassan et al. [27, 28]. There are recognized comparison theorems that compare the first-order differential equations with the second-order (1.4).

In this paper, we derive the various conditions for oscillation of (1.1) using Riccati transformations and comparison principles. The main results are demonstrated with examples.

This paper is structured as follows: We provide the researched equation and the general circumstances required to arrive at the paper's key conclusions in the first section (Introduction). We also give a summary of relevant subjects and the purpose of this research. We provide a few relationships and findings in Section 2 that will be utilized to arrive at the oscillation results covered in the "Oscillation results" paragraph. To demonstrate the importance of the results acquired, we offer a few instances in Section 3. In Section 4, we conclude by summarizing the paper's key findings and highlighting an unanswered question that might be of interest to scholars working in this area.

For ease of use, we indicate that

$$\begin{aligned}
w(\eta) &:= y(\eta) + \varpi(\eta)y(\zeta(\eta)), \\
\pi_{\eta_0}(\eta) &:= \int_{\eta_0}^{\eta} a^{-1/\beta}(s)\,\mathrm{d}s, \\
\widetilde{\pi}_{\eta_0}(\eta) &:= \pi_{\eta_0}(\eta) + \frac{1}{\beta}\int_{\eta_0}^{\eta} \pi_{\eta_1}(s)\pi_{\eta_0}^{\beta}(\xi(s))\varsigma(s)(1-\varpi(\xi(s)))^{\beta}\,\mathrm{d}s, \\
\widehat{\pi}(\eta) &:= \exp\left(-\beta\int_{\xi(\eta)}^{\eta} \frac{\mathrm{d}s}{\widetilde{\pi}_{\eta_0}(s)a^{1/\beta}(s)}\right),
\end{aligned}$$

and

$$E(\eta) := \varsigma(\eta)(1-\varpi(\xi(\eta)))^{\beta}\widehat{\pi}(\eta).$$

2. Main results

2.1. Supplementary lemmas

Over the years, the study of first-order equation oscillation underwent several stages of growth before becoming more theoretically and scientifically clear and intelligible. Examine the differential equation of the first order.

$$\omega'(\eta)+\widetilde{\pi}_{\eta_1}^{\beta}(\xi(\eta))\varsigma(\eta)(1-\varpi(\xi(\eta)))^{\beta}\,\omega(\xi(\eta))=0. \tag{2.1}$$

We now outline several auxiliary lemmas and conditions that we will use to achieve the main results:

Lemma 1. *[23] If y is an eventually positive solution of (1.1), then*

$$w(\eta)>0,\ w'(\eta)>0,\left(a(\eta)(w'(\eta))^{\beta}\right)'\leq 0, \tag{2.2}$$

for $\eta\geq\eta_1$.

Lemma 2. *[29] Let* $f(y)=Gy-By^{(\beta+1)/\beta}$ *where* $G,B>0$ *are constants, and*

$$\max_{y\in a} f=f(y^*)=\beta^{\beta}(\beta+1)^{-(\beta+1)}\frac{G^{\beta+1}}{B^{\beta}}, \tag{2.3}$$

where $y^*=(\beta G/((\beta+1)B))^{\beta}$.

Lemma 3. *Let y be an eventually positive solution of (1.1). Then*

$$\left(a(\eta)(w'(\eta))^{\beta}\right)'\leq-\varsigma(\eta)(1-\varpi(\xi(\eta)))^{\beta}\,w^{\beta}(\xi(\eta)), \tag{2.4}$$

and

$$w(\eta)\geq\widetilde{\pi}_{\eta_1}(\eta)\,a^{1/\beta}(\eta)\,w'(\eta), \tag{2.5}$$

also,

$$\left(a(\eta)(w'(\eta))^{\beta}\right)'\leq-\varsigma(\eta)(1-\varpi(\xi(\eta)))^{\beta}\widehat{\pi}(\eta)\,w^{\beta}(\eta). \tag{2.6}$$

Proof. Let y be an eventually positive solution of (1.1). (2.2) holds according to Lemma 1. Therefore, using the definition of$w(\eta)$, we obtain

$$\begin{aligned} y(\eta) &= w(\eta)-\varpi(\eta)\,y(\zeta(\eta)) \\ &\geq w(\eta)-\varpi(\eta)\,w(\zeta(\eta)) \\ &\geq w(\eta)(1-\varpi(\eta)). \end{aligned}$$

This suggests that (1.1)

$$\left(a(\eta)(w'(\eta))^{\beta}\right)'\leq-\varsigma(\eta)\,w^{\beta}(\xi(\eta))(1-\varpi(\xi(\eta)))^{\beta}.$$

Since $w'(\eta)>0$ and $\frac{\partial}{\partial s}\xi(\eta)>0$, we obtain $w(\xi(\eta))>w(\xi(\eta))$ and so

$$\left(a(\eta)(w'(\eta))^{\beta}\right)'\leq-\varsigma(\eta)(1-\varpi(\xi(\eta)))^{\beta}\,w^{\beta}(\xi(\eta)).$$

