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Research article

Certain domains of a new matrix constructed by Euler totient and its summation function

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Abstract: With the aid of the Euler totient function φ and its summation function $\top$, a new matrix $\Delta(\varphi,\top) = (\delta(\varphi,\top)_{nk})$, where

$$\delta(\varphi,\top)_{nk} = \begin{cases} \frac{(-1)^{n-k}\top(k)}{\varphi(n)}, & n-1 \le k \le n, \\ 0, & \text{otherwise} \end{cases}$$

is constructed to define the domains $\ell_p(\Delta(\varphi,\top))$, $\ell_\infty(\Delta(\varphi,\top))$, and $\ell_1(\Delta(\varphi,\top))$. After obtaining the norms on these domains, it is proved that these spaces are linearly isomorphic to classical ones. Also, their dual spaces are determined. Finally, characterizations of several matrix mappings are stated and proved.

Keywords: arithmetic divisor sum function; matrix domain; dual space; matrix mapping
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1. Introduction

A sequence space is a linear subspace of the space ω of all sequences. The space of all finitely non-zero sequences ω_0, the space of all bounded sequences ℓ_∞, the space of all convergent sequences c, the space of all null sequences c_0, and the space of all absolutely p- summable sequences ℓ_p are the examples for the classical sequence spaces. Also, ℓ_1 is the space of all absolutely summable sequences, consisting of sequences $u = (u_n)$ with $\sum_{n=1}^{\infty} |u_n| < \infty$. Unless otherwise stated, $\sum_n$ is a brief demonstration of $\sum_{n=1}^{\infty}$. The spaces c_0, c, and ℓ_∞ are complete normed spaces with $\|u\|_{\ell_\infty} = \|u\|_c = \|u\|_{c_0} = \sup_{n\in\mathbb{N}} |u_n|$ and the space ℓ_p is a complete normed space with $\|u\|_{\ell_p} = (\sum_n |u_n|^p)^{1/p}$, where $\mathbb{N} = \{1, 2, 3, ...\}$. Also, $\sup_n$ stands for $\sup_{n\in\mathbb{N}}$. Unless stated otherwise, assume that $1 < p < \infty$ and $q = \frac{p}{p-1}$ is the conjugate of p. If $p = 1$, then $q = \infty$.

A linear topological sequence space $\mathcal{U}$ is called a K space provided that each functional $p_m : \mathcal{U} \to$

$\mathbb{K}$, $p_m(u) = u_m$ is continuous for all $m \in \mathbb{N}$, where $\mathbb{K}$ is a real or complex field. If a K space $\mathcal{U}$ is a complete linear metric space, then it is called an FK space. If the topology of an FK space is normable, then it is called a BK space. Let $e = (e_r)$ be the sequence with the term $e_r = 1$ for all r and $e^{(n)} = (e_r^{(n)})$ $(n \in \mathbb{N})$ be the sequence with terms 1 if $n = r$ and 0 if $n \neq r$. Given any FK space $\mathcal{U} \supset \omega_0$ and a sequence $u = (u_r)$ in $\mathcal{U}$, it is said that the sequence u satisfies the AK property if $(u^{[n]})$ converges to u, where $u^{[n]} = \sum_{r=1}^{n} u_r e^{(r)}$.

Let $S = (s_{nk})$ be an infinite matrix and S_n be the sequence in the nth row of S. The S-transform of a sequence $u = (u_n) \in \omega$ is the sequence Su obtained by the usual matrix product, and its terms are written as

$$S_n(u) = \sum_k s_{nk} u_k$$

provided that the series is convergent for each $n \in \mathbb{N}$. If the sequence Su exists and $Su \in \mathcal{V}$ for all $u \in \mathcal{U}$, then S is called a matrix mapping from the sequence space $\mathcal{U}$ into the sequence space $\mathcal{V}$. $(\mathcal{U}, \mathcal{V})$ denotes the class of all infinite matrices from $\mathcal{U}$ into $\mathcal{V}$.

If the sequence Su converges, then it is said that the matrix S defines a summability method. Φ-summability is a special type of summability method introduced by Schoenberg [1] for the purpose of examining the Riemann integrability of a generalized Dirichlet function on the interval $[0, 1]$. This method is also called φ-convergence, which is weaker than usual convergence. A sequence (u_n) is said to be φ-convergent if the limit of (v_n) with

$$v_n = \frac{1}{n} \sum_{k|n} \varphi(k) u_k \tag{1.1}$$

exits.

The sequence space $\mathcal{U}_S$ called the (matrix) domain of S in the space $\mathcal{U}$ is the set

$$\mathcal{U}_S = \{u \in \omega : Su \in \mathcal{U}\}.$$

By using this concept for special summability matrices, several sequence spaces are obtained (see [2–6]).

The $\alpha-, \beta-, \gamma$-duals of a sequence space $\mathcal{U}$ are the sets

$$\mathcal{U}^\alpha = \Big\{a = (a_k) \in \omega : \sum_k |a_k u_k| < \infty \text{ for all } u = (u_k) \in \mathcal{U}\Big\},$$

$$\mathcal{U}^\beta = \Big\{a = (a_k) \in \omega : \sum_k a_k u_k \text{ converges for all } u = (u_k) \in \mathcal{U}\Big\},$$

$$\mathcal{U}^\gamma = \Big\{a = (a_k) \in \omega : \sup_n \Big|\sum_{k=1}^{n} a_k u_k\Big| < \infty \text{ for all } u = (u_k) \in \mathcal{U}\Big\},$$

respectively.

It is known that an infinite matrix can be considered as a linear operator between two sequence spaces. The theory of matrix transformations between sequence spaces has aroused interest over the years due to its significant implications in summability theory. For relevant literature on matrix transformations and their applications in summability theory, refer to [7–11].

2. Preliminaries and background

2.1. Some arithmetical functions

In the theory of numbers, an arithmetical function is a function such that it has natural numbers as the domain and real or complex numbers as the range. Some examples of arithmetical functions are the divisor sum function σ_r (with order r), the Möbius function μ, the Euler totient function φ, and the Jordan totient function J_r (with order r).

According to the Fundamental Theorem of Arithmetic, it is known that a natural number n can be represented uniquely as the product of powers of different primes. (1 is represented as the empty product.)

