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Research article

Generalized $*$-Ricci soliton on Kenmotsu manifolds

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Abstract: In the present paper, we examine generalized $*$-Ricci solitons on Kenmotsu manifolds. To illustrate our findings, we present an example of a five-dimensional Kenmotsu manifold that admits the generalized $*$-Ricci soliton.

Keywords: generalized $*$-Ricci solitons; Einstein manifolds; Kenmotsu manifolds
Mathematics Subject Classification: 53C15, 53C25, 53C44, 53D10

1. Introduction

In [16], Kenmotsu introduced Kenmotsu manifolds (or KMs for short), which are a family of almost contact manifolds and intricately linked to warped product manifolds. Recently, the Einstein metrics have been generalized, and the Ricci soliton (or RS) is one of them, which was introduced by Hamilton [12].

In 2017, Cattino et al. [4] defined the Einstein-type manifold, also known as the generalized Ricci soliton (GRS), as an extension of Einstein manifolds. Due to the interesting and important subject of studying the GRSs in geometry and physics, many researchers have researched this topic. These solitons pertain to geometric flows and illustrate characteristics of particular manifolds. For more details, see [8, 14, 17, 22]. In [3], Calvaruso investigated the GRS equation both in Riemannian and Lorentzian settings on some Lie groups. In [1], the second author classified GRSs on Lie groups of three dimensions related to some connections. Azami examined the Kobayashi-Nomizu connections and canonical connections within Lie groups of three dimensions, successfully identifying all the GRSs associated with these structures. Recently, many authors have studied generalized η-RSs, almost RSs, Ricci-Yamabe solitons, $*$-Ricci-Yamabe solitons, and their properties on KMs [18–20]. Sharma [28] studied the RS on contact manifolds. Then Dey [9] investigated $*$-η-Ricci-Yamabe solitons on contact geometry. Yoldas [36] examined η-Ricci-Yamabe solitons on KMs. Chen [5] recently demonstrated the existence of a real hypersurface within a non-flat complex space form that fulfills the criteria of a $*$-RS. Moreover, Wang [32] proved that if a three-dimensional KM M satisfies a $*$-RS, then the manifold M becomes locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$. Additional studies on $*$-Ricci solitons and generalized Ricci solitons are available in [10, 13, 23–25].

The $*$-Ricci tensor, as referenced in [11, 30], is defined by the equation

$$S^*(Z_1, Z_2) = \frac{1}{2}\left(\text{trace}\,\{R(Z_1, \phi Z_2) \circ \phi\}\right) = \frac{1}{2}\text{trace}\,\{Z_3 \to R(Z_1, \phi Z_2)\phi Z_3\},$$

for all vector fields Z_1, Z_2, and Z_3 on the manifold M where R is the Riemannian curvature and ϕ represents a $(1,1)$-tensor field. A manifold (M, g) is classified as $*$-η-Einstein if functions a and b exist that satisfy the equation

$$S^* = ag + b\eta \otimes \eta.$$

Additionally, the manifold M is designated as a $*$-Einstein manifold when $b = 0$.

In the following discussion, we present the idea of the generalized $*$-Ricci soliton (or $*$-GRS), highlighting its similarities to the well-known concept of GRSs.

Definition 1.1. *A pseudo-Riemannian manifold (M^n, g) characterized by a $*$-Ricci tensor S^* and a $*$-scalar curvature defined as $r^* = \mathrm{Tr}(S^*)$ is referred to as a $*$-GRS whenever there are a vector field V, a smooth function λ, and constants α, β, μ, ρ such that the following equation holds:*

$$\alpha S^* + \frac{\beta}{2}\mathcal{L}_V g + \mu V^\flat \otimes V^\flat = (\rho r^* + \lambda)g, \tag{1.1}$$

where $\mathcal{L}_V$ denotes the Lie derivative indirection V, and $V^\flat(U) = g(V, U)$. The constants (α, β, μ) cannot all be zero simultaneously. A $*$-GRS is called expanding, steady, or shrinking if λ, is negative, zero, or positive, respectively.

