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Research article

Fractional stochastic functional differential equations with non-Lipschitz condition

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Abstract: This article investigates fractional stochastic functional differential equations (FSFDEs) with a non-Lipschitz condition. The analysis explores the boundedness of solutions. Within this framework, results on the existence and uniqueness of solutions are presented. Furthermore, we derive error estimates between the Picard approximate solutions $y^n(t)$, $n \geq 1$, and the exact solution $y(t)$. Finally, it is demonstrated that the solutions exhibit mean square stability. To illustrate the applicability of the proposed theory, a detailed example is presented.

Keywords: stability; existence-uniqueness; boundedness; error estimates; fractional SFDEs
Mathematics Subject Classification: 60H35, 60H20, 60H10, 62L20

1. Introduction

Fractional stochastic functional differential equations (FSFDEs) provide a robust framework for modeling systems characterized by memory effects and long-range dependencies. These equations extend stochastic functional differential equations (SFDEs) by integrating fractional calculus, stochastic processes, and functional dependencies. They are particularly effective for modeling complex systems with memory and time-dependent behaviors. However, their study requires sophisticated mathematical tools, particularly in the realms of fractional calculus. FSFDEs have broad applications across disciplines, including finance, physics, and biology [1–3]. A thorough understanding of their mathematical foundations, such as various forms of fractional derivatives and stochastic processes, is vital for their successful implementation in real-world modeling [4]. The concept of fractional derivatives dates back to the 19th century when Joseph Liouville first introduced it [5]. Later, the Riemann-Liouville and Caputo formulations, developed in 1967, became pivotal milestones, paving the way for modern research in fractional calculus [6]. Igor Podlubny, recognized as a key figure in the field, significantly advanced fractional differential equations (FDEs) in 1974 [7]. Additionally, Oldham and Spanier (1999) contributed foundational theories to the field [8], while Bertram Øksendal (2003) enriched the domain of stochastic differential equations (SDEs), incorporating both fractional and stochastic perspectives [9]. F. Biagini, Y. Hu, B. Øksendal, and T. Zhang (2008) made significant contributions to the study of fractional Brownian motion and its applications to stochastic processes and differential equations [10]. In 2017, J. Z. Zhang and G. A. Chechkin further explored solutions to fractional stochastic differential equations (FSDEs), highlighting their connections to fractional Brownian motion [11]. Also see [12]. Chang et al. investigated the existence and uniqueness of solutions for fractional stochastic functional differential equations (FSFDEs) with Lipschitz continuous coefficients [13]. Later on, Saci et al. examined the stated theory for solutions to FSFDEs with Lipschitz continuous coefficients in the framework of G-Brownian motion [14]. The relatively exact controllability of fractional stochastic delay systems driven by Lévy noise is studied in [15], and the averaging principle of Caputo-type fractional delay stochastic differential equations with Brownian motion is investigated in [16]. Also see [17, 18]. Addressing non-Lipschitz conditions, however, presents substantial analytical challenges, particularly in establishing the existence, uniqueness, and stability of solutions. This article examines fractional stochastic functional differential equations, particularly those of the form:

$$D_t^r y(t) = \delta(t, y_t) + \lambda(t, y_t)\frac{dB(t)}{dt}, \tag{1.1}$$

with initial data $\zeta(0) \in \mathbb{R}^d$, $t \in [0, T]$, and $y_t = \{y(t+\theta), -\infty < \theta \leq 0\}$ is a $BC([-\infty, 0]; \mathbb{R}^d)$-valued stochastic process. The space $\mathbb{M}^2((-\infty, T]$ indicates the collection of the process $\{\zeta(t)\}_{t\leq 0}$ in $L^p([-\infty, 0]; \mathbb{R}^d)$ such that $\mathbb{E}\int_{-\infty}^0 |\zeta|^2 dt < \infty$ a.s. and $BC([-\infty, 0]; \mathbb{R}^d)$ represents the space of bounded continuous mappings. D_t^r is the Caputo fractional derivative of order $r \in (1, 2)$ and $B(t)$ is an m-dimensional standard Brownian motion, $\delta : [0, T] \times BC([-\infty, 0]; \mathbb{R}^d) \to \mathbb{R}^d$, and $\lambda : [0, T] \times BC([-\infty, 0]; \mathbb{R}^d) \to \mathbb{R}^{d\times m}$ for every $y \in \mathbb{R}^d$. The following is the initial data with (1.1).

$$y_0 = \zeta = \{\zeta(\theta) : -\infty < \theta \leq 0\}, \tag{1.2}$$

is $\mathcal{F}_0$-measurable, $BC((-\infty, 0]; \mathbb{R}^d)$-value random variable such that $\zeta \in \mathbb{M}^2((-\infty, T]; \mathbb{R}^d)$ and $y_0' = \zeta' = \frac{d\zeta}{d\theta} \in \mathbb{M}^2((-\infty, T]; \mathbb{R}^d)$. Its integral form is given as the following [13]:

$$y(t) = \zeta(0) + \zeta'(0)t + \frac{1}{\Gamma(r)}\int_0^t (t-u)^{r-1}\delta(u, y_u)du + \frac{1}{\Gamma(r)}\int_0^t (t-u)^{r-1}\lambda(u, y_u)dB(u), \tag{1.3}$$

where ζ' is the derivative of ζ, $r \in (1, 2)$ and $t \in [0, T]$.

Definition 1.1. [13] A solution of (1.1) given initial data (1.2) is defined as an $\mathbb{R}^d$-valued stochastic process $\{y(t)\}_{-\theta<t\leq T}$ if

a. $\{y(t)\}_{0\leq t\leq T}$ is $\mathcal{F}_t$-adopted and is continuous.
b. $f(t, y) \in L^1([0, T]; \mathbb{R}^d)$ and $h(t, y) \in L^2([0, T]; \mathbb{R}^{d\times m})$.
c. $y_{t_0} = \zeta$ and for every $t \in [0, T]$,

$$y(t) = \xi(0) + \xi'(0)t + \frac{1}{\Gamma(r)}\int_0^t (t-u)^{r-1}\delta(u, y_u)du + \frac{1}{\Gamma(r)}\int_0^t (t-u)^{r-1}\lambda(u, y_u)dB(u), \quad a.s.$$

If the following equality holds for any other solution $z(t)$ of systems (1.1) and (1.2), then the solution $y(t)$ is considered unique

$$\mathbb{P}\{y(t) = z(t), \quad -\theta < t \leq T\} = 1.$$

The structure of the remainder of the article is organized as follows: Section 2 presents some fundamental results. Section 3 discusses the boundedness of solutions. Section 4 addresses the existence and uniqueness of solutions. Section 5 provides the derivation of error estimates and stability. Finally, Section 6 includes the conclusion.

