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Research article

A robust uniform practical stability approach for Caputo fractional hybrid systems

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Abstract: This work introduces a novel approach for analyzing the uniform practical stability (UPS) and strong uniform practical stability (SUPS) of Caputo fractional dynamic equations on time scales, using two measures (m, m_0). The method employs an extended derivative, the Caputo fractional delta Dini derivative (CFrΔDiD) of order $\zeta \in (0, 1)$, addressing the gap in unified stability frameworks for fractional hybrid systems that span both continuous and discrete time domains. This generalized framework not only unifies various stability concepts but also makes it applicable to hybrid systems with both gradual and abrupt changes. The UPS and SUPS results are demonstrated through illustrative examples.

Keywords: fractional calculus; uniform practical stability; Caputo fractional derivative; time scales; vector Lyapunov functions; dynamic systems
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Abbreviations

r-d: right dense; l-d: left dense; r-s: right scattered; l-s: left scattered; UPS: uniform practical stability; SUPS: strongly uniform practical stability; LF: Lyapunov function; CFrΔD: Caputo fractional delta derivative; CFrΔDiD: Caputo fractional delta Dini derivative; GLFrΔD: Grunwald–Letnikov fractional delta derivative; GLFrΔDiD: Grunwald–Letnikov fractional delta Dini derivative; CFrDy$\mathbb{T}$: Caputo fractional dynamic equations on time scales; CFrD: Caputo fractional derivative

1. Introduction

In the field of dynamic systems, stability analysis is essential for understanding system responses to various initial conditions and disturbances. Traditional Lyapunov stability theory has made significant contributions here, focusing on convergence to the equilibrium [1, 2]. However, this strict convergence requirement can be overly restrictive in real-world applications where slight deviations are often tolerable. UPS offers a more flexible framework, allowing systems to operate within an acceptable range around the equilibrium rather than requiring exact convergence (see [3, 4]).

The recent growth of fractional calculus has introduced valuable tools for stability analysis. By extending differentiation and integration to arbitrary order, fractional calculus has proven to be highly effective in modeling complex systems with memory effects and non-local interactions [5–7]. Among these tools, the Caputo fractional derivative (CFrD) has shown practical relevance for various applications (see [8, 9]). However, most studies have relied on scalar Lyapunov functions (LFs), which evaluate stability on an individual-variable basis [10, 11]. It is widely recognized that as the complexity of a dynamical system increases, identifying a suitable LF becomes more challenging. This difficulty often leads to the use of multiple LFs, forming a vector LF, where each component reveals insights into different parts of the system's behavior. By this approach, certain complex elements can be simplified into interconnected subsystems, allowing for more relaxed conditions. Thus, the method of vector LFs provides a highly adaptable framework for analysis [12–14].

Previous studies, including [15–17], have concentrated largely on stability aspects like uniform, asymptotic, and variational stability within delay and impulsive settings, primarily for continuous-time systems. In contrast, research such as [18] has examined stability in discrete settings. This paper introduces an innovative approach using vector LFs for Caputo fractional dynamic equations on time scales (CFrDy$\mathbb{T}$), establishing UPS in terms of two measures (m_0, m), and presenting a versatile framework well-suited for real-world applications. This approach ensures that systems remain within specific bounds despite small disturbances, reflecting a balance between rigorous stability and practical adaptability.

With the introduction of time scale calculus in [19], a unified approach for analyzing systems evolving in both discrete and continuous time domains has emerged, extending traditional continuous analysis techniques to discrete cases. Integrating this calculus with fractional dynamics enables a more sophisticated framework for understanding systems that undergo both gradual and abrupt changes, common in hybrid systems. When combined with fractional calculus, time scale calculus provides a versatile platform for analyzing dynamic behaviors in both continuous and discrete settings.

Building on these advancements, this paper extends the results in [20, 21], presenting a novel methodology for examining the (m_0, m)-uniform practical stability (UPS) and (m_0, m)-strongly uniform practical stability (SUPS) of CFrDy$\mathbb{T}$. To further extend the scope of practical stability analysis [22], this work applies the novel derivatives introduced in [23], the Caputo fractional delta derivative (CFrΔD) and Caputo fractional delta Dini derivative (CFrΔDiD) of order $\zeta \in (0, 1)$, which enables unified stability analysis, creating a robust framework for hybrid systems exhibiting both gradual and abrupt changes.

Let us examine the system

$$\begin{aligned} {}^{C}\Delta^{\zeta}\mathfrak{y} &= \mathfrak{D}(t, \mathfrak{y}), \quad t \in \mathbb{T}, \\ \mathfrak{y}(t_0) &= \mathfrak{y}_0, \qquad t_0 \geq 0. \end{aligned} \tag{1.1}$$

Here,

$$\mathfrak{D} \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N]$$

is a function satisfying

$$\mathfrak{D}(t, 0) \equiv 0,$$

and ${}^{C}\Delta^{\zeta}\mathfrak{y}$ denotes the $CFr\Delta D$ of $\mathfrak{y} \in \mathbb{R}^N$ of order ζ. Let

$$\mathfrak{y}(t) = \mathfrak{y}(t, t_0, \mathfrak{y}_0) \in C^{\zeta}_{rd}[\mathbb{T}, \mathbb{R}^N]$$

represent a solution of (1.1). Assuming that the function $\mathfrak{D}$ possesses sufficient smoothness to ensure the existence, uniqueness, and continuous dependence of solutions (see [24, 25]), this paper examines the (m_0, m)-UPS and (m_0, m)-SUPS of the system (1.1).

To achieve this, we use the comparison system of the form

$$ {}^{C}\Delta^{\zeta}\kappa = \Theta(t, \kappa), \quad \kappa(t_0) = \kappa_0 \geq 0, \tag{1.2}$$

where

$$\Theta \in C_{rd}[\mathbb{T} \times \mathbb{R}^n_+, \mathbb{R}^n_+].$$

The system (1.2) is a simpler system of lesser dimension than (1.1), whose qualitative properties including the existence and uniqueness of its solution

$$\kappa(t) = \kappa(t; t_0, \kappa_0)$$

and practical stability are already known or easy to find, see [26].

The structure of this research is outlined as follows: In Section 2, necessary terminologies, key definitions, remarks, and foundational lemmas that support the later developments are given. In Section 3, we develop and give details of core findings and theoretical contributions of our research. In Section 4, we present practical examples to illustrate the relevance and applicability of our approach. Lastly, in Section 5, we present a summary of the main findings and discuss their implications.

2. Preliminaries and definitions

Definition 2.1. *[27] If $t \in \mathbb{T}$, then*

$$\varpi(t) = \inf\{s \in \mathbb{T} : s > t\}$$

is called the forward jump operator and

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

is called the backward jump operator. So that $t \in \mathbb{T}$ is said to be: right dense (r-d) if

$$\varpi(t) = t,$$

right scattered (r-s) if $\varpi(t) > t$, left dense (l-d) if $\rho(t) = t$, and left scattered if $\rho(t) < t$. Also the function

$$\tau(t) = \varpi(t) - t$$

is called the graininess function.

Definition 2.2. *[28] A function*

$$h : \mathbb{T} \to \mathbb{R}$$

is said to be r-d continuous represented by C_{rd}, if it remains continuous at every r-d point in $\mathbb{T}$, and has finite left-sided limits at each l-d point of $\mathbb{T}$.