Let a natural number $n = p_1^{c_1} p_2^{c_2} ... p_m^{c_m}$, where $p_1 < p_2 < ... < p_m$ are primes and $c_1, c_2, ..., c_m$ are positive integers.

$$\sigma_r(n) = \sum_{k|n} k^r,$$

$$\mu(n) = \begin{cases} (-1)^m, & \text{if } c_1 = c_2 = ... = c_m = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)...\left(1 - \frac{1}{p_m}\right),$$

$$J_r(n) = n^r\left(1 - \frac{1}{p_1^r}\right)\left(1 - \frac{1}{p_2^r}\right)...\left(1 - \frac{1}{p_m^r}\right).$$

Note that the Euler totient function gives the number of positive integers less than n that are coprime to n, and the Jordan totient function gives the number of r-tuples of positive integers all less than or equal to n that form a coprime $(r + 1)$-tuple together with n. The Jordan function generalizes the Euler function since $\varphi(n) = J_1(n)$ holds.

In this study, the function $\top$ defined by $\top(n) = \sum_{k=1}^{n} \varphi(k)$ denotes the Euler totient summation function. It gives the number of coprime integer pairs p_1, p_2 with $1 \leq p_1 \leq p_2 \leq n$.

2.2. Some special matrix operators

The Euler totient matrix operator $\Phi = (\phi_{nk})$ is given in [12] as

$$\phi_{nk} = \begin{cases} \frac{\varphi(k)}{n}, & \text{if } k \mid n, \\ 0, & \text{if } k \nmid n. \end{cases}$$

In [12], the complete normed spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ are introduced as the domains of the special matrix Φ in the spaces ℓ_p $(1 \leq p < \infty)$ and ℓ_∞. Schoenberg [1] proved that this transformation is regular. That is, it preserves limit, and it is a mapping from c into c. Subsequently, in [13], the complete normed spaces c_0^Φ and c^Φ are studied as the matrix domain of Φ in c_0 and c. Following these studies, by using the Euler totient matrix, Demiriz et al. [14] defined almost convergence and core theorems; İlkhan [15] introduced paranormed sequence spaces; Demiriz and Erdem [16] defined and studied double Euler totient matrix and double sequence spaces. Also, in [17–21], the authors studied Euler totient series spaces and some other sequence spaces.

Motivated by the study of Schoenberg [1], the authors in [22] have constructed a new regular matrix $\Upsilon^r = (\upsilon^r_{nk})$ as

$$\upsilon^r_{nk} = \begin{cases} \frac{J_r(k)}{n^r}, & \text{if } k \mid n, \\ 0, & \text{if } k \nmid n \end{cases}$$

for each $r \in \mathbb{N}$. This special operator is obtained by using the Jordan totient function instead of the Euler totient function in transformation (1.1). Also, in [22], a space consisting of sequences whose Υ^r-transforms are in the space ℓ_p $(1 \leq p < \infty)$ is introduced and studied. Later, the domain of the matrix Υ^r in the spaces ℓ_∞, c, and c_0 are studied in [23]. In the meantime, the compact operators on the resulting Banach spaces are characterized in [24]. For more on Jordan totient sequence spaces, see [25–29].

In [30], the authors have constructed a new matrix $\mathcal{S} = (s_{nk})$ using the arithmetic divisor sum function σ_1 as

$$s_{nk} = \begin{cases} \frac{k}{\sigma_1(n)}, & \text{if } k \mid n, \\ 0, & \text{if } k \nmid n. \end{cases}$$

They have introduced and studied several Banach sequence spaces by utilizing the concept of summability with the aid of this matrix. Later on, the authors in [31] generalized this study by using the divisor sum function of order r. A recent matrix $\mathcal{D} = (d_{nk})$ has been constructed as

$$d_{nk} = \begin{cases} \frac{k^r}{\sigma_r(n)}, & \text{if } k \mid n, \\ 0, & \text{if } k \nmid n. \end{cases}$$

2.3. Motivation and objectives of the study

In some cases, the most general linear operator between two sequence spaces is given by an infinite matrix. So the theory of matrix transformations has always been of great interest in the study of sequence spaces. The study of the general theory of matrix transformations was motivated by special results in summability theory.

The theory of sequence spaces is fundamental to summability. Summability is a wide field of mathematics, mainly in analysis and functional analysis, and has many applications, for instance, in numerical analysis to speed up the rate of convergence, in operator theory, the theory of orthogonal series, and approximation theory.

The classical summability theory deals with the generalization of the convergence of sequences or series of real or complex numbers. The idea is to assign a limit of some sort to divergent sequences or series by considering a transform of a sequence or series rather than the original sequence or series.

The references [30, 31] are recent studies in the field of sequence spaces. They have become the starting point of our study to construct a new band matrix $\Delta(\varphi, \top)$. Clearly, there is no relation between the matrices in [30, 31] and $\Delta(\varphi, \top)$.

Inspired by the studies [30, 31], we have constructed the matrix $\Delta(\varphi, \top)$ by utilizing the arithmetic Euler totient function and its sum function. By the concept of matrix domain, we have aimed to introduce complete normed sequence spaces $\ell_p(\Delta(\varphi, \top))$ and $\ell_\infty(\Delta(\varphi, \top))$. Since the matrix $\Delta(\varphi, \top)$ is lower triangular, its inverse can be calculated easily. By using the inverse matrix method, we have proved that the resulting spaces are linearly isomorphic to classical ones, and also we have determined

α-, β-, and γ-duals of the spaces. Finally, we have characterized several matrix classes between newly defined spaces and some classical spaces. For this purpose, we have used the necessary and sufficient conditions on an infinite matrix belonging to the class $(\mathcal{U}, \mathcal{V})$, where $\mathcal{U} \in \{\ell_p, \ell_\infty, \ell_1\}$ and $\mathcal{V} \in \{\ell_1, c, c_0, \ell_\infty, \}$.

3. Domain of $\Delta(\varphi, \top)$ in ℓ_p and ℓ_∞

In the present section, we introduce the sequence spaces $\ell_p(\Delta(\varphi, \top))$ and $\ell_\infty(\Delta(\varphi, \top))$ by using the novel matrix $\Delta(\varphi, \top)$, where $1 \leq p < \infty$. Also, we present some theorems that give inclusion relations concerning these spaces.