Using basic computation and the chain rule, it is evident that

$$\begin{aligned} \pi_{\eta_1}(\eta)\left(a(\eta)(w'(\eta))^{\beta}\right)' &= \beta\left(a^{1/\beta}(\eta)\,w'(\eta)\right)^{\beta-1}\pi_{\eta_1}(\eta)\left(a^{1/\beta}(\eta)\,w'(\eta)\right)' \\ &= -\beta\left(a^{1/\beta}(\eta)\,w'(\eta)\right)^{\beta-1}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(w(\eta)-\pi_{\eta_1}(\eta)\,a^{1/\beta}(\eta)\,w'(\eta)\right). \end{aligned} \tag{2.7}$$

Combining (2.4) and (2.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(w(\eta)-\pi_{\eta_1}(\eta)\,a^{1/\beta}(\eta)\,w'(\eta)\right)\geq\frac{1}{\beta}\pi_{\eta_1}(\eta)\left(a^{1/\beta}(\eta)\,w'(\eta)\right)^{1-\beta}\varsigma(\eta)\,(1-\varpi(\xi(\eta)))^{\beta}\,w^{\beta}(\xi(\eta)).$$

Integrating this inequality from η_1 to η, we have

$$w(\eta)\geq\pi_{\eta_1}(\eta)\,a^{1/\beta}(\eta)\,w'(\eta)+\frac{1}{\beta}\int_{\eta_1}^{\eta}\pi_{\eta_1}(s)\,\varsigma(\eta)\,(1-\varpi(\xi(s)))^{\beta}\left(a^{1/\beta}(s)\,w'(s)\right)^{1-\beta}w^{\beta}(\xi(s))\,\mathrm{d}s. \quad (2.8)$$

From the monotonicity of $a^{1/\beta}(\eta)\,w'(\eta)$, we have

$$w(\eta)=w(\eta_1)+\int_{\eta_1}^{\eta}\frac{1}{a^{1/\beta}(s)}\left(a^{1/\beta}(s)\,w'(s)\right)\mathrm{d}s\geq\pi_{\eta_1}(\eta)\,a^{1/\beta}(\eta)\,w'(\eta).$$

So, by $\left(a^{1/\beta}(\eta)\,w'(\eta)\right)'\leq 0$, (2.8) becomes

$$\begin{aligned}
w(\eta) &\geq \pi_{\eta_1}(\eta)\,a^{1/\beta}(\eta)\,w'(\eta)\\
&\quad+\frac{1}{\beta}\int_{\eta_1}^{\eta}\pi_{\eta_1}(s)\,\varsigma(s)\,(1-\varpi(\xi(s)))^{\beta}\left(a^{1/\beta}(s)\,w'(s)\right)^{1-\beta}\pi_{\eta_1}^{\beta}(\xi(s))\left[a(\xi(s))\,(w'(\xi(s)))^{\beta}\right]\mathrm{d}s.\\
&\geq \pi_{\eta_1}(\eta)\,a^{1/\beta}(\eta)\,w'(\eta)\\
&\quad+\frac{1}{\beta}\int_{\eta_1}^{\eta}\left(a^{1/\beta}(s)\,w'(s)\right)^{1-\beta}\pi_{\eta_1}(s)\,\pi_{\eta_1}^{\beta}(\xi(s))\,\varsigma(s)\,(1-\varpi(\xi(s)))^{\beta}\left[a^{1/\beta}(s)\,w'(s)\right]^{\beta}\mathrm{d}s\\
&\geq a^{1/\beta}(\eta)\,w'(\eta)\left[\pi_{\eta_1}(\eta)+\frac{1}{\beta}\int_{\eta_1}^{\eta}\pi_{\eta_1}(s)\,\pi_{\eta_1}^{\beta}(\xi(s))\,\varsigma(s)\,(1-\varpi(\xi(s)))^{\beta}\,\mathrm{d}s\right].\\
&\geq \widetilde{\pi}_{\eta_1}(\eta)a^{1/\beta}(\eta)\,w'(\eta),
\end{aligned}$$

or

$$\frac{w'(\eta)}{w(\eta)}\leq\frac{1}{\widetilde{\pi}_{\eta_1}(\eta)a^{1/\beta}(\eta)}.$$

