AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(3): 6934–6951.
DOI: 10.3934/math.2025317
Received: 08 December 2024
Revised: 23 February 2025
Accepted: 18 March 2025
Published: 26 March 2025

Research article

Arbitrary finite spectrum assignment and stabilization of bilinear systems with multiple lumped and distributed delays in state

Vasilii Zaitsev* **and Inna Kim**

Laboratory of Mathematical Control Theory, Udmurt State University, Izhevsk, 426034, Russia

* **Correspondence:** Email: verba@udm.ru.

Abstract: We consider a bilinear control system defined by a linear time-invariant system of differential equations with multiple lumped and distributed delays in the state variable. A problem of finite spectrum assignment is studied. One needs to construct control vectors such that the characteristic function of the closed-loop system is equal to a polynomial with arbitrary given coefficients. We obtain conditions on coefficients of the system under which the criterion was found for solvability of this finite spectrum assignment problem. This criterion is expressed in terms of rank conditions for matrices of the special form. Corollaries on stabilization of a bilinear system with delays are obtained. An illustrative example is presented.

Keywords: linear time-invariant differential system; time-delay systems; spectrum assignment problem; stabilization; bilinear system
Mathematics Subject Classification: 93B55, 93D23, 93C43, 34H15

1. Introduction

Bilinear systems appear in the control of various real problems and have found a rich area of applications in physics, biochemistry, agriculture, engineering, economics, etc. (see, e.g., [1] and the references therein). The problems of stabilization of bilinear systems using the second (direct) Lyapunov method were studied in early works [2–5] and in later and recent works (see [6–10] and references therein). The Jurdjevic–Quinn stabilization technique [2] was extended to bilinear periodic systems [11–13], infinite-dimensional bilinear systems [14–16], bilinear systems with delays [17–19], and complex-valued bilinear systems [20–22]. See also the recent papers [23, 24].

Another approach for solving problems of stabilization and control over asymptotic behavior of bilinear systems is based on the first Lyapunov method (eigenvalue approach). The problems of assigning the spectrum of eigenvalues were studied in [25–29]. The eigenvalue approach for stabilizing systems with delays is presented in [30].

In the presented work we use the first Lyapunov method (eigenvalue approach). We consider a bilinear time-invariant system with multiple lumped and distributed delays. We are studying the problem of assigning a polynomial with given coefficients to the characteristic function.

The paper is organized as follows. In Section 2, we introduce notations and present some background to the issue and describe previously obtained results. In Section 3, we give the statement of the problem and present the main result of the work, namely, Theorem 1. In Section 4, we present auxiliary statements that are required to prove the main result. In Section 5 we give a proof of Theorem 1. In Section 6 we derive corollaries from the main result and note that Theorem 1 generalizes some results obtained earlier. In Section 7 we show that the main result can be extended to systems that have a slightly more general form. In Section 8, we provide modeling examples that illustrate the results obtained. In Section 9, we give a conclusion.

2. Notations and preliminaries

Relations $\alpha := \beta$ and $\beta =: \alpha$ mean that α is assumed, by definition, equal to β. Denote $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$; $\mathbb{K}^n = \{x = \operatorname{col}(x_1, \ldots, x_n) \colon x_i \in \mathbb{K}\}$ is the linear space of column vectors over $\mathbb{K}$; $M_{m,n}(\mathbb{K})$ is the space of $m \times n$-matrices over $\mathbb{K}$; $M_n(\mathbb{K}) := M_{n,n}(\mathbb{K})$; $I \in M_n(\mathbb{K})$ is the identity matrix; $O \in M_n(\mathbb{K})$ is the zero matrix; T is the transposition of a vector or a matrix; $*$ is the Hermitian conjugation, i.e., $A^* = \overline{A}^T$; $\operatorname{Sp} H$ is the trace of a matrix $H \in M_n(\mathbb{K})$; $\chi(H; \lambda) \overset{def}{=} \det(\lambda I - H)$; we use the denotation $A^0 := I$ for any matrix $A \in M_n(\mathbb{K})$; the notation $i = \overline{\alpha, \beta}$ (where $\alpha, \beta \in \mathbb{Z}$ and $\alpha \leq \beta$) means that i runs from α to β.

Consider a bilinear control system defined by a time-invariant differential system with multiple lumped and distributed delays in the state:

$$\begin{aligned}
\dot{x}(t) &= A_{00}x(t) + u_{01}A_{01}x(t) + \ldots + u_{0r_0}A_{0r_0}x(t) \\
&+ A_{10}x(t-\omega_1) + u_{11}A_{11}x(t-\omega_1) + \ldots + u_{1r_1}A_{1r_1}x(t-\omega_1) + \ldots \\
&+ A_{\ell 0}x(t-\omega_\ell) + u_{\ell 1}A_{\ell 1}x(t-\omega_\ell) + \ldots + u_{\ell r_\ell}A_{\ell r_\ell}x(t-\omega_\ell) \\
&+ \int_{-\omega_1}^{0} (C_{10}(\tau) + w_{11}(\tau)C_{11} + \ldots + w_{1q_1}(\tau)C_{1q_1})x(t+\tau)\,d\tau + \ldots \\
&+ \int_{-\omega_\ell}^{-\omega_{\ell-1}} (C_{\ell 0}(\tau) + w_{\ell 1}(\tau)C_{\ell 1} + \ldots + w_{\ell q_\ell}(\tau)C_{\ell q_\ell})x(t+\tau)\,d\tau, \quad t > 0,
\end{aligned} \tag{2.1}$$