The matrix $\Delta(\varphi, \top) = (\delta(\varphi, \top)_{nk})$ is defined as

$$\delta(\varphi, \top)_{nk} = \begin{cases} \frac{(-1)^{n-k}\top(k)}{\varphi(n)}, & n-1 \leq k \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

that is,

$$\Delta(\varphi, \top) = \begin{bmatrix} \frac{\top(1)}{\varphi(1)} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{-\top(1)}{\varphi(2)} & \frac{\top(2)}{\varphi(2)} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{-\top(2)}{\varphi(3)} & \frac{\top(3)}{\varphi(3)} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{-\top(3)}{\varphi(4)} & \frac{\top(4)}{\varphi(4)} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{-\top(4)}{\varphi(5)} & \frac{\top(5)}{\varphi(5)} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

The inverse $\Delta(\varphi, \top)^{-1} = (\delta(\varphi, \top)^{-1}_{nk})$ is calculated as

$$\delta(\varphi, \top)^{-1}_{nk} = \begin{cases} \frac{\varphi(k)}{\top(n)}, & \text{if } 1 \leq k \leq n, \\ 0, & \text{if } k > n, \end{cases}$$

that is,

$$\Delta(\varphi, \top)^{-1} = \begin{bmatrix} \frac{\varphi(1)}{\top(1)} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{\varphi(1)}{\top(2)} & \frac{\varphi(2)}{\top(2)} & 0 & 0 & 0 & 0 & \cdots \\ \frac{\varphi(1)}{\top(3)} & \frac{\varphi(2)}{\top(3)} & \frac{\varphi(3)}{\top(3)} & 0 & 0 & 0 & \cdots \\ \frac{\varphi(1)}{\top(4)} & \frac{\varphi(2)}{\top(4)} & \frac{\varphi(3)}{\top(4)} & \frac{\varphi(4)}{\top(4)} & 0 & 0 & \cdots \\ \frac{\varphi(1)}{\top(5)} & \frac{\varphi(2)}{\top(5)} & \frac{\varphi(3)}{\top(5)} & \frac{\varphi(4)}{\top(5)} & \frac{\varphi(5)}{\top(5)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Now, we introduce the sequence spaces $\ell_p(\Delta(\varphi, \top))$ and $\ell_\infty(\Delta(\varphi, \top))$ by

$$\ell_p(\Delta(\varphi, \top)) = \left\{ u = (u_n) \in \omega : \sum_n \left| \sum_{k=n-1}^{n} (-1)^{(n-k)} \frac{\top(k)}{\varphi(n)} u_k \right|^p < \infty \right\} \ (1 \leq p < \infty),$$

and

$$\ell_\infty(\Delta(\varphi, \top)) = \left\{ u = (u_n) \in \omega : \sup_n \left| \sum_{k=n-1}^{n} (-1)^{(n-k)} \frac{\top(k)}{\varphi(n)} u_k \right| < \infty \right\}.$$

As the notation of matrix domain, the sequence spaces $\ell_p(\Delta(\varphi, \top))$ and $\ell_\infty(\Delta(\varphi, \top))$ may be represented by

$$\ell_p(\Delta(\varphi, \top)) = (\ell_p)_{\Delta(\varphi, \top)} \ \ (1 \leq p < \infty) \ \text{ and } \ \ell_\infty(\Delta(\varphi, \top)) = (\ell_\infty)_{\Delta(\varphi, \top)}.$$

Unless otherwise stated, $v = (v_n)$ will be the $\Delta(\varphi, \top)$-transform of a sequence $u = (u_n)$, that is,

$$v_n = \Delta(\varphi, \top)_n(u) = \sum_{k=n-1}^{n} (-1)^{(n-k)} \frac{\top(k)}{\varphi(n)} u_k$$

for all $n \in \mathbb{N}$.

Theorem 3.1. *The spaces $\ell_p(\Delta(\varphi, \top))$ and $\ell_\infty(\Delta(\varphi, \top))$ are linear spaces with the co-ordinatewise addition and scalar multiplication, which are the BK spaces with the norms*

$$\|u\|_{\ell_p(\Delta(\varphi, \top))} = \left(\sum_n \left| \sum_{k=n-1}^{n} (-1)^{(n-k)} \frac{\top(k)}{\varphi(n)} u_k \right|^p \right)^{1/p},$$

and

$$\|u\|_{\ell_\infty(\Delta(\varphi, \top))} = \sup_{n \in \mathbb{N}} \sum_n \left| \sum_{k=n-1}^{n} (-1)^{(n-k)} \frac{\top(k)}{\varphi(n)} u_k \right|,$$

respectively.

Proof. It is a routine verification to prove that $\ell_p(\Delta(\varphi, \top))$ and $\ell_\infty(\Delta(\varphi, \top))$ are linear spaces. The fact that these spaces are BK-spaces comes from Theorem 4.3.2 of Wilansky [32, p. 61]. □

Theorem 3.2. *The spaces $\ell_p(\Delta(\varphi, \top))$ and $\ell_\infty(\Delta(\varphi, \top))$ are linearly isomorphic to ℓ_p and ℓ_∞, respectively.*

Proof. Let T be a mapping defined from $\ell_p(\Delta(\varphi, \top))$ to ℓ_p such that $T(u) = \Delta(\varphi, \top)u$ for all $u \in \ell_p(\Delta(\varphi, \top))$. It is clear that T is linear. Also, it is injective since the kernel of T consists of only zero. To prove that T is surjective, consider the sequence $u = (u_n)$ whose terms are

$$u_n = \sum_{k=1}^{n} \frac{\varphi(k)}{\top(n)} v_k,$$

for all $n \in \mathbb{N}$, where $v = (v_k)$ is any sequence in ℓ_p. It follows that

$$\begin{aligned}
\Delta(\varphi, \top)_n(u) &= \sum_{k=n-1}^{n} (-1)^{(n-k)} \frac{\top(k)}{\varphi(n)} u_k = \sum_{k=n-1}^{n} (-1)^{(n-k)} \frac{\top(k)}{\varphi(n)} \left(\sum_{j=1}^{k} \frac{\varphi(j)}{\top(k)} v_j \right) \\
&= \sum_{k=n-1}^{n} \sum_{j=1}^{k} (-1)^{(n-k)} \frac{\varphi(j)}{\varphi(n)} v_j = v_n,
\end{aligned}$$

and so, $u = (u_n) \in \ell_p(\Delta(\varphi, \top))$. T preserves norms since the equality $\|u\|_{\ell_p(\Delta(\varphi, \top))} = \|Tu\|_{\ell_p}$ holds.