2. Notations and basic notions

This section introduces key definitions, concepts, and results that underpin the research presented in this article. The d-dimensional Euclidean space with the norm $|\cdot|$ is indicated by $\mathbb{R}^d$. The transpose of a matrix D is represented by the notation D^T, and its trace norm is $|D| = \sqrt{\text{trace}(D^T D)}$. Assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ that satisfies the standard requirements: $\mathcal{F}_0$ covers all P-null sets, and the filtration is right-continuous and increasing. Furthermore, the σ-field generated by $\{B(t) - B(t_0) : t_0 \leq t \leq T\}$ does not affect $\mathcal{F}_0$. With the norm $|\psi| = \sup_{-\infty<\theta\leq 0} |\psi(\theta)|$, let $BC((-\infty, 0]; \mathbb{R}^d)$ represent the space of all continuous and bounded $\mathbb{R}^d$-valued mappings ψ defined on $(-\infty, 0]$.

Definition 2.1. [8] Let $r > 0$ and $t > 0$. The functional integral of order r for a mapping h denoted by $I^r h(t)$ is defined as the following:

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_0^t (t-u)^{r-1} h(u) du. \tag{2.1}$$

Definition 2.2. [8] Let $t > 0$ and $0 \leq n-1 < r < n$. For a mapping h represented by $D_t^r h(t)$, the Caputo derivative of order r is defined as follows:

$$D_t^r h(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-u)^{n-r-1} h^{(n)}(u) = I^{n-r} h^{(n)}(t). \tag{2.2}$$

According to Definitions 2.1 and 2.2, problem (1.1) with initial conditions (1.2) has the following equivalent form [13]:

$$y(t) = \zeta(0) + \zeta'(0)t + \frac{1}{\Gamma(r)} \int_0^t (t-u)^{r-1} \delta(u, y_u) du + \frac{1}{\Gamma(r)} \int_0^t (t-u)^{r-1} \lambda(u, y_u) dB(u), \tag{2.3}$$

where ζ' is the derivative of ζ, $r \in (1, 2)$ and $t \in [0, T]$.

Lemma 2.3. *[19] Let $a(t)$ and $u(t)$ be locally integrable non-negative mappings on $0 \leq t \leq T$ satisfying*

$$u(t) \leq a(t) + b \int_0^t (t-u)^{r-1} u(u) du,$$

then

$$u(t) \leq a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(b\Gamma(r))^n}{\Gamma(nr)} (t-u)^{nr-1} a(u) \Big] du, \quad t \in [0, \ T],$$

where $r > 0$ and $b \geq 0$.

Lemma 2.4. *[19] Suppose that the hypotheses of Lemma 2.3 are met and $a(t)$ is a non-decreasing mapping on $0 \leq t \leq T$. Then the following result holds:*

$$u(t) \leq a(t)E_r[b\Gamma(r)t^r],$$

where E_r is the Mittang-Leffler mapping defined by $E_r(y) = \sum_{n=1}^{\infty} \frac{y^n}{\Gamma(nr+1)}$.

The following result is known as the Bihari inequality [20].

Lemma 2.5. *Let $u_0 \geq 0$ and $T \geq 0$. Suppose $\sigma(t)$ and $s(t)$ are continuous mappings on $[0\ T]$. Assume $\kappa(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, concave, non-decreasing mapping, where $\kappa(u) > 0$ for $u > 0$. If*

$$u(t) \leq s(0) + \int_0^T \sigma(u)\kappa(s(u))du, \quad t \in [0\ T],$$

then for every $t \in [0\ T]$, it follows that

$$s(t) \leq \mu^{-1}\Big(\mu(s_0) + \int_t^T \sigma(u)du\Big),$$

where $\mu(s_0) + \int_t^T \sigma(u)du \in Dom(\mu^{-1})$,

$$\mu(q) = \int_t^q \frac{1}{\kappa(u)}du, \quad q \geq 0,$$

and μ^{-1} is the inverse mapping of μ.

Lemma 2.6. *[20] Let the conditions of Lemma 2.5 be satisfied, and $\sigma(t) \geq 0$, for $t \in [0\ T]$. If for all $\epsilon > 0$, there is a $t_1 \geq 0$ such that for $0 \leq s_0 \leq \epsilon$, the following inequality holds:*

$$\int_{t_1}^T \sigma(u)du \leq \int_{u_0}^T \frac{1}{\kappa(u)}du,$$

then for each $t_1 \in [0\ T]$ the inequality

$$s(t) \leq \epsilon,$$

holds.

Refer to [20] for the definition of mean square stability of solutions to stochastic differential equations (SDEs).

Definition 2.7. Consider the solution of systems (1.1) and (1.2) to be $\phi(t)$. With initial data $\varsigma \in M^2([-\theta, 0] : \mathbb{R}^d)$, let $\psi(t)$ be another solution of problem (1.1). If for $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ satisfying

$$\mathbb{E}|\zeta - \varsigma|^2 + \mathbb{E}|(\zeta' - \varsigma')T|^2 \leq \delta(\epsilon) \quad \Rightarrow \quad \mathbb{E}|\phi(t) - \psi(t)|^2 < \epsilon,$$

for every $t \geq 0$, then $\phi(t)$ is known as a mean square stable solution of the systems (1.1) and (1.2).

3. Boundedness of solutions

The current section, examines the boundedness of solutions to the systems (1.1) and (1.2). To this end, we introduce the Picard iteration sequence and analyze its boundedness. The study is based on the following hypotheses.