with initial conditions $x(\tau) = x_0(\tau)$, $\tau \in [-\omega_\ell, 0]$; here $0 = \omega_0 < \omega_1 < \ldots < \omega_\ell$ are constant delays; $x_0 \colon [-\omega_\ell, 0] \to \mathbb{K}^n$ is a continuous function; $x \in \mathbb{K}^n$ is a state vector; $u_\mu = \operatorname{col}(u_{\mu 1}, \ldots, u_{\mu r_\mu}) \in \mathbb{K}^{r_\mu}$, $\mu = \overline{0, \ell}$, are control vectors; $w_\xi(\tau) = \operatorname{col}(w_{\xi 1}(\tau), \ldots, w_{\xi q_\xi}(\tau)) \in \mathbb{K}^{q_\xi}$, $\xi = \overline{1, \ell}$, $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$, are control vector functions; $A_{\mu\nu} \in M_n(\mathbb{K})$, $\mu = \overline{0, \ell}$, $\nu = \overline{0, r_\mu}$; $C_{\xi 0} \colon [-\omega_\xi, -\omega_{\xi-1}] \to M_n(\mathbb{K})$ are integrable functions, $C_{\xi\zeta} \in M_n(\mathbb{K})$, $\xi = \overline{1, \ell}$, $\zeta = \overline{1, q_\xi}$.

In [31], we considered a control system defined by a linear time-invariant differential equation of nth order with multiple not necessarily commensurate lumped and distributed delays in the state variable $x \in \mathbb{K}$; the input is a linear combination of m variables and its derivatives up to the order $\leq n - p$, and the output is a k-dimensional vector of linear combinations of the state x and its derivatives up to the order $\leq p - 1$:

$$x^{(n)}(t) + \sum_{i=1}^{n}\sum_{j=0}^{s} a_{ij}x^{(n-i)}(t-h_j) + \sum_{i=1}^{n}\sum_{\eta=1}^{s}\int_{-h_\eta}^{-h_{\eta-1}} g_{i\eta}(\tau)x^{(n-i)}(t+\tau)\,d\tau = \sum_{\alpha=1}^{m}\sum_{l=p}^{n} b_{l\alpha}\mathrm{u}_\alpha^{(n-l)}(t), \tag{2.2}$$

$$y_\beta(t)=\sum_{\nu=1}^{p}c_{\nu\beta}x^{(\nu-1)}(t),\quad \beta=\overline{1,k}, \tag{2.3}$$

$t>0$; here $x^{(n-i)}(\tau)=x_{i0}(\tau)$ are initial conditions, $\tau\in[-h_s,0]$; $0=h_0<h_1<\ldots<h_s$ are constant delays, $x_{i0}:[-h_s,0]\to\mathbb{K}$ are continuous functions; $a_{ij},b_{l\alpha},c_{\nu\beta}\in\mathbb{K}$, $i=\overline{1,n}$, $j=\overline{0,s}$, $l=\overline{p,n}$, $\alpha=\overline{1,m}$, $\nu=\overline{1,p}$, $\beta=\overline{1,k}$; $g_{i\eta}:[-h_\eta,-h_{\eta-1}]\to\mathbb{K}$ are integrable functions ($i=\overline{1,n},\eta=\overline{1,s}$); $\mathfrak{u}=\operatorname{col}(\mathfrak{u}_1,\ldots,\mathfrak{u}_m)\in\mathbb{K}^m$ is a control vector, $y=\operatorname{col}(y_1,\ldots,y_k)\in\mathbb{K}^k$ is an output vector; $p\in\{\overline{1,n}\}$. The controller in the system (2.2), (2.3) was constructed as linear static output feedback with lumped and distributed delays:

$$\mathfrak{u}(t)=\sum_{\rho=0}^{\theta}Q_\rho y(t-\sigma_\rho)+\sum_{\varkappa=1}^{\theta}\int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}}R_\varkappa(\tau)y(t+\tau)\,d\tau, \tag{2.4}$$

where $y(t)=0$, $t<-h_s$; here $\theta\geq 0$ is an integer, $0=\sigma_0<\sigma_1<\ldots<\sigma_\theta$ are constant delays; $Q_\rho=\{\mathfrak{q}^\rho_{\alpha\beta}\}\in M_{m,k}(\mathbb{K})$ are constant matrices ($\rho=\overline{0,\theta}$), $R_\varkappa(\tau)=\{\mathfrak{r}^\varkappa_{\alpha\beta}(\tau)\}\in M_{m,k}(\mathbb{K})$, $\mathfrak{r}^\varkappa_{\alpha\beta}:[-\sigma_\varkappa,-\sigma_{\varkappa-1}]\to\mathbb{K}$ are integrable functions ($\varkappa=\overline{1,\theta}$), $\alpha=\overline{1,m}$, $\beta=\overline{1,k}$. The problem of assignment of arbitrary coefficients to the characteristic function for the closed-loop system (2.2)–(2.4) has been solved in [31]; in particular, the arbitrary finite spectrum assignment problem has been solved.