□

Remark 3.3. *The space* $\ell_2(\Delta(\varphi,\top))$ *is an inner product space with the inner product defined as* $\langle u,\tilde{u}\rangle_{\ell_2(\Delta(\varphi,\top))} = \langle \Delta(\varphi,\top)u, \Delta(\varphi,\top)\tilde{u}\rangle_{\ell_2}$, *where* $\langle .,.\rangle_{\ell_2}$ *is the inner product on* ℓ_2, *which induces* $\|.\|_{\ell_2}$.

Theorem 3.4. *The space* $\ell_p(\Delta(\varphi,\top))$ *is not an inner product space for* $p \neq 2$.

Proof. Consider the sequence $u = (u_n)$ and $\widetilde{u} = (\widetilde{u_n})$, where

$$u_n = \begin{cases} \frac{\varphi(1)}{\top(1)}, & n = 1, \\ \frac{\varphi(1)+\varphi(2)}{\top(n)}, & n \neq 1, \end{cases}$$

and

$$\widetilde{u}_n = \begin{cases} \frac{\varphi(1)}{\top(1)}, & n = 1, \\ \frac{\varphi(1)-\varphi(2)}{\top(n)}, & n \neq 1, \end{cases}$$

for all $n \in \mathbb{N}$. Then, we have $\Delta(\varphi,\top)u = (1,1,0,\dots,0,\dots) \in \ell_p$ and $\Delta(\varphi,\top)\widetilde{u} = (1,-1,0,\dots,0,\dots) \in \ell_p$. Hence, one easily observes that

$$\|u+\widetilde{u}\|_{\ell_p(\Delta(\varphi,\top))} + \|u-\widetilde{u}\|_{\ell_p(\Delta(\varphi,\top))} \neq 2(\|u\|_{\ell_p(\Delta(\varphi,\top))} + \|\widetilde{u}\|_{\ell_p(\Delta(\varphi,\top))}).$$

□

Theorem 3.5. *The inclusion* $\ell_p(\Delta(\varphi,\top)) \subset \ell_q(\Delta(\varphi,\top))$ *strictly holds for* $(1 \leq p < q < \infty)$.

Proof. It is clear that the inclusion $\ell_p(\Delta(\varphi,\top)) \subset \ell_q(\Delta(\varphi,\top))$ holds since $\ell_p \subset \ell_q$ for $(1 \leq p < q < \infty)$. Also, $\ell_p \subset \ell_q$ is strict, and so there exists a sequence $w = (w_n)$ in $\ell_p \setminus \ell_q$. By defining a sequence $u = (u_n)$ as

$$u_n = \sum_{k=1}^{n} \frac{\varphi(k)}{\top(n)} w_k,$$

for all $n \in \mathbb{N}$, we conclude that $u \in \ell_q(\Delta(\varphi,\top)) \setminus \ell_p(\Delta(\varphi,\top))$. Hence, the desired inclusion is strict. □

Theorem 3.6. *The inclusion* $\ell_p(\Delta(\varphi,\top)) \subset \ell_\infty(\Delta(\varphi,\top))$ *strictly holds for* $1 \leq p < \infty$.

Proof. The inclusion is obvious since $\ell_p \subset \ell_\infty$ holds for $1 \leq p < \infty$. Let $u = (u_n)$ be a sequence such that

$$u_n = \sum_{k=1}^{n} (-1)^k \frac{\varphi(k)}{\top(n)},$$

for all $n \in \mathbb{N}$. We obtain that

$$\Delta(\varphi,\top)u = \left(\sum_{k=n-1}^{n} (-1)^{n-k} \frac{\top(k)}{\varphi(n)} \sum_{j=1}^{k} (-1)^j \frac{\varphi(j)}{\top(k)} \right) = ((-1)^n) \in \ell_\infty \setminus \ell_p,$$

which implies that $u \in \ell_\infty((\Delta(\varphi,\top)) \setminus \ell_p(\Delta(\varphi,\top))$ for all $1 \leq p < \infty$. □

4. The α-β-γ- duals of the space $\ell_p(\Delta(\varphi, \top))$

In this section, we determine the α-β-γ- duals of the sequence space $\ell_p(\Delta(\varphi, \top))$ $(1 \leq p \leq \infty)$. The following lemmas are required to prove our main results in this section. Here and in what follows, $\mathcal{K}$ denotes the family of all finite subsets of $\mathbb{N}$.

Lemma 4.1. *[33] The following statements hold:*

$S = (s_{nk}) \in (\ell_p, \ell_1)$ if and only if

$$\sup_{F\in\mathcal{K}} \sum_k \left| \sum_{n\in F} s_{nk} \right|^q < \infty \tag{4.1}$$

holds, where $1 < p < \infty$.

$S = (s_{nk}) \in (\ell_\infty, \ell_1)$ if and only if (4.1) holds with $q = 1$.

$S = (s_{nk}) \in (\ell_1, \ell_1)$ if and only if

$$\sup_k \sum_n \left| s_{nk} \right| < \infty \tag{4.2}$$

holds.

$S = (s_{nk}) \in (\ell_p, c)$ if and only if

$$\lim_{n\to\infty} s_{nk} \text{ exists for each } k \in \mathbb{N}, \tag{4.3}$$

and

$$\sup_n \sum_k \left| s_{nk} \right|^q < \infty \tag{4.4}$$

holds, where $1 < p < \infty$.

$S = (s_{nk}) \in (\ell_\infty, c)$ if and only if (4.3), and

$$\lim_{n\to\infty} \sum_k \left| s_{nk} \right| = \sum_k \left| \lim_{n\to\infty} s_{nk} \right| \tag{4.5}$$

hold.

$S = (s_{nk}) \in (\ell_1, c)$ if and only if (4.3), and

$$\sup_{n,k} \left| s_{nk} \right| < \infty \tag{4.6}$$

hold.