When integrated from $\xi(\eta)$ to η, we find that

$$\frac{w(\xi(\eta))}{w(\eta)}\geq\exp\left(-\int_{\xi(\eta)}^{\eta}\frac{\mathrm{d}\eta}{\widetilde{\pi}_{\eta_1}(\eta)a^{1/\beta}(\eta)}\right),$$

which, with (2.4), gives

$$\begin{aligned}
\frac{\left(a(\eta)\,(w'(\eta))^{\beta}\right)'}{w^{\beta}(\eta)} &\leq -\varsigma(\eta)\,(1-\varpi(\xi(\eta)))^{\beta}\left(\frac{w(\xi(\eta))}{w(\eta)}\right)^{\beta}\\
&\leq -\varsigma(\eta)\,(1-\varpi(\xi(\eta)))^{\beta}\,\widehat{\pi}(\eta).
\end{aligned}$$

The proof is complete. □

Lemma 4. *Let (1.1) have a positive solution. If*

$$D(\eta)=x(\eta)\,a(\eta)\left(\frac{w'(\eta)}{w(\eta)}\right)^{\beta},>0, \quad (2.9)$$

then

$$D'(\eta)\leq\frac{x'_{+}(\eta)}{x(\eta)}D(\eta)-x(\eta)\varsigma(\eta)\,(1-\varpi(\xi(\eta)))^{\beta}\,\widehat{\pi}(\eta)-\frac{\beta}{(x(\eta)\,a(\eta))^{1/\beta}}D^{(\beta+1)/\beta}(\eta). \quad (2.10)$$

Proof. Let y be a positive solution of Eq (1.1). From Lemma 3, we have (2.6) holds. Thus, when we differentiate $D(\eta)$ we obtain

$$D'(\eta) = \frac{x'(\eta)}{x(\eta)} D(\eta) + x(\eta) \frac{(a(\eta) w'(\eta))'}{w^{\beta}(\eta)} - \beta x(\eta) a(\eta) \left(\frac{w'(\eta)}{w(\eta)} \right)^{\beta+1}.$$

From (2.6) and (2.9), we deduce that

$$D'(\eta) \leq \frac{x'_{+}(\eta)}{x(\eta)} D(\eta) - 1 x(\eta) \varsigma(\eta) (1 - \varpi(\xi(\eta)))^{\beta} \widehat{\pi}(\eta) - \frac{\beta}{(x(\eta) a(\eta))^{1/\beta}} D^{(\beta+1)/\beta}(\eta).$$

The proof is complete. □

2.2. *Oscillation results*

Theorem 1. *If Eq (2.1) is oscillatory, then (1.1) is oscillatory.*

Proof. Let $y(\eta) > 0$, that is, $y(\zeta(\eta)) > 0$ and $y(\xi(\eta)) > 0$. From Lemma 3, we have (2.4) and (2.5) hold. Using (2.4) and (2.5), we find $\omega(\eta) = a(\eta)(w'(\eta))^{\beta}$ is a positive solution of

$$\omega'(\eta) + \widetilde{\pi}^{\beta}_{\eta_1}(\xi(\eta)) \varsigma(\eta) (1 - \varpi(\xi(\eta)))^{\beta} \omega(\xi(\eta)) \leq 0.$$

By [13, Theorem 1], then also, the solution of the associated Eq (2.1) is positive, and this a contradiction. The proof is complete. □

Corollary 1. *Let*

$$\limsup_{\eta\to\infty} \int_{\xi(\eta)}^{\eta} \widetilde{\pi}^{\beta}_{\eta_1}(\xi(s)) \varsigma(s) (1 - \varpi(\xi(s)))^{\beta} \, \mathrm{d}s > 1, \ \frac{\partial}{\partial\eta} \xi(\eta) \geq 0 \tag{2.11}$$

or

$$\liminf_{\eta\to\infty} \int_{\xi(\eta)}^{\eta} \widetilde{\pi}^{\beta}_{\eta_1}(\xi(s)) \varsigma(s) (1 - \varpi(\xi(s)))^{\beta} \, \mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.12}$$

then all solutions of (1.1) is oscillatory.

Proof. As may be shown from [9, Theorem 2.1.1], (2.11) or (2.12) guarantee oscillation of (2.1). □

Lemma 5. *Suppose ξ is strictly growing in relation to η and*

$$\liminf_{\eta\to\infty} \int_{\xi(\eta)}^{\eta} \widetilde{\pi}^{\beta}_{\eta_1}(\xi(s)) \varsigma(s) (1 - \varpi(\xi(s)))^{\beta} \, \mathrm{d}s \geq \delta, \tag{2.13}$$

for some $\delta > 0$, and (1.1) has an eventually positive solution y. Then,

$$\frac{H(\xi(\eta))}{H(\eta)} \geq z_n(\delta), n \geq 0, \tag{2.14}$$

where $H(\eta) := a(\eta)(w'(\eta))^{\beta}$, and

$$z_0(\eta) := 1 \text{ and } z_n(\eta) := \exp(\rho z_{n-1}(\eta)). \tag{2.15}$$

Proof. Let $y(\eta) > 0$, $y(\zeta(\eta)) > 0$ and $y(\xi(\eta)) > 0$ for $\eta \geq \eta_1$. We conclude that ω is a positive solution of (2.1) by following the same procedure as in the proof of Theorem 1. We can demonstrate that (2.14) holds in a manner akin to that used in the proof of Lemma 1 in [29]. □