Definition 2.3. *[28] We define the following class of functions:*

$\mathcal{K} = \{\psi \in [[0, r], [0, \infty)]\}$*: $\psi(t)$ is strictly increasing on $[0, r]$ and $\psi(0) = 0$;*

$C\mathcal{K} = \{\mathfrak{a} \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+] : \mathfrak{a}(t, s) \in \mathcal{K}$ *for each t*$\}$*;*

$\mathfrak{M} = \{m \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+] : \inf\limits_{(t,x)} m(t, x) = 0\}$.

We now define the derivative of LF utilizing the CFrΔDiD as presented in [23].

Definition 2.4. *Let*

$$[t_0, T)_{\mathbb{T}} = [t_0, T) \cap \mathbb{T},$$

where $T > t_0$. Then a Lyapunov-like function

$$\daleth(t, \mathfrak{y}) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$$

is defined as the generalized CFrΔDiD relative to (1.1) as follows: given $\epsilon > 0$, there exists a neighborhood $P(\epsilon)$ of $t \in \mathbb{T}$ such that

$$\frac{1}{\tau^{\zeta}}\Big\{\daleth(\varpi(t), \mathfrak{y}(\varpi(t)) - \daleth(t_0, \mathfrak{y}_0) - \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]} (-1)^{\iota+1}({}^{\zeta}C_{\iota})[\daleth(\varpi(t) - \iota\tau, \mathfrak{y}(\varpi(t) - \tau^{\zeta}\mathfrak{D}(t, \mathfrak{y}(t)) - \daleth(t_0, \mathfrak{y}_0)]\Big\}$$
$$< {}^{C}\Delta^{\zeta}_{+}\daleth(t, \mathfrak{y}) + \epsilon,$$

for each $s \in P(\epsilon)$ and $s > t$, and

$$\mathfrak{y}(t) = \mathfrak{y}(t, t_0, \mathfrak{y}_0)$$

is any solution of (1.1), where $0 < \zeta < 1$, ϖ is the forward jump operator as defined in Definition 2.1,

$$\tau = \varpi - t, \quad {}^{\zeta}C_{\iota} = \frac{\zeta(\zeta-1)...(\zeta-\iota+1)}{\iota!}$$

and $[\frac{(t-t_0)}{\tau}]$ denotes the integer part of the fraction $\frac{(t-t_0)}{\tau}$.

Definition 2.5. *[23] Using Definition 2.6, we state the CFrΔDiD of the LF $\mathfrak{I}(t, \mathfrak{y})$ as*

$$^{C}\Delta_{+}^{\zeta}\mathfrak{I}(t, \mathfrak{y}) = \limsup_{\tau \to 0^+} \frac{1}{\tau^{\zeta}} \Big[\sum_{\iota=0}^{[\frac{t-t_0}{\tau}]} (-1)^{\iota} ({}^{\zeta}C_{\iota}) [\mathfrak{I}(\varpi(t) - \iota\tau, \mathfrak{y}(\varpi(t)) - \tau^{\zeta}\mathfrak{D}(t, \mathfrak{y}(t)) - \mathcal{L}(t_0, \mathfrak{y}_0)] \Big], \tag{2.1}$$

which is equivalent to

$$\begin{aligned} ^{C}\Delta_{+}^{\zeta}\mathfrak{I}(t, \mathfrak{y}) = \limsup_{\tau \to 0^+} \frac{1}{\tau^{\zeta}} \Big\{ & \mathfrak{I}(\varpi(t), \mathfrak{y}(\varpi(t)) - \mathfrak{I}(t_0, \mathfrak{y}_0) \\ & - \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]} (-1)^{\iota+1} ({}^{\zeta}C_{\iota}) [\mathfrak{I}(\varpi(t) - \iota\tau, \mathfrak{y}(\varpi(t)) - \tau^{\zeta}\mathfrak{D}(t, \mathfrak{y}(t)) - \mathfrak{I}(t_0, \mathfrak{y}_0)] \Big\} \end{aligned}$$

and

$$\begin{aligned} ^{C}\Delta_{+}^{\zeta}\mathfrak{I}(t, \mathfrak{y}) = \limsup_{\tau \to 0^+} \frac{1}{\tau^{\zeta}} & \left\{ \mathfrak{I}(\varpi(t), \mathfrak{y}(\varpi(t)) + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]} (-1)^{\iota} ({}^{\zeta}C_{\iota}) \left[\mathfrak{I}(\varpi(t) - \iota\tau, \mathfrak{y}(\varpi(t)) - \tau^{\zeta}\mathfrak{D}(t, \mathfrak{y}(t)) \right] \right\} \\ & - \frac{\mathfrak{I}(t_0, \mathfrak{y}_0)(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)}, \end{aligned} \tag{2.2}$$

where $t \in \mathbb{T}$, and $\mathfrak{y}, \mathfrak{y}_0 \in \mathbb{R}^n$, and

$$\mathfrak{y}(\varpi(t)) - \tau^{\zeta}\mathfrak{D}(t, \mathfrak{y}) \in \mathbb{R}^N.$$

For discrete times, Eq (2.1) *becomes*

$$^{C}\Delta_{+}^{\zeta}\mathfrak{I}(t, \mathfrak{y}) = \frac{1}{\tau^{\zeta}} \Big[\sum_{\iota=0}^{[\frac{t-t_0}{\tau}]} (-1)^{\iota} ({}^{\zeta}C_{\iota}) (\mathfrak{I}(\varpi(t), \mathfrak{y}(\varpi(t))) - \mathfrak{I}(t_0, \mathfrak{y}_0)) \Big],$$

and for continuous times ($\mathbb{T} = \mathbb{R}$), Eq (2.1) becomes

$$\begin{aligned} ^{C}\Delta_{+}^{\zeta}\mathfrak{I}(t, \mathfrak{y}) = {}^{C}D_{+}^{\zeta}\mathfrak{I}(t, \mathfrak{y}) = \limsup_{\kappa \to 0^+} \frac{1}{\kappa^{\zeta}} \Big\{ & \mathfrak{I}(t, \mathfrak{y}(t)) - \mathfrak{I}(t_0, \mathfrak{y}_0) \\ & - \sum_{\iota=1}^{[\frac{t-t_0}{\kappa}]} (-1)^{\iota+1} ({}^{\zeta}C_{\iota}) [\mathfrak{I}(t - \iota\kappa, \mathfrak{y}(t)) - \kappa^{\zeta}\mathfrak{D}(t, \mathfrak{y}(t)) - \mathfrak{I}(t_0, \mathfrak{y}_0)] \Big\}. \end{aligned} \tag{2.3}$$

Notice that (2.3) *is the same in [5], where $\kappa > 0$.*

Definition 2.6. *Let*

$$\mathfrak{y}(t) = \mathfrak{y}(t; t_0, \mathfrak{y}_0)$$

be any solution of (1.1)*, then system* (1.1) *is said to be:*

(UP_1) *(m_0, m)-uniformly practically stable if, given* $(\lambda, A) \in \mathbb{R}_+$ *with* $0 < \lambda < A$*, we have* $m_0(t_0, \mathfrak{y}_0) < \lambda$ *implies* $m(\mathfrak{y}(t)) < A$, $\forall\ t_0 \in \mathbb{T}$*;*

(UP_2) *(m_0, m)-uniformly practically quasi-stable if given* $(\lambda, B, T) > 0$ *and* $t_0 \in \mathbb{T}$*, we have* $m_0(t_0, \mathfrak{y}_0) < \lambda \implies m(\mathfrak{y}(t)) < B$*, for* $t \geq t_0 + T$*;*

