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Research article

On centrally-extended n-homoderivations on rings

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Abstract: In this article, we explored the commutativity of a ring Λ that is equipped with a unique class of mappings called centrally extended n-homoderivations, where n is an integer. These mappings generalize the concepts of derivations and homoderivations. Furthermore, we investigated specific properties of the center of such rings.

Keywords: prime and semiprime rings; homoderivation; CE-derivation; CE-n-homoderivation
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1. Introduction

In this work, Λ refers to an associative ring, and $\mathcal{Z}(\Lambda)$ denotes the center of Λ. The symbols $[r, s]$ and $r \circ s$ stand for $rs - sr$ and $rs + sr$, respectively. If $\mathcal{S} \subseteq \Lambda$, define $\mathcal{F} : \Lambda \to \Lambda$ to be centralizing on $\mathcal{S}$ if $[s, \mathcal{F}(s)] \in \mathcal{Z}(\Lambda)$ for all $s \in \mathcal{S}$; and is commuting on $\mathcal{S}$ if $[s, \mathcal{F}(s)] = 0$ for all $s \in \mathcal{S}$. A mapping $\mathcal{F}$ is said to be strong commutativity-preserving on $\mathcal{S}$ if $[s, t] = [\mathcal{F}(s), \mathcal{F}(t)]$ for all $s, t \in \mathcal{S}$.

Λ is considered prime if $r\Lambda s = \{0\}$, where r and s are both in Λ, implying that either r is zero or s is zero. This prime ring definition is equivalent to: (i) The product of two non-zero two-sided ideals of Λ is not zero. (ii) The left annihilator of a non-zero left ideal is zero; for further information, see [1], page 47. Λ is considered semiprime if $r\Lambda r = \{0\}$, then $r = 0$. If $\mathcal{D}(rs) = \mathcal{D}(r)s + r\mathcal{D}(s)$ holds for every r, s in Λ, then the additive map $\mathcal{D}$ is said to be a derivation.

In a recent publication [2], Bell and Daif introduced the idea of a ring's centrally extended derivation (CE-derivation). Suppose that $\mathcal{D}$ is a mapping of a ring Λ. If $\mathcal{D}(s+u)-\mathcal{D}(s)-\mathcal{D}(u) \in \mathcal{Z}(\Lambda)$ and $\mathcal{D}(su) - \mathcal{D}(s)u - s\mathcal{D}(u) \in \mathcal{Z}(\Lambda)$ for every $s, u \in \Lambda$, then $\mathcal{D}$ is known as a CE-derivation. The CE-(ρ, σ)-derivation on Λ has been described by Tammam et al. [3] as a map $\mathcal{D}$ on Λ achieves, for each

$s, u \in \Lambda$, both

$$\mathcal{D}(s+u) - \mathcal{D}(s) - \mathcal{D}(u) \text{ and } \mathcal{D}(su) - \mathcal{D}(s)\rho(u) - \sigma(s)\mathcal{D}(u) \text{ are in } \mathcal{Z}(\Lambda).$$

The concept of homoderivations of rings was first introduced by El-Soufi in 2000 [4]. A mapping $\hbar$ on a ring Λ is defined as a homoderivation if it satisfies the relation $\hbar(su) = s\hbar(u) + \hbar(s)u + \hbar(s)\hbar(u)$ for all $s, u \in \Lambda$, provided that $\hbar$ is additive.

The following are a few instances of homoderivations:

Example 1.1. [4] *Let Λ be a ring and ϕ be an endomorphism of Λ. Then, the mapping $\hbar : \Lambda \to \Lambda$ defined by $\hbar(u) = \phi(u) - u$ is a homoderivation of Λ.*

Example 1.2. [4] *Let Λ be a ring. Then, the additive mapping $\hbar : \Lambda \to \Lambda$ defined by $\hbar(u) = -u$ is a homoderivation of Λ.*

Example 1.3. [4] *Let $\Lambda = \mathbb{Z}(\sqrt{2})$, a ring of all the real numbers of the form $u + s\sqrt{2}$ such that $u, s \in \mathbb{Z}$, the set of all the integers, under the usual addition and multiplication of real numbers. Then, the map $\hbar : \Lambda \to \Lambda$ defined by $\hbar(u + s\sqrt{2}) = -2s\sqrt{2}$ is a homoderivation of Λ.*

Melaibari et al. [5] demonstrated the commutativity of a prime ring Λ in 2016 by admitting a non-zero homomoderivation $\hbar$ that satisfies any one of the following requirements: i. $[v, u] = [\hbar(v), \hbar(u)]$, for all $v, u \in U$, non-zero ideal of Λ; ii. $\hbar([v, u]) = 0$, for all $v, u \in U$, a non-zero ideal of Λ; or iii. $\hbar([v, u]) \in (\Lambda)$, for all $v, u \in \Lambda$.

According to Alharfie et al. [6], a prime ring Λ is commutative if any of the following requirements are met: For all $v, u \in I$, i. $v\hbar(u) \pm vu \in \mathcal{Z}(\Lambda)$, ii. $v\hbar(u) \pm uv \in \mathcal{Z}(\Lambda)$, or iii. $v\hbar(u) \pm [v, u] \in \mathcal{Z}(R)$. $\hbar$ is a homoderivation of Λ, and I is a non-zero left ideal of Λ.

The commutativity of a semiprime (prime) ring admitting a homoderivation meeting certain identities on a ring was investigated in 2019 by Alharfie et al. [7] and Rehman et al. [8].

Over the past few years, researchers [9–11] have obtained many significant results pertaining to different aspects of homoderivations.

In 2022, Tammam et al. [12] extended the concept of homoderivations by introducing the notion of n-homoderivations, where n is an integer. A map $\hbar_n$ is known as an n-homoderivation if it fulfills the requirement $\hbar_n(su) = s\hbar_n(u) + \hbar_n(s)u + n\hbar_n(s)\hbar_n(u)$ for all $s, u \in \Lambda$, provided $\hbar_n$ is additive.

We draw inspiration from Bell and Daif's study [2], building on the new concept of n-homoderivations introduced in [12]. We focus on exploring the notion of a centrally extended n-homoderivation (CE-n-homoderivation), where $n \in \mathbb{Z}$, as an extension of the traditional definition of homoderivations. Moreover, we explore several results regarding the ring commutativity of a ring equipped with a CE-n-homoderivation fulfilling specific conditions.

Definition 1.1. *Let s and u be any two elements in Λ and n be an integer, and let $\mathcal{H}_n$ be a mapping on a ring Λ. If $\mathcal{H}_n$ achieves*

$$\mathcal{H}_n(s+u) - \mathcal{H}_n(s) - \mathcal{H}_n(u) \in \mathcal{Z}(\Lambda), \text{ and } \mathcal{H}_n(su) - \mathcal{H}_n(s)u - s\mathcal{H}_n(u) - n\mathcal{H}_n(s)\mathcal{H}_n(u) \in \mathcal{Z}(\Lambda),$$

then $\mathcal{H}_n$ is called a CE-n-homoderivation.

It is clear that the previous definition generalizes the idea of centrally extended derivations (CE-derivations) presented by Bell and Daif [2] to the general case of centrally extended homoderivations of the type n (CE-n-homoderivations).

Chung was the first to develop the idea of nil and nilpotent derivations in [13]. Consider a ring Λ that has a derivation δ. δ is considered to be nil if $k = k(r) \in \mathbb{Z}^+$ occurs for every $r \in \Lambda$ with $\delta^{k(r)} = 0$. If the integer k can be freely taken out of r, then the derivation δ is said to be nilpotent.