$S = (s_{nk}) \in (\ell_p, c_0)$ if and only if (4.4), holds and

$$\lim_{n\to\infty} s_{nk} = 0 \text{ for each } k \in \mathbb{N}, \tag{4.7}$$

where $1 < p < \infty$.

$S = (s_{nk}) \in (\ell_\infty, c_0)$ if and only if

$$\lim_{n\to\infty} \sum_k \left| s_{nk} \right| = 0 \tag{4.8}$$

hold.

$S = (s_{nk}) \in (\ell_1, c_0)$ if and only if (4.6) and (4.7) hold.

$S = (s_{nk}) \in (\ell_p, \ell_\infty)$ if and only if (4.4) holds, where $1 < p < \infty$.

$S = (s_{nk}) \in (\ell_\infty, \ell_\infty)$ if and only if (4.4) holds with $q = 1$.

$S = (s_{nk}) \in (\ell_1, \ell_\infty)$ if and only if (4.6) holds.

In the following theorem, we determine the α- duals of the spaces $\ell_p(\Delta(\varphi, \top))$ $(1 < p < \infty)$, $\ell_\infty(\Delta(\varphi, \top))$, and $\ell_1(\Delta(\varphi, \top))$.

Theorem 4.2. *The α- duals of the speaces $\ell_p(\Delta(\varphi, \top))$ $(1 < p < \infty), \ell_\infty(\Delta(\varphi, \top))$ and $\ell_1(\Delta(\varphi, \top))$ follows:*

$$(\ell_p(\Delta(\varphi, \top)))^\alpha = \Big\{a = (a_n) \in \omega : \sup_{F \in \mathcal{K}} \sum_k \Big| \sum_{n \in F} \frac{\varphi(k)}{\top(n)} a_n \Big|^q < \infty \Big\},$$

$$(\ell_\infty(\Delta(\varphi, \top)))^\alpha = \Big\{a = (a_n) \in \omega : \sup_{F \in \mathcal{K}} \sum_k \Big| \sum_{n \in F} \frac{\varphi(k)}{\top(n)} a_n \Big| < \infty \Big\},$$

$$(\ell_1(\Delta(\varphi, \top)))^\alpha = \Big\{a = (a_n) \in \omega : \sup_k \sum_n \Big| \frac{\varphi(k)}{\top(n)} a_n \Big| < \infty \Big\}.$$

Proof. Consider the matrix $C = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} \frac{\varphi(k)}{\top(n)} a_n, & 1 \le k \le n, \\ 0, & k > n, \end{cases}$$

for any sequence $a = (a_n) \in \omega$. Hence, given any $u = (u_n) \in \ell_p(\Delta(\varphi, \top))$ for $1 \le p \le \infty$, we have $a_n u_n = C_n(v)$ for all $n \in \mathbb{N}$. This implies that $au \in \ell_1$ with $u \in \ell_p(\Delta(\varphi, \top))$ if and only if $Cv \in \ell_1$ with $v \in \ell_p$. It follows that $a \in (\ell_p(\Delta(\varphi, \top)))^\alpha$ if and only if $C \in (\ell_p, \ell_1)$, which completes the proof in view of Lemma 4.1. □

Lemma 4.3. *[34, Theorem 3.1] Let $B = (b_{nk})$ be defined via a sequence $a = (a_k) \in \omega$, and the inverse matrix $D^{-1} = (d^{-1}_{nk})$ of the triangle matrix $D = (d_{nk})$ by*

$$b_{nk} = \sum_{j=k}^{n} a_j d^{-1}_{jk},$$

for all $n, k \in \mathbb{N}$. Then,

$$\mathcal{U}^\beta_D = \{a = (a_n) \in \omega : B \in (\mathcal{U}, c)\},$$
$$\mathcal{U}^\gamma_D = \{a = (a_n) \in \omega : B \in (\mathcal{U}, \ell_\infty)\}.$$

Consequently,we have the following theorem.

Theorem 4.4. *Let's define the following sets:*

$$\aleph_1 = \{a = (a_n) \in \omega : \lim_{n \to \infty} \sum_{j=k}^{n} a_j \frac{\varphi(k)}{\top(j)} \text{ exists for each } k \in \mathbb{N}\},$$

$$\aleph_2 = \{a = (a_n) \in \omega : \sup_n \sum_k \Big| \sum_{j=k}^{n} a_j \frac{\varphi(k)}{\top(j)} \Big|^q < \infty\},$$

$$\aleph_3 = \{a = (a_n) \in \omega : \lim_{n\to\infty} \sum_k \Big| \sum_{j=k}^{n} a_j \frac{\varphi(k)}{\top(j)} \Big| = \sum_k \Big| \sum_{j=k}^{\infty} a_j \frac{\varphi(k)}{\top(j)} \Big| \},$$

$$\aleph_4 = \{a = (a_n) \in \omega : \sup_{n,k} \Big| \sum_{j=k}^{n} a_j \frac{\varphi(k)}{\top(j)} \Big| < \infty\}.$$

The β- and γ- duals of the spaces $\ell_p(\Delta(\varphi,\top))$ $(1 < p < \infty)$, $\ell_\infty(\Delta(\varphi,\top))$, and $\ell_1(\Delta(\varphi,\top))$ are as follows:

$$(\ell_p(\Delta(\varphi,\top)))^\beta = \aleph_1 \cap \aleph_2, (\ell_\infty(\Delta(\varphi,\top)))^\beta = \aleph_1 \cap \aleph_3, (\ell_1(\Delta(\varphi,\top)))^\beta = \aleph_1 \cap \aleph_4,$$

$$(\ell_p(\Delta(\varphi,\top)))^\gamma = \aleph_2, (\ell_\infty(\Delta(\varphi,\top)))^\gamma = \aleph_2 \text{ with } q = 1, (\ell_1(\Delta(\varphi,\top)))^\gamma = \aleph_4.$$

Proof. Let $a = (a_n) \in \omega$, $\mathcal{U} \in \{\ell_p, \ell_\infty, \ell_1\}$ and $B = (b_{nk})$ be an infinite matrix with terms

$$b_{nk} = \begin{cases} \sum_{j=k}^{n} a_j \frac{\varphi(k)}{\top(j)}, & \text{if } 1 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