(i) Let $\mu(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing concave mapping so that for $u > 0$, $\mu(u) > 0$, $\mu(0) = 0$, and

$$\int_{0+} \frac{du}{\mu(u)} = \infty. \tag{3.1}$$

Consider Eq (1.1). Let $t \in [0, T]$ and $U, V \in BC([-\theta, 0]; \mathbb{R}^d)$,

$$|\delta(t, U) - \delta(t, V)|^2 + |\lambda(t, U) - \lambda(t, V)|^2 \le \mu(|U - V|^2). \tag{3.2}$$

We notice that for all $u \ge 0$, $\mu(u) \le a + bu$, where $a > 0$ and $b > 0$ are real numbers.

(ii) Letting $\delta(t, 0)$, $\lambda(t, 0) \in \mathbb{L}^2$ and $t \in [0\ T]$, then

$$|\delta(t, 0)|^2 + |\lambda(t, 0)|^2 \le c, \tag{3.3}$$

where $c > 0$ is a real number.

We demonstrate that any solution $y(t)$ of problem (1.1) is bounded in the following lemma, specifically, $y(t) \in \mathbb{M}^2([-\theta, T]; \mathbb{R}^n)$.

Lemma 3.1. *Consider the solution to problems* (1.1) *and* (1.2) *to be* $y(t)$*. Assume that conditions* (i) *and* (ii) *are met. Then, for every* $n \ge 1$*,*

$$\mathbb{E}\Big[\sup_{-\theta \le t \le T} |y(t)|^2\Big] \le \alpha_4,$$

where $\alpha_4 = \mathbb{E}[|\zeta(0)|^2] + \alpha_3$, $\alpha_3 = \alpha_1 E_{2r-1}\Big(\alpha_2 \Gamma(2r-1) T^{2r-1}\Big)$, $\alpha_1 = 4\mathbb{E}[|\zeta(0)|^2]\Big[1 + \frac{4bm_3 T^{2r-1}}{(2r-1)\Gamma^2(r)}\Big] + \frac{8m_3 T^{2r-1}}{(2r-1)\Gamma^2(r)}(c + 2a)$, $\alpha_2 = \frac{16bm_3}{\Gamma^2(r)}$ *and* $m_3 = T + m_2$ *are positive constants.*

Proof. By utilizing the basic inequality of calculus $|\sum_{k=1}^4 c_k|^2 \le 4\sum_{k=1}^4 |c_k|^2$, we obtain from (3.4) that

$$\begin{aligned} |y(t)|^2 \le\ & 4|\zeta(0)|^2 + 4|\zeta'(0)T|^2 + \frac{4}{\Gamma^2(r)}\Big|\int_0^t (t-u)^{r-1}\delta(u, y_u)du\Big|^2 \\ & + \frac{4}{\Gamma^2(r)}\Big|\int_0^t (t-u)^{r-1}\lambda(u, y_u)dB(u)\Big|^2. \end{aligned}$$

By employing the expectation on both sides and utilizing the Burkholder-Davis-Gundy (BDG) inequality [20] along with H$\ddot{o}$lder's inequality, we obtain:

$$\begin{aligned}
\mathbb{E}[|y(t)|^2] &\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{4T}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}|\delta(u,y_u)|^2\Big]du \\
&+ \frac{4m_2}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}|\lambda(u,y_u)|^2\Big]du \\
&\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8T}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}(|\delta(u,y_u)-\delta(u,0)|^2 + |\delta(u,0)|^2)\Big]du \\
&+ \frac{8m_2}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}(|\lambda(u,y_u)-\lambda(u,0)|^2 + |\lambda(u,0)|^2)\Big]du \\
&\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8T}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}|\delta(u,0)|^2\Big]du \\
&+ \frac{8m_2}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}|\lambda(u,0)|^2\Big]du + \frac{8T}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}(|\delta(u,y_u)-\delta(u,0)|^2)\Big]du \\
&+ \frac{8m_2}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}(|\lambda(u,y_u)-\lambda(u,0)|^2)\Big]du.
\end{aligned}$$

Under conditions (i) and (ii), the above inequality can be expressed as:

$$\begin{aligned}
\mathbb{E}[|y(t)|^2] &\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8Tc}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}\Big]du \\
&+ \frac{8cm_2}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}\Big]du + \frac{8T}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}\mu(|y_u|^2)\Big]du \\
&+ \frac{8m_2}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}\mu(|y_u|^2)\Big]du \\
&= 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8cT^{2r-1}}{(2r-1)\Gamma^2(r)}(T+m_2) \\
&+ \frac{8}{\Gamma^2(r)}(T+m_2)\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}\mu(|y_u|^2)\Big]du \\
&\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8cm_3T^{2r-1}}{(2r-1)\Gamma^2(r)} + \frac{16am_3T^{2r-1}}{(2r-1)\Gamma^2(r)} \\
&+ \frac{16bm_3}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}|y_u|^2\Big]du,
\end{aligned}$$

where $m_3 = (T + m_2)$. Noticing the following

$$\sup_{0\le u\le t}|y_u|^2 \le \sup_{0\le u\le t}\sup_{-\theta\le v\le 0}|y(u+v)|^2 \le \sup_{-\theta\le e\le t}|y(e)|^2 \le |\zeta|^2 + \sup_{0\le e\le t}|y(e)|^2,$$

we compute

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le s\le t}|y(s)|^2\Big] &\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8cm_3T^{2r-1}}{(2r-1)\Gamma^2(r)} + \frac{16am_3T^{2r-1}}{(2r-1)\Gamma^2(r)} \\
&+ \frac{16bm_3}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t [|\zeta(0)|^2(t-u)^{2r-2} + (t-u)^{2r-2}\sup_{0\le e\le u}|y(e)|^2]\Big]du \\
&\le \alpha_1 + \alpha_2\int_0^t \Big[(t-u)^{2r-2}\mathbb{E}\Big[\sup_{0\le e\le u}|y(e)|^2\Big]\Big]du,
\end{aligned}$$

where $\alpha_1 = 4\mathbb{E}[|\zeta(0)|^2]\Big[1+\frac{4bm_3T^{2r-1}}{(2r-1)\Gamma^2(r)}\Big] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8m_3T^{2r-1}}{(2r-1)\Gamma^2(r)}(c+2a)$ and $\alpha_2 = \frac{16bm_3}{\Gamma^2(r)}$. At this stage, Lemmas 2.3 and 2.4 give:

$$\mathbb{E}\Big[\sup_{0\le t\le T}|y(t)|^2\Big] \le \alpha_3,$$

where $\alpha_3 = \alpha_1 E_{2r-1}\Big(\alpha_2\Gamma(2r-1)T^{2r-1}\Big)$. Noticing that

$$\mathbb{E}\Big[\sup_{-\theta\le t\le T}|y(t)|^2\Big] \le \mathbb{E}[|\zeta(0)|^2] + \mathbb{E}\Big[\sup_{0\le t\le T}|y(t)|^2\Big],$$

we get the required result. The proof is complete. □

Assume that for $0 \le t \le T$, $y^0(t) = \zeta(0)$. For every $n = 1, 2, \ldots$, we fix $y_0^n = \zeta$ and introduce the following Picard approximation sequence for Eq (1.1):

$$y^n(t) = \zeta(0) + \zeta'(0)t + \frac{1}{\Gamma(r)}\int_0^t (t-u)^{r-1}\delta(u, y_u^{n-1})du + \frac{1}{\Gamma(r)}\int_0^t (t-u)^{r-1}\lambda(u, y_u^{n-1})dB(u), \tag{3.4}$$

where $t \in [0, T]$. Let us determine that $y^n(t)$, $n \ge 1$, is a bounded sequence; particularly, $y^n(t) \in \mathbb{M}^2([-\theta, T]; \mathbb{R}^n)$.

Lemma 3.2. *Presume that* (*i*) *and* (*ii*) *are satisfied. Then, the following inequality is true for each* $n \ge 1$*:*

$$\sup_{-\theta\le t\le T}\mathbb{E}\Big[|y^n(t)|^2\Big] \le \hat{\beta}_2,$$

where $\hat{\beta}_2 = \mathbb{E}[|\zeta|^2] + \alpha E_{2r-1}(\alpha_2\Gamma(2r-1))$, $\alpha = 4\mathbb{E}[|\zeta(0)|^2]\Big(1+\frac{8bm_3T^{2r-1}}{(2r-1)(\Gamma(r))^2}\Big) + 8\frac{m_3T^{2r-1}}{(2r-1)(\Gamma(r))^2}\Big(c+2a\Big)$ *and* $\alpha_2 = \frac{16bm_3}{(\Gamma(r))^2}$, $m_3 = T + m_2$ *are positive constants.*

Proof. Obviously, $y^0(\cdot) \in \mathbb{M}^2([-\theta, T]; \mathbb{R}^n)$. Considering the inequality $|\sum_{k=1}^4 c_k|^2 \le 4\sum_{k=1}^4 |c_k|^2$, then from (3.4), we derive

$$\begin{aligned}|y^n(t)|^2 &\le 4|\zeta(0)|^2 + 4|\zeta'(0)T|^2 + \frac{4}{\Gamma^2(r)}\Big|\int_0^t (t-u)^{r-1}\delta(u, y_u^{n-1})du\Big|^2\\ &\quad + \frac{4}{\Gamma^2(r)}\Big|\int_0^t (t-u)^{r-1}\lambda(u, y_u^{n-1})dB(u)\Big|^2.\end{aligned}$$

Applying expectation to both sides and utilizing Hölder's and BDG inequalities, in a similar fashion as in Lemma 3.1, we derive,

$$\begin{aligned}\mathbb{E}[|y^n(t)|^2] &\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8T}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}|\delta(u,0)|^2\Big]du\\ &\quad + \frac{8m_2}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}|\lambda(u,0)|^2\Big]du\\ &\quad + \frac{8T}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}(|\delta(u, y_u^{n-1}) - \delta(u,0)|^2)\Big]du\\ &\quad + \frac{8m_2}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}(|\lambda(u, y_u^{n-1}) - \lambda(u,0)|^2)\Big]du.\end{aligned}$$

Under conditions (i) and (ii), the above inequality implies

$$\begin{aligned}\mathbb{E}[|y^n(t)|^2] &\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8cm_3T^{2r-1}}{(2r-1)\Gamma^2(r)} + \frac{16am_3T^{2r-1}}{(2r-1)\Gamma^2(r)} \\ &+ \frac{16bm_3}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t (t-u)^{2r-2}|y_u^{n-1}|^2\Big]du,\end{aligned}$$

where $m_3 = (T + m_2)$. Observing the following inequality

$$\sup_{0\le u\le t} |y_u^n|^2 \le \sup_{0\le u\le t}\sup_{-\theta\le v\le 0} |y^n(u+v)|^2 \le \sup_{-\theta\le e\le t} |y^n(e)|^2 \le |\zeta|^2 + \sup_{0\le e\le t} |y^n(e)|^2,$$

it follows

$$\begin{aligned}\sup_{0\le u\le t}\mathbb{E}[|y^n(u)|^2] &\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8cm_3T^{2r-1}}{(2r-1)\Gamma^2(r)} + \frac{16am_3T^{(2r-1)}}{(2r-1)\Gamma^2(r)} \\ &+ \frac{16bm_3}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t [|\zeta(0)|^2(t-u)^{2r-2} + (t-u)^{2r-2}\sup_{0\le e\le u}|y^{n-1}(e)|^2]\Big]du \\ &\le 4\mathbb{E}[|\zeta(0)|^2] + \frac{8cm_3T^{2r-1}}{(2r-1)\Gamma^2(r)} + \frac{16am_3T^{(2r-1)}}{(2r-1)\Gamma^2(r)} \\ &+ \frac{16bm_3T^{2r-1}}{(2r-1)\Gamma^2(r)}\mathbb{E}[|\zeta(0)|^2] + \frac{16bm_3}{\Gamma^2(r)}\mathbb{E}\Big[\int_0^t [(t-u)^{2r-2}\sup_{0\le e\le u}|y^{n-1}(e)|^2]\Big]du.\end{aligned}$$