These results were generalized in [32] for the case when a control system is defined by a system of differential equations with lumped and distributed delays. In [32], we considered the following system of delay differential equations with constant coefficients:

$$\dot{x}(t)=\sum_{j=0}^{s}\Phi_j x(t-h_j)+\sum_{\eta=1}^{s}\int_{-h_\eta}^{-h_{\eta-1}}\Psi_\eta(\tau)x(t+\tau)\,d\tau+\Theta\mathfrak{u}(t),\quad t>0, \tag{2.5}$$

$$y(t)=\Xi^* x(t), \tag{2.6}$$

with an initial condition $x(\tau)=x_0(\tau)$, $\tau\in[-h_s,0]$; here $0=h_0<h_1<\ldots<h_s$ are constant delays, $x_0\colon[-h_s,0]\to\mathbb{K}^n$ is a continuous function; $\Phi_j\in M_n(\mathbb{K})$ ($j=\overline{0,s}$), $\Theta\in M_{n,m}(\mathbb{K})$, and $\Xi\in M_{n,k}(\mathbb{K})$ are constant matrices; $\Psi_\eta:[-h_\eta,-h_{\eta-1}]\to M_n(\mathbb{K})$ ($\eta=\overline{1,s}$) are integrable functions; $x\in\mathbb{K}^n$ is a state vector, $\mathfrak{u}\in\mathbb{K}^m$ is a control vector, and $y\in\mathbb{K}^k$ is an output vector.

For the system (2.5), (2.6), the controller is constructed in [32] as linear static output feedback with lumped and distributed delays (2.4). The closed-loop system (2.4)–(2.6) has the form

$$\begin{aligned}\dot{x}(t)&=(\Phi_0+\Theta Q_0\Xi^*)x(t)+\sum_{j=1}^{s}\Phi_j x(t-h_j)+\sum_{\rho=1}^{\theta}\Theta Q_\rho\Xi^* x(t-\sigma_\rho)\\&+\sum_{\eta=1}^{s}\int_{-h_\eta}^{-h_{\eta-1}}\Psi_\eta(\tau)x(t+\tau)\,d\tau+\sum_{\varkappa=1}^{\theta}\int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}}\Theta R_\varkappa(\tau)\Xi^* x(t+\tau)\,d\tau.\end{aligned} \tag{2.7}$$

For system (2.5), (2.6), the problem of arbitrary coefficient assignment to the characteristic function of the scalar type by linear static output delayed feedback (2.4) has been solved in [32]; in particular, sufficient conditions have been obtained for assigning an arbitrary finite spectrum for the system (2.7).

The system (2.7) can be considered as a special case of the system (2.1). In fact, let system (2.7) be given, where

$$\Theta=[\Theta_1,\ldots,\Theta_m],\quad \Xi=[\Xi_1,\ldots,\Xi_k],\quad \Theta_\alpha,\Xi_\beta\in\mathbb{K}^n,$$

$$Q_\rho = \{\mathfrak{q}^\rho_{\alpha\beta}\}, \quad R_\varkappa(\tau) = \{\mathfrak{r}^\varkappa_{\alpha\beta}(\tau)\}, \quad \mathfrak{q}^\rho_{\alpha\beta} \in \mathbb{K}, \quad \mathfrak{r}^\varkappa_{\alpha\beta}\colon [-\sigma_\varkappa, -\sigma_{\varkappa-1}] \to \mathbb{K},$$
$$\rho = \overline{0,\theta}, \quad \varkappa = \overline{1,\theta}, \quad \alpha = \overline{1,m}, \quad \beta = \overline{1,k}.$$

Denote

$$T_1 := \{h_1, \ldots, h_s\}, \quad T_2 := \{\sigma_1, \ldots, \sigma_\theta\},$$
$$S_1 := \{j \in \{\overline{1,s}\} : h_j \in T_1 \setminus T_2\}, \quad S_2 := \{j \in \{\overline{1,s}\} : h_j \in T_1 \cap T_2\},$$
$$S_3 := \{\rho \in \{\overline{1,\theta}\} : \sigma_\rho \in T_1 \cap T_2\}, \quad S_4 := \{\rho \in \{\overline{1,\theta}\} : \sigma_\rho \in T_2 \setminus T_1\}.$$

Set $T := T_1 \cup T_2$, $\ell := |T|$. Set $\omega_0 := 0$. Denote the elements of the set T as $\omega_1 < \omega_2 < \ldots < \omega_\ell$. Let

$$K_1 := \{\mu \in \{\overline{1,\ell}\} : \exists\, j \in S_1 \;\; \omega_\mu = h_j\},$$
$$K_2 := \{\mu \in \{\overline{1,\ell}\} : \exists\, j \in S_2 \;\; \exists \rho \in S_3 \;\; \omega_\mu = h_j = \sigma_\rho\},$$
$$K_3 := \{\mu \in \{\overline{1,\ell}\} : \exists \rho \in S_4 \;\; \omega_\mu = \sigma_\rho\}.$$