(UP_3) *(m_0, m)-SUPS if* (UP_1) *and* (UP_2) *hold together.*

Definition 2.7. *Corresponding to Definition 2.6, the solution*

$$\kappa(t) = \kappa(t; t_0, \kappa_0)$$

of (1.2)*, is called uniformly practically stable if given* $0 < \lambda < A$*, we have*

$$\sum_{i=1}^{n} \kappa_0 < \lambda \implies \sum_{i=1}^{n} \kappa_i(t; t_0, \kappa_0) < A, \quad t \geq t_0, \tag{2.4}$$

for some $t_0 \in \mathbb{T} \cap \mathbb{R}_+$*.*

Lemma 2.1. *[23] Let*

$$\mathcal{R}, \mathcal{B} \in C_{rd}[\mathbb{T}, \mathbb{R}^n].$$

Assume $\exists\ t_1 > t_0$*, with*

$$t_1 \in \mathbb{T} : \mathcal{R}(t_1) = \mathcal{B}(t_1)$$

and

$$\mathcal{R}(t) < \mathcal{B}(t)$$

for $t_0 \leq t < t_1$*. If the CFrΔDiD of* $\mathcal{R}$ *and* $\mathcal{B}$ *are defined at* t_1*, then the inequality*

$${}^{C}\Delta_+^{\zeta}\mathcal{R}(t_1) >^{C} \Delta_+^{\zeta}\mathcal{B}(t_1)$$

holds.

Lemma 2.2. *(Comparison theorem) Let*

(*i*) $\Theta \in C_{rd}[\mathbb{T} \times \mathbb{R}_+^n, \mathbb{R}_+^n]$ *and* $\Theta(t, \kappa)\tau$ *is non-decreasing in* κ*.*

(*ii*) $\daleth(t, \mathfrak{y}) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+^N]$ *be locally Lipschitzian in* $\mathfrak{y}$ *and*

$${}^{C}\Delta_+^{\zeta}\daleth(t, \mathfrak{y}) \leq \Theta(t, \daleth(t, \mathfrak{y})), (t, \mathfrak{y}) \in \mathbb{T} \times \mathbb{R}^N.$$

(*iii*) $\varrho(t) = \varrho(t; t_0, \kappa_0)$ *be the maximal solution of (1.2) existing on* $\mathbb{T}$*.*

Then

$$\daleth(t, \mathfrak{y}(t)) \leq \varrho(t), \quad t \geq t_0 \tag{2.5}$$

provided that

$$\daleth(t_0, \mathfrak{y}_0) \leq \kappa_0,$$

where

$$\mathfrak{y}(t) = \mathfrak{y}(t; t_0, \mathfrak{y}_0)$$

is any solution of (1.1)*,* $t \in \mathbb{T}$*,* $t \geq t_0$*.*

Proof. We shall make this proof by the principle of induction. For an assertion

$$\mathfrak{S}(\mathrm{t}) : \daleth(t, \mathfrak{y}(t)) \leq \varrho(t), \quad t \in \mathbb{T},$$

the following holds:

(i) $\mathfrak{S}(\mathrm{t})$ is true since $\daleth(t_0, \mathfrak{y}_0) \leq \kappa_0$.

(ii) Assuming t is r-s and $\mathfrak{S}(\mathrm{t})$ is true. Then we prove that $\mathfrak{S}(\varpi(t))$ is true; i.e.,

$$\daleth(\varpi(t), \mathfrak{y}(\varpi(t))) \leq \varrho(\varpi(t)), \tag{2.6}$$

set

$$\varkappa(t) = \daleth(t, \mathfrak{y}(t)),$$

and then

$$\varkappa(\varpi(t)) = \daleth(\varpi(t), \mathfrak{y}(\varpi(t))),$$

then from Definition 2.6, we obtain

$${}^{C}\Delta_{+}^{\zeta}\varkappa(t) = \limsup_{\tau\to0+} \frac{1}{\tau^{\zeta}} \left\{ \sum_{\iota=0}^{[\frac{(t-t_0)}{\tau}]} (-1)^{\iota\zeta} C_{\iota}[\varkappa(\varpi(t) - \iota\tau) - \varkappa(t_0)] \right\},$$

also,

$${}^{C}\Delta_{+}^{\zeta}\varrho(t) = \limsup_{\tau\to0+} \frac{1}{\tau^{\zeta}} \left\{ \sum_{\iota=0}^{[\frac{(t-t_0)}{\tau}]} (-1)^{\iota\zeta} C_{\iota}[\varrho(\varpi(t) - \iota\tau) - \varrho(t_0)] \right\}, \quad t \geq t_0,$$

then,

$$\begin{aligned} {}^{C}\Delta_{+}^{\zeta}\varrho(t) -{}^{C} \Delta_{+}^{\zeta}\varkappa(t) &= \limsup_{\tau\to0+} \frac{1}{\tau^{\zeta}} \left\{ \sum_{\iota=0}^{[\frac{(t-t_0)}{\tau}]} (-1)^{\iota\zeta} C_{\iota}[\varrho(\varpi(t) - \iota\tau) - \varrho(t_0)] \right\} \\ &\quad - \limsup_{\tau\to0+} \frac{1}{\tau^{\zeta}} \left\{ \sum_{\iota=0}^{[\frac{(t-t_0)}{\tau}]} (-1)^{\iota\zeta} C_{\iota}[\varkappa(\varpi(t) - \iota\tau) - \varkappa(t_0)] \right\}, \\ \left({}^{C}\Delta_{+}^{\zeta}\varrho(t) -{}^{C} \Delta_{+}^{\zeta}\varkappa(t)\right)\tau^{\zeta} &= \limsup_{\tau\to0+} \left\{ \sum_{\iota=0}^{[\frac{(t-t_0)}{\tau}]} (-1)^{\iota\,\zeta} C_{\iota}\Big[[\varrho(\varpi(t) - \iota\tau) - \varrho(t_0)] \right. \\ &\quad \left. - [\varkappa(\varpi(t) - \iota\tau) - \varkappa(t_0)]\Big]\right\} \\ &\leq [\varrho(\varpi(t)) - \varrho(t_0)] - [\varkappa(\varpi(t)) - \varkappa(t_0)] \\ &\leq [\varrho(\varpi(t)) - \varkappa(\varpi(t))] - [\varrho(t_0) - \varkappa(t_0)], \end{aligned}$$

then,

$$[\varrho(\varpi(t)) - \varkappa(\varpi(t))] \geq \left({}^{C}\Delta_{+}^{\zeta}\varrho(t) -{}^{C} \Delta_{+}^{\zeta}\varkappa(t)\right)\tau^{\zeta} + [\varrho(t_0) - \varkappa(t_0)],$$

so that,

$$[\varkappa(\varpi(t)) - \varrho(\varpi(t))] \leq \Big({}^{C}\Delta_{+}^{\zeta}\varrho(t) -{}^{C}\Delta_{+}^{\zeta}\varkappa(t)\Big)\tau^{\zeta} + [\varkappa(t_0) - \varrho(t_0)]$$
$$\leq \Big(\Theta(t,\varkappa(t)) - \Theta(t,\varrho(t))\Big)\tau^{\zeta} + [\varkappa(t_0) - \varrho(t_0)].$$

By the non-decreasing property of $\Theta(t,\kappa)\tau$ and since $\mathfrak{S}$(t) is true, then

$$\varkappa(\varpi(t)) - \varrho(\varpi(t)) \leq 0,$$

so Eq (2.6) holds.