The generalized $*$-Ricci soliton is a generalization of

(1) the $*$-$V^\flat$-Einstein equation (if $\alpha \neq 0$ and $\beta = 0$),
(2) the $*$-RS [15, 31] (if $\alpha = \beta = 1$ and $\rho = \mu = 0$),
(3) the $*$-Ricci-Yamabe soliton [9] (if $\beta \neq 0$ and $\mu = 0$).

Motivated by [1, 3, 21, 29] and the works presented above, we study $*$-GRSs on KMs. We present an example of $*$-GRS on a five-dimensional KM.

The structure of the article is organized in a specific manner. Section 2 introduces key concepts and formulas related to KMs, which are referenced in subsequent sections of the paper. In the final section, we outline the primary results, accompanied by their proofs, and provide an illustrative example.

2. Preliminaries

Consider a Riemannian manifold (M, g) of $(2n + 1)$ dimensions. This manifold is referred to as an almost contact metric manifold characterized by the structure (ϕ, ξ, η, g) if it possesses a vector field ξ on M, a $(1,1)$-tensor field ϕ, and a 1-form η that satisfy the following conditions:

$$\eta \circ \phi = 0, \quad \phi^2(U) = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \tag{2.1}$$

$$\eta(U_1) = g(U_1, \xi), \quad g(\phi U, \phi U_1) = g(U, U_1) - \eta(U)\eta(U_1), \quad \forall U, U_1 \in \mathcal{X}(M). \tag{2.2}$$

In addition, it is called a KM [16] whenever

$$(\nabla_{Z_1}\phi)Z_2 = -g(Z_1, \phi Z_2)\xi - \eta(Z_1)\phi Z_2, \tag{2.3}$$

$$\nabla_{Z_1}\xi = Z_1 - \eta(Z_1)\xi. \tag{2.4}$$

The symbol ∇ represents the Levi-Civita connection of g. For a KM with a Riemannian curvature tensor R the following equations are true [2, 26]:

$$R(Z_1, U_1)\xi = \eta(Z_1)U_1 - \eta(U_1)Z_1, \tag{2.5}$$

$$R(Z_1, \xi)U_1 = -\eta(U_1)Z_1 + g(Z_1, U_1)\xi, \tag{2.6}$$

$$\eta(R(Z_1, U_1)U) = g(Z_1, U)\eta(U_1) - g(U_1, U)\eta(Z_1), \tag{2.7}$$

for all vector fields Z_1, U_1, U. We also have

$$S(Z_1, \xi) = -2n\eta(Z_1), \tag{2.8}$$

$$S(\phi Z_1, \phi Z_2) = S(Z_1, Z_2) + 2n\eta(Z_1)\eta(Z_2), \tag{2.9}$$

$$(\nabla_{Z_1}\eta)Z_2 = g(Z_1, Z_2) - \eta(Z_1)\eta(Z_2), \tag{2.10}$$

for all vector fields Z_1, Z_2 where S is the Ricci tensor of g. By the definition of a Lie derivative, it follows that

$$(\mathcal{L}_\xi g)(Z_1, Z_2) = g(\nabla_{Z_1}\xi, Z_2) + g(Z_1, \nabla_{Z_2}\xi). \tag{2.11}$$

Applying (2.4) to (2.11), we infer

$$(\mathcal{L}_\xi g) = 2\left[g - \eta \otimes \eta\right]. \tag{2.12}$$

In the following, we recall the formula that expresses the $*$-Ricci tensor in terms of the Ricci tensor, and we need it to prove our results. In [31], Venkatesha et al. proved the following proposition by the Bianchi identity.