Definition 1.2. *Assume that $\mathcal{S} \subseteq \Lambda$ and that $\mathcal{H}$ and ϕ are two maps on a ring Λ. For some $k \in \mathbb{Z}^+ - \{1\}$, $\mathcal{H}$ is considered nilpotent on $\mathcal{S}$ if $\mathcal{H}^k(\mathcal{S}) = \{0\}$. If $\phi(\mathcal{H}(s)) = \mathcal{H}(\phi(s))$, for every $s \in \mathcal{S}$, then two mappings $\mathcal{H}$ and ϕ are said to be commute on $\mathcal{S}$.*

Remark 1.1. *According to our definition of a CE-n-homoderivation, we assert that*

(1) *Any CE-0-homoderivation of Λ is a CE-derivation on Λ.*

(2) *Any CE-1-homoderivation of Λ is a CE-homoderivation on Λ.*

(3) *Any n-homoderivation is a CE-n-homoderivation, but the inverse (in general) is not true.*

Remark 1.2. *$\theta_{\mathcal{H}_n}(r, s, +)$ and $\theta_{\mathcal{H}_n}(r, s, \cdot)$ refer to the central elements generated through the influence of $\mathcal{H}_n$ on the sum $r + s$ and the product $r \cdot s$, respectively, for any two elements $r, s \in \Lambda$.*

Theorem 1.1. *Given a ring Λ, let n be any arbitrary non-zero integer. If the following centrally additive map $\gamma_n : \Lambda \to \Lambda$ satisfies*

$$\gamma_n(st) = \gamma_n(s)t + s\gamma_n(t) + n\gamma_n(s)\gamma_n(t) + \theta_{\mathcal{H}_n}(s, t, \cdot), \tag{1.1}$$

for each $s, t \in \Lambda$, $\theta_{\mathcal{H}_n}(s, t, \cdot) \in \mathcal{Z}(\Lambda)$, then there exists a centrally extended homomorphism $\phi_n : \Lambda \to \Lambda$ such that $\phi_n(s) = s + n\gamma_n(s)$ for each $s \in \Lambda$.

Proof. Clearly, since γ_n is a centrally additive, ϕ_n is centrally additive. Multiplying (1.1) with n leads to

$$n\gamma_n(st) = n\gamma_n(s)t + ns\gamma_n(t) + n\gamma_n(s)n\gamma_n(t) + n\theta_{\mathcal{H}_n}(s, t, \cdot) \text{ for all } s, t \in \Lambda.$$

If we add st to both sides of this equation, then

$$n\gamma_n(st) + st = n\gamma_n(s)t + ns\gamma_n(t) + n\gamma_n(s)n\gamma_n(t) + st + n\theta_{\mathcal{H}_n}(s, t, \cdot),$$

for all $s, t \in \Lambda$. Observe however that

$$n\gamma_n(s)t + ns\gamma_n(t) + n\gamma_n(s)n\gamma_n(t) + st = (n\gamma_n(s) + s)(n\gamma_n(t) + t),$$

for all $s, t \in \Lambda$; revealing precisely that the mapping $\phi_n : \Lambda \to \Lambda$ specified by $\phi_n(s) = n\gamma_n(s) + s$ for all $s, t \in \Lambda$ is a centrally extended homomorphism. □

Few adoptions on the proof of [12] Lemma 1 asserts that

Lemma 1.1. *Let $\mathcal{K}$ be a non-zero left ideal and Λ be a semi-prime ring. $\mathcal{H}_n$ is commuting on $\mathcal{K}$ if it is a centralizing CE-n-homoderivation on $\mathcal{K}$.*

2. Examples of CE-n-homoderivations

In this section, we confirm the presence of CE-n-homoderviation maps in the instances listed below.

Example 2.1. *Let $\Lambda = M_2(\mathbb{Z})$, the ring of 2×2 integer matrices, and let $\mathcal{K}$ be a nonzero central ideal of Λ. Suppose that $f_n : \Lambda \to \mathcal{K}$ is any additive map and $\hbar_n : \Lambda \to \Lambda$ is any n-homoderivation of Λ. Therefore, the map $\mathcal{H}_n : \Lambda \to \Lambda$ such that $\mathcal{H}_n(x) = \hbar_n(x) + f_n(x)$, for all $x \in \Lambda$, is a CE n-homoderivation but it is not n-homoderivation.*

Example 2.2. *Let Λ_1 be a commutative domain, Λ_2 a noncommutative prime ring with an n-homoderivation $\hbar_n$, and $\Lambda = \Lambda_1 \oplus \Lambda_2$. Define $\mathcal{H}_n : \Lambda \to \Lambda$ by $\mathcal{H}_n((s, u)) = (g(s), \hbar_n(u))$, where $g : \Lambda_1 \to \Lambda_1$ is a map that is not an n-homoderivation. Then, Λ is a semiprime ring, and $\mathcal{H}_n$ is a CE-n-homoderivation that is not an n-homoderivation. Furthermore, $\Lambda_1 \oplus \{0\}$ is an ideal that is contained in the center of Λ.*

3. Rings with centrally extended n-homoderivations

In this section, we explore the conditions under which a CE-n-homoderivation fulfills the requirements of an n-homoderivation. Additionally, it delves into the fundamental properties of CE-n-homoderivations.

Throughout, $\mathcal{H}_n$ is a centrally extended n-homoderivation of a ring Λ, and $n \in \mathbb{Z}$, ϕ_n will be the related CE-homomorphism to $\mathcal{H}_n$ defined in Theorem 1.1.

Theorem 3.1. *Let Λ be any ring containing no non-zero ideals in its center. Then, each nilpotent CE-n-homoderivation $\mathcal{H}_n$ on Λ is additive. Also, every CE-n-homoderivation $\mathcal{H}_n$ on Λ related to an epimorphism ϕ_n is additive.*

Proof. (i) If $\mathcal{H}_n$ is nilpotent:

Let $s, u \in \Lambda$ be two fixed elements. By assumption,

$$\mathcal{H}_n(s + u) = \mathcal{H}_n(s) + \mathcal{H}_n(u) + \theta_{\mathcal{H}_n}(s, u, +). \tag{3.1}$$

So, for each $v \in \Lambda$, we obtain

$$\begin{aligned}\mathcal{H}_n((s + u)v) &= (s + u)\mathcal{H}_n(v) + \mathcal{H}_n(s + u)v + n\mathcal{H}_n(s + u)\mathcal{H}_n(v) + \theta_{\mathcal{H}_n}(s + u, v, \cdot)\\ &= (\mathcal{H}_n(s) + \mathcal{H}_n(u) + \theta_{\mathcal{H}_n}(s, u, +))(v + n\mathcal{H}_n(v)) + u\mathcal{H}_n(v) + s\mathcal{H}_n(v)\\ &\quad + \theta_{\mathcal{H}_n}(s + u, v, \cdot).\end{aligned} \tag{3.2}$$

However, we also have

$$\begin{aligned}\mathcal{H}_n((s + u)v) &= \mathcal{H}_n(sv + uv)\\ &= \mathcal{H}_n(sv) + \mathcal{H}_n(uv) + \theta_{\mathcal{H}_n}(sv, uv, +)\\ &= \mathcal{H}_n(s)v + s\mathcal{H}_n(v) + n\mathcal{H}_n(s)\mathcal{H}_n(v) + u\mathcal{H}_n(v) + \mathcal{H}_n(u)v\\ &\quad + n\mathcal{H}_n(u)\mathcal{H}_n(v) + \theta_{\mathcal{H}_n}(sv, uv, +) + \theta_{\mathcal{H}_n}(s, v, \cdot) + \theta_{\mathcal{H}_n}(u, v, \cdot).\end{aligned} \tag{3.3}$$

Comparing (3.2) and (3.3), we get

$$(v + n\mathcal{H}_n(v))\theta_{\mathcal{H}_n}(s,u,+) \in \mathcal{Z}(\Lambda), \text{ for all } v \in \Lambda. \tag{3.4}$$