(iii) For r-d points $t^* \in \mathcal{U}$, where $\mathcal{U}$ is a right neighborhood $t \in \mathbb{T}$. We can clearly see that $\mathfrak{S}(t^*)$ is true immediately from [5, Lemma 3] since at every r-d point of $\mathbb{T}$, $\daleth(t,\mathfrak{y}(t))$ is continuous and the domain is $\mathbb{R}$.

(iv) Let t be l-d, and let $\mathfrak{S}(s)$ be true for all $s > t$. We want to show that $\mathfrak{S}(t)$ is true. This immediately follows r-d continuity of $\daleth(t,\mathfrak{y}(t))$, $\mathfrak{y}(t)$, and the maximal solution $\varrho(t)$. Thus, by the principle of induction, the assertion $\mathfrak{S}(t)$ holds for all $t \in \mathbb{T}$, thereby concluding the proof.

The proof is completed. □

3. Main results

Now, we present the (m_0,m)-UPS and (m_0,m)-SUPS results for the fractional dynamic system (1.1).

Theorem 3.1. *((m,m_0)-UPS) Assume that*

($\mathfrak{m}_1$) $0 < \lambda < A$;

($\mathfrak{m}_2$) $m_0, m \in \mathfrak{M}$, *and* m_0 *is uniformly finer than* m, *implying* $m(t,\mathfrak{y}) \leq \phi(m_0(t,\mathfrak{y}))$, $\phi \in \mathcal{K}$, *whenever* $m_0(t,\mathfrak{y}) < \lambda$;

($\mathfrak{m}_3$) *There exists*

$$\daleth \in C_{rd}[\mathbb{T}\times\mathbb{R}^N, \mathbb{R}_+^N]$$

and

$$\mathfrak{Q} \in C_{rd}[\mathbb{R}_+^n, \mathbb{R}_+^n],$$

such that if

$$\mathfrak{Q}(\daleth(t,\mathfrak{y})) \equiv \mathfrak{V}(t,\mathfrak{y}),$$

$\mathfrak{V}(t,\mathfrak{y})$ *is locally Lipschitz in* $\mathfrak{y}$, *and for*

$$(t,\mathfrak{y}) \in \mathfrak{S}(m,A) = \{(t,\mathfrak{y}) \in \mathbb{T}\times\mathbb{R}^N : m(t,\mathfrak{y}) < A\},$$

then,

$$\begin{aligned} \mathfrak{b}(m(t,\mathfrak{y})) &\leq \sum_{i=0}^{n}\mathfrak{V}_i(t,\mathfrak{y}), \quad \text{if} \quad m(t,\mathfrak{y}) < A \\ \sum_{0}^{\infty}\mathfrak{V}_i(t,\mathfrak{y}) &\leq \mathfrak{a}(m_0(t,\mathfrak{y})), \quad \text{if} \quad m_0(t,\mathfrak{y}) < \lambda; \end{aligned} \tag{3.1}$$

($\mathfrak{m}_4$) *for* $(t, \mathfrak{y}) \in \mathfrak{S}(m, A)$

$$^{C}\Delta_{+}^{\zeta}\mathfrak{V}(t, \mathfrak{y}) \leq \Theta(t, \mathfrak{V}(t, \mathfrak{y})), \tag{3.2}$$

where

$$\Theta \in C_{rd}[\mathbb{T} \times \mathbb{R}_{+}^{n}, \mathbb{R}_{+}^{n}],$$

$\Theta(t, \kappa)$ *is quasimonotone nondecreasing in* κ *with*

$$\Theta(t, \kappa) \equiv 0,$$

$\mathfrak{a} \in \mathcal{CK}$*, and for each i,* $1 \leq i \leq n$, $\Theta_i(t, \kappa)\tau(t) + \kappa_i$ *is nondecreasing in* κ *for all* $t \in \mathbb{T}$*;*

($\mathfrak{m}_5$) $\phi(\lambda) < A$ *and* $\mathfrak{a}(\lambda) < \mathfrak{b}(A)$ *hold;*

Then the UPS properties of (1.2) *imply the corresponding* (m_0, m)*-UPS properties of* (1.1).

Proof. Since (1.2) is UPS, then we deduce from (2.4) that $\forall\, t_0 \in \mathbb{T}$ and $0 < \lambda < A$,

$$\sum_{i=0}^{n} \kappa_{0_i} < \mathfrak{a}(\lambda) \implies \sum_{i=1}^{n} \kappa_i(t; t_0, \kappa_0) < \mathfrak{b}(A), \quad t \geq t_0. \tag{3.3}$$

We can easily assert that (1.1) is (m, m_0)-UPS with respect to (λ, A).

However, if the assertion were false, then for any solution

$$\mathfrak{y}(t) = \mathfrak{y}(t; t_0, \mathfrak{y}_0)$$

of (1.1) with

$$m_0(t_0, \mathfrak{y}_0) < \lambda,$$

there would be a point $t_1 > t_0$ such that

$$m(t_1, \mathfrak{y}(t_1)) = A \;\text{ and }\; m(t, \mathfrak{y}(t)) \leq A \;\text{ for } t \in [t_0, t_1). \tag{3.4}$$

Setting

$$\kappa_{0_i} = \mathfrak{V}_i(t, \mathfrak{y}_0),$$

and from (2.5), we have

$$\mathfrak{V}(t, \mathfrak{y}(t)) \leq \varrho(t, t_0, \kappa_0), \tag{3.5}$$

where $\varrho(t)$ is the maximal solution of (1.2).

Also, from assumptions ($\mathfrak{m}_2$) and ($\mathfrak{m}_5$), it is clear to see that

$$m(t_0, \mathfrak{y}_0) \leq \phi(m_0, (t_0, \mathfrak{y}_0)) < \phi(\lambda) < A.$$

Then, we deduce that,

$$\sum_{i=1}^{n} \kappa_{0_i} < \sum_{0}^{\infty} \mathfrak{V}_i(t_0, \mathfrak{y}_0) \leq a(m_0(t_0, \mathfrak{y}_0)) < a(\lambda). \tag{3.6}$$

Combining (3.3)–(3.6), we obtain

$$\mathfrak{b}(A) = b(m(t_1, \mathfrak{y}(t_1))) \leq \sum_{i=1}^{n} \mathfrak{V}_i(t_1, \mathfrak{y}(t_1)) \leq \sum_{i=1}^{n} \varrho_i(t_1; t_0, \kappa_0) < b(A). \tag{3.7}$$

Equation (3.7) is a contradiction, so the statement that (1.1) is (m_0, m)-UPS is true. □

Theorem 3.2. *((m, m_0)-SUPS) Assume that conditions* $(\mathfrak{m}_1)$–$(\mathfrak{m}_4)$ *of Theorem 3.1 is satisfied, and Eq* (1.2) *is SUPS; then, Eq* (1.1) *is* (m, m_0)*-SUPS.*