Once more, we note that for an arbitrary $j \ge n$,

$$\max_{1\le n\le j}\mathbb{E}[|y^{n-1}(e)|^2] \le \mathbb{E}[|\zeta(0)|^2] + \max_{1\le n\le j}\mathbb{E}[|y^n(e)|^2],$$

we deduce

$$\begin{aligned}\max_{1\le n\le j}\sup_{0\le u\le t}\mathbb{E}[|y^n(u)|^2] &\le 4\mathbb{E}[|\zeta(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8cm_3T^{2r-1}}{(2r-1)\Gamma^2(r)} + \frac{16am_3T^{2r-1}}{(2r-1)\Gamma^2(r)} \\ &+ \frac{16bm_3T^{2r-1}}{(2r-1)\Gamma^2(r)}\mathbb{E}[|\zeta(0)|^2] \\ &+ \frac{16bm_3}{\Gamma^2(r)}\int_0^t (t-u)^{2r-2}\Big(\mathbb{E}[|\zeta(0)|^2] + \max_{1\le n\le j}\sup_{0\le e\le u}\mathbb{E}[|y^n(e)|^2]\Big)du \\ &\le 4\mathbb{E}[|\xi(0)|^2] + 4\mathbb{E}[|\zeta'(0)T|^2] + \frac{8cm_3T^{2r-1}}{(2r-1)\Gamma^2(r)} + \frac{16am_3T^{2r-1}}{(2r-1)\Gamma^2(r)} \\ &+ \frac{32bm_3T^{2r-1}}{(2r-1)\Gamma^2(r)}\mathbb{E}[|\zeta(0)|^2] \\ &+ \frac{16bm_3}{\Gamma^2(r)}\int_0^t (t-u)^{2r-2}\max_{1\le n\le j}\sup_{0\le e\le u}\mathbb{E}[|y^n(e)|^2]du \\ &= \alpha + \alpha_2\int_0^t (t-u)^{2r-2}\max_{1\le n\le j}\sup_{0\le e\le u}\mathbb{E}[|y^n(e)|^2]du,\end{aligned}$$

where $\alpha = 4\mathbb{E}[|\zeta(0)|^2]\Big(1 + \frac{8bm_3T^{2r-1}}{(2r-1)\Gamma^2(r)}\Big) + 4\mathbb{E}[|\zeta'(0)T|^2] + 8\frac{m_3T^{2r-1}}{(2r-1)\Gamma^2(r)}\Big(c + 2a\Big)$ and $\alpha_2 = \frac{16bm_3}{\Gamma^2(r)}$. At this stage, Lemmas 2.3 and 2.4 give:

$$\max_{1\le n\le j}\sup_{0\le t\le T}\mathbb{E}[|y^n(t)|^2] \le \alpha E_{2r-1}\Big(\alpha_2\Gamma(2r-1)T^{2r-1}\Big),$$

since j is arbitrary, we deduce

$$\sup_{0\le t\le T}\mathbb{E}[|y^n(t)|^2]\le \alpha E_{2r-1}\Big(\alpha_2\Gamma(2r-1)T^{2r-1}\Big).$$

Consequently for each $t\in[0\ \ T]$,

$$\sup_{-\theta\le t\le T}\mathbb{E}[|y^n(t)|^2]\le\hat{\beta}_2,$$

where $\hat{\beta}_2=\mathbb{E}[|\zeta|^2]+\alpha E_{2r-1}\Big(\alpha_2\Gamma(2r-1)$. □

Lemma 3.3. *Presume that* (i) *and* (ii) *are satisfied. For all integers* $n\ge 1$ *and* $m\ge 1$*,*

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le t}|y^{n+m}(u)-y^n(u)|^2\Big]&\le\beta_1\int_0^t(t-u)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\le e\le u}|y^{n+m-1}(e)-y^{n-1}(e)|^2\Big]\Big)du\\&\le\gamma_1 t^{2r-1},\end{aligned}$$

where $\gamma_1=\frac{\beta_1\mu(4\beta_2)}{2r-1}$, $\beta_1=\frac{3m_3}{\Gamma^2(r)}$, $m_3=T+m_2$ *and* $m_2>0$ *are real numbers.*

Proof. Our deduction from (3.4) based on the inequality $|\sum_{k=1}^2 a_k|^2\le 2\sum_{k=1}^2|a_k|^2$ yields the following:

$$\begin{aligned}|y^{n+m}(t)-y^n(t)|^2\le&\frac{2}{\Gamma^2(r)}\Big|\int_0^t(t-u)^{r-1}[\delta(u,y_u^{n+m-1})-\delta(u,y_u^{n-1})]du\Big|^2\\&+\frac{2}{\Gamma^2(r)}\Big|\int_0^t(t-u)^{r-1}[\lambda(u,y_u^{n+m-1})-\lambda(u,y_u^{n-1})]dB(u)\Big|^2.\end{aligned}$$

By employing the expectation on both sides, the Jensen inequality, assumptions (i), and (ii), we calculate

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le t}|y^{n+m}(u)-y^n(u)|^2\Big]\le&\frac{2T}{\Gamma^2(r)}\int_0^t(t-u)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\le e\le u}|y^{n+m-1}(e)-y^{n-1}(e)|^2\Big]\Big)du\\&+\frac{2m_2}{\Gamma^2(r)}\int_0^t(t-u)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\le e\le u}|y^{n+m-1}(e)-y^{n-1}(e)|^2\Big]\Big)du\\\le&\beta_1\int_0^t(t-u)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\le e\le s}|y^{n+m-1}(e)-y^{n-1}(e)|^2\Big]\Big)du,\end{aligned}$$

where $\beta_1=\frac{3m_3}{\Gamma^2(r)}$, $m_3=T+m_2$. Finally, by invoking Lemma 4.1, we obtain

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le t}|y^{n+m}(u)-y^n(u)|^2\Big]&\le\beta_1\int_0^t(t-u)^{2r-2}\mu\Big(2\mathbb{E}\Big[\sup_{0\le e\le u}|y^{n+m-1}(e)|^2\Big]+2\mathbb{E}\Big[\sup_{0\le e\le u}|y^{n-1}(e)|^2\Big]\Big)du\\&\le\beta_1\int_0^t(t-u)^{2r-2}\mu(4\beta_2)du\\&=\frac{\beta_1\mu(4\beta_2)}{2r-1}t^{2r-1}=\gamma_1 t^{2r-1},\end{aligned}$$

where $\gamma_1=\frac{\beta_1\mu(4\beta_2)}{2r-1}$. This concludes the proof. □

Section 3 investigates several key findings, including the boundedness of Picard's approximate solutions and the actual solutions, among others. These results are of significant importance and will serve as a valuable foundation for further research in this direction.