Hence, it follows that

$$\sum_{k=1}^{n} a_k u_k = \sum_{k=1}^{n} a_k \left(\sum_{j=1}^{k} \frac{\varphi(j)}{\top(k)} v_j \right) = \sum_{k=1}^{n} \left(\sum_{j=k}^{n} a_j \frac{\varphi(k)}{\top(j)} \right) v_k = B_n(v),$$

for any $u = (u_n) \in \mathcal{U}(\Delta(\varphi,\top))$. This equality yields that $au \in cs$ for $u \in \mathcal{U}(\Delta(\varphi,\top))$ if and only if $Bv \in c$ for $v \in \mathcal{U}$. That is, $a \in (\mathcal{U}(\Delta(\varphi,\top)))^\beta$ if and only if $B \in (\mathcal{U}, c)$. Hence, by Lemma 4.1, it is concluded that

$$(\ell_p(\Delta(\varphi,\top)))^\beta = \aleph_1 \cap \aleph_2, (\ell_\infty(\Delta(\varphi,\top)))^\beta = \aleph_1 \cap \aleph_3, (\ell_1(\Delta(\varphi,\top)))^\beta = \aleph_1 \cap \aleph_4.$$

This equality also yields that $au \in bs$ for $u \in \mathcal{U}(\Delta(\varphi,\top))$ if and only if $Bv \in \ell_\infty$ for $v \in \mathcal{U}$. That is, $a \in (\mathcal{U}(\Delta(\varphi,\top)))^\gamma$ if and only if $B \in (\mathcal{U}, \ell_\infty)$. Hence, by Lemma 4.1, it is concluded that

$$(\ell_p(\Delta(\varphi,\top)))^\gamma = \aleph_2, (\ell_\infty(\Delta(\varphi,\top)))^\gamma = \aleph_2 \text{ with } q = 1, (\ell_1(\Delta(\varphi,\top)))^\gamma = \aleph_4.$$

□

5. Some matrix transformations related to the sequence spaces $\ell_p(\Delta(\varphi,\top))$

The matrix classes from $\ell_p(\Delta(\varphi,\top))$, $\ell_\infty(\Delta(\varphi,\top))$, and $\ell_1(\Delta(\varphi,\top))$ into ℓ_1, c, c_0, and ℓ_∞ are characterized by the aid of the following theorem and Lemma 4.1.

Theorem 5.1. *Let* $1 < p < \infty$, $\mathcal{U} \in \{\ell_p, \ell_\infty, \ell_1\}$ *and* $\mathcal{V} \subset \omega$. *Then,* $S = (s_{nk}) \in (\mathcal{U}_{\Delta(\varphi,\top)}, \mathcal{V})$ *if and only if* $E^n = (e_{mk}^{(n)}) \in (\mathcal{U}, c)$ *for each fixed* $n \in \mathbb{N}$ *and* $F = (f_{nk}) \in (\mathcal{U}, \mathcal{V})$, *where*

$$e_{mk}^{(n)} = \begin{cases} \sum_{j=k}^{m} s_{nj} \frac{\varphi(k)}{\top(j)}, & 1 \le k \le m, \\ 0, & k > m. \end{cases}$$

$$f_{nk} = \sum_{j=k}^{\infty} s_{nj} \frac{\varphi(k)}{\top(j)}.$$

Proof. Let $S \in (\mathcal{U}_{\Delta(\varphi,\top)}, \mathcal{V})$ and $u \in \mathcal{U}_{\Delta(\varphi,\top)}$. Then, the equality

$$\sum_{k=1}^{m} s_{nk}u_k = \sum_{k=1}^{m} s_{nk}\Big(\sum_{j=1}^{k} \frac{\varphi(j)}{\top(k)}v_j\Big) = \sum_{k=1}^{m}\Big(\sum_{j=k}^{m} s_{nj}\frac{\varphi(k)}{\top(j)}\Big)v_k = \sum_{k=1}^{m} e_{mk}^{(n)}v_k$$

holds. Since Su exists, it follows that $E^n \in (\mathcal{U}, c)$ for each fixed $n \in \mathbb{N}$. From the last equality, one can easily observe that $Su = Fv$ as $m \to \infty$. Hence, $Su \in \mathcal{V}$ implies that $Fv \in \mathcal{V}$; that is, $F \in (\mathcal{U}, \mathcal{V})$.

For the converse, let $E^n \in (\mathcal{U}, c)$ for each fixed $n \in \mathbb{N}$ and $F \in (\mathcal{U}, \mathcal{V})$. Given any $u \in \mathcal{U}_{\Delta(\varphi,\top)}$, $f_{nk} \in \mathcal{U}^{\beta}$ for each fixed $n \in \mathbb{N}$. This yields that $(s_{nk}) \in \mathcal{U}^{\beta}_{\Delta(\varphi,\top)}$ for each fixed $n \in \mathbb{N}$. Hence, Su exists. The equality $Su = Fv$ as $m \to \infty$ proves that $S \in (\mathcal{U}_{\Delta(\varphi,\top)}, \mathcal{V})$. □

Hence, the following results are obtained.

Theorem 5.2. *Let $S = (s_{nk})$ be an infinite matrix.*

1) $S \in (\ell_\infty(\Delta(\varphi, \top)), \ell_\infty)$ if and only if

$$\lim_{m} \sum_{j=k}^{m} s_{nj}\frac{\varphi(k)}{\top(j)} \text{ exists for each fixed } n, k \in \mathbb{N}, \tag{5.1}$$

$$\lim_{m} \sum_{k} \Big| \sum_{j=k}^{m} s_{nj}\frac{\varphi(k)}{\top(j)} \Big| = \sum_{k} \Big| \sum_{j=k}^{\infty} s_{nj}\frac{\varphi(k)}{\top(j)} \Big|, \tag{5.2}$$

and

$$\sup_{n} \sum_{k} \Big| \sum_{j=k}^{\infty} s_{nj}\frac{\varphi(k)}{\top(j)} \Big| < \infty. \tag{5.3}$$