Proof. To make this proof, we need (1.1) to be (m_0, m)-UPS and (m_0, m)-uniformly practically quasi-stable together. Now, (1.1) is (m_0, m)-UPS by Theorem 3.1, so we need to prove only (m_0, m)-uniform practical quasi-stability. By the assumption of the theorem, Eq (1.2) is (m_0, m)-SUPS for

$$(\mathfrak{a}(\lambda), \mathfrak{b}(\lambda), \mathfrak{b}(B), T) > 0,$$

so that for all $t_0 \in \mathbb{T}$,

$$\sum_{i=1}^{n} \kappa_{0_i} < \mathfrak{a}(\lambda) \implies \sum_{i=1}^{n} \kappa_i(t, t_0, \kappa_0) < \mathfrak{b}(B), \quad t \geq t_0 + T, \tag{3.8}$$

where $\kappa_i(t, t_0, \kappa_0)$ is any solution of (1.2). By the UPS of (1.2), we can comfortably make the assumption that

$$m_0(t_0, \mathfrak{y}_0) < \lambda,$$

so that

$$m(t, \mathfrak{y}(t)) < A$$

for $t \geq t_0$. Setting

$$\kappa_{0_i} = \mathfrak{V}_i(t_0, \mathfrak{y}_0),$$

and by Lemma 2.2, we have the estimate

$$\mathfrak{V}(t, \mathfrak{y}(t)) \leq \varrho(t, t_0, \kappa_0), \tag{3.9}$$

where $\varrho(t)$ is the maximal solution of (1.2). Following this argument closely, and from (3.8) and (3.9), we obtain

$$\mathfrak{b}(m(t, \mathfrak{y}(t))) \leq \sum_{i=1}^{n} \mathfrak{V}_i(t, \mathfrak{y}(t)) \leq \sum_{i=1}^{n} \varrho_i(t, t_0, \kappa_0) < \mathfrak{b}(B), \quad t \geq t_0 + T.$$

So that

$$m(t, \mathfrak{y}(t)) < B$$

whenever

$$m_0(t_0, \mathfrak{y}_0) < \lambda,$$

for $t \geq t_0 + T$. Therefore, we conclude that (1.1) is (m_0, m)-SUPS. □

4. Illustrations

4.1. Illustration 1

Given the following system

$$\begin{aligned} {}^{C}\Delta^{\zeta}\wp_1(t) &= -2\wp_1 - \wp_1^2 \exp(\wp_1) - \wp_2^2 \exp(\wp_1), \\ {}^{C}\Delta^{\zeta}\wp_2(t) &= -\wp_1^2 \exp(\wp_2) - 2\wp_2 - \wp_2^2 \exp(\wp_2), \end{aligned} \tag{4.1}$$

with initial conditions

$$\wp_1(t_0) = \wp_{10}$$

and

$$\wp_2(t_0) = \wp_{20},$$

for

$$(\wp_1, \wp_2) \in \mathbb{R}^2.$$

Choose the LF for (4.1) to be

$$\daleth = (\mathfrak{V}_1, \mathfrak{V}_2)^T,$$

where

$$\mathfrak{V}_1(t, \wp_1, \wp_2) = |\wp_1|$$

and

$$\mathfrak{V}_2(t, \wp_1, \wp_2) = |\wp_2|.$$

Using (2.5), the CFrΔDiD for

$$\mathfrak{V}_1(t, \wp_1, \wp_2) = |\wp_1|$$

is computed as

$$\begin{aligned}
{}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_1(t, \wp_1) &= \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}}\Big\{\mathfrak{V}_1(\varpi(t), \wp_1(\varpi(t)) \\
&\quad + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota})\Big[\mathfrak{V}_1(\varpi(t) - \iota\tau, \wp_1(\varpi(t)) - \tau^{\zeta}\mathfrak{D}_1(t, \wp_1(t))\Big]\Big\} - \frac{\mathfrak{V}_1(t_0, \wp_{10})(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \\
&= \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}}\left\{|\wp_1(\varpi(t))| + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota})[|\wp_1(\varpi(t)) - \tau^{\zeta}\mathfrak{D}_1(t, \wp_1)|]\right\} - \frac{|\wp_{10}|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \\
&\le \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}}\left\{|\wp_1(\varpi(t))| + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota})[|\wp_1(\varpi(t))| + |\tau^{\zeta}\mathfrak{D}_1(t, \wp_1)|]\right\} - \frac{|\wp_{10}|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \\
&\le \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}}\left\{|\wp_1(\varpi(t))| + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota})|\wp_1(\varpi(t))| + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota})|\tau^{\zeta}\mathfrak{D}_1(t, \wp_1)|\right\} - \frac{|\wp_{10}|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \\
&\le |\wp_1(\varpi(t))| \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}} \sum_{\iota=0}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota}) + |\mathfrak{D}_1(t, \wp_1)| \limsup_{\tau\to 0^+} \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota}) - \frac{|\wp_{10}|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)}.
\end{aligned}$$

Applying (2.12) and (2.14) in [23], we obtain

$$\begin{aligned}
{}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_1(t, \wp_1) &= \frac{|\wp_1(\varpi(t))|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} - |\mathfrak{D}_1(t; \wp_1)| - \frac{|\wp_{10}|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)}, \\
{}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_1 &\le \frac{|\wp_1(\varpi(t))|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} - |\mathfrak{D}_1(t; \wp_1)|.
\end{aligned}$$

When

$$t \to \infty, \quad \frac{|\wp_1(\varpi(t))|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \to 0,$$

then

$$\begin{aligned} {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_1 &\le -|\mathfrak{D}_1(t;\wp_1)| \\ &= -\Big[| - 2\wp_1 - \wp_1^2\exp(\wp_1) - \wp_2^2\exp(\wp_1)|\Big] \\ &= -\Big[| - 2\wp_1 - (\wp_1^2 + \wp_2^2)\exp(\wp_1)|\Big] \\ &\le -[| - 2\wp_1|] \\ &\le -[2|\wp_1|], \\ {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_1 &\le -2\mathfrak{V}_1 + 0\mathfrak{V}_2. \end{aligned} \tag{4.2}$$

Similarly, compute the CFrΔDiD for

$$\begin{aligned} \mathfrak{V}_2(t,\wp_1,\wp_2) = |\wp_2| &= \limsup_{\tau\to0^+}\frac{1}{\tau^{\zeta}}\left\{|\wp_2(\varpi(t))| + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota})[|\wp_2(\varpi(t)) - \tau^{\zeta}\mathfrak{D}_2(t,\wp_2)|]\right\} - \frac{|\wp_{20}|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \\ &\le \limsup_{\tau\to0^+}\frac{1}{\tau^{\zeta}}\left\{|\wp_2(\varpi(t))| + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota})[|\wp_2(\varpi(t))| + |\tau^{\zeta}\mathfrak{D}_2(t,\wp_2)|]\right\} - \frac{|\wp_{20}|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \\ &\le \limsup_{\tau\to0^+}\frac{1}{\tau^{\zeta}}\left\{|\wp_2(\varpi(t))| + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota})|\wp_2(\varpi(t))| + \sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota})|\tau^{\zeta}\mathfrak{D}_2(t,\wp_2)|\right\} - \frac{|\wp_{20}|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \\ &\le |\wp_2(\varpi(t))|\limsup_{\tau\to0^+}\frac{1}{\tau^{\zeta}}\sum_{\iota=0}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota}) + |\mathfrak{D}_2(t,\wp_2)|\limsup_{\tau\to0^+}\sum_{\iota=1}^{[\frac{t-t_0}{\tau}]}(-1)^{\iota}({}^{\zeta}C_{\iota}) - \frac{|\wp_{20}|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)}. \end{aligned}$$

Applying (2.12) and (2.14) in [23], we obtain

$${}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_2 \le \frac{|\wp_2(\varpi(t))|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} - |\mathfrak{D}_2(t;\wp_2)|.$$

When

$$t \to \infty, \quad \frac{|\wp_2(\varpi(t))|(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \to 0,$$

then,

$$\begin{aligned} {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_2 &\le -|\mathfrak{D}_2(t;\wp_2)| \\ &= -\Big[| - \wp_1^2\exp(\wp_2) - 2\wp_2 - \wp_2^2\exp(\wp_2)|\Big] \\ &= -\Big[| - 2\wp_2 - (\wp_1^2 - \wp_2^2)\exp(\wp_2)|\Big] \\ &\le -[| - 2\wp_2|] \\ &\le -2|\wp_2|. \end{aligned}$$