Set

$$r_\mu := mk, \quad \mu = \overline{0,\ell}; \quad q_\xi := mk, \quad \xi = \overline{1,\ell}; \quad A_{00} := \Phi_0;$$
$$A_{0,\beta} := \Theta_1 \Xi^*_\beta, \quad A_{0,k+\beta} := \Theta_2 \Xi^*_\beta, \quad \ldots, \quad A_{0,(m-1)k+\beta} := \Theta_m \Xi^*_\beta, \quad \beta = \overline{1,k}, \tag{2.8}$$
$$u_0 := \operatorname{col}(\mathfrak{q}^0_{11}, \mathfrak{q}^0_{12}, \ldots, \mathfrak{q}^0_{1k}, \mathfrak{q}^0_{21}, \ldots, \mathfrak{q}^0_{2k}, \ldots, \mathfrak{q}^0_{m1}, \ldots, \mathfrak{q}^0_{mk}).$$

Next, we set

$$A_{\mu 0} := \begin{cases} \Phi_j, & \text{if } \mu \in K_1 \cup K_2 \text{ and } \omega_\mu = h_j, \\ O, & \text{if } \mu \in K_3. \end{cases} \tag{2.9}$$

Next, we set

$$A_{\mu\nu} := \begin{cases} A_{0\nu}, & \text{if } \mu \in K_2 \cup K_3 \text{ and } \omega_\mu = \sigma_\rho, \\ O, & \text{if } \mu \in K_1, \end{cases} \qquad \nu = \overline{1, r_\mu},$$
$$u_\mu := \begin{cases} \operatorname{col}(\mathfrak{q}^\rho_{11}, \mathfrak{q}^\rho_{12}, \ldots, \mathfrak{q}^\rho_{1k}, \mathfrak{q}^\rho_{21}, \ldots, \mathfrak{q}^\rho_{2k}, \ldots, \mathfrak{q}^\rho_{m1}, \ldots, \mathfrak{q}^\rho_{mk}), & \text{if } \mu \in K_2 \cup K_3 \text{ and } \omega_\mu = \sigma_\rho, \\ \operatorname{col}(0, \ldots, 0), & \text{if } \mu \in K_1. \end{cases} \tag{2.10}$$

Next, we set

$$C_{\xi 0}(\tau) := \begin{cases} \Psi_\eta(\tau), & \text{if } \xi \in K_1 \cup K_2 \text{ and } \omega_\xi = h_\eta, \\ O, & \text{if } \xi \in K_3. \end{cases} \tag{2.11}$$

Next, we set

$$C_{\xi\zeta} := \begin{cases} A_{0\zeta}, & \text{if } \xi \in K_2 \cup K_3 \text{ and } \omega_\xi = \sigma_\varkappa, \\ O, & \text{if } \xi \in K_1, \end{cases} \qquad \zeta = \overline{1, q_\xi},$$
$$w_\xi(\tau) := \begin{cases} \operatorname{col}(\mathfrak{r}^\varkappa_{11}(\tau), \ldots, \mathfrak{r}^\varkappa_{1k}(\tau), \ldots, \mathfrak{r}^\varkappa_{m1}(\tau), \ldots, \mathfrak{r}^\varkappa_{mk}(\tau)), & \text{if } \xi \in K_2 \cup K_3 \text{ and } \omega_\xi = \sigma_\varkappa, \\ \operatorname{col}(0, \ldots, 0), & \text{if } \xi \in K_1. \end{cases} \tag{2.12}$$

Then system (2.7) takes the form (2.1) with coefficients (2.8)–(2.12).

In the present paper, we obtain sufficient conditions for arbitrary finite spectrum assignment for the system (2.1). These results generalize some of the results of [32] (namely, Corollary 3 on assigning arbitrary finite spectrum and Corollary 4 on stabilization by linear static delayed output feedback) from system (2.7) to system (2.1) and extend the results of [33, 34].

3. Main result

System (2.1) can be written in the form

$$\begin{aligned}\dot{x}(t) = &\sum_{\mu=0}^{\ell}\Big(A_{\mu 0}x(t-\omega_\mu) + \sum_{\nu=1}^{r_\mu} u_{\mu\nu}A_{\mu\nu}x(t-\omega_\mu)\Big)\\ &+\sum_{\xi=1}^{\ell}\Big(\int_{-\omega_\xi}^{-\omega_{\xi-1}}\Big(C_{\xi 0}(\tau)+\sum_{\zeta=1}^{q_\xi} w_{\xi\zeta}(\tau)C_{\xi\zeta}\Big)x(t+\tau)\,d\tau\Big).\end{aligned} \tag{3.1}$$

Let us denote by $\psi(\lambda)$ the characteristic function of the system (3.1), i.e.,

$$\psi(\lambda) = \det\Big[\lambda I - \sum_{\mu=0}^{\ell}\Big(A_{\mu 0} + \sum_{\nu=1}^{r_\mu} u_{\mu\nu}A_{\mu\nu}\Big)e^{-\lambda\omega_\mu} - \sum_{\xi=1}^{\ell}\int_{-\omega_\xi}^{-\omega_{\xi-1}}\Big(C_{\xi 0}(\tau)+\sum_{\zeta=1}^{q_\xi} w_{\xi\zeta}(\tau)C_{\xi\zeta}\Big)e^{\lambda\tau}\,d\tau\Big]. \tag{3.2}$$