2) $S \in \ell_\infty((\Delta(\varphi, \top)), c)$ if and only if (5.1), (5.2) hold and

$$\lim_{n} \sum_{j=k}^{\infty} s_{nj}\frac{\varphi(k)}{\top(j)} \text{ exists for each fixed } k \in \mathbb{N}, \tag{5.4}$$

$$\lim_{n} \sum_{k} \Big| \sum_{j=k}^{\infty} s_{nj}\frac{\varphi(k)}{\top(j)} \Big| = \sum_{k} \Big| \lim_{n} \sum_{j=k}^{\infty} s_{nj}\frac{\varphi(k)}{\top(j)} \Big|.$$

3) $S \in (\ell_\infty(\Delta(\varphi, \top)), c_0)$ if and only if (5.1), (5.2) hold and

$$\lim_{n} \sum_{k} \Big| \sum_{j=k}^{\infty} s_{nj}\frac{\varphi(k)}{\top(j)} \Big| = 0. \tag{5.5}$$

4) $S \in (\ell_\infty(\Delta(\varphi, \top)), \ell_1)$ if and only if (5.1), (5.2) hold and

$$\sup_{F\in\mathcal{K}} \sum_{k} \Big| \sum_{n\in F} \sum_{j=k}^{\infty} s_{nj}\frac{\varphi(k)}{\top(j)} \Big| < \infty. \tag{5.6}$$

Corollary 5.3. *Let $S = (s_{nk})$ be an infinite matrix.*

1) $S \in (\ell_\infty(\Delta(\varphi, \top)), bs)$ if and only if (5.1), (5.2) hold and

$$\sup_n \sum_k \left| \sum_{m=1}^{n} \sum_{j=k}^{\infty} s_{mj} \frac{\varphi(k)}{\top(j)} \right| < \infty. \tag{5.7}$$

2) $S \in (\ell_\infty(\Delta(\varphi, \top)), cs)$ if and only if (5.1), (5.2) hold and

$$\lim_n \sum_{m=1}^{n} \sum_{j=k}^{\infty} s_{mj} \frac{\varphi(k)}{\top(j)} \text{ exists for each } k \in \mathbb{N}, \tag{5.8}$$

$$\lim_n \sum_k \left| \sum_{m=1}^{n} \sum_{j=k}^{\infty} s_{mj} \frac{\varphi(k)}{\top(j)} \right| = \sum_k \left| \lim_n \sum_{m=1}^{n} \sum_{j=k}^{\infty} s_{mj} \frac{\varphi(k)}{\top(j)} \right|.$$

3) $S \in (\ell_\infty(\Delta(\varphi, \top)), cs_0)$ if and only if (5.1), (5.2) hold and

$$\lim_n \sum_k \left| \sum_{m=1}^{n} \sum_{j=k}^{\infty} s_{mj} \frac{\varphi(k)}{\top(j)} \right| = 0. \tag{5.9}$$

Theorem 5.4. *Let $S = (s_{nk})$ be an infinite matrix and $1 < p < \infty$.*

1) $S \in (\ell_p(\Delta(\varphi, \top)), \ell_\infty)$ if and only if (5.1) hold, and

$$\sup_m \sum_{k=1}^{m} \left| \sum_{j=k}^{m} s_{nj} \frac{\varphi(k)}{\top(j)} \right|^q < \infty \text{ for each fixed } n \in \mathbb{N}, \tag{5.10}$$

$$\sup_n \sum_k \left| \sum_{j=k}^{\infty} s_{nj} \frac{\varphi(k)}{\top(j)} \right|^q < \infty. \tag{5.11}$$

2) $S \in (\ell_p(\Delta(\varphi, \top)), c)$ if and only if (5.1), (5.10), (5.4), (5.11) hold.
3) $S \in (\ell_p(\Delta(\varphi, \top)), c_0)$ if and only if (5.1), (5.10), (5.11) hold, and

$$\lim_n \sum_{j=k}^{\infty} s_{nj} \frac{\varphi(k)}{\top(j)} = 0 \text{ for each } k \in \mathbb{N}. \tag{5.12}$$

4) $S \in (\ell_p(\Delta(\varphi, \top)), \ell_1)$ if and only if (5.1), (5.10) hold, and

$$\sup_{F \in \mathcal{K}} \sum_k \left| \sum_{n \in F} \sum_{j=k}^{\infty} s_{nj} \frac{\varphi(k)}{\top(j)} \right|^q < \infty.$$

Corollary 5.5. *Let $S = (s_{nk})$ be an infinite matrix.*

1) $S \in (\ell_p(\Delta(\varphi, \top)), bs)$ if and only if (5.1), (5.10) hold, and

$$\sup_n \sum_k \left| \sum_{m=1}^{n} \sum_{j=k}^{\infty} s_{mj} \frac{\varphi(k)}{\top(j)} \right|^q < \infty. \tag{5.13}$$

2) $S \in (\ell_p(\Delta(\varphi, \top)), cs)$ if and only if (5.1), (5.10), (5.8), (5.13) hold.
3) $S \in (\ell_p(\Delta(\varphi, \top)), cs_0)$ if and only if (5.1), (5.10), (5.13) hold, and

$$\lim_n \sum_{m=1}^{n} \sum_{j=k}^{\infty} s_{mj} \frac{\varphi(k)}{\top(j)} = 0 \text{ for each } k \in \mathbb{N}. \tag{5.14}$$

Theorem 5.6. *Let* $S = (s_{nk})$ *be an infinite matrix.*

1) $S \in (\ell_1(\Delta(\varphi, \top)), \ell_\infty)$ if and only if (5.1) hold, and

$$\sup_{m,k} \left| \sum_{j=k}^{m} s_{nj} \frac{\varphi(k)}{\top(j)} \right| < \infty \text{ for each fixed } n \in \mathbb{N}, \tag{5.15}$$

$$\sup_{n,k} \left| \sum_{j=k}^{\infty} s_{nj} \frac{\varphi(k)}{\top(j)} \right| < \infty. \tag{5.16}$$

2) $S \in (\ell_1(\Delta(\varphi, \top)), c)$ if and only if (5.1), (5.15), (5.4), (5.16) hold.
3) $S \in (\ell_1(\Delta(\varphi, \top)), c_0)$ if and only if (5.1), (5.15), (5.12), (5.16) hold.
4) $S \in (\ell_1(\Delta(\varphi, \top)), \ell_1)$ if and only if (5.1), (5.15) hold, and

$$\sup_k \sum_n \left| \sum_{j=k}^{\infty} s_{nj} \frac{\varphi(k)}{\top(j)} \right| < \infty. \tag{5.17}$$

Corollary 5.7. *Let* $S = (s_{nk})$ *be an infinite matrix.*

1) $S \in (\ell_1(\Delta(\varphi, \top)), bs)$ if and only if (5.1), (5.15) hold and

$$\sup_{n,k} \left| \sum_{m=1}^{n} \sum_{j=k}^{\infty} s_{mj} \frac{\varphi(k)}{\top(j)} \right| < \infty. \tag{5.18}$$

2) $S \in (\ell_1(\Delta(\varphi, \top)), cs)$ if and only if (5.1), (5.15), (5.8), (5.18) hold.
3) $S \in (\ell_1(\Delta(\varphi, \top)), cs_0)$ if and only if (5.1), (5.15), (5.14), (5.18) hold.