4. Existence-uniqueness results

In this section, a specific approximation technique is introduced, and the existence and uniqueness of solutions are established. We introduce the following symbols to state the main result. Select $T_1 \in [0, T]$ in such a way that for every $t \in [0, T_1]$

$$\beta_1 \mu(\gamma_1 t^{2r-1}) \leq (2r-1)\gamma_1. \tag{4.1}$$

For each $n, m \geq 1$, the recursive mapping is given by

$$\Lambda_1(t) = \gamma_1 t^{2r-1}, \tag{4.2}$$

$$\begin{aligned} \Lambda_{n+1}(t) &= \beta_1 \int_0^t (t-s)^{2r-2} \mu(\Lambda_n(s)) ds, \\ \Lambda_{n,m}(t) &= \mathbb{E}\Big[\sup_{0 \leq e \leq t} |y^{n+m}(e) - y^n(e)|^2\Big]. \end{aligned} \tag{4.3}$$

Lemma 4.1. *Assume conditions* (*i*) *and* (*ii*) *are satisfied. Then, there exists* $T_1 \in [0, T]$ *such that for all* $m \geq 1$ *and* $n \geq 1$*,*

$$0 \leq \Lambda_{n,m}(t) \leq \Lambda_n(t) \leq \Lambda_{n-1}(t) \leq ... \leq \Lambda_n(t), \quad t \in [0 \;\; T_1]. \tag{4.4}$$

Proof. To establish the inequality (4.4), we use mathematical induction. By leveraging the definition of the mapping $\Lambda(\cdot)$ and applying Lemma 3.3, we obtain:

$$\Lambda_{1,m}(t) = \mathbb{E}\Big[\sup_{0 \leq e \leq t} |y^{1+m}(e) - y^1(e)|^2\Big] \leq \gamma_1 t^{2r-1} = \Lambda_1(t).$$

$$\begin{aligned} \Lambda_{2,m}(t) &= \mathbb{E}\Big[\sup_{0 \leq e \leq t} |y^{2+m}(e) - y^2(e)|^2\Big] \\ &\leq \beta_1 \int_0^t (t-u)^{2r-1} \mu\Big(\mathbb{E}\Big[\sup_{0 \leq e \leq t} |y^{1+m}(e) - y^1(e)|^2\Big]\Big) du \\ &\leq \beta_1 \int_0^t (t-u)^{2r-1} \mu(\Lambda_1(u)) du = \Lambda_2(t). \end{aligned}$$

In view of (4.1), it follows

$$\begin{aligned} \Lambda_2(t) &= \beta_1 \int_0^t (t-u)^{2r-2} \mu(\Lambda_1(u)) du \\ &= \int_0^t (t-u)^{2r-2} \beta_1 \mu(\gamma_1 t^{2r-1}) du \\ &\leq \int_0^t (t-u)^{2r-2} (2r-1)\gamma_1 du \\ &= \frac{1}{(2r-1)} t^{2r-1} (2r-1)\gamma_1 = \gamma_1 t^{2r-1} = \Lambda_1(t). \end{aligned}$$

Therefore, it follows that for every $t \in [0, T_1]$, $\Lambda_{2,m}(t) \leq \Lambda_2(t) \leq \Lambda_1(t)$. Now, let (4.4) be true for $n \geq 1$. We proceed to compute that (4.4) holds for $n + 1$ as the following:

$$\begin{aligned}\Lambda_{n+1,m}(t) &= \mathbb{E}\Big[\sup_{0\leq e\leq t} |y^{n+m+1}(e) - y^{n+1}(e)|^2\Big] \\ &\leq \beta_1 \int_0^t (t-u)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\leq e\leq u} |y^{n+m}(e) - y^n(e)|^2\Big]\Big)du \\ &\leq \beta_1 \int_0^t (t-u)^{2r-2}\mu\Big(\Lambda_{n,m}(u)\Big)du \\ &= \Lambda_{n+1}(t).\end{aligned}$$

Also,

$$\Lambda_{n+1}(t) = \beta_1 \int_0^t (t-u)^{2r-2}\mu\Big(\Lambda_{n,m}(u)\Big)du \leq \beta_1 \int_0^t (t-u)^{2r-2}\mu\Big(\Lambda_{n-1}(u)\Big)du = \Lambda_n(u).$$

Consequently, for every $t \in [0, T_1]$, we have $\Lambda_{n+1,m}(t) \leq \Lambda_{n+1}(t) \leq \Lambda_n(t)$, verifying that result (4.4) holds for $n + 1$. The proof is finished with this. □

Theorem 4.2. *Assume that* $r \in (\frac{3}{2}, 2)$ *holds and that conditions* (i) *and* (ii) *are true. Then Eq* (1.1) *admits a maximum of one solution.*

Proof. The proof is carried out as follows: First, we demonstrate uniqueness, and then we establish existence. Consider the problem (1.1), and let $z(t)$ and $y(t)$ indicate two solutions. In a similar fashion as before, we derive

$$\begin{aligned}|z(t) - y(t)|^2 \leq\ & \frac{2}{\Gamma^2(r)}\Big|\int_0^t (t-u)^{r-1}[\delta(u, z_u) - \delta(u, y_u)]du\Big|^2 \\ &+ \frac{2}{\Gamma^2(r)}\Big|\int_0^t (t-u)^{r-1}[\lambda(u, z_u) - \lambda(u, y_u)]dB(u)\Big|^2.\end{aligned}$$

Using a similar procedure as before, it follows:

$$\mathbb{E}\Big[\sup_{0\leq e\leq t} |z(e) - y(e)|^2\Big] \leq \beta_1 \int_0^t (t-u)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\leq e\leq u} |z(e) - y(e)|^2\Big]\Big)du.$$