System (3.1) has the form

$$\dot{x}(t) = D_0x(t) + \sum_{\mu=1}^{\ell} D_\mu x(t-\omega_\mu) + \int_{-\omega}^{0} F(\tau)x(t+\tau)\,d\tau. \tag{3.3}$$

Here $\omega = \omega_\ell$,

$$D_\mu = A_{\mu 0} + \sum_{\nu=1}^{r_\mu} u_{\mu\nu}A_{\mu\nu}, \quad \mu = \overline{0,\ell},$$

$$F(\tau) = F_\xi(\tau), \quad \tau \in [-\omega_\xi, -\omega_{\xi-1}],$$

$$F_\xi(\tau) = C_{\xi 0}(\tau) + \sum_{\zeta=1}^{q_\xi} w_{\xi\zeta}(\tau)C_{\xi\zeta}, \quad \xi = \overline{1,\ell}.$$

Let us introduce the following denotations:

$$\Sigma_1 = \Big\{\mathfrak{s} : \mathfrak{s} = \sum_{i=1}^{1}\omega_{\varepsilon_i},\ \varepsilon_i \in \{0,1,\dots,\ell\}\Big\} = \{\omega_0,\omega_1,\dots,\omega_\ell\} =: \{\mathfrak{s}_0^1,\mathfrak{s}_1^1,\dots,\mathfrak{s}_{\pi_1}^1\},$$

$$\Sigma_2 = \Big\{\mathfrak{s} : \mathfrak{s} = \sum_{i=1}^{2}\omega_{\varepsilon_i},\ \varepsilon_i \in \{0,1,\dots,\ell\}\Big\} =: \{\mathfrak{s}_0^2,\mathfrak{s}_1^2,\dots,\mathfrak{s}_{\pi_2}^2\},$$

$\dots,$

$$\Sigma_n = \Big\{\mathfrak{s} : \mathfrak{s} = \sum_{i=1}^{n}\omega_{\varepsilon_i},\ \varepsilon_i \in \{0,1,\dots,\ell\}\Big\} =: \{\mathfrak{s}_0^n,\mathfrak{s}_1^n,\dots,\mathfrak{s}_{\pi_n}^n\}.$$

Here $\pi_i := |\Sigma_i| - 1$, $i = \overline{1,n}$; the i in the notation $\mathfrak{s}_k^i$ means the index, not the power. We suppose that the elements $\mathfrak{s}_k^i$ of the set Σ_i are sorted in ascending order, i.e., $\mathfrak{s}_k^i < \mathfrak{s}_{k+1}^i$. We have $\mathfrak{s}_0^1 = \ldots = \mathfrak{s}_0^n = 0$. Set $\pi_0 := 0$, $\mathfrak{s}_0^0 := 0$.

The characteristic function $\psi(\lambda)$ of the system (3.3) has the form

$$\psi(\lambda) = \lambda^n + \sum_{i=1}^{n} \lambda^{n-i}\Big(\sum_{k=0}^{\pi_i} \delta_{i0k} \exp(-\lambda \mathfrak{s}_k^i) + \sum_{\upsilon=1}^{i}\sum_{k=0}^{\pi_{i-\upsilon}} \int_{-\omega}^{0} \dots \int_{-\omega}^{0} \delta_{i\upsilon k}(\tau_1,\dots,\tau_\upsilon) \exp\Big(\lambda\Big(\sum_{l=1}^{\upsilon} \tau_l - \mathfrak{s}_k^{i-\upsilon}\Big)\Big) d\tau_1 \dots d\tau_\upsilon\Big). \quad (3.4)$$

Here the numbers δ_{i0k} ($i = \overline{1,n}$, $k = \overline{0,\pi_i}$) and the functions $\delta_{i\upsilon k}(\tau_1,\dots,\tau_\upsilon)$ $i = \overline{1,n}$, $\upsilon = \overline{1,i}$, $k = \overline{0,\pi_{i-\upsilon}}$, $\tau_l \in [-\omega, 0]$, $l = \overline{1,\upsilon}$) depend on coefficients D_μ ($\mu = \overline{0,\ell}$) and $F(\tau)$ ($\tau \in [-\omega, 0]$) of the system (3.3).

The set $\Lambda = \{\lambda \in \mathbb{C} : \psi(\lambda) = 0\}$ of the roots of the function (3.2) forms the spectrum of the system (3.1). In general, the spectrum Λ of the system (3.1) is infinite.

Suppose that, in (3.4), $\delta_{i0k} = 0$ for all $i = \overline{1,n}$ and $k = \overline{1,\pi_i}$, and $\delta_{i\upsilon k}(\tau_1,\dots,\tau_\upsilon) \equiv 0$ for all $i = \overline{1,n}$, $\upsilon = \overline{1,i}$, $k = \overline{1,\pi_{i-\upsilon}}$, $\tau_l \in [-\omega, 0]$, $l = \overline{1,\upsilon}$. Then the characteristic function turns into a polynomial, and the spectrum Λ is finite. Consider the problem of assigning an arbitrary finite spectrum Λ for the system (3.1).