Proposition 2.1. *On a KM of* $2n + 1$ *dimensions the* $*$*-Ricci tensor is determined as follows:*

$$S^*(Z_1, U_1) = S(Z_1, U_1) + (2n - 1)g(Z_1, U_1) + \eta(Z_1)\eta(U_1). \tag{2.13}$$

Let $\{e_i\}_{i=1}^{2n+1}$ be a local orthonormal frame. By considering $Z_1 = e_i$ and $U_1 = e_i$ in (2.13) and summing over i from 1 to $2n + 1$, we deduce

$$r^* = r + 4n^2. \tag{2.14}$$

Notice that the generalized $*$-Ricci soliton is a generalized form of the $* - \eta$-Ricci soliton when $V = \xi$, $\rho = 0$, $\alpha = \frac{1}{2}$, and $\beta = 1$.

3. Main results and their proofs

In this section, we present our main results, along with their proofs.

Theorem 3.1. *Let a KM of* $2n+1$ *dimensions be a* $*$*-GRS* $(g, \xi, \alpha, \beta, \mu, \rho, \lambda)$*, where* ξ *represents the Reeb vector field. The* $*$*-GRS becomes shrinking, steady, or expanding if* $\mu - \rho(r + 4n^2)$ *is positive, zero, or negative, respectively.*

Proof. Suppose M is a KM of $2n+1$ dimensions. We plug $V = \xi$ into the identity (1.1) on M to achieve

$$\alpha S^*(Z_1, U_1) + \frac{\beta}{2}\mathcal{L}_\xi g(Z_1, U_1) + \mu\xi^\flat(Z_1)\xi^\flat(U_1) = (\rho r^* + \lambda)g(Z_1, U_1), \tag{3.1}$$

for all vector fields Z_1, U_1. If we use the equations $\xi^\flat(Z_1)\xi^\flat(U_1) = \eta(Z_1)\eta(U_1)$, (2.12), and (2.13), Eq (3.1) becomes

$$\alpha S(Z_1, U_1) + [(2n-1)\alpha + \beta - \rho r^* - \lambda]\, g(Z_1, U_1) + [\alpha - \beta + \mu]\, \eta(Z_1)\eta(U_1) = 0. \tag{3.2}$$

Now, we consider $U_1 = \xi$ in Eq (3.2) and use (2.1) and (2.8), thus arriving at

$$[\mu - \rho r^* - \lambda]\, \eta(Z_1) = 0. \tag{3.3}$$

Since Z_1 is arbitrary, we conclude that $\lambda = \mu - \rho r^*$. Using (2.14), we obtain

$$\lambda = \mu - \rho(r + 4n^2). \tag{3.4}$$

This completes the proof of the theorem.

Now, from Theorem 3.1, we get the following corollary.

Corollary 3.1. *If a KM of* $2n+1$ *dimensions is a* $*$*-GRS* $(g, \xi, \alpha, \beta, \mu, \rho, \lambda)$ *with* $\alpha \neq 0$ *and a Reeb vector field* ξ*, then KM is* $*$*-*η*-Einstein.*

Proof. If $\alpha \neq 0$, then from (3.1), we obtain

$$S^*(Z_1, U_1) = \frac{1}{\alpha}(\rho r^* + \lambda - \beta)g(Z_1, U_1) + \frac{1}{\alpha}(\beta - \mu)\eta(Z_1)\eta(U_1).$$

This proves that KM is $*$-η-Einstein.

Theorem 3.2. *If a KM of* $2n+1$ *dimensions possesses a* $*$*-GRS* $(g, \nabla f, \alpha, \beta, \mu, \rho, \lambda)$ *such that* $\beta \neq 0$ *then*

$$\Delta f + \frac{\mu}{\beta}|\nabla f|^2 = -\frac{\alpha}{\beta}(r + 4n^2) + \frac{1}{\beta}(\rho(r + 4n^2) + \lambda)(2n + 1). \tag{3.5}$$

Proof. If we take the trace from the sides of Eq (1.1), it follows that

$$\alpha r^* + \beta \mathrm{div} V + \mu|V|^2 = (\rho r^* + \lambda)(2n + 1). \tag{3.6}$$

If $V = \nabla f$, then we deduce that $\mathrm{div} V = \Delta f$. From (3.6), we can derive (3.5).