Therefore,

$$ {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_2 \leq 0\mathfrak{V}_1 - 2\mathfrak{V}_2. \tag{4.3} $$

From (4.2) and (4.3), it follows that

$$ {}^{C}\Delta_{+}^{\zeta}\gimel \leq \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \mathfrak{V}_1 \\ \mathfrak{V}_2 \end{pmatrix} = \Theta(t, \mathfrak{V}). \tag{4.4} $$

Next, we choose a comparison system

$$ {}^{C}\Delta_{+}^{\zeta}\kappa = \Theta(t, \kappa) = M\kappa \tag{4.5} $$

with

$$ M = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}. $$

Clearly, Figure 1 below shows the stability of system (4.2). The vector inequality (4.4), along with all the other requirements of Theorem 3.1, holds when the matrix M has eigenvalues with negative real parts. Given that the eigenvalues of M are both -2, it can be concluded that (4.1) is not only (m, m_0)-UPS but also (m, m_0)-SUPS.

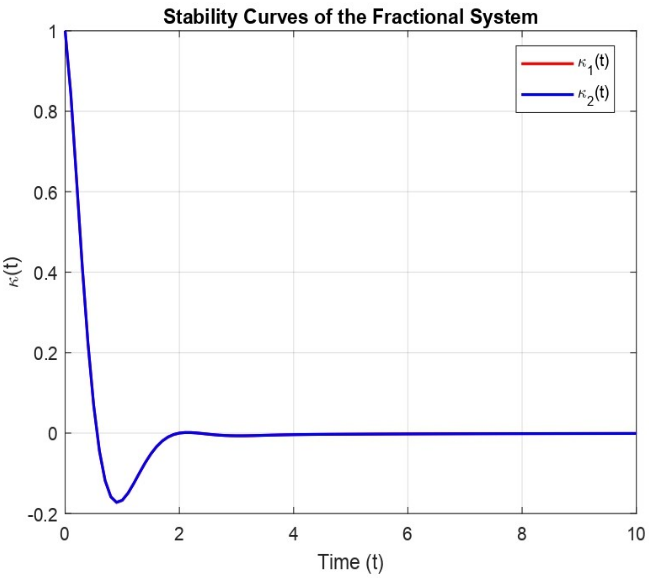

Figure 1. Stability of system (4.5).

4.2. *Illustration 2*

Given the following system

$$ \begin{aligned} {}^{C}\Delta^{\zeta}\aleph_1(t) &= 4\aleph_1 - \frac{\aleph_2^2 + 3\aleph_3^2}{\aleph_1}, \\ {}^{C}\Delta^{\zeta}\aleph_2(t) &= -\frac{3\aleph_1^2 +}{\aleph_2} + 4\aleph_2 - \frac{\aleph_3^2}{\aleph_2}, \\ {}^{C}\Delta^{\zeta}\aleph_3(t) &= -\frac{\aleph_1^2}{\aleph_3} + 3\aleph_3, \end{aligned} \tag{4.6} $$

with initial conditions

$$\aleph_1(t_0) = \aleph_{10}, \quad \aleph_2(t_0) = \aleph_{20}, \quad \text{and} \quad \aleph_3(t_0) = \aleph_{30},$$

for

$$\aleph = (\aleph_1, \aleph_2, \aleph_3) \in \mathbb{R}^3.$$

Choose a vector LF

$$\beth = (\mathfrak{V}_1, \mathfrak{V}_2, \mathfrak{V}_3)^T : \mathfrak{V}_1 = \aleph_1^2, \quad \mathfrak{V}_2 = \aleph_2^2, \quad \mathfrak{V}_3 = \aleph_3^2$$

and

$$\beth_0(\aleph_1, \aleph_2, \aleph_3) = \sum_{i=1}^{2} \mathfrak{V}_i(\aleph_1, \aleph_2, \aleph_3) = \aleph_1^2 + \aleph_2^2 + \aleph_3^2.$$

Computing the CFrΔDiD for

$$\mathfrak{V}_1 = \aleph_1^2,$$

we obtain:

$$\begin{aligned}
{}^{C}\Delta_+^{\zeta}\mathfrak{V}_1 &= \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}} \Big\{ \left[(\aleph_1(\varpi(t)))^2\right] - \left[(\aleph_{10})^2\right] \\
&+ \sum_{\iota=1}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) [(\aleph_1(\varpi(t)) - \tau^{\zeta}\mathfrak{D}_1(t,\aleph))^2] - [((\aleph_{10})^2] \Big\} \\
&= \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}} \Big\{ \left[(\aleph_1(\varpi(t)))^2\right] - \left[(\aleph_{10})^2\right] \\
&+ \sum_{\iota=1}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) [(\aleph_1(\varpi(t)))^2 - 2\aleph_1(\varpi(t))\tau^{\zeta}\mathfrak{D}_1(t,\aleph_1,\aleph_2,\aleph_3) \\
&+ \tau^{2\zeta}(\mathfrak{D}_1(t,\aleph_1,\aleph_2,\aleph_3))^2] - \left[(\aleph_{10})^2\right] \Big\} \\
&= -\limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}} \left\{ \sum_{\iota=0}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) \left[(\aleph_{10})^2\right] \right\} + \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}} \left\{ \sum_{\iota=0}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) \left[(\aleph_1(\varpi(t)))^2\right] \right\} \\
&- \limsup_{\tau\to 0^+} \Big\{ \sum_{\iota=1}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) [2\aleph_1(\varpi(t))\tau^{\zeta}\mathfrak{D}_1(t,\aleph_1,\aleph_2,\aleph_3)] \Big\}.
\end{aligned}$$

From (2.12) and (2.14) in [23], we obtain

$${}^{C}\Delta_+^{\zeta}\mathfrak{V}_1 \leq \frac{(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \left[(\aleph_1(\varpi(t)))^2\right] - [2\aleph_1(\varpi(t))\mathfrak{D}_1(t,\aleph_1,\aleph_2,\aleph_3)].$$

When

$$t \to \infty, \quad \frac{(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \left[(\aleph_1(\varpi(t)))^2\right] \to 0,$$

so that we obtain

$$ {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_1 \leq -2[\aleph_1(\varpi(t))\mathfrak{D}_1(t,\aleph_1,\aleph_2,\aleph_3)]. $$

Using

$$ \aleph(\varpi(t)) \leq \tau^{C}\Delta^{\zeta}\aleph(t) + \aleph(t), $$

we obtain

$$ \begin{aligned} {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_1 &= -2\Big[\tau(t)\mathfrak{D}_1^2(t,\aleph_1,\aleph_2,\aleph_3) + \aleph_1(t)\mathfrak{D}_1(t,\aleph_1,\aleph_2,\aleph_3)\Big] \\ &= -2\left[\tau(t)\left(4\aleph_1 - \frac{\aleph_2^2+3\aleph_3^2}{\aleph_1}\right)^2 + \aleph_1\left(4\aleph_1 - \frac{\aleph_2^2+3\aleph_3^2}{\aleph_1}\right)\right] \\ &= -2\tau(t)\left[\left(4\aleph_1 - \frac{\aleph_2^2+3\aleph_3^2}{\aleph_1}\right)^2\right] - 2\aleph_1\left[4\aleph_1 - \frac{\aleph_2^2+3\aleph_3^2}{\aleph_1}\right]. \end{aligned} \tag{4.7} $$