Lemmas 2.5 and 2.6 allow us to obtain the following for $t \in [0, T]$:

$$\mathbb{E}\Big[\sup_{0\leq e\leq t} |z(e) - y(e)|^2\Big] = 0.$$

This concludes the proof of uniqueness. Next, we proceed to establish existence. Observe that $\Lambda_n(t)$ is continuous for $t \in [0, T_1]$. Furthermore, on $t \in [0, T_1]$ and for $n \geq 1$, $\Lambda_n(t)$ is decreasing. By the Dominated Convergence Theorem, the mapping $\Lambda(t)$ is defined as the following:

$$\Lambda(t) = \lim_{n\to\infty} \Lambda_n(t) = \lim_{n\to\infty} \beta_1 \int_0^t (t-u)^{2r-2}\mu\Big(\Lambda_{n-1}(u)\Big)du = \beta_1 \int_0^t (t-u)^{2r-2}\mu\Big(\Lambda(u)\Big)du, \quad t \in [0\ \ T_1].$$

So,

$$\Lambda(t) \leq \Lambda(0) + \beta_1 \int_0^t (t-u)^{2r-2} \mu\Big(\Lambda(u)\Big) du.$$

Therefore, for every $t \in [0, T_1]$, Lemmas 2.5 and 2.6 imply that $\Lambda(t) = 0$. From Lemma 4.1, we obtain $\Lambda_{n,m}(u) \leq \Lambda_n(u) \to 0$ as $n \to \infty$, which leads to $\mathbb{E}|y^{n+m}(t) - y^n(t)|^2 \to 0$ as $n \to \infty$. Utilizing the properties of the mapping $\mu(\cdot)$, conditions (i), (ii), and the completeness of $\mathbb{L}^2$, it implies that for every $t \in [0, T_1]$,

$$\delta(t, y_t^n) \to \delta(t, y_t), \lambda(t, y_t^n) \to \lambda(t, y_t), \text{ in } \mathbb{L}^2 \text{ as } n \to \infty.$$

Consequently for each $t \in [0 \ \ T_1]$,

$$\lim_{n\to\infty} y^n(t) = \zeta(0) + \zeta'(0)t + \lim_{n\to\infty} \frac{1}{\Gamma(r)} \int_0^t (t-u)^{r-1} \delta(u, y_u^{n-1}) du + \lim_{n\to\infty} \frac{1}{\Gamma(r)} \int_0^t (t-u)^{r-1} \lambda(u, y_u^{n-1}) dB(u),$$

that is,

$$y(t) = \zeta(0) + \zeta'(0)t + \frac{1}{\Gamma(r)} \int_0^t (t-u)^{r-1} \delta(u, y_u) du + \frac{1}{\Gamma(r)} \int_0^t (t-u)^{r-1} \lambda(u, y_u) dB(u).$$

Thus, $y(t)$ is the unique solution to problem (1.1) on $t \in [0 \ \ T_1]$. By applying this argument iteratively, it can be shown that the problem (1.1) admits at most one solution on $t \in [0 \ \ T]$. This concludes the proof. □

We have shown that the solutions of SDEs of type (1.1) exist are unique even if the coefficients are not Lipschitz continuous.

5. Error estimation and stability

In this section, the error estimates between the exact solution $y(t)$ and the Picard approximate solution $y^n(t)$, $n \geq 1$ are initially determined. The mean-square stability of the solutions to systems (1.1) and (1.2) is next examined.

Theorem 5.1. *Suppose that conditions* (i) *and* (ii) *are satisfied. Let* $y(t)$ *be the unique solution of problem* (1.1) *and* $y^n(t)$ *be the Picard approximation defined by* (3.4)*; then*

$$\mathbb{E}\Big[\sup_{0\leq u\leq t} |y^n(u) - y(u)|^2\Big] \leq \frac{1}{2r-1} \beta_1 \mu\Big(2(\alpha_1 + \alpha) E_{2r-1}(\alpha_2 \Gamma(2r-1) T^{2r-1})\Big) T^{2r-1},$$

where $\beta_1 = \frac{3m_3}{\Gamma^2(r)}$, $\alpha_2 = \frac{16b^2m_3}{\Gamma^2(r)}$, $\alpha_1 = 4\mathbb{E}[|\xi(0)|^2]\Big[1 + \frac{4bm_3T^{2r-1}}{(2r-1)\Gamma^2(r)}\Big] + \frac{8m_3T^{2r-1}}{(2r-1)\Gamma^2(r)}(c + 2a)$, $\alpha = 4\mathbb{E}[|\xi(0)|^2]\Big(1 + \frac{8bm_3T^{2r-1}}{(2r-1)\Gamma^2(r)}\Big) + 8\frac{m_3T^{2r-1}}{(2r-1)\Gamma(r)}\Big(c + 2a\Big)$ *and* $m_3 = T + m_2$ *are positive constants.*

Proof. The deduction from (3.4), based on the inequality $|\sum_{k=1}^2 a_k|^2 \leq 2\sum_{k=1}^2 |a_k|^2$, yields the following:

$$|y^n(t) - y(t)|^2 \leq \frac{2}{\Gamma^2(r)} \Big| \int_0^t (t-u)^{r-1} [\delta(u, y_u^{n-1}) - \delta(u, y_u)] du \Big|^2 + \frac{2}{\Gamma^2(r)} \Big| \int_0^t (t-u)^{r-1} [\lambda(u, y_u^{n-1}) - \lambda(u, y_u)] dB(u) \Big|^2.$$

By employing the expectation on both sides and in view of the Jensen inequality, assumptions (i) and (ii), we calculate

$$\mathbb{E}\Big[\sup_{0\le u\le t}|y^n(u)-y(u)|^2\Big] \le \frac{2T}{\Gamma^2(r)}\int_0^t (t-s)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\le e\le u}|y^{n-1}(e)-y(e)|^2\Big]\Big)du$$
$$+\frac{2m_2}{\Gamma^2(r)}\int_0^t (t-u)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\le e\le u}|y^{n-1}(e)-y(e)|^2\Big]\Big)du$$
$$\le \beta_1\int_0^t (t-u)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\le e\le u}|y^{n-1}(e)-y(e)|^2\Big]\Big)du,$$

where $\beta_1=\frac{2m_3}{\Gamma^2(r)}$, $m_3=T+m_2$. Finally, by invoking Lemmas 3.1 and 3.3, we obtain

$$\mathbb{E}\Big[\sup_{0\le u\le t}|y^n(u)-y(u)|^2\Big] \le \beta_1\int_0^t (t-u)^{2r-2}\mu\Big(2\mathbb{E}\Big[\sup_{0\le e\le u}|y^{n-1}(e)|^2\Big]+2\mathbb{E}\Big[\sup_{0\le e\le u}|y(e)|^2\Big]\Big)du$$
$$\le \beta_1\int_0^t (t-u)^{2r-2}\mu\Big(2(\alpha_1+\alpha)E_{2r-1}(\alpha_2\Gamma(2r-1)T^{2r-1})\Big)du$$
$$\le \frac{1}{2r-1}\beta_1\mu\Big(2(\alpha_1+\alpha)E_{2r-1}(\alpha_2\Gamma(2r-1)T^{2r-1})\Big)T^{2r-1}.$$