Theorem 2. *Suppose ξ is strictly growing in relation to η and (2.13) holds. If $\varphi \in C^1(I,(0,\infty))$ such that*

$$\limsup_{\eta\to\infty} \int_{\eta_1}^{\eta} \left(\varphi(s)\varsigma(s)(1-\varpi(\xi(s)))^{\beta} - \frac{(\varphi'_+(s))^{\beta+1} a(\xi(s))}{(\beta+1)^{\beta+1} z_n(\delta)\varphi^{\beta}(s)(\xi'(s))^{\beta}} \right) = \infty, \tag{2.16}$$

for some $\delta < 0$ and $n \geq 0$, where $\varphi'_+(\eta) = \max\{0,\varphi'(\eta)\}$ and $z_n(\delta)$ is defined as (2.15), then all solutions of (1.1) are oscillatory.

Proof. Suppose $y(\eta) > 0$, $y(\zeta(\eta)) > 0$, and $y(\xi(\eta)) > 0$.

From Lemma 3, we obtain (2.4) holds. By Lemma 5, we find

$$\frac{w'(\xi(\eta))}{w'(\eta)} \geq \left(\frac{z_n(\delta)a(\eta)}{a(\xi(\eta))} \right)^{1/\beta}. \tag{2.17}$$

Let

$$\sigma(\eta) := \varphi(\eta)a(\eta)\left(\frac{w'(\eta)}{w(\xi(\eta))} \right)^{\beta} > 0. \tag{2.18}$$

Differentiating (2.18), we obtain

$$\sigma'(\eta) = \frac{\varphi'(\eta)}{\varphi(\eta)}\sigma(\eta) + \varphi(\eta)\frac{(a(\eta)(w'(\eta))^{\beta})'}{w^{\beta}(\xi(\eta))} - \beta\varphi(\eta)a(\eta)\left(\frac{w'(\eta)}{w(\xi(\eta))} \right)^{\beta}\left(\frac{w'(\xi(\eta))}{w(\xi(\eta))} \right)\xi'(\eta).$$

From (2.4), (2.18), and (2.17), we obtain

$$\sigma'(\eta) \leq -\varphi(\eta)\varsigma(\eta)(1-\varpi(\xi(\eta)))^{\beta} + \frac{\varphi'_+(\eta)}{\varphi(\eta)}\sigma(\eta) - \frac{\beta z_n^{1/\beta}(\delta)\xi'(\eta)}{(\varphi(\eta)a(\xi(\eta)))^{1/\beta}}\sigma^{(\beta+1)/\beta}(\eta). \tag{2.19}$$

Using Lemma 2 with $B = \beta z_n^{1/\beta}(\delta) / (\varphi(\eta)a(\xi(\eta)))^{-1/\beta}$ and $G = \varphi'_+(\eta)/\varphi(\eta)$, (2.19) yield

$$\sigma'(\eta) \leq -\varphi(\eta)\varsigma(\eta)(1-\varpi(\xi(\eta)))^{\beta} + \frac{\varphi'_+(\eta)^{\beta+1} a(\xi(\eta))}{(\beta+1)^{\beta+1} z_n(\delta)\varphi^{\beta}(\eta)(\xi'(\eta))^{\beta}}.$$

Integrating this inequality from η_1 to η, we find

$$\int_{\eta_1}^{\eta} \left(\varphi(s)\varsigma(s)(1-\varpi(\xi(s)))^{\beta} - \frac{(\varphi'_+(s))^{\beta+1} a(\xi(s))}{(\beta+1)^{\beta+1} z_n(\delta)\varphi^{\beta}(s)(\xi'(s))^{\beta}} \right) \mathrm{d}s \leq \sigma(\eta).$$

A contradiction with condition (2.16) is then discovered. The proof is finished. □

Theorem 3. *If*

$$\limsup_{\eta\to\infty} \int_{\eta_1}^{\eta} \left(x(s)\varsigma(s)(1-\varpi(\xi(s)))^{\beta}\widehat{\pi}(s) - \frac{a(s)(x'_+(s))^{\beta+1}}{(\beta+1)^{\beta+1} x^{\beta}(s)} \right) \mathrm{d}s = \infty, \tag{2.20}$$

where $x \in C^1(I,(0,\infty))$ and $x'_+(\eta) = \max\{0,\psi'(\eta)\}$, then (1.1) is oscillatory.