If $\mathbb{T}=\mathbb{R}$ then $\tau=0$, reducing (4.7) to the form

$$ \begin{aligned} {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_1(\aleph_1,\aleph_2,\aleph_3) &= -2\aleph_1\left[4\aleph_1 - \frac{\aleph_2^2+3\aleph_3^2}{\aleph_1}\right] \\ &= -8\aleph_1^2 + 2\aleph_2^2 + 6\aleph_3^2 \\ &= (-8\ \ 2\ \ 6)\cdot(\mathfrak{V}_1\ \ \mathfrak{V}_2\ \ \mathfrak{V}_3)^T. \end{aligned} \tag{4.8} $$

When $\mathbb{T}=\mathbb{N}_0$ then, $\tau=1$, reducing (4.7) to

$$ \begin{aligned} {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_1(\aleph_1,\aleph_2,\aleph_3) &= -2\left[\left(4\aleph_1 - \frac{\upsilon_2^2+3\aleph_3^2}{\aleph_1}\right)^2\right] - 2\aleph_1\left[4\aleph_1 - \frac{\aleph_2^2+3\aleph_3^2}{\aleph_1}\right] \\ &\leq -2\aleph_1\left[4\aleph_1 - \frac{\aleph_2^2+3\aleph_3^2}{\aleph_1}\right], \end{aligned} $$

resulting in the same outcome as (4.8). Evidently, this applies to any other discrete time as well.

Also, computing the CFrΔDiD for

$$ \mathfrak{V}_2(\aleph) = \aleph_2^2, $$

we obtain:

$$ \begin{aligned} {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_2(\aleph) &= \limsup_{\tau\to 0^+}\frac{1}{\tau^{\zeta}}\Big\{\Big[(\aleph_2(\varpi(t)))^2\Big] - \Big[(\aleph_{20})^2\Big] \\ &+ \sum_{\iota=1}^{\left[\frac{t-t_0}{\tau}\right]}(-1)^{\iota}\left({}^{\zeta}C_{\iota}\right)[(\aleph_2(\varpi(t)) - \tau^{\zeta}\mathfrak{D}_2(t,\aleph))^2] - [(\aleph_{20})^2]\Big\} \\ &= \limsup_{\tau\to 0^+}\frac{1}{\tau^{\zeta}}\Big\{\Big[(\aleph_2(\varpi(t)))^2\Big] - \Big[(\aleph_{20})^2\Big] + \sum_{\iota=1}^{\left[\frac{t-t_0}{\tau}\right]}(-1)^{\iota}\left({}^{\zeta}C_{\iota}\right)[(\aleph_2(\varpi(t)))^2 \\ &- 2\aleph_2(\varpi(t))\tau^{\zeta}\mathfrak{D}_2(t,\aleph_1,\aleph_2,\aleph_3) + \tau^{2\zeta}(\mathfrak{D}_2(t,\aleph_1,\aleph_2,\aleph_3))^2] - \Big[(\aleph_{20})^2\Big]\Big\} \end{aligned} $$

$$= -\limsup_{\tau\to 0^+}\frac{1}{\tau^\zeta}\left\{\sum_{\iota=0}^{\left[\frac{t-t_0}{\tau}\right]}(-1)^\iota\left({}^\zeta C_\iota\right)\left[(\aleph_{20})^2\right]\right\}+\limsup_{\tau\to 0^+}\frac{1}{\tau^\zeta}\left\{\sum_{\iota=0}^{\left[\frac{t-t_0}{\tau}\right]}(-1)^\iota\left({}^\zeta C_\iota\right)\left[(\aleph_2(\varpi(t)))^2\right]\right\}$$
$$-\limsup_{\tau\to 0^+}\left\{\sum_{\iota=1}^{\left[\frac{t-t_0}{\tau}\right]}(-1)^\iota\left({}^\zeta C_\iota\right)[2\aleph_2(\varpi(t))\tau^\zeta\mathfrak{D}_2(t,\aleph_1,\aleph_2,\aleph_3)]\right\}.$$

From (2.12) and (2.14) in [23], we obtain

$${}^C\Delta_+^\zeta\mathfrak{V}_2(\aleph)\le\frac{(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)}\left[(\aleph_2(\varpi(t)))^2\right]-[2\aleph_2(\varpi(t))\mathfrak{D}_2(t,\aleph_1,\aleph_2,\aleph_3)].$$

When

$$t\to\infty,\quad \frac{(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)}\left[(\aleph_2(\varpi(t)))^2\right]\to 0,$$

so that

$${}^C\Delta_+^\zeta\mathfrak{V}_2(\aleph)\le -2[\aleph_2(\varpi(t))\mathfrak{D}_2(t,\aleph_1,\aleph_2,\aleph_3)],$$

using

$$\aleph(\varpi(t))\le\tau^C\Delta^\zeta\aleph(t)+\aleph(t),$$

we obtain

$$\begin{aligned}{}^C\Delta_+^\zeta\mathfrak{V}_2(\aleph)&=-2\Big[\tau(t)\mathfrak{D}_2^2(t,\aleph_1,\aleph_2,\aleph_3)+\aleph_2(t)\mathfrak{D}_2(t,\aleph_1,\aleph_2,\aleph_3)\Big]\\&=-2\left[\tau(t)\left(-\frac{3\aleph_1^2+}{\aleph_2}+4\aleph_2-\frac{\aleph_3^2}{\aleph_2}\right)^2+\aleph_2\left(-\frac{3\aleph_1^2+}{\aleph_2}+4\aleph_2-\frac{\aleph_3^2}{\aleph_2}\right)\right]\\&=-2\tau(t)\left[\left(-\frac{3\aleph_1^2+}{\aleph_2}+4\aleph_2-\frac{\aleph_3^2}{\aleph_2}\right)^2\right]-2\aleph_1\left[-\frac{3\aleph_1^2+}{\aleph_2}+4\aleph_2-\frac{\aleph_3^2}{\aleph_2}\right].\end{aligned}\tag{4.9}$$

If $\mathbb{T}=\mathbb{R}$, then $\tau=0$, reducing (4.9) to

$$\begin{aligned}{}^C\Delta_+^\zeta\mathfrak{V}_2(\aleph_1,\aleph_2)&=-2\aleph_2\left[-\frac{3\aleph_1^2+}{\aleph_2}+4\aleph_2-\frac{\aleph_3^2}{\aleph_2}\right]\\&=6\aleph_1^2-8\aleph_2^2+\aleph_3^2\\&=(6\ \ -8\ \ 1)\cdot(\mathfrak{V}_1\ \ \mathfrak{V}_2\ \ \mathfrak{V}_3)^T.\end{aligned}\tag{4.10}$$

When $\mathbb{T}=\mathbb{N}_0$, then $\tau=1$, and (4.9) reduces to:

$$\begin{aligned}{}^C\Delta_+^\zeta\mathfrak{V}_2(\aleph)&=-2\left[\left(-\frac{3\aleph_1^2+}{\aleph_2}+4\aleph_2-\frac{\aleph_3^2}{\aleph_2}\right)^2\right]-2\aleph_1\left[-\frac{3\aleph_1^2+}{\aleph_2}+4\aleph_2-\frac{\aleph_3^2}{\aleph_2}\right]\\&\le -2\aleph_1\left[-\frac{3\aleph_1^2+}{\aleph_2}+4\aleph_2-\frac{\aleph_3^2}{\aleph_2}\right],\end{aligned}$$

resulting in (4.10).