Due to the fact that $\mathcal{H}_n$ is nilpotent, $\exists k \in \mathbb{Z}, k > 1$ so that $\mathcal{H}_n^k(s) = 0$ for all $s \in \Lambda$. By putting $\mathcal{H}_n^{k-1}(v)$ instead of v in (3.4), the result is

$$\mathcal{H}_n^{k-1}(v)\theta_{\mathcal{H}_n}(s,u,+) \in \mathcal{Z}(\Lambda), \quad \text{for each } v \in \Lambda. \tag{3.5}$$

Putting $\mathcal{H}_n^{k-2}(v)$ instead of v in (3.4), we get

$$(\mathcal{H}_n^{k-2}(v) + n\mathcal{H}_n^{k-1}(v))\theta_{\mathcal{H}_n}(s,u,+) \in \mathcal{Z}(\Lambda), \quad \text{for each } v \in \Lambda. \tag{3.6}$$

Once more, using (3.5), we get

$$\mathcal{H}_n^{k-2}(v)\theta_{\mathcal{H}_n}(s,u,+) \in \mathcal{Z}(\Lambda), \quad \text{for each } v \in \Lambda. \tag{3.7}$$

Hence, we may repeat the preceding procedure to achieve

$$\mathcal{H}_n(v)\theta_{\mathcal{H}_n}(s,u,+) \in \mathcal{Z}(\Lambda), \quad \text{for each } v \in \Lambda. \tag{3.8}$$

Using (3.4) and (3.8), we get $v\theta_{\mathcal{H}_n}(s,u,+) \in \mathcal{Z}(\Lambda)$, for all $v \in \Lambda$. Thus, $v\theta_{\mathcal{H}_n}(s,u,+) \in \mathcal{Z}(\Lambda)$, for all $v \in \Lambda$. Therefore, $\Lambda\theta_{\mathcal{H}_n}(s,u,+) \subseteq \mathcal{Z}(\Lambda)$. Thus, $\Lambda\theta_{\mathcal{H}_n}(s,u,+) = \{0\}$. If $Ann(\Lambda)$ is the 2-sided annihilator of Λ, then $\theta_{\mathcal{H}_n}(s,u,+) \in Ann(\Lambda)$. However, $Ann(\Lambda)$ is an ideal on Λ contained in $\mathcal{Z}(\Lambda)$, so $\theta_{\mathcal{H}_n}(s,u,+) = 0$. Therefore, using (3.1), $\mathcal{H}_n(s+u) = \mathcal{H}_n(s) + \mathcal{H}_n(u)$.

(*ii*) If ϕ_n is an epimorphism:

Rewriting (3.4) in the form

$$\phi_n(v)\theta_{\mathcal{H}_n}(s,u,+) \in \mathcal{Z}(\Lambda),$$

i.e., $\phi_n(v)\alpha = \beta \in \mathcal{Z}(\Lambda)$, where $\alpha = \theta_{\mathcal{H}_n}(s,u,+) \in \mathcal{Z}(\Lambda)$, and $\beta \in \mathcal{Z}(\Lambda)$. Since ϕ_n is an epimorphism, we get $\Lambda\alpha$ is an ideal contained in $\mathcal{Z}(\Lambda)$ and therefore $\Lambda\alpha = \{0\}$. If $\mathcal{K}(\Lambda)$ is the two-sided annihilator of Λ, then, we have $\alpha \in \mathcal{K}(\Lambda)$. But $\mathcal{K}(\Lambda)$ is an ideal contained in $\mathcal{Z}(\Lambda)$, so $\alpha = 0$ and using (3.1), $\mathcal{H}_n(s+u) = \mathcal{H}_n(s) + \mathcal{H}_n(u)$ □

Applying the previous theorem, when $n = 0$, we obtain the following special case

Corollary 3.1. *Assume Λ is a ring. If Λ containing no non-zero ideals in the center, then every nilpotent CE-derivation $\mathcal{D}$ is additive.*

Also, when $n = 1$, we get the case of ordinary CE-homoderivation as a special case.

Corollary 3.2. *Let Λ be any ring containing no non-zero ideals in the center. Then, every nilpotent CE-homoderivation $\mathcal{H}$ on Λ and every CE-homoderivation $\mathcal{H}$ on Λ related to an epimorphism $\phi_1(t) = t + \mathcal{H}_1(t)$ is additive.*

Theorem 3.2. *If the semiprime ring Λ has no non-zero ideals in its center, then each CE-n-homoderivation $\mathcal{H}_n$ on Λ related to an epimorphism ϕ_n is an n-homoderivation.*

Proof. Let $s, u, t \in \Lambda$ be arbitrary elements. Then,

$$\mathcal{H}_n((su)t) - n\mathcal{H}_n(su)\mathcal{H}_n(t) - su\mathcal{H}_n(t) - \mathcal{H}_n(su)t \in \mathcal{Z}(\Lambda) \text{ and}$$
$$\mathcal{H}_n(s(ut)) - n\mathcal{H}_n(s)\mathcal{H}_n(ut) - s\mathcal{H}_n(ut) - \mathcal{H}_n(s)ut \in \mathcal{Z}(\Lambda). \tag{3.9}$$

Subtracting, we get

$$-\mathcal{H}_n(su)\phi_n(t) - su\mathcal{H}_n(t) + \phi_n(s)\mathcal{H}_n(ut) + \mathcal{H}_n(s)ut \in \mathcal{Z}(\Lambda). \tag{3.10}$$

Let

$$\mathcal{H}_n(su) = \phi_n(s)\mathcal{H}_n(u) + \mathcal{H}_n(s)u + \theta_{\mathcal{H}_n}(s, u, \cdot), \quad \theta_{\mathcal{H}_n}(s, u, \cdot) \in \mathcal{Z}(\Lambda) \text{ and}$$
$$\mathcal{H}_n(ut) = \mathcal{H}_n(u)\phi_n(t) + u\mathcal{H}_n(t) + \theta_{\mathcal{H}_n}(u, t, \cdot), \quad \theta_{\mathcal{H}_n}(u, t, \cdot) \in \mathcal{Z}(\Lambda). \tag{3.11}$$

Using (3.11) in (3.10), we obtain

$$-\{\phi_n(s)\mathcal{H}_n(u) + \mathcal{H}_n(s)u + \theta_{\mathcal{H}_n}(s, u, \cdot)\}\phi_n(t) - su\mathcal{H}_n(t)$$
$$+\phi_n(s)\{\mathcal{H}_n(u)\phi_n(t) + u\mathcal{H}_n(t) + \theta_{\mathcal{H}_n}(u, t, \cdot)\} + \mathcal{H}_n(s)ut \in \mathcal{Z}(\Lambda),$$

which can simplify to

$$-\theta_{\mathcal{H}_n}(s, u, \cdot)\phi_n(t) + \phi_n(s)\theta_{\mathcal{H}_n}(u, t, \cdot) \in \mathcal{Z}(\Lambda). \tag{3.12}$$

This gives

$$[\phi_n(s)\theta_{\mathcal{H}_n}(u, t, \cdot), \phi_n(t)] = [\phi_n(s), \phi_n(t)]\theta_{\mathcal{H}_n}(u, t, \cdot) = 0.$$

Since ϕ_n is an epimorphism, we have

$$[s, \phi_n(t)]\theta_{\mathcal{H}_n}(u, t, \cdot) = 0, \quad \text{for all } s, t, u \in \Lambda. \tag{3.13}$$

Replacing s by sr, $r \in \Lambda$, and using (3.13) and (3.11) we have

$$[s, \phi_n(t)]r\{\mathcal{H}_n(ut) - n\mathcal{H}_n(u)\mathcal{H}_n(t) - u\mathcal{H}_n(t) - \mathcal{H}_n(u)t\} = 0, \text{ for all } r, s, t, u \in \Lambda. \tag{3.14}$$

Thus,

$$[s, \phi_n(t)]\Lambda\{\mathcal{H}_n(ut) - n\mathcal{H}_n(u)\mathcal{H}_n(t) - u\mathcal{H}_n(t) - \mathcal{H}_n(u)t\} = \{0\}. \tag{3.15}$$