This completes the proof. □

Theorem 5.2. *Assume that* $r\in(\frac{3}{2},2)$ *and that criteria* (i) *and* (ii) *are met. The initial conditions* ζ *and* ξ *correspond to the two solutions to problem* (1.1)*, denoted by* $z(t)$ *and* $y(t)$*, respectively. A* $\delta(\epsilon)>0$ *exists for all* $t\in[0,T]$ *and every* $\epsilon>0$ *such that if*

$$\mathbb{E}|\zeta-\xi|^2+\mathbb{E}|(\zeta'-\xi')T|^2<\delta(\epsilon),\qquad \textit{then},\qquad \mathbb{E}|z(t)-y(t)|^2\le\epsilon.$$

Proof. Observe that $z(t)$ and $y(t)$ are both solutions of Eq (1.1). Accordingly, for $t\in[0,T]$, we have

$$z(t)-y(t)=\zeta(0)-\xi(0)+\zeta'(0)t-\xi'(0)t+\frac{1}{\Gamma(r)}\int_0^t (t-u)^{r-1}[\delta(u,z_u)-\delta(u,y_u)]du$$
$$+\frac{1}{\Gamma(r)}\int_0^t (t-u)^{r-1}[\lambda(u,z_u)-\lambda(u,y_u)]dB(u).$$

Using the inequality $|\sum_{i=1}^4 a_i|^2\le 4\sum_{i=1}^4|a_i|^2$ and employing similar reasoning as before, it follows that

$$\mathbb{E}\Big[\sup_{0\le u\le t}|z(u)-y(u)|^2\Big]\le 4\mathbb{E}|\zeta(0)-\xi(0)|^2+4\mathbb{E}|(\zeta'(0)-\xi'(0))T|^2$$
$$+\frac{4T}{\Gamma^2(r)}\int_0^t (t-u)^{2r-2}\mathbb{E}\Big[\sup_{0\le u\le t}|\delta(u,z_u)-\delta(u,y_u)|^2\Big]du$$
$$+\frac{4m_2}{\Gamma^2(r)}\int_0^t (t-u)^{2r-2}\mathbb{E}\Big[\sup_{0\le u\le t}|\lambda(u,z_u)-\lambda(u,y_u)|^2\Big]du,$$

it gives

$$\mathbb{E}\Big[\sup_{0\le u\le t}|z(u)-y(u)|^2\Big]\le 4\mathbb{E}|\zeta(0)-\xi(0)|^2+4\mathbb{E}|(\zeta'(0)-\xi'(0))T|^2$$
$$+2\beta_1\int_0^t (t-u)^{2r-2}\mu\Big(\mathbb{E}\Big[\sup_{0\le e\le u}|z(e)-y(e)|^2\Big]\Big)du.$$

Lemmas 2.5 and 2.6 are therefore applied, and we obtain

$$\mathbb{E}\Big[|z(t) - y(t)|^2\Big] \leq \epsilon,$$

for $t \in [0, T]$. This concludes the proof. □

Example 5.3. Consider the following Eq (5.1), in which the coefficients do not satisfy the Lipschitz continuity condition.

$$D_t^r y(t) = a\sqrt{y_t}dt + c\sqrt{y_t}dB(t), \;\; r \in (\frac{3}{2}, 2), \tag{5.1}$$

where $y(0) = \zeta_0$ and a, b, c, and d are all positive constants. A non-decreasing and concave function on $[0, \infty)$, $\mu(y) = \sqrt{y}$, ensures that $\mu(0) = 0$, $\mu(y) > 0$ for $y > 0$, and $B(t)$ is a 1-dimensional Brownian motion.

$$\int_{0+} \frac{dy}{\mu(y)} = \lim_{\delta \to 0^+} \int_\delta^\infty y^{-\frac{1}{2}} dy = 2 \lim_{\delta \to 0^+} \left| y^{\frac{1}{2}} \right|_\delta^\infty = \infty. \tag{5.2}$$

Consequently, the conditions of Theorems 4.2 and 5.2 are satisfied, guaranteeing that Eq (5.1) possesses a unique mean-square stable solution.

6. Conclusions

The connection between fractional stochastic functional differential equations (SFDEs), partial differential equations (PDEs), and chemotaxis models lies in their shared mathematical structures, modeling objectives, and the types of phenomena they seek to describe [21, 22]. This paper provides a comprehensive analysis of fractional stochastic functional differential equations (FSFDEs) under non Lipschitz condition. The authors successfully establish the boundedness, existence, and uniqueness of solutions within this framework. Additionally, they derive error estimates that compare the exact solution with the Picard approximate solutions, providing valuable insights into the accuracy of the approximation process. The results also confirm the mean square stability of the solutions, demonstrating the robustness of the proposed theory. To further illustrate the practical applicability of their findings, the authors present a detailed example, showcasing the relevance of their approach to real-world problems. The existence, uniqueness, and stability of solutions to fractional SDEs driven by Lévy processes remain unresolved. Furthermore, investigating fractional stochastic dynamical systems with the G-framework under non-Lipschitz condition, as well as fractional stochastic dynamical systems driven by G-Lévy processes under both Lipschitz [23–25] and non-Lipschitz conditions [26, 27], remains an open challenge. We hope that the ideas developed in this article will play a pivotal role in advancing research in these areas. The future work includes studying appropriate numerical simulation graphs for the considered FDEs with time-varying coefficients [28].

Author contributions

All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest that may influence the publication of this paper.

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