Definition 1. *We say that the system* (3.1) *is an arbitrary finite spectrum assignable if, for any $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, there exist constant vectors $u_\mu \in \mathbb{K}^{r_\mu}$ ($\mu = \overline{0,\ell}$) and integrable vector functions $w_\xi : [-\omega_\xi, -\omega_{\xi-1}] \to \mathbb{K}^{q_\xi}$ ($\xi = \overline{1,\ell}$) such that the characteristic function $\psi(\lambda)$ of the system* (3.1) *satisfies the equality*

$$\psi(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_n.$$

In [34], sufficient conditions for the solvability of the finite spectrum assignment problem were obtained for the system (3.1) containing only lumped delays and no distributed delays (i.e., in the case where $C_{\xi\zeta} \equiv O$, $\xi = \overline{1,\ell}$, $\zeta = \overline{0,q_\xi}$). In [33], sufficient conditions for the solvability of the finite spectrum assignment problem were obtained for the system (3.1) containing only one lumped and one distributed delay ($\ell = 1$). Here, the results of [33, 34] are generalized to systems with multiple lumped and distributed delays.

Suppose that the coefficients of the system (3.1) have the following special form: The matrix A_{00} has the lower Hessenberg form with non-zero superdiagonal entries; for some $p \in \{1,\dots,n\}$, the first $p-1$ rows and the last $n-p$ columns of matrices $A_{\mu\nu}$, $\mu = \overline{0,\ell}$, $\nu = \overline{0,r_\mu}$ ($(\mu,\nu) \neq (0,0)$), and of matrices $C_{\xi\zeta}$, $\xi = \overline{1,\ell}$, $\zeta = \overline{0,q_\xi}$, are equal to zero, i.e.,

$$A_{00} = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1,1} & a_{n-1,2} & \dots & \dots & a_{n-1,n} \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{bmatrix}, \qquad a_{i,i+1} \neq 0, \quad i = \overline{1,n-1}, \quad (3.5)$$

$$A_{\mu\nu} = \begin{bmatrix} 0 & 0 \\ \widehat{A}_{\mu\nu} & 0 \end{bmatrix}, \quad \widehat{A}_{\mu\nu} \in M_{n-p+1,p}(\mathbb{K}), \quad \mu = \overline{0,\ell},\ \nu = \overline{0,r_\mu},\ (\mu,\nu) \neq (0,0), \quad (3.6)$$

$$C_{\xi 0}(\tau) = \begin{bmatrix} 0 & 0 \\ \widehat{C}_{\xi 0}(\tau) & 0 \end{bmatrix}, \quad C_{\xi\zeta} = \begin{bmatrix} 0 & 0 \\ \widehat{C}_{\xi\zeta} & 0 \end{bmatrix}, \quad (3.7)$$

$$\widehat{C}_{\xi 0}(\tau), \widehat{C}_{\xi\zeta} \in M_{n-p+1,p}(\mathbb{K}), \quad \xi = \overline{1,\ell},\ \zeta = \overline{1,q_\xi},\ \tau \in [-\omega_\xi, -\omega_{\xi-1}].$$

From system (3.1) we construct the matrices $\mathit{\Gamma}_\mu \in M_{n,r_\mu}(\mathbb{K})$ $(\mu = \overline{0,\ell})$, $\mathcal{X}_\mu \in M_{n,1}(\mathbb{K})$ $(\mu = \overline{1,\ell})$ and matrices $\mathcal{Y}_\xi \in M_{n,q_\xi}(\mathbb{K})$, $\mathcal{Z}_\xi(\tau) \in M_{n,1}(\mathbb{K})$, $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$, $\xi = \overline{1,\ell}$:

$$\mathit{\Gamma}_0 = \begin{bmatrix} \mathrm{Sp}\,(A_{01}) & \mathrm{Sp}\,(A_{02}) & \dots & \mathrm{Sp}\,(A_{0r_0}) \\ \mathrm{Sp}\,(A_{01}A_{00}) & \mathrm{Sp}\,(A_{02}A_{00}) & \dots & \mathrm{Sp}\,(A_{0r_0}A_{00}) \\ \dots & \dots & \dots & \dots \\ \mathrm{Sp}\,(A_{01}A_{00}^{n-1}) & \mathrm{Sp}\,(A_{02}A_{00}^{n-1}) & \dots & \mathrm{Sp}\,(A_{0r_0}A_{00}^{n-1}) \end{bmatrix}, \tag{3.8}$$

$$\mathit{\Gamma}_\mu = \begin{bmatrix} \mathrm{Sp}\,(A_{\mu1}) & \mathrm{Sp}\,(A_{\mu2}) & \dots & \mathrm{Sp}\,(A_{\mu r_\mu}) \\ \mathrm{Sp}\,(A_{\mu1}A_{00}) & \mathrm{Sp}\,(A_{\mu2}A_{00}) & \dots & \mathrm{Sp}\,(A_{\mu r_\mu}A_{00}) \\ \dots & \dots & \dots & \dots \\ \mathrm{Sp}\,(A_{\mu1}A_{00}^{n-1}) & \mathrm{Sp}\,(A_{\mu2}A_{00}^{n-1}) & \dots & \mathrm{Sp}\,(A_{\mu r_\mu}A_{00}^{n-1}) \end{bmatrix}, \tag{3.9}$$