Proof. Let $y(\eta) > 0$, that is, $y(\zeta(\eta))$ and $y(\xi(\eta))$ are positive on $[\eta_0, \infty)$. From Lemma 3, we have (2.4)–(2.6) hold. Next, we arrive at (2.10) using Lemma 2 with $G = x'_+(\eta)/x(\eta)$ and $B = \beta(x(\eta)a(\eta))^{-1/\beta}$ (Lemma 4), (2.10) becomes

$$D'(\eta) \leq -x(\eta)\varsigma(\eta)(1 - \varpi(\xi(\eta)))^{\beta}\widehat{\pi}(\eta) + \frac{a(\eta)(x'_+(\eta))^{\beta+1}}{(\beta+1)^{\beta+1}x^{\beta}(\eta)}.$$

Integrating this inequality from η_1 to η, we have

$$\int_{\eta_1}^{\eta}\left(x(s)\varsigma(s)(1 - \varpi(\xi(s)))^{\beta}\widehat{\pi}(s) - \frac{a(s)\left(x'_+(s)^{\beta+1}\right)}{(\beta+1)^{\beta+1}x^{\beta}(s)}\right)\mathrm{d}\eta \leq D(\eta),$$

This contradicts the condition (2.20). The proof is finished. □

Now, we obtain some oscillation results for Eq (1.1) using other methods.

Theorem 4. *Let*

$$\int_{\eta_0}^{\infty} E(s)\,\mathrm{d}s = \infty, \tag{2.21}$$

then, Eq (1.1) is oscillatory.

Proof. Suppose $y(\eta) > 0$, $y(\zeta(\eta)) > 0$ and $y(\xi(\eta)) > 0$, we can infer from Lemma 4 that (2.10) holds. if we set $x(\eta) := 1$, then (2.10) becomes

$$D'(\eta) + E(\eta) + \beta/(a(\eta))^{1/\beta}D^{\frac{\beta+1}{\beta}}(\eta) \leq 0 \tag{2.22}$$

or

$$D'(\eta) + E(\eta) \leq 0. \tag{2.23}$$

Integrating (2.23) from η_3 to η and using (2.21), we arrive at

$$D(\eta) \leq D(\eta_3) - \int_{\eta_3}^{\eta} E(\eta)\,\mathrm{d}s \to \infty \quad \text{as} \quad \eta \to \infty.$$

This contradicts the conclusion that the evidence is complete because $D(\eta) > 0$. □

Now, assume that the series of functions $\{\vartheta_n(\eta)\}_{n=0}^{\infty}$ is defined as

$$\vartheta_n(\eta) = \int_{\eta}^{\infty} \beta/(a(s))^{1/\beta}\vartheta_{n-1}^{\frac{\beta+1}{\beta}}(s)\,\mathrm{d}s + \vartheta_0(\eta), \quad \eta \geq \eta_0, \quad n = 1, 2, 3, ... \tag{2.24}$$

and

$$\vartheta_0(\eta) = \int_{\eta}^{\infty} E(s)\,\mathrm{d}s, \quad \eta \geq \eta_0,$$

where $\vartheta_n(\eta) \leq \vartheta_{n+1}(\eta)$, $\eta \geq \eta_0$.

Lemma 6. *Let y be a solution of Eq (1.1) that becomes positive for suffciently large η. Then $D(\eta) \geq \vartheta_n(\eta)$ where $\lim_{n\to\infty}\vartheta_n(\eta) = \vartheta(\eta)$ for $\eta \geq \eta \geq \eta_0$ when $\vartheta(\eta)$ on $[\eta, \infty)$ and*

$$\vartheta(\eta) = \int_{\eta}^{\infty} \beta/(a(s))^{1/\beta}\,\vartheta^{\frac{\beta+1}{\beta}}(s)\,\mathrm{d}s + \vartheta_0(\eta),\ \eta \geq \eta. \tag{2.25}$$

Proof. Let y be a solution of Eq (1.1) that becomes positive for suffciently large η. We obtain to (2.22) by using the same steps as in the proof of Theorem 4. The result of integrating (2.22) from η to η' is

$$D(\eta') - D(\eta) + \int_{\eta}^{\eta'} E(s)\,ds + \int_{\eta}^{\eta'} D^{\frac{\beta+1}{\beta}}(s)\beta/(a(s))^{1/\beta}\,\mathrm{d}s \leq 0.$$

This implies

$$D(\eta') - D(\eta) + \int_{\eta}^{\eta'} D^{\frac{\beta+1}{\beta}}(s)\beta/(a(s))^{1/\beta}\,\mathrm{d}s \leq 0.$$

Then, we conclude that

$$\int_{\eta}^{\infty} D^{\frac{\beta+1}{\beta}}(s)\beta/(a(s))^{1/\beta}\,\mathrm{d}s < \infty \text{ for } \eta \geq \eta, \tag{2.26}$$