Presume that the ring Λ has a collection of prime ideals $\{\mathcal{K}_\lambda \mid \lambda \in \Omega\}$ such that $\bigcap \mathcal{K}_\lambda = \{0\}$, and let $\mathcal{K}$ denote a typical $\mathcal{K}_\lambda$. Let $\overline{\Lambda} = \Lambda/\mathcal{K}$ and $\overline{\mathcal{Z}(\Lambda)}$ the center of $\overline{\Lambda}$, and let $\overline{r} = r + \mathcal{K}$ be a typical element of $\overline{\Lambda}$. Fix u and t above, and let s vary. Then $\theta_{\mathcal{H}_n}(u, t, \cdot)$ is fixed but $\theta_{\mathcal{H}_n}(s, u, \cdot)$ depends on s. As seen from (3.15), either

(*i*) $[s, \phi_n(t)] \in \mathcal{K}$ for all $s \in \Lambda$,

or

(*ii*) $\theta_{\mathcal{H}_n}(u, t, \cdot) = \mathcal{H}_n(ut) - \mathcal{H}_n(u)\mathcal{H}_n(t) - u\mathcal{H}_n(t) - \mathcal{H}_n(u)t \in \mathcal{K}$, hence $\overline{\phi_n(t)} \in \overline{\mathcal{Z}(\Lambda)}$ or $\overline{\theta_{\mathcal{H}_n}(u,t,\cdot)} = \overline{0}$. It follows from (3.12) that for each $s \in \Lambda$, $-\overline{\theta_{\mathcal{H}_n}(s,u,\cdot)}\,\overline{\phi_n(t)}+\overline{\phi_n(s)}\,\overline{\theta_{\mathcal{H}_n}(u,t,\cdot)} \in \overline{\mathcal{Z}(\Lambda)}$ so that if $\overline{\phi_n(t)} \in \overline{\mathcal{Z}(\Lambda)}$, $\overline{\Lambda\theta_{\mathcal{H}_n}(u,t,\cdot)} \subseteq \overline{\mathcal{Z}(\Lambda)}$. On the other hand, if $\overline{\theta_{\mathcal{H}_n}(u,t,\cdot)} = \overline{0}$, Certainly, it is true that $\overline{\Lambda\theta_{\mathcal{H}_n}(u,t,\cdot)} \subseteq \overline{\mathcal{Z}(\Lambda)}$. Thus $[r\theta_{\mathcal{H}_n}(u,t,\cdot),u] \in \mathcal{K}$ for all $r,u \in \Lambda$; and since $\bigcap \mathcal{K}_\lambda = \{0\}$. This provides the conclusion that $\Lambda\theta_{\mathcal{H}_n}(u,t,\cdot)$ is a central ideal of Λ, therefore $\Lambda\theta_{\mathcal{H}_n}(u,t,\cdot) = \{0\}$. Thus, letting $\mathcal{K}(\Lambda)$ be the two-sided annihilator of Λ, we have $\theta_{\mathcal{H}_n}(u,t,\cdot) \in \mathcal{K}(\Lambda)$. However, $\mathcal{K}(\Lambda)$ is a central ideal, so $\theta_{\mathcal{H}_n}(u,t,\cdot) = 0$. Since $\mathcal{H}_n$ is additive by Theorem 3.1, then $\mathcal{H}_n$ is an n-homoderivation. □

Corollary 3.3. *Every CE-homoderivation $\mathcal{H}$ on Λ related to an epimorphism $\phi_1(t) = t + \mathcal{H}_1(t)$, for each $t \in \Lambda$, is also a homoderivation if the only central ideal in the semiprime ring is the zero ideal.*

Corollary 3.4. *Every CE-derivation $\mathcal{D}$ on Λ is also a homoderivation if the only central ideal in the semiprime ring is the zero ideal.*

Theorem 3.2, Examples 2.1 and 2.2 together provide the following result.

Theorem 3.3. *A semiprime ring Λ admits a CE-n-homoderivation $\mathcal{H}_n$ on Λ related to an epimorphism ϕ_n which is not an n-homoderivation if and only if the only ideal in the center of Λ is the zero ideal.*

Theorem 3.4. *If a semiprime ring Λ has no non-zero central ideals, then every nilpotent CE-n-homoderivation $\mathcal{H}_n$ on Λ must be an n-homoderivation.*

Proof. Theorem 3.1 states that $\mathcal{H}_n$ is additive. For any u, s and t in Λ. From (3.12), we have

$$\theta_{\mathcal{H}_n}(u,s,\cdot)(t + n\mathcal{H}_n(t)) - \theta_{\mathcal{H}_n}(s,t,\cdot)(u + n\mathcal{H}_n(u)) \in \mathcal{Z}(\Lambda). \tag{3.16}$$

Therefore,

$$\theta_{\mathcal{H}_n}(u,s,\cdot)[t + n\mathcal{H}_n(t), u + n\mathcal{H}_n(u)] = 0. \tag{3.17}$$

Replacing t by $\mathcal{H}_n^{k-1}(t)$ in (3.17), we have

$$\theta_{\mathcal{H}_n}(u,s,\cdot)[\mathcal{H}_n^{k-1}(t), u + n\mathcal{H}_n(u)] = 0. \tag{3.18}$$

Replacing t by $\mathcal{H}_n^{k-2}(t)$ in (3.17) and using (3.18), we have

$$\theta_{\mathcal{H}_n}(u,s,\cdot)[\mathcal{H}_n^{k-2}(t), u + n\mathcal{H}_n(u)] = 0. \tag{3.19}$$

By repeating the previous procedures, we obtain

$$\theta_{\mathcal{H}_n}(u,s,\cdot)[\mathcal{H}_n(t), u + n\mathcal{H}_n(u)] = 0. \tag{3.20}$$

From (3.17) and (3.20), we obtain

$$\theta_{\mathcal{H}_n}(u,s,\cdot)[t, u + n\mathcal{H}_n(u)] = 0. \tag{3.21}$$

Substituting tx for t in (3.21), we obtain

$$\theta_{\mathcal{H}_n}(u,s,\cdot)t[x, u + n\mathcal{H}_n(u)] = 0.$$

Therefore,

$$\theta_{\mathcal{H}_n}(u,s,\cdot)\Lambda[x, n\mathcal{H}_n(u) + u] = \{0\}.$$

Let $\mathcal{K} = \{\mathcal{K}_\lambda \mid \Omega \in \Lambda, \mathcal{K}_\lambda \text{be a prime ideal in } \Lambda\}$ and $\cap\mathcal{K}_\lambda = \{0\}$. Suppose that $\mathcal{K}$ represents a standard $\mathcal{K}_\lambda$ in $\mathcal{K}$. For each $u \in \Lambda$, we have either $\theta_{\mathcal{H}_n}(u, s, .) \in \mathcal{K}$, for all $s \in \Lambda$ or $[x, n\mathcal{H}_n(u) + u] \in \mathcal{K}$, for all $x \in \Lambda$. First, if $\theta_{\mathcal{H}_n}(u, s, .) \in \mathcal{K}$, for all $s \in \Lambda$, then $\mathcal{K} + \theta_{\mathcal{H}_n}(u, s, .) = \mathcal{K}$, for all $s \in \Lambda$. Thus, $\mathcal{K} + \Lambda\theta_{\mathcal{H}_n}(u, s, .) = \mathcal{K}$, for all $s \in \Lambda$. So, $(\mathcal{K} + \Lambda\theta_{\mathcal{H}_n}(u, s, .))(\mathcal{K} + r) = (\mathcal{K} + r)(\mathcal{K} + \Lambda\theta_{\mathcal{H}_n}(u, s, .))$, for all $s, r \in \Lambda$. Therefore, $\mathcal{K} + [\Lambda\theta_{\mathcal{H}_n}(u, s, .), r] = \mathcal{K}$, for all $s, r \in \Lambda$. Thus, $[\Lambda\theta_{\mathcal{H}_n}(u, s, .), r] \in \cap\mathcal{K}_\lambda = \{0\}$, for all $s, r \in \Lambda$. That is $\Lambda\theta_{\mathcal{H}_n}(u, s, .) \subseteq \mathcal{Z}(\Lambda)$, for all $s \in \Lambda$. So, $\theta_{\mathcal{H}_n}(u, s, .) = 0$ for all $s \in \Lambda$. In the other case, if $[x, n\mathcal{H}_n(u) + u] \in \mathcal{K}$, for each $x \in \Lambda$, then $[x, n\mathcal{H}_n(u) + u] + \mathcal{K} = \mathcal{K}$, for each $x \in \Lambda$. Therefore,

$$[x + \mathcal{K}, (n\mathcal{H}_n(u) + u) + \mathcal{K}] = \mathcal{K}, \quad \text{for each } x \in \Lambda. \tag{3.22}$$