Since $\Delta(e^{\frac{\mu}{\beta}f}) = \frac{\mu}{\beta}(\Delta f + \frac{\mu}{\beta}|\nabla f|^2)e^{\frac{\mu}{\beta}f}$, we can rewrite (3.5) as follows:

$$\Delta(e^{\frac{\mu}{\beta}f}) = \frac{\mu}{\beta}\left\{-\frac{\alpha}{\beta}(r + 4n^2) + \frac{1}{\beta}(\rho(r + 4n^2) + \lambda)(2n + 1)\right\} e^{\frac{\mu}{\beta}f}. \tag{3.7}$$

Now consider a closed KM of $2n + 1$ dimensions that admits a $*$-GRS $(g, \nabla f, \alpha, \beta, \mu, \rho, \lambda)$ such that $\beta \neq 0$ and $\mu \neq 0$. In this case, by integrating (3.7) and using $\int_M \Delta(e^{\frac{\mu}{\beta}f})dv = 0$, we deduce that

$$\int_M \left\{(2n + 1)\lambda + (\rho(2n + 1) - \alpha)r + 4n^2(2n + 1)\rho - 4n^2\alpha\right\} e^{\frac{\mu}{\beta}f} dv = 0. \tag{3.8}$$

Let $G = (2n + 1)\lambda + (\rho(2n + 1) - \alpha)r + 4n^2(2n + 1)\rho - 4n^2\alpha$. Since $e^{\frac{\mu}{\beta}f} > 0$, we can deduce the following corollary.

Corollary 3.2. *If a closed KM of* $2n+1$ *dimensions admits a* $*$*-GRS* $(g, \nabla f, \alpha, \beta, \mu, \rho, \lambda)$ *such that* $\beta \neq 0$ *and* $\mu \neq 0$*, then* $G = 0$ *or* $G < 0$ *for some points* $G \neq e^{\frac{\mu}{\beta}f}$.

Definition 3.1. *A conformal Killing vector field (or CKVF) W is defined as a vector field that satisfies the equation*

$$\mathcal{L}_W g = 2hg, \tag{3.9}$$

where h is a smooth function. The CKVF W is called: Proper if h is non-constant, homothetic if h is constant, and Killing if $h = 0$.

Theorem 3.3. *If a KM of* $2n + 1$ *dimensions possesses a* $*$*-GRS* $(g, W, \alpha, \beta, \mu, \rho, \lambda)$*, in which W is a CKVF such that* $\mathcal{L}_W g = 2hg$*, then the following equation holds:*

$$\beta h\xi + \mu\eta(W)W - \rho(r + 4n^2)\xi - \lambda\xi = 0. \tag{3.10}$$

Proof. We have $W^\flat(\xi) = g(W, \xi) = \eta(W)$ and $W^\flat(Z_1) = g(W, Z_1)$. Therefore, we deduce that

$$\mu W^\flat(Z_1)W^\flat(\xi) = g(\mu\eta(W)W, Z_1).$$

Suppose that the vector field W is a CKVF and satisfies (3.9). By (3.9), (2.13), and (1.1), we have

$$\alpha(S(Z_1, U_1) + (2n - 1)g(Z_1, U_1) + \eta(Z_1)\eta(U_1)) + \beta hg(Z_1, U_1) + \mu W^\flat(Z_1)W^\flat(U_1) = (\rho r^* + \lambda)g(Z_1, U_1). \tag{3.11}$$

By inserting $U_1 = \xi$ in (3.11) and using (2.8), we obtain

$$g(\beta h\xi + \mu\eta(W)W - \rho(r + 4n^2)\xi - \lambda\xi, Z_1) = 0. \tag{3.12}$$

Since Z_1 is arbitraray, Eq (3.12) yields Eq (3.10).