Otherwise, when $\eta' \to \infty$, $D(\eta') \leq D(\eta) - \int_{\eta}^{\eta'} D^{\frac{\beta+1}{\beta}}(s)\beta/(a(s))^{1/\beta}\,\mathrm{d}s \to -\infty$, which contradicts $D(\eta) > 0$. Given that $D(\eta) > 0$ and $D'(\eta) > 0$, (2.22) indicates that

$$D(\eta) \geq \int_{\eta}^{\infty} E(s)\,\mathrm{d}s + \int_{\eta}^{\infty} D^{\frac{\beta+1}{\beta}}(s)\beta/(a(s))^{1/\beta}\,\mathrm{d}s = \vartheta_0(\eta) + \int_{\eta}^{\infty} D^{\frac{\beta+1}{\beta}}(s)\beta/(a(s))^{1/\beta}\,\mathrm{d}s, \tag{2.27}$$

or

$$D(\eta) \geq \int_{\eta}^{\infty} E(s)\,\mathrm{d}s := \vartheta_0(\eta).$$

Consequently, $D(\eta) \geq \vartheta_n(\eta)$, where $n = 1, 2, 3, \ldots$. We obtain that $\vartheta_n \to \vartheta$ as $n \to \infty$ since $\{\vartheta_n(\eta)\}_{n=0}^{\infty}$ is growing and bounded above. The monotone convergence theorem of Lebesgue shows that when $n \to \infty$, (2.24) becomes (2.25). □

Theorem 5. *If*

$$\liminf_{\eta\to\infty} \frac{1}{\vartheta_0(\eta)} \int_{\eta}^{\infty} \vartheta_0^{\frac{\beta+1}{\beta}}(s)\beta/(a(s))^{1/\beta}\,\mathrm{d}s > \frac{\beta}{(\beta+1)^{\frac{\beta+1}{\beta}}}, \tag{2.28}$$

then all solutions (1.1) are oscillatory.

Proof. Assume that $y(\eta) > 0$, meaning that both $y(\zeta(\eta))$ and $y(\xi(\eta))$ are positive. Following the same steps as in the Lemma 6 proof, we obtain (2.27). Using (2.27), we discover

$$\frac{D(\eta)}{\vartheta_0(\eta)} \geq 1 + \frac{1}{\vartheta_0(\eta)} \int_{\eta}^{\infty} \vartheta_0^{\frac{\beta+1}{\beta}}(s)\beta/(a(s))^{1/\beta} \left(\frac{D(s)}{\vartheta_0(s)}\right)^{\frac{\beta+1}{\beta}} \mathrm{d}s. \tag{2.29}$$

If we consider $\mu = inf_{\eta\geq\eta}(D(\eta)/\vartheta_0(\eta))$, then $\mu \geq 1$, of course. We can observe using (2.28) and (2.29) that

$$\mu \geq 1 + \beta\left(\frac{\mu}{\beta+1}\right)^{\frac{\beta+1}{\beta}}$$

or

$$\frac{\beta}{\beta+1}\left(\frac{\mu}{\beta+1}\right)^{\frac{\beta+1}{\beta}}+\frac{1}{\beta+1}\leq\frac{\mu}{\beta+1}.$$

It defies the predicted value of μ and β; hence, the proof is finished. □

Theorem 6. *Let*

$$\limsup_{\eta\to\infty}\vartheta_n(\eta)\left(\int_{\eta_0}^{\eta}a^{-\frac{1}{\beta}}(s)\mathrm{d}s\right)^{\beta}>1, \tag{2.30}$$

then every solution of (1.1) is oscillatory.

Proof. Assume that $y(\eta)>0$, meaning that both $y(\zeta(\eta))$ and $y(\xi(\eta))$ are positive. From (2.9), we obtain

$$\begin{aligned}\frac{1}{D(\eta)} &= \frac{1}{a(\eta)}\left(\frac{w(\eta)}{w'(\eta)}\right)^{\beta}=\frac{1}{a(\eta)}\left(\frac{w(\eta)+\int_{\eta}^{\eta}a^{-1/\beta}(s)\,a^{1/\beta}(s)\,w'(s)\,\mathrm{d}s}{w'(\eta)}\right)^{\beta}\\ &\geq \frac{1}{a(\eta)}\left(\frac{a^{1/\beta}(\eta)\,w'(\eta)\int_{\eta}^{\eta}a^{-1/\beta}(s)\,\mathrm{d}s}{w'(\eta)}\right)^{\beta}\\ &= \left(\int_{\eta}^{\eta}a^{-1/\beta}(s)\,\mathrm{d}s\right)^{\beta},\end{aligned} \tag{2.31}$$

for $\eta\geq\eta$. So, from (2.31) we find

$$D(\eta)\left(\int_{\eta_0}^{\eta}a^{-1/\beta}(s)\,\mathrm{d}s\right)^{\beta}\leq\left(\frac{\int_{\eta_0}^{\eta}a^{-1/\beta}(s)\,\mathrm{d}s}{\int_{\eta}^{\eta}a^{-1/\beta}(s)\,\mathrm{d}s}\right)^{\beta},$$