From (3.16) and (3.22), we have

$$\begin{aligned} \mathcal{K} &= [\theta_{\mathcal{H}_n}(u, s, .)(n\mathcal{H}_n(t) + t) + \mathcal{K} - \theta_{\mathcal{H}_n}(s, t, .)(n\mathcal{H}_n(u) + u) + \mathcal{K}, x + \mathcal{K}] \\ &= [\theta_{\mathcal{H}_n}(u, s, .)(n\mathcal{H}_n(t) + t) + \mathcal{K}, x + \mathcal{K}] \ \text{ for each } s, t, x \in \Lambda. \end{aligned} \tag{3.23}$$

As above in Eq (3.17) we get $\mathcal{K} = [\theta_{\mathcal{H}_n}(u, s, .)t + \mathcal{K}, x + \mathcal{K}] = [\theta_{\mathcal{H}_n}(u, s, .)t, x] + \mathcal{K}$, for each $s, t, x \in \Lambda$. Thus, $[\theta_{\mathcal{H}_n}(u, s, .)t, x] \in \mathcal{K}$, for each $s, t, x \in \Lambda$. Thus, we achieve $[\theta_{\mathcal{H}_n}(u, s, .)t, x] \in \cap\mathcal{K}_\lambda = \{0\}$, for each $u, s, t, x \in \Lambda$. Again, $\theta_{\mathcal{H}_n}(u, s, .) = 0$, for all $s \in \Lambda$. Moreover, we have $\theta_{\mathcal{H}_n}(u, s, .) = 0$, for all $u, s \in \Lambda$. From (3.11), we have

$$\mathcal{H}_n(us) = \mathcal{H}_n(u)s + u\mathcal{H}_n(s) + n\mathcal{H}_n(u)\mathcal{H}_n(s).$$

Therefore, $\mathcal{H}_n$ is an n-homoderivation of Λ. □

Corollary 3.5. *Any nilpotent CE-homoderivation is also a homoderivation if the only central ideal in the semiprime ring is the zero ideal.*

Corollary 3.6. *Any nilpotent CE-derivation is also a derivation if the only central ideal in the semiprime ring is the zero ideal.*

Theorem 3.4, Examples 2.1 and 2.2 together provide the following result

Theorem 3.5. *A semiprime ring Λ admits a CE-n-homoderivation $\mathcal{H}_n$ on Λ which is not an n-homoderivation if and only if Λ contains a non-zero ideal that is a subset of its center.*

4. Center invariance with CE-n-homoderivations

A map $\mathcal{F} : \Lambda \to \Lambda$ preserves the subset $\mathcal{S} \subseteq \Lambda$ if $\mathcal{F}(S) \subseteq \mathcal{S}$. Our purpose of this section is to study preservation of $\mathcal{Z}(\Lambda)$ by CE-n-homoderivations. It is necessary to show that not all CE-n-homoderivations preserve $\mathcal{Z}(\Lambda)$. Here is an example for a CE-n-homoderivation, with $\mathcal{H}_n(\mathcal{Z}(\Lambda)) \nsubseteq \mathcal{Z}(\Lambda)$.

Example 4.1. *Let Λ_2 be a noncommutative ring satisfying $\Lambda_2^2 \subseteq \mathcal{Z}(\Lambda_2)$, for example a noncommutative ring with $\Lambda_2^3 = \{0\}$. Let Λ_1 be a zero ring with $(\Lambda_1, +) \cong (\Lambda_2, +)$. Let $f : (\Lambda_1, +) \to (\Lambda_2, +)$ be an isomorphism. Let $\Lambda = \Lambda_1 \oplus \Lambda_2$, and let $\mathcal{H}_n : \Lambda \to \Lambda$ given by $\mathcal{H}_n((x, y)) = (0, f(x))$, where $x \in \Lambda_1, y \in \Lambda_2$. It is clear that $\mathcal{Z}(\Lambda) = (\Lambda_1, \mathcal{Z}(\Lambda_2))$. Thus, $\mathcal{H}_n$ is a CE-n-homoderivation, but $\mathcal{H}_n(\mathcal{Z}(\Lambda))$ is generally not central unless $f(x)$ is zero. Moreover, $\Lambda_1 \oplus \{0\}$ is a two-sided ideal in Λ, and $\Lambda_1 \oplus \{0\} \subseteq Z(\Lambda)$, but $\mathcal{H}_n(\Lambda_1 \oplus \{0\}) \nsubseteq Z(\Lambda)$.*

A CE-n-homoderivation preserves the center under certain conditions, according to the following theorem.

Theorem 4.1. *Let Λ be a ring with center $\mathcal{Z}(\Lambda)$, and assume that zero is the only nilpotent element in $\mathcal{Z}(\Lambda)$. Then every CE-n-homoderivation $\mathcal{H}_n$ on Λ associated with an epimorphism ϕ_n, or every nilpotent CE-n-homoderivation $\mathcal{H}_n$ on Λ, preserves $\mathcal{Z}(\Lambda)$.*

Proof. (i) The first case, when $\mathcal{H}_n$ is related to an epimorphism ϕ_n.

Let $\xi \in \mathcal{Z}(\Lambda)$ and $r \in \Lambda$. Then

$$\mathcal{H}_n(\xi r) - n\mathcal{H}_n(\xi)\mathcal{H}_n(r) - \mathcal{H}_n(\xi)r - \xi\mathcal{H}_n(r) \in \mathcal{Z}(\Lambda)$$

and

$$\mathcal{H}_n(r\xi) - n\mathcal{H}_n(r)\mathcal{H}_n(\xi) - \mathcal{H}_n(r)\xi - r\mathcal{H}_n(\xi) \in \mathcal{Z}(\Lambda),$$

and by subtracting, we obtain

$$[\phi_n(r), \mathcal{H}_n(\xi)] = [r, \mathcal{H}_n(\xi)] + [n\mathcal{H}_n(r), \mathcal{H}_n(\xi)] \in \mathcal{Z}(\Lambda) \ \ for\, all\, r \in \Lambda. \tag{4.1}$$

Since ϕ_n is an epimorphism on Λ, we get

$$[r, \mathcal{H}_n(\xi)] \in \mathcal{Z}(\Lambda) \ \ for\, all\, r \in \Lambda. \tag{4.2}$$

Replacing r by $r\mathcal{H}_n(\xi)$ in (4.2) gives $[r, \mathcal{H}_n(\xi)]\mathcal{H}_n(\xi) \in \mathcal{Z}(\Lambda)$, so

$$[[r, \mathcal{H}_n(\xi)]\mathcal{H}_n(\xi), r] = 0 = [r, \mathcal{H}_n(\xi)]^2 \ \ for\, all\, r \in \Lambda. \tag{4.3}$$

Since there is no nontrivial nilpotent elements in $\mathcal{Z}(\Lambda)$, (4.2) and (4.3) give $[r, \mathcal{H}_n(\xi)] = 0$ for all $r \in \Lambda$, i.e., $\mathcal{H}_n(\xi) \in \mathcal{Z}(\Lambda)$.