Corollary 3.3. *If a KM of* $2n + 1$ *dimensions possesses a* $*$*-GRS* $(g, W, \alpha, \beta, \mu, \rho, \lambda)$*, where W is perpendicular to* ξ *and is a CKVF such that* $\mathcal{L}_W g = 2hg$*, then the following equation holds:*

$$\lambda = \beta h - \rho(r + 4n^2).$$

Proof. If W is perpendicular to ξ, then $\eta(W) = 0$, and Eq (3.10) leads to $(\beta h - \rho(r + 4n^2) - \lambda)\xi = 0$. Since $\xi \neq 0$, we obtain $\beta h - \rho(r + 4n^2) - \lambda = 0$.

Corollary 3.4. *If a KM of* $2n+1$ *dimensions possesses a* $*$*-GRS* $(g, W, \alpha, \beta, \mu, \rho, \lambda)$*, where W is a CKVF and* $\alpha \neq 0$*, then the KM is* $*$*-*$W^{\flat}$*-Einstein.*

Proof. From (3.11), if $\alpha \neq 0$, then we get the following equation:

$$S^{*}(Z_1, U_1) = \frac{1}{\alpha}\left\{(\rho r^{*} + \lambda - \beta h)g(Z_1, U_1) - \mu W^{\flat}(Z_1)W^{\flat}(U_1)\right\},$$

which shows that the KM is $*$-$W^{\flat}$-Einstein.

Definition 3.2. *A torse-forming vector field W (TFVF) [34] is defined as a vector field that satisfies the equation*

$$\nabla_{Z_1} W = hZ_1 + \vartheta(Z_1)W, \tag{3.13}$$

where h is a smooth function and ϑ *is a 1-form. The TFVF becomes concircular [7,33], concurrent [27, 35], parallel, and torqued [6]* ϑ *vanishes identically,* $h = 1$*,* $h = \vartheta = 0$ $\vartheta(W) = 0$*, respectively.*

Theorem 3.4. *If a KM of* $2n+1$ *dimensions satisfies a* $*$*-GRS* $(g, W, \alpha, \beta, \mu, \rho, \lambda)$ *such that W is a TFVF and admits (3.13), then*

$$\lambda = \frac{1}{2n+1}\left[\alpha(r+1) + \beta\vartheta(W) + \mu|W|^2\right] + \alpha(2n-1) + \beta h - \rho(r + 4n^2). \tag{3.14}$$

Proof. Suppose a KM of $2n+1$ dimensions satisfies a $*$-GRS $(g, W, \alpha, \beta, \mu, \rho, \lambda)$ such that W is a TFVF and admits (3.13). Then from (1.1) and (2.13), we get

$$\begin{aligned}&\alpha(S(Z_1, U_1) + (2n-1)g(Z_1, U_1) + \eta(Z_1)\eta(U_1)) + \frac{\beta}{2}(\mathcal{L}_W g)(Z_1, U_1)\\&+\mu W^{\flat}(Z_1)W^{\flat}(U_1) = (\rho r^{*} + \lambda)g(Z_1, U_1).\end{aligned} \tag{3.15}$$

On the other hand, by using (3.13), we arrive at

$$(\mathcal{L}_W g)(Z_1, U_1) = g(\nabla_{Z_1} W, U_1) + g(W, \nabla_{Z_1} U_1) = 2hg(Z_1, U_1) + \vartheta(Z_1)g(W, U_1) + \vartheta(U_1)g(W, Z_1), \tag{3.16}$$

for all vector fields Z_1, U_1. Substituting (3.16) into (3.15), we infer that

$$\begin{aligned}&\alpha S(Z_1, U_1) + \left[\alpha(2n-1) + \beta h - \rho(r + 4n^2) - \lambda\right]g(Z_1, U_1) + \alpha\eta(Z_1)\eta(U_1)\\&= -\frac{\beta}{2}\left[\vartheta(Z_1)g(W, U_1) + \vartheta(U_1)g(W, Z_1)\right] - \mu g(W, Z_1)g(W, U_1).\end{aligned} \tag{3.17}$$

Taking the trace of Eq (3.17), one gets

$$\alpha r + \left[\alpha(2n-1) + \beta h - \rho(r + 4n^2) - \lambda\right](2n+1) + \alpha = -\beta\vartheta(W) - \mu|W|^2. \tag{3.18}$$

Solving Eq (3.18) with respect to λ yields Eq (3.14).