$$\mathcal{X}_\mu = \begin{bmatrix} \mathrm{Sp}\,(A_{\mu0}) \\ \mathrm{Sp}\,(A_{\mu0}A_{00}) \\ \dots\dots \\ \mathrm{Sp}\,(A_{\mu0}A_{00}^{n-1}) \end{bmatrix}, \tag{3.10}$$

$$\mathcal{Y}_\xi = \begin{bmatrix} \mathrm{Sp}\,(C_{\xi1}) & \mathrm{Sp}\,(C_{\xi2}) & \dots & \mathrm{Sp}\,(C_{\xi q_\xi}) \\ \mathrm{Sp}\,(C_{\xi1}A_{00}) & \mathrm{Sp}\,(C_{\xi2}A_{00}) & \dots & \mathrm{Sp}\,(C_{\xi q_\xi}A_{00}) \\ \dots & \dots & \dots & \dots \\ \mathrm{Sp}\,(C_{\xi1}A_{00}^{n-1}) & \mathrm{Sp}\,(C_{\xi2}A_{00}^{n-1}) & \dots & \mathrm{Sp}\,(C_{\xi q_\xi}A_{00}^{n-1}) \end{bmatrix}, \tag{3.11}$$

$$\mathcal{Z}_\xi(\tau) = \begin{bmatrix} \mathrm{Sp}\,(C_{\xi0}(\tau)) \\ \mathrm{Sp}\,(C_{\xi0}(\tau)A_{00}) \\ \dots\dots\dots \\ \mathrm{Sp}\,(C_{\xi0}(\tau)A_{00}^{n-1}) \end{bmatrix}. \tag{3.12}$$

Construct the matrices $\mathit{\Delta}_\mu = [\mathit{\Gamma}_\mu, \mathcal{X}_\mu] \in M_{n,r_\mu+1}(\mathbb{K})$ $(\mu = \overline{1,\ell})$ and the matrices $\Omega_\xi(\tau) = [\mathcal{Y}_\xi, \mathcal{Z}_\xi(\tau)] \in M_{n,q_\xi+1}(\mathbb{K})$, $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$, $\xi = \overline{1,\ell}$.

Theorem 1. *Suppose that the matrices of the system* (3.1) *have the special form* (3.5)–(3.7)*. Then the system* (3.1) *is arbitrary finite spectrum assignable if and only if the following conditions hold:*

($C1$) $\operatorname{rank} \mathit{\Gamma}_0 = n$;

($C2$) $\operatorname{rank} \mathit{\Gamma}_\mu = \operatorname{rank} \mathit{\Delta}_\mu$ *for all* $\mu = \overline{1,\ell}$;

($C3$) $\operatorname{rank} \mathcal{Y}_\xi = \operatorname{rank} \Omega_\xi(\tau)$ *a.e.* $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$ *for all* $\xi = \overline{1,\ell}$.

4. Auxiliary statements

For proving Theorem 1, we need auxiliary statements. Let $\chi(A_{00}; \lambda) =: \lambda^n + \mathfrak{a}_1\lambda^{n-1} + \ldots + \mathfrak{a}_n$. Set $\mathfrak{a}_0 := 1$. From the matrix A_{00}, we construct the matrices

$$N_i := \mathfrak{a}_0 A_{00}^i + \mathfrak{a}_1 A_{00}^{i-1} + \ldots + \mathfrak{a}_i I, \quad i = \overline{0, n-1}. \tag{4.1}$$

Lemma 1. *Suppose that a matrix* A_{00} *has the form* (3.5) *and a matrix* $H \in M_n(\mathbb{K})$ *has the following form for some* $p \in \{1, \ldots, n\}$*:*

$$H = \begin{bmatrix} 0 & 0 \\ H_1 & 0 \end{bmatrix}, \quad H_1 \in M_{n-p+1,p}(\mathbb{K}). \tag{4.2}$$

Let $\chi(A_{00} + H; \lambda) =: \lambda^n + \mathfrak{g}_1\lambda^{n-1} + \ldots + \mathfrak{g}_n$. *Then* $\mathfrak{g}_i = \mathfrak{a}_i - \mathrm{Sp}\,(HN_{i-1})$ *for all* $i = \overline{1,n}$.

Lemma 1 is given in [32, Lemma 1]. The proof is given in [35, Lemma 1].

Lemma 2. *Let* $\tau_0 < \tau_1 < \ldots < \tau_k$. *Let* $\beta : [-\tau_k, -\tau_0] \to \mathbb{K}$ *be an integrable function. Consider a function*

$$f(\lambda) = \alpha_0 e^{-\lambda\tau_0} + \alpha_1 e^{-\lambda\tau_1} + \ldots + \alpha_k e^{-\lambda\tau_k} + \int_{-\tau_k}^{-\tau_0} \beta(\tau)e^{\lambda\tau}d\tau, \quad \lambda \in \mathbb{K}.$$

Suppose that $f(\lambda) = 0$ *for any* $\lambda \in \mathbb{K}$. *Then*

$$\alpha_0 = \alpha_1 = \ldots = \alpha_k = 0, \qquad \beta(\tau) \equiv 0 \text{ a.e. } \tau \in [-\tau_k, -\tau_0].$$

Lemma 2 with the proof is given in [32, Lemma 4].