and so

$$\limsup_{\eta\to\infty}D(\eta)\left(\int_{\eta_0}^{\eta}a^{-\frac{1}{\beta}}(s)\mathrm{d}s\right)^{\beta}\leq 1,$$

which contradicts (2.30). Hence, the proof is finished. □

Corollary 2. *If*

$$\int_{\eta_0}^{\infty}E(u)\exp\left(\int_{\eta_0}^{\eta}\vartheta_n^{\frac{1}{\beta}}(s)\beta/(a(s))^{1/\beta}\,\mathrm{d}s\right)\mathrm{d}u=\infty \tag{2.32}$$

or

$$\int_{\eta_0}^{\infty}\beta/(a(u))^{1/\beta}\,\vartheta_n^{\frac{1}{\beta}}(u)\vartheta_0(u)\exp\left(\int_{\eta_0}^{\eta}\beta/(a(s))^{1/\beta}\,\vartheta_n^{\frac{1}{\beta}}(s)\mathrm{d}s\right)\mathrm{d}u=\infty, \tag{2.33}$$

then all solutions (1.1) are oscillatory.

Proof. Suppose that $y(\eta)>0$, meaning that both $y(\zeta(\eta))$ and $y(\xi(\eta))$ are positive on $[\eta_0,\infty)$. (2.25) holds according to Lemma 6. (2.25) gives us

$$\vartheta'(\eta) = -\beta/(a(\eta))^{1/\beta}\,\vartheta^{\frac{\beta+1}{\beta}}(\eta)-E(\eta)$$

$$\leq \quad -\beta/\left(a\left(\eta\right)\right)^{1/\beta}\vartheta_{n}^{\frac{1}{\beta}}\left(\eta\right)\vartheta\left(\eta\right)-E\left(\eta\right). \tag{2.34}$$

Hence,

$$\int_{\eta_0}^{\eta}E\left(s\right)\exp\left(\int_{\eta}^{s}\vartheta_{n}^{\frac{1}{\beta}}(u)\beta/\left(a\left(u\right)\right)^{1/\beta}\mathrm{d}u\right)\mathrm{d}s\leq\vartheta\left(\eta\right)<\infty,$$

which contradicts (2.32).

Next, let $M\left(\eta\right)=\int_{\eta}^{\infty}\beta/\left(a\left(s\right)\right)^{1/\beta}\vartheta^{\frac{\beta+1}{\beta}}\left(s\right)\mathrm{d}s$. Then, we obtain

$$\begin{aligned}M'\left(\eta\right) &= -\beta/\left(a\left(\eta\right)\right)^{1/\beta}\vartheta^{\frac{\beta+1}{\beta}}\left(\eta\right)\\ &\leq -\beta/\left(a\left(\eta\right)\right)^{1/\beta}\vartheta_{n}^{\frac{1}{\beta}}\left(\eta\right)\vartheta\left(\eta\right)\\ &= -\beta/\left(a\left(\eta\right)\right)^{1/\beta}\vartheta_{n}^{\frac{1}{\beta}}\left(\eta\right)\left(M\left(\eta\right)+\vartheta_0\left(\eta\right)\right).\end{aligned}$$

Consequently, we discover

$$\int_{\eta_0}^{\infty}\beta/\left(a\left(u\right)\right)^{1/\beta}\vartheta_{n}^{\frac{1}{\beta}}(u)\vartheta_0(\eta)\exp\left(\int_{\eta_0}^{u}\beta/\left(a\left(s\right)\right)^{1/\beta}\vartheta_{n}^{\frac{1}{\beta}}(s)\mathrm{d}s\right)\mathrm{d}u<\infty,$$

This runs counter to (2.33). The proof is finished. □

3. Examples

Example 1. *Let the equation*

$$\left(\left(\left(y\left(\eta\right)+\varpi_0 y\left(\zeta_0\eta\right)\right)'\right)^{\beta}\right)'+\frac{\varsigma_0}{\eta^{\beta+1}}y^{\beta}\left(\varepsilon\eta\right)=0, \tag{3.1}$$

where $\varepsilon,\zeta_0\in(0,1)$. *Let* $a\left(\eta\right)=1,\varpi\left(\eta\right)=\varpi_0,\zeta\left(\eta\right)=\zeta_0\eta,\varsigma\left(\eta\right)=\frac{\varsigma_0}{\eta^{\beta+1}}$ *and* $\xi\left(\eta\right)=\varepsilon\eta$.