In the following, we present an example of a $*$-GRS that describes some of the our theorems.

Example 3.1. *We denote the standard coordinates of* $\mathbb{R}^5$ *as* $(y_1, y_2, y_3, y_4, y_5)$ *and assume that* $M = \mathbb{R}^5$. *We consider the vector fields*

$$u_1 = e^{-y_5}\frac{\partial}{\partial y_1}, \quad u_2 = e^{-y_5}\frac{\partial}{\partial y_2}, \quad u_3 = e^{-y_5}\frac{\partial}{\partial y_3}, \quad u_4 = e^{-y_5}\frac{\partial}{\partial y_4}, \quad u_5 = \frac{\partial}{\partial y_5},$$

which are linearly independent. The metric tensor g *on the manifold* M *is characterized by the following expression:* $g(u_i, u_i) = 1$ *if* $i \in \{1, \dots, 5\}$*; otherwise,* $g(u_i, u_j) = 0$. *The configuration* (ϕ, ξ, η) *on the manifold* M *is defined as follows:*

$$\phi = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \xi = u_5, \quad \eta(Z_1) = g(Z_1, u_5), \quad \forall Z_1 \in \mathcal{X}(M).$$

Notice that, $\eta(\xi) = 1$, $\phi^2(Z_1) = -Z_1 + \eta(Z_1)\xi$, *and* $g(\phi Z_1, \phi Y_2) = g(Z_1, Y_2) - \eta(Z_1)\eta(Y_2)$. *We also find* $[u_i, u_5] = u_i$ *for* $i = 1, 2, 3, 4$, *and the other brackets are equal to zero. Then* $\nabla_{u_i} u_i = -u_5$ *and* $\nabla_{u_i} u_5 = u_1$ *for* $i = 1, 2, 3, 4$, *and the other connections are equal to zero.*

Identities (2.3) and (2.4) hold, and thus (M, ϕ, ξ, η, g) *denotes a KM. Hence, we obtain*

$$S = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} = -4g,$$

and $r = -20$. *We also get*

$$S^* = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -14 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -g + \eta \otimes \eta,$$

and $r^* = -4$. *We get* $\mathcal{L}_\xi g = 2g - 2\eta\otimes\eta$. *Therefore,* $(g, \xi, \alpha, \beta, \mu = \beta - \alpha, \lambda = \beta - \alpha + 4\rho)$ *is a* $*$*-GRS on the KM* M. *It is also shrinking, steady, or expanding if* $\beta + 4\rho > \alpha$, $\beta + 4\rho = \alpha$, *or* $\beta + 4\rho < \alpha$, *respectively.*

4. Conclusions

In this paper, we consider $*$-GRSs on KMs. We prove if a KM of $2n + 1$ dimensions admits the $*$-GRS $(g, \xi, \alpha, \beta, \mu, \rho, \lambda)$, where ξ represents the Reeb vector field, then it is shrinking, steady, or expanding when $\mu - \rho(r + 4n^2)$ is positive, zero, or negative, respectively. Moreover, in this case, it is a $*$-η-Einstein manifold. We then establish if a KM of $2n + 1$ dimensions which admits a $*$-GRS $(g, \nabla f, \alpha, \beta, \mu, \rho, \lambda)$, then we obtain the Laplacian of f in terms of a soliton structure. We also study KMs that admit $*$-GRSs such that their potential vector fields are CKVFs or TFVFs. To illustrate our results, we give an example of a five-dimensional Kenmotsu manifold that admits the generalized $*$-Ricci soliton.

Author contributions

Yanlin Li: Methodology, writing-review and editing; Shahroud Azami: Writing-original draft, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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