(ii) Now, we are in a position to prove the second case when $\mathcal{H}_n$ is nilpotent.

From (4.1), we have

$$[n\mathcal{H}_n(r) + r, \mathcal{H}_n(\xi)] \in \mathcal{Z}(\Lambda), \ \text{for all } r \in \Lambda. \tag{4.4}$$

Putting $\mathcal{H}_n^{k-1}(r)$ instead of r in (4.4), we get

$$[\mathcal{H}_n^{k-1}(r), \mathcal{H}_n(\xi)] \in \mathcal{Z}(\Lambda), \ \text{for all } r \in \Lambda. \tag{4.5}$$

Once more, substituting $\mathcal{H}_n^{k-2}(r)$ for r in (4.4) and using (4.5), we achieve

$$[\mathcal{H}_n^{k-2}(r), \mathcal{H}_n(\xi)] \in \mathcal{Z}(\Lambda), \ \ \text{for each } r \in \Lambda. \tag{4.6}$$

Using the same procedure as before, we get

$$[\mathcal{H}_n(r), \mathcal{H}_n(\xi)] \in \mathcal{Z}(\Lambda), \ \ \text{for each } r \in \Lambda. \tag{4.7}$$

From (4.4) and (4.7) we have

$$[r, \mathcal{H}_n(\xi)] \in \mathcal{Z}(\Lambda), \ \ \text{for each } r \in \Lambda. \tag{4.8}$$

In (4.8), replacing r with $r\mathcal{H}_n(\xi)$ gives

$$[r\mathcal{H}_n(\xi), \mathcal{H}_n(\xi)] = [r, \mathcal{H}_n(\xi)]\mathcal{H}_n(\xi) \in \mathcal{Z}(\Lambda), \quad \text{for each } r \in \Lambda. \tag{4.9}$$

Thus, we get $[[r, \mathcal{H}_n(\xi)]\mathcal{H}_n(\xi), r] = 0$, for all $r \in \Lambda$. Therefore,

$$[r, \mathcal{H}_n(\xi)]^2 = 0, \quad \text{for each } r \in \Lambda. \tag{4.10}$$

However, the nilpotent elements in the center $\mathcal{Z}(\Lambda)$ are zero, so we can deduce that $[r, \mathcal{H}_n(\xi)] = 0$, for all $r \in \Lambda$ from (4.8) and (4.10). Hence, $\mathcal{H}_n(\xi) \in \mathcal{Z}(\Lambda)$, i.e., $\mathcal{H}_n$ preserves the center. □

Naturally, the following consequence follows.

Corollary 4.1. *Let Λ be a ring with center $\mathcal{Z}(\Lambda)$ that has no non-zero nilpotent central elements. Then every CE-homoderivation $\mathcal{H}$ on Λ associated with an epimorphism ϕ_n, or every nilpotent CE-derivation $\mathcal{D}$ on Λ, preserves $\mathcal{Z}(\Lambda)$.*

CE-n-homoderivations that preserve $\mathcal{Z}(\Lambda)$ may also preserve subsets of $\mathcal{Z}(\Lambda)$, namely the set $K(\Lambda) = \{\xi \in \mathcal{Z}(\Lambda) \mid \xi\Lambda \subseteq \mathcal{Z}(\Lambda)\}$. It is readily seen that $K(\Lambda)$ is the maximal central ideal, a central ideal that contains all other central ideals.

Theorem 4.2. *If $\mathcal{H}_n$ is a CE-n-homoderivations on a ring Λ which preserves $\mathcal{Z}(\Lambda)$, then $\mathcal{H}_n$ preserves $K(\Lambda)$.*

Proof. Let $\xi \in K(\Lambda)$. Since $K(\Lambda) \subseteq \mathcal{Z}(\Lambda)$, $\mathcal{H}_n(\xi) \in \mathcal{Z}(\Lambda)$. For arbitrary $s \in \Lambda$,

$$\mathcal{H}_n(\xi s) - n\mathcal{H}_n(\xi)\mathcal{H}_n(s) - \xi\mathcal{H}_n(s) - \mathcal{H}_n(\xi)s \in \mathcal{Z}(\Lambda);$$

and since $\mathcal{H}_n(\xi s) \in \mathcal{Z}(\Lambda)$, $\mathcal{H}_n(\xi)\mathcal{H}_n(s) \in \mathcal{Z}(\Lambda)$, and $\xi\mathcal{H}_n(s) \in \mathcal{Z}$, and $\mathcal{H}_n(\xi)s \in \mathcal{Z}(\Lambda)$. Therefore $\mathcal{H}_n(\xi) \in K(\Lambda)$. □

Corollary 4.2. *Every CE-homoderivation $\mathcal{H}$ or every CE-derivation $\mathcal{D}$ on a ring Λ that preserves $\mathcal{Z}(\Lambda)$, then $\mathcal{H}_n$ and $\mathcal{D}$ preserve $K(\Lambda)$.*

5. CE-n-homoderivations and commutativity of prime rings

In this section, our main objective is to illustrate the requirements that ensure a prime or semiprime ring is commutative when it admits a CE-n-homoderivation.

Theorem 5.1. *If $\mathcal{H}_n$ is not an $n-$homoderivation of a prime ring Λ, then Λ is commutative.*

Proof. If Λ includes no non-zero central ideals, according to Theorem 3.4, $\mathcal{H}_n$ is an $n-$homoderivation on Λ, which is a contradiction. As a consequence, Λ has a non-zero ideal that is contained in the center $\mathcal{Z}(\Lambda)$. Thus, Λ is commutative using [14, Lemma 1(b)]. □

Theorem 5.2. *Let Λ be a prime ring and $\mathcal{H}_n$ be a CE-n-homoderivation. If $\mathcal{H}_n(0) \neq 0$, then Λ is commutative.*

Proof. Let $\mathcal{H}_n$ be a CE-n-homoderivation with $\mathcal{H}_n(0) \neq 0$. Since $\mathcal{H}_n(0+0) - \mathcal{H}_n(0) - \mathcal{H}_n(0) \in \mathcal{Z}(\Lambda)$, we have $\mathcal{H}_n(0) \in \mathcal{Z}(\Lambda)$. Since $\mathcal{H}_n(0t) - n\mathcal{H}_n(0)\mathcal{H}_n(t) - \mathcal{H}_n(0)t - 0\mathcal{H}_n(t) \in \mathcal{Z}(\Lambda)$, we now get $\mathcal{H}_n(0)\phi_n(t) \in \mathcal{Z}(\Lambda)$ for all $t \in \Lambda$. But $\phi_n(t)$ is epimorphism of Λ, then we get $\mathcal{H}_n(0)t \in \mathcal{Z}(\Lambda)$ for all $t \in \Lambda$. Therefore, $[\mathcal{H}_n(0)t, v] = 0$, for all $t, v \in \Lambda$. Since $\mathcal{H}_n(0) \in \mathcal{Z}(\Lambda)$, we get $\mathcal{H}_n(0)[t, v] = 0$, for all $v, t \in \Lambda$. Replacing t by wt, we arrive at $\mathcal{H}_n(0)w[t, v] = 0$, for each $v, t, w \in \Lambda$. So, $\mathcal{H}_n(0)\Lambda[t, v] = 0$, for all $v, t \in \Lambda$. Using the primeness of Λ and $\mathcal{H}_n(0) \neq 0$, $[t, v] = 0$, for all $v, t \in \Lambda$, i.e., Λ is commutative. □

Theorem 5.3. *Let Λ be a prime ring endowed with either a non-zero nilpotent CE-n-homoderivation $\mathcal{H}_n$, or a non-zero CE-n-homoderivation $\mathcal{H}_n$ associated with an epimorphism ϕ_n. If $\mathcal{H}_n([u, s]) = 0$ or $\mathcal{H}_n(u \circ s) = 0$, for each $u, s \in \Lambda$, then Λ is commutative.*