It is easy to verify that

$$\begin{aligned}&\varsigma\left(\eta\right)\left(1-\varpi\left(\xi\left(\eta\right)\right)\right)^{\beta}\\ =\ &\frac{\varsigma_0}{\eta^{\beta+1}}\left(1-\varepsilon\right)\left[1-\varpi_0\right]^{\beta},\ \pi_{\eta_0}\left(\eta\right)=\eta \text{ and } \widetilde{\pi}_{\eta_0}(\eta)=M\eta,\end{aligned}$$

where

$$M:=1+\varepsilon^{\beta}\frac{\varsigma_0}{\beta}\left(1-\varepsilon\right)\left[1-\varpi_0\right]^{\beta}.$$

By Corollary 1, we find (3.1) is oscillatory if

$$\left(M^{\beta}\varepsilon^{\beta}\varsigma_0\left(1-\varepsilon\right)\left[1-\varpi_0\right]^{\beta}\right)\ln\frac{1}{\varepsilon}>\frac{1}{\mathrm{e}}$$

or

$$\beta\left(M-1\right)M^{\beta}\ln\frac{1}{\varepsilon}>\frac{1}{\mathrm{e}}. \tag{3.2}$$

Also, we find that

$$\widehat{\pi}_{\eta_1}\left(\eta\right)=\varepsilon^{1/M},\ E\left(\eta\right)=\frac{N}{\eta^{\beta+1}}\varepsilon^{\beta/M},\ \int_{\eta}^{\infty}E\left(s\right)\mathrm{d}s=\frac{N\varepsilon^{\beta/M}}{\beta}\frac{1}{\eta^{\beta+1}},$$

where $N = \varsigma_0 (1 - \varpi_0)^{\beta} (1 - \varepsilon)$. *From Theorem 5, (3.1) is oscillatory if*

$$\left(\frac{N}{\beta}\varepsilon^{\beta/M}\right)^{1/\beta} > \beta(\beta+1)^{-(\beta+1)/\beta}.$$

Example 2. *Consider the differential equation*

$$\left(y(\eta) + \frac{1}{2}y(\zeta_0\eta)\right)'' + \frac{\varsigma_0}{\eta^2}y(\varepsilon\eta) = 0, \tag{3.3}$$

where $\varepsilon, \zeta_0 \in (0,1)$. *Let* $a(\eta) = 1, \varpi(\eta) = \frac{1}{2}, \zeta(\eta) = \zeta_0\eta, \varsigma(\eta) = \frac{\varsigma_0}{\eta^2}$ *and* $\xi(\eta) = \varepsilon\eta$.

Based on the findings in Example 1, Eq (3.3) is oscillatory if

$$\varepsilon\frac{\varsigma_0}{2}\left(1 + \frac{1}{2}\varepsilon\varsigma_0\right)\ln\frac{1}{\varepsilon} > \frac{1}{\mathrm{e}}, \tag{3.4}$$

where $\varepsilon = 1/3$, *conditions (3.4) reduce to* $\varsigma_0 > 1.588$.

Example 3. *Consider the differential equation*

$$\left(y(\eta) + \frac{1}{2}y\left(\frac{\eta}{3}\right)\right)'' + \frac{\varsigma_0}{\eta^2}y\left(\frac{\eta}{2}\right) = 0, \tag{3.5}$$

where $\varsigma_0 > 0$. *Let* $a(\eta) = 1, \varpi(\eta) = \frac{1}{2}, \zeta(\eta) = \frac{\eta}{3}, \varsigma(\eta) = \frac{\varsigma_0}{\eta^2}$ *and* $\xi(\eta) = \frac{\eta}{2}$.

It is easy to verify that

$$\pi_{\eta_0}(\eta) = \eta,$$

and

$$\widetilde{\pi}_{\eta_0} = \eta + \frac{\varsigma_0}{4}\int_{\eta_0}^{\eta} dy = \eta\left(1 + \frac{\varsigma_0}{4}\right).$$

Using Corollary 1, if

$$\frac{\varsigma_0}{4}\left(1 + \frac{\varsigma_0}{4}\right)\ln 2 > \frac{1}{e},$$

then (3.5) is oscillatory.

4. Conclusions

This study examined the oscillatory characteristics of a set of differential equations of second order with a neutral term. In order to meet the oscillation criteria of Eq (1.1), we first provide several features pertaining to oscillatory solutions. There are no prerequisites for the oscillation criterion discussed in this study. We discover that the majority of earlier research has focused on outcomes that are not applicable to our broader equation, specifically when it comes to neutral terms. In light of the aforementioned, the findings of this work represent an expansion, enhancement, and completion of the earlier findings.

It is anticipated that studying the following equation would greatly advance oscillation theory in upcoming scientific domains: $\left(a(\eta)\left(\left(y(\eta) + \sum_{i=1}^{n}\varpi_i(\eta)\,y(\zeta_i(\eta))\right)'\right)^{\beta}\right)' + \sum_{i=1}^{n}\varsigma_i(\eta)\,y^{\beta}(\xi_i(\eta)) = 0$. Additionally, it will be interesting for scholars to discuss the consequences of Eq (1.1) when $\xi_i(\eta) \geq \eta$ or if Eq (1.1) with a damping term is present.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no competing interests

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