Finally, for the CFrΔDiD of

$$\mathfrak{V}_3(\aleph_1, \aleph_2, \aleph_3) = \aleph_3^2,$$

we obtain

$$\begin{aligned}
{}^{C}\Delta_+^{\zeta}\mathfrak{V}_3 &= \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}}\Big\{ \Big[(\aleph_3(\varpi(t)))^2\Big] - \Big[(\aleph_{30})^2\Big] \\
&+ \sum_{\iota=1}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) [(\aleph_3(\varpi(t)) - \tau^{\zeta}\mathfrak{D}_3(t,\aleph))^2] - [(\aleph_{30})^2]\Big\} \\
&= \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}}\Big\{ \Big[(\aleph_3(\varpi(t)))^2\Big] - \Big[(\aleph_{30})^2\Big] \\
&+ \sum_{\iota=1}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) [(\aleph_3(\varpi(t)))^2 - 2\aleph_3(\varpi(t))\tau^{\zeta}\mathfrak{D}_3(t,\aleph_1,\aleph_2,\aleph_3) \\
&+ \tau^{2\zeta}(\mathfrak{D}_3(t,\aleph_1,\aleph_2,\aleph_3))^2] - \Big[(\aleph_{30})^2\Big]\Big\} \\
&= -\limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}} \left\{ \sum_{\iota=0}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) \Big[(\aleph_{30})^2\Big] \right\} + \limsup_{\tau\to 0^+} \frac{1}{\tau^{\zeta}} \left\{ \sum_{\iota=0}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) \Big[(\aleph_3(\varpi(t)))^2\Big] \right\} \\
&- \limsup_{\tau\to 0^+} \Big\{ \sum_{\iota=1}^{\left[\frac{t-t_0}{\tau}\right]} (-1)^{\iota} \left({}^{\zeta}C_{\iota}\right) [2\aleph_3(\varpi(t))\tau^{\zeta}\mathfrak{D}_3(t,\aleph_1,\aleph_2,\aleph_3)]\Big\}.
\end{aligned}$$

From (2.12) and (2.14) in [23], we obtain

$${}^{C}\Delta_+^{\zeta}\mathfrak{V}_3 \quad \leq \quad \frac{(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \Big[(\aleph_3(\varpi(t)))^2\Big] - [2\aleph_1(\varpi(t))\mathfrak{D}_3(t,\aleph_1,\aleph_2,\aleph_3)].$$

As

$$t \to \infty, \quad \frac{(t-t_0)^{-\zeta}}{\Gamma(1-\zeta)} \Big[(\aleph_3(\varpi(t)))^2\Big] \to 0,$$

then

$${}^{C}\Delta_+^{\zeta}\mathfrak{V}_3 \quad \leq \quad -2[\aleph_3(\varpi(t))\mathfrak{D}_3(t,\aleph_1,\aleph_2,\aleph_3)].$$

Using

$$\aleph(\varpi(t)) \leq \tau^{C}\Delta^{\zeta}\aleph(t) + \aleph(t),$$

we obtain

$$\begin{aligned}
{}^{C}\Delta_+^{\zeta}\mathfrak{V}_3 &= -2\Big[\tau(t)\mathfrak{D}_3^2(t,\aleph_3,\aleph_2,\aleph_3) + \upsilon_3(t)\mathfrak{D}_3(t,\aleph_1,\aleph_2,\aleph_3)\Big] \\
&= -2\left[\tau(t)\left(-\frac{\aleph_1^2}{\aleph_3} + 3\aleph_3\right)^2 + \aleph_3\left(-\frac{\aleph_1^2}{\aleph_3} + 3\aleph_3\right)\right] \\
&= -2\tau(t)\left[\left(-\frac{\aleph_1^2}{\aleph_3} + 3\aleph_3\right)^2\right] - 2\aleph_3\left[-\frac{\aleph_1^2}{\aleph_3} + 3\aleph_3\right].
\end{aligned} \tag{4.11}$$

If $\mathbb{T}$ is continuous, then $\tau = 0$, and (4.11) reduces to:

$$\begin{aligned} {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_3(\aleph_1, \aleph_2, \aleph_3) &= -2\aleph_3 \left[-\frac{\aleph_1^2}{\aleph_3} + 3\aleph_3 \right] \\ &= 2\aleph_1^2 - 6\aleph_3^2 \\ &= (2 \;\; 0 \;\; -6) \cdot (\mathfrak{V}_1 \;\; \mathfrak{V}_2 \;\; \mathfrak{V}_3)^T. \end{aligned} \tag{4.12}$$

Otherwise, for discrete $\mathbb{T}$ $\tau = 1$, (4.11) reduces to:

$$\begin{aligned} {}^{C}\Delta_{+}^{\zeta}\mathfrak{V}_3 &= -2\left[\left(-\frac{\aleph_1^2}{\aleph_3} + 3\aleph_3 \right)^2 \right] - 2\aleph_1 \left[-\frac{\aleph_1^2}{\aleph_3} + 3\aleph_3 \right] \\ &\leq -2\aleph_1 \left[-\frac{\aleph_1^2}{\aleph_3} + 3\aleph_3 \right], \end{aligned}$$

then we also obtain (4.12).

From (4.8), (4.10), and (4.12), we obtain

$${}^{C}\Delta_{+}^{\zeta}\mathfrak{V} \leq \begin{pmatrix} -8 & 2 & 6 \\ 6 & -8 & 1 \\ 2 & 0 & -6 \end{pmatrix} \begin{pmatrix} \mathfrak{V}_1 \\ \mathfrak{V}_2 \\ \mathfrak{V}_3 \end{pmatrix} = \Theta(t, \mathfrak{V}). \tag{4.13}$$

Next, we choose a comparison system of the form

$${}^{C}\Delta_{+}^{\zeta}\kappa = \Theta(t, \kappa) = M\kappa$$

with

$$M = \begin{pmatrix} -8 & 2 & 6 \\ 6 & -8 & 1 \\ 2 & 0 & -6 \end{pmatrix}.$$

The vector inequality (4.13), along with all the conditions outlined in Theorems 3.1 and 3.2, hold true if the matrix M has eigenvalues with negative real parts. Given that the eigenvalues of M are

$$\lambda_1 = -12.433, \quad \lambda_2 = -7.199, \quad \lambda_3 = -2.369,$$

it follows that (4.6) is uniformly practically stable. Thus, we conclude that system (4.6) exhibits UPS.

5. Conclusions

In conclusion, this paper develops a unified approach for UPS analysis of CFrDy$\mathbb{T}$, utilizing vector LFs and a two-measure framework. By applying the CFrΔD and the CFrΔDiD, this study extends stability analysis methods across both continuous and discrete domains, making it versatile for hybrid systems exhibiting both gradual and abrupt changes. The use of vector LFs over scalar LFs enables a broader and more robust stability analysis, accommodating complex system dynamics with improved precision and adaptability. Illustrative examples, Eqs (4.1) and (4.6), demonstrate the practical application of the proposed framework, underscoring its effectiveness and relevance in capturing the essential stability characteristics of fractional dynamic systems. This work offers significant contributions to stability theory by enhancing the tools available for fractional dynamic systems, with implications for fields such as engineering, control theory, and applied sciences.

Author contributions

Michael Ineh: conceptualization, methodology, software, formal analysis, writing—original draft preparation, writing—review and editing; Umar Ishtiaq: conceptualization, validation, investigation, resources, visualization, funding acquisition; Jackson Ante: conceptualization, methodology, supervision; Mubariz Garayev: validation, investigation, resources, visualization, funding acquisition; Ioan-Lucian Popa: conceptualization, software, validation, investigation, project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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