Proof. If Λ has a non-zero central ideal, then by [14, Lemma 1(b)] Λ is commutative. Now, assume that the only central ideal in Λ is the zero ideal. Due to Theorem 3.1, $\mathcal{H}_n$ is additive. First, assume that $\mathcal{H}_n([u, s]) = 0$, for all $u, s \in \Lambda$. Substituting su for u, we get $\mathcal{H}_n([su, s]) = 0 = \mathcal{H}_n(s[u, s])$, for each u, s in Λ. Thus, we get

$$\mathcal{H}_n(u)[s, u] \in \mathcal{Z}(\Lambda), \text{ for all } u, s \in \Lambda. \tag{5.1}$$

In (5.1), putting su instead of s, the result is $\mathcal{H}_n(u)[s, u]u \in \mathcal{Z}(\Lambda)$, for all $u, s \in \Lambda$. Thus,

$$[t, \mathcal{H}_n(u)[s, u]u] = 0, \text{ for all } u, s, t \in \Lambda,$$

which leads to

$$\mathcal{H}_n(u)[s, u][t, u] = 0, \text{ for all } u, s, t \in \Lambda. \tag{5.2}$$

Putting tw in place t in (5.2) and using (5.2), we get

$$\mathcal{H}_n(u)[s, u]t[w, u] = 0, \text{ for all } u, s, w, t \in \Lambda. \tag{5.3}$$

Using the primeness of Λ, for each $u \in \Lambda$ either $u \in \mathcal{Z}(\Lambda)$ or $\mathcal{H}_n(u)[s, u] = 0$, for all $s \in \Lambda$. Assume that $u \in \Lambda$ with $\mathcal{H}_n(u)[s, u] = 0$ for all $s \in \Lambda$. Replacing s by st, we get $\mathcal{H}_n(u)s[t, u] = 0$, for all $t, s \in \Lambda$. Thus, for each $u \in \Lambda$ either $u \in \mathcal{Z}(\Lambda)$ or $\mathcal{H}_n(u) = 0$. Consider that

$$\mathcal{A} = \{u \in \Lambda : u \in \mathcal{Z}(\Lambda)\},$$

and

$$\mathcal{B} = \{u \in \Lambda : \mathcal{H}_n(u) = 0\}.$$

Then, $(\mathcal{A}, +)$ and $(\mathcal{B}, +)$ are additive subgroups of the group $(\Lambda, +)$, and the union of $\mathcal{A}$ and $\mathcal{B}$ gives the whole ring Λ. So either $\mathcal{A} = \Lambda$ implies Λ is commutative or $\mathcal{B} = \Lambda$ implies $\mathcal{H}_n = 0$.

Second, let $\mathcal{H}_n(u \circ s) = 0$, for all u, s in Λ. Putting su instead of u in $\mathcal{H}_n(u \circ s) = 0$, then $\mathcal{H}_n(su \circ s) = \mathcal{H}_n(s(u \circ s)) = 0$ for all $u, s \in \Lambda$. So,

$$\mathcal{H}_n(s)(u \circ s) \in \mathcal{Z}(\Lambda), \text{ for all } u, s \in \Lambda. \tag{5.4}$$

Substituting us for u in (5.4), we get

$$\mathcal{H}_n(s)(u \circ s)s \in \mathcal{Z}(\Lambda), \text{ for all } u, s \in \Lambda.$$

By [15, Lemma 4] for each $s \in \Lambda$, either $\mathcal{H}_n(s)(u \circ s) = 0$ for all $u \in \Lambda$ or $s \in \mathcal{Z}(\Lambda)$. Assume that $s \in \Lambda$ where

$$\mathcal{H}_n(s)(u \circ s) = 0 \text{ for all } u \in \Lambda. \tag{5.5}$$

Putting tu instead of u in (5.5) and using (5.5), we get $\mathcal{H}_n(s)t[u, s] = 0$ for all $u \in \Lambda$. By the primeness of Λ, either $\mathcal{H}_n(s) = 0$ or $s \in \mathcal{Z}(\Lambda)$. Therefore, for each $s \in \Lambda$, there are two cases: Either $\mathcal{H}_n(s) = 0$ or $s \in \mathcal{Z}(\Lambda)$. Thus, $\mathcal{H}_n = 0$ or Λ is commutative. □

Theorem 5.4. *Let Λ be a semiprime ring and $\mathcal{K}$ a non-zero left ideal of Λ. If Λ admits a CE-n-homoderivation, which is non-zero on $\mathcal{K}$ and centralizing on $\mathcal{K}$, then Λ contains a non-zero central ideal.*

Proof. By Theorem 3.3, Λ has a non-zero central ideal or $\mathcal{H}_n$ is an n-homoderivation; and if $\mathcal{H}_n$ is an n-homoderivation, our theorem reduces to Tammam et al (2022), Theorem 2, which was an extension to Bell and Martindale [16] (1987), Theorem 3. □

As a demonstration of our findings, we achieve the subsequent result:

Corollary 5.1. *A prime ring Λ with either a nilpotent CE-homoderivation $\mathcal{H}$ or a nilpotent CE-derivation $\mathcal{D}$ is commutative if any of the following conditions hold.*

(1) *$\mathcal{H}_n$ is not a homoderivation.*
(2) *$\mathcal{H}_n(0)$ is not zero.*
(3) *$\mathcal{H}_n([u, t]) = 0$ (or $\mathcal{H}_n(u \circ t) = 0$) for each $u, t \in \Lambda$.*

It is essential that a semiprime ring Λ be commutative if it admits a derivation $\mathcal{D}$ such that $[s, t] = [\mathcal{D}(t), \mathcal{D}(s)]$, for all $s, t \in \Lambda$. we conclude with a commutativity theorem with hypotheses using CE-n-homoderivations. (For further details, see [17], Theorem 3.3; [18], Corollary 1.3.)

Theorem 5.5. *Let Λ be a semiprime ring and $\mathcal{H}_n$ a CE-n-homoderivation on Λ such that $[u, t] = [\mathcal{H}_n(t), \mathcal{H}_n(u)]$ for all $u, t \in \Lambda$. If $\mathcal{H}_n$ is centralizing CE-n-homoderivation on Λ related with an epimorphism ϕ_n or $\mathcal{H}_n$ is nilpotent, then Λ is commutative.*

Proof. (i) If $\mathcal{H}_n$ is centralizing, then by Lemma 1.1, $\mathcal{H}_n$ is commuting. Thus, we have

$$[\mathcal{H}_n(t), t] = 0 \ for\, all\, t \in \Lambda. \tag{5.6}$$

Now, our assumption assert that

$$[u, t] = [\mathcal{H}_n(t), \mathcal{H}_n(u)] \ for\, all\, u, t \in \Lambda. \tag{5.7}$$

Replacing u by tu in (5.7) and using (5.6) and (5.7), we obtain

$$\mathcal{H}_n(t)[\phi_n(u), \mathcal{H}_n(t)] = 0 \ for\, all\, u, t \in \Lambda.$$

Since ϕ_n is surjective, we obtain

$$\mathcal{H}_n(t)[u, \mathcal{H}_n(t)] = 0 \ for\, all\, u, t \in \Lambda. \tag{5.8}$$

We now replace u by uw in (5.8), thereby obtaining

$$\mathcal{H}_n(t)u[w,\mathcal{H}_n(t)],=0 \; for\, all\, u,t,w\in\Lambda.$$

i.e.,

$$[\mathcal{H}_n(t),w]\Lambda[\mathcal{H}_n(t),w]=\{0\};$$

and Λ is semiprime, gives

$$[w,\mathcal{H}_n(t)]=0 \; for\, all\, w,t\in\Lambda. \tag{5.9}$$

Hence $\mathcal{H}_n(\Lambda)\subseteq\mathcal{Z}(\Lambda)$ and therefore Λ is commutative by (5.7).