Now, the matrix classes from ℓ_1, c, c_0, and ℓ_∞ into $\ell_p(\Delta(\varphi, \top))$, $\ell_\infty(\Delta(\varphi, \top))$, and $\ell_1(\Delta(\varphi, \top))$ are characterized.

Theorem 5.8. *Let* $S = (s_{nk})$ *be an infinite matrix and* $1 < p < \infty$.

(a) $S \in (\ell_\infty, \ell_p(\Delta(\varphi, \top))) = (c, \ell_p(\Delta(\varphi, \top))) = (c_0, \ell_p(\Delta(\varphi, \top)))$ if and only if

$$\sup_{F\in\mathcal{K}} \sum_n \left| \sum_{k\in F} \sum_{m=n-1}^{n} \frac{(-1)^{n-m}\top(m)}{\varphi(n)} s_{mk} \right|^p < \infty.$$

(b) $S \in (\ell_1, \ell_p(\Delta(\varphi, \top)))$ if and only if

$$\sup_j \sum_i \left| \sum_{m=n-1}^{n} \frac{(-1)^{n-m}\top(m)}{\varphi(n)} s_{mk} \right|^p < \infty.$$

Proof. Let $S = (s_{nk}) \in (\ell_\infty, \ell_p(\Delta(\varphi, \top)))$. Define the matrix $\hat{S} = (\hat{s}_{nk})$ as

$$\hat{s}_{nk} = \sum_{m=n-1}^{n} \frac{(-1)^{n-m}\top(m)}{\varphi(n)} s_{mk}$$

for all $n, k \in \mathbb{N}$. Then, for any $u = (u_k) \in \ell_\infty$, the equality

$$\sum_k \hat{s}_{nk} u_k = \sum_{m=n-1}^{n} \frac{(-1)^{n-m}\top(m)}{\varphi(n)} \sum_k s_{mk} u_k$$

means that $(\hat{S}u)_n = (\Delta(\varphi, \top)(Su))_n$ for all $n \in \mathbb{N}$. This implies that $Su \in \ell_p(\Delta(\varphi, \top))$ for $u = (u_j) \in \ell_\infty$ if and only if $\hat{S}u \in \ell_p$ for $u = (u_k) \in \ell_\infty$. Hence, we conclude from Lemma 4.1 that

$$\sup_{F\in\mathcal{K}} \sum_n \left| \sum_{k\in F} \sum_{m=n-1}^{n} \frac{(-1)^{n-m}\top(m)}{\varphi(n)} s_{mk} \right|^p < \infty.$$

The other case can be proved with a similar method. □

Theorem 5.9. *Let $S = (s_{nk})$ be an infinite matrix.*

(a) $S \in (\ell_\infty, \ell_\infty(\Delta(\varphi, \top))) = (c, \ell_\infty(\Delta(\varphi, \top))) = (c_0, \ell_\infty(\Delta(\varphi, \top)))$ if and only if

$$\sup_n \sum_k \left| \sum_{m=n-1}^{n} \frac{(-1)^{n-m}\top(m)}{\varphi(n)} s_{mk} \right| < \infty.$$

(b) $S \in (\ell_1, \ell_\infty(\Delta(\varphi, \top)))$ if and only if

$$\sup_{n,k} \left| \sum_{m=n-1}^{n} \frac{(-1)^{n-m}\top(m)}{\varphi(n)} s_{mk} \right| < \infty.$$

(c) $S \in (\ell_\infty, \ell_1(\Delta(\varphi, \top))) = (c, \ell_1(\Delta(\varphi, \top))) = (c_0, \ell_1(\Delta(\varphi, \top)))$ if and only if

$$\sup_{F\in\mathcal{K}} \sum_n \left| \sum_{k\in F} \sum_{m=n-1}^{n} \frac{(-1)^{n-m}\top(m)}{\varphi(n)} s_{mk} \right| < \infty.$$

(d) $S \in (\ell_1, \ell_1(\Delta(\varphi, \top)))$ if and only if

$$\sup_k \sum_n \left| \sum_{m=n-1}^{n} \frac{(-1)^{n-m}\top(m)}{\varphi(n)} s_{mk} \right| < \infty.$$

Proof. One can follow the same technique with the proof of Theorem 5.8. □

6. Conclusions

Classical summability theory is concerned with the generalization of the concept for convergence of series or sequences by assigning limit to non-convergent series or sequences. For this purpose, infinite special matrices are used. One of the main purposes of this paper is to introduce a new triangular matrix whose terms are obtained by using the Euler totient function and its summation function. The domains of this matrix are studied within the spaces of p-absolutely summable sequences and bounded sequences, which construct BK spaces and are linearly isomorphic to classical ones. After determining α, β, and γ-duals for these BK spaces, certain classes of matrices are characterized as an application of matrix transformations.

By using this new matrix operator and resulting sequence spaces, many fascinating results can be obtained in the theory of sequence spaces and matrix transformations. One can study compact operators on matrix domains as the interesting application of Hausdorff measure of noncompactness in the theory of sequence spaces. New paranormed sequence spaces or series spaces can be defined. It is planned to study the domains of this matrix in the spaces of convergent and null sequences. In the next stage, a novel matrix will be obtained by using the Jordan totient function and its summation function. The final study will be the advancement of the previous ones.

Author contributions

Merve İlkhan Kara and Dilek Aydın: Conceptualization, methodology, validation, writing-original draft, writing-review & editing. The authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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