(*ii*) The second case, if $\mathcal{H}_n$ is nilpotent:
Replacing u by tu in (5.7) and using (5.7), we obtain

$$[\mathcal{H}_n(t),u+n\mathcal{H}_n(u)]\mathcal{H}_n(t)+\mathcal{H}_n(u)[\mathcal{H}_n(t),t]=0 \; for\, all\, u,t\in\Lambda. \tag{5.10}$$

In (5.10), replacing u by $\mathcal{H}_n^{k-1}(u)$, we obtain

$$[\mathcal{H}_n(t),\mathcal{H}_n^{k-1}(u)]\mathcal{H}_n(t)=0 \; for\, all\, u,t\in\Lambda, \tag{5.11}$$

using (5.7), gives

$$[u,\mathcal{H}_n^{k-2}(t)]\mathcal{H}_n(t)=0 \; for\, all\, u,t\in\Lambda, \tag{5.12}$$

replacing u by uw, gives

$$[u,\mathcal{H}_n^{k-2}(t)]w\mathcal{H}_n(t)=0 \; for\, all\, u,t\in\Lambda, \tag{5.13}$$

replacing w by $w\mathcal{H}_n^{k-1}(u)$, gives

$$[u,\mathcal{H}_n^{k-2}(t)]w\mathcal{H}_n^{k-1}(u)\mathcal{H}_n(t)=0 \; for\, all\, u,t\in\Lambda. \tag{5.14}$$

Commuting (5.13) with $\mathcal{H}_n^{k-1}(u)$, we get

$$[[u,\mathcal{H}_n^{k-2}(t)]w\mathcal{H}_n(t),\mathcal{H}_n^{k-1}(u)]=0, \tag{5.15}$$

which gives

$$[u,\mathcal{H}_n^{k-2}(t)]w[\mathcal{H}_n(t),\mathcal{H}_n^{k-1}(u)]+[[u,\mathcal{H}_n^{k-2}(t)]w,\mathcal{H}_n^{k-1}(u)]\mathcal{H}_n(t)=0. \tag{5.16}$$

using (5.13) and (5.14) in (5.16), we get

$$[u,\mathcal{H}_n^{k-2}(t)]w[\mathcal{H}_n(t),\mathcal{H}_n^{k-1}(u)]=0. \tag{5.17}$$

using (5.7) in (5.17), we obtain

$$[u,\mathcal{H}_n^{k-2}(t)]w[u,\mathcal{H}_n^{k-2}(t)]=0. \tag{5.18}$$

By semi-primness of Λ, we obtain

$$[u,\mathcal{H}_n^{k-2}(t)]=0. \tag{5.19}$$

Thus, $\mathcal{H}_n^{k-2}(t)\in\mathcal{Z}(\Lambda)$. Now, in (5.7), replacing t by $\mathcal{H}_n^{k-3}(t)$, we get $\mathcal{H}_n^{k-3}(t)\in\mathcal{Z}(\Lambda)$. We repeat this until we get $[u,t]=0$, which gives the commutativity of Λ. □

We conclude the article by presenting the following open question: Can the results derived in this manuscript be extended to a more general framework, such as non-associative structures, specifically alternative rings and algebras? For recent publications in this area, refer to [19–21].

6. Conclusions

The commutativity of a ring Λ with a special class of mappings known as centrally extended n-homoderivations, where n is an integer, is investigated in this article. The ideas of derivations and homoderivations are expanded upon by these maps. We also looked into certain characteristics of the center of these rings.

Author contributions

M. S. Tammam: Conceptualization, methodology, validation, formal analysis, investigation, data curation, writing-original draft preparation, writing-review and editing, supervision; M. Almulhem: Validation, formal analysis, writing-review and editing, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. I. N. Herstein, *Topics in ring theory*, Chicago: University of Chicago Press, 1969.
2. H. E. Bell, M. N. Daif, On centrally-extended maps on rings, *Beitr. Algebra Geom.*, **57** (2016), 129–136. https://doi.org/10.1007/s13366-015-0244-8
3. M. S. Tammam El-Sayiad, N. M. Muthana, Z. S. Alkhamisi, On right generalized (α,β)-derivations in prime rings, *East-West J. Math.*, **18** (2016), 47–51.
4. M. M. El-Soufi, *Rings with some kinds of mappings*, Cairo University, 2000.
5. A. Melaibari, N. Muthana, A. Al-Kenani, Homoderivations on rings, *G. Math. Notes*, **35** (2016), 1–8.
6. E. F. Alharfie, N. M. Mthana, The commutativity of prime rings with homoderivations, *Int. J. Adv. Appl. Sci.*, **5** (2018), 79–81. https://doi.org/10.21833/ijaas.2018.05.010
7. E. F. Alharfie, N. M. Mthana, On homoderivations and commutativity of rings, *Bull. Int. Math. Virtual. Inst.*, **9** (2019), 301–304.
8. N. Rehman, M. R. Mozumder, A. Abbasi, Homoderivations on ideals of prime and semiprime rings, *Aligarh Bull. Math.*, **38** (2019), 77–87.
9. M. S. T. El-Sayiad, M. Almulhem, On centrally extended mappings that are centrally extended additive, *AIMS Math.*, **9** (2024), 33254–33262. https://doi.org/10.3934/math.20241586

10. M. M. El-Soufi, A. Ghareeb, Centrally-extended α-homoderivations on prime and semiprime rings, *J. Math.*, **2022** (2022), 5. https://doi.org/10.1155/2022/2584177

11. A. Boua, E. Koç Sögütcü, Semiprime rings with generalized homoderivations, *Bol. da Soc. Paran. Matematica* **41** (2023), 8. https://doi.org/10.5269/bspm.62479

12. M. S. T. El-Sayiad, A. Ageeb, A. Ghareeb, Centralizing n-homoderivations of semiprime rings, *J. Math.*, **2022** (2022), 8. https://doi.org/10.1155/2022/1112183

13. L. O. Chung, Nil derivations, *J. Algebra*, **95** (1985), 20–30. https://doi.org/10.1016/0021-8693(85)90089-4

14. H. E. Bell, M. N. Daif, On commutativity and strong commutativity preserving maps, *Canad. Math. Bull.*, **37** (1994), 443–447. https://doi.org/10.4153/cmb-1994-064-x

15. J. H. Mayne, Centralizing mappings of prime rings, *Canad. Math. Bull.*, **26** (1984), 122–126. https://doi.org/10.4153/cmb-1984-018-2

16. H. E. Bell, W. S. Martindale, Centralizing mappings of semiprime rings, *Can. Math. Bull.*, **30** (1987), 92–101. https://doi.org/10.4153/CMB-1987-014-x

17. S. Ali, S. Huang, On derivations in semiprime rings, *Algebras Rep. Theory*, **15** (2012), 1023–1033. https://doi.org/10.1007/s10468-011-9271-9

18. C. K. Liu, On skew derivations in semiprime rings, *Algebras Rep. Theory*, **16** (2013), 1561–1576. https://doi.org/10.1007/s10468-012-9370-2

19. B. L. M. Ferreira, H. Guzzo, R. N. Ferreira, An approach between the multiplicative and additive structure of a Jordan ring, *Bull. Iran. Math. Soc.,* **47** (2021), 961–975. https://doi.org/10.1007/s41980-020-00423-4

20. J. C. M. Ferreira, B. L. M. Ferreira, Additivity of n-multiplicative maps on alternative rings, *Comm. Algebra*, **44** (2016), 1557–1568. https://doi.org/10.1080/00927872.2015.1027364

21. B. L. M. Ferreira, H. Julius, D. Smigly, Commuting maps and identities with inverse on alternative division rings, *J. Algebra*, **638** (2024), 488–505. https://doi.org/10.1016/j.jalgebra.2023.09.022

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