5. Proof of Theorem 1

Let the matrices of the system (3.1) have the form (3.5)–(3.7). We will solve the problem of assigning an arbitrary finite spectrum. Let a polynomial

$$\mathfrak{p}(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_n \tag{5.1}$$

with numbers $\gamma_i \in \mathbb{K}$ be given. We need to find $u_\mu \in \mathbb{K}^{r_\mu}$ ($\mu = \overline{0,\ell}$) and $w_\xi : [-\omega_\xi, -\omega_{\xi-1}] \to \mathbb{K}^{q_\xi}$ ($\xi = \overline{1,\ell}$) such that the characteristic function (3.2) of the system (3.1) satisfies the equality

$$\psi(\lambda) = \mathfrak{p}(\lambda). \tag{5.2}$$

Denote

$$H = \sum_{\nu=1}^{r_0} u_{0\nu}A_{0\nu} + \sum_{\mu=1}^{\ell}\Big(A_{\mu 0} + \sum_{\nu=1}^{r_\mu} u_{\mu\nu}A_{\mu\nu}\Big)e^{-\lambda\omega_\mu} + \sum_{\xi=1}^{\ell}\int_{-\omega_\xi}^{-\omega_{\xi-1}}\Big(C_{\xi 0}(\tau) + \sum_{\zeta=1}^{q_\xi} w_{\xi\zeta}(\tau)C_{\xi\zeta}\Big)e^{\lambda\tau}\,d\tau. \tag{5.3}$$

We have

$$\psi(\lambda) = \det\big(\lambda I - (A_0 + H)\big) = \chi(A_0 + H; \lambda). \tag{5.4}$$

From conditions (3.6) and (3.7), it follows that the matrix (5.3) has the form (4.2). Taking into account (5.1), (5.2), (5.4), and condition (3.5), and applying Lemma 1, we obtain that the system (3.1) is an arbitrary finite spectrum assignable if and only if there exist $u_\mu \in \mathbb{K}^{r_\mu}$ ($\mu = \overline{0,\ell}$) and w_ξ : $[-\omega_\xi, -\omega_{\xi-1}] \to \mathbb{K}^{q_\xi}$ ($\xi = \overline{1,\ell}$) such that the following equalities hold:

$$\gamma_i = \mathfrak{a}_i - \mathrm{Sp}\,(HN_{i-1}), \quad i = \overline{1,n}. \tag{5.5}$$

Taking into account (5.3), we obtain that equalities (5.5) are equivalent to the following equalities:

$$\begin{aligned}\gamma_i = \mathfrak{a}_i - \mathrm{Sp}\left(\Big(\sum_{\nu=1}^{r_0} u_{0\nu}A_{0\nu}\Big)N_{i-1}\right) - \sum_{\mu=1}^{\ell}\mathrm{Sp}\left(\Big(A_{\mu 0} + \sum_{\nu=1}^{r_\mu} u_{\mu\nu}A_{\mu\nu}\Big)N_{i-1}\right)e^{-\lambda\omega_\mu} \\ - \sum_{\xi=1}^{\ell}\int_{-\omega_\xi}^{-\omega_{\xi-1}}\mathrm{Sp}\left(\Big(C_{\xi 0}(\tau) + \sum_{\zeta=1}^{q_\xi} w_{\xi\zeta}(\tau)C_{\xi\zeta}\Big)N_{i-1}\right)e^{\lambda\tau}d\tau, \quad i = \overline{1,n}.\end{aligned} \tag{5.6}$$

By Lemma 2, we obtain that (5.6) are equivalent to the following equalities:

$$\gamma_i = \mathfrak{a}_i - \mathrm{Sp}\left(\Big(\sum_{\nu=1}^{r_0} u_{0\nu}A_{0\nu}\Big)N_{i-1}\right), \quad i=\overline{1,n},$$
$$\mathrm{Sp}\left(\Big(A_{\mu 0}+\sum_{\nu=1}^{r_\mu} u_{\mu\nu}A_{\mu\nu}\Big)N_{i-1}\right)=0, \quad \mu=\overline{1,\ell}, \quad i=\overline{1,n}, \tag{5.7}$$
$$\mathrm{Sp}\left(\Big(C_{\xi 0}(\tau)+\sum_{\zeta=1}^{q_\xi} w_{\xi\zeta}(\tau)C_{\xi\zeta}\Big)N_{i-1}\right)=0, \quad \text{a.e. } \tau\in[-\omega_\xi,-\omega_{\xi-1}], \quad \xi=\overline{1,\ell}, \quad i=\overline{1,n}.$$

From the definition (4.1) of the matrices N_i, it follows that, for any $i=\overline{1,n}$,

$$N_{i-1}=\sum_{\eta=0}^{i-1}\mathfrak{a}_{i-1-\eta}A_{00}^{\eta}. \tag{5.8}$$

Substituting (5.8) into (5.7), we get that (5.7) are equivalent to the following systems:

$$\sum_{\eta=0}^{i-1}\mathfrak{a}_{i-1-\eta}\Big(\sum_{\nu=1}^{r_0} u_{0\nu}\,\mathrm{Sp}\,(A_{0\nu}A_{00}^{\eta})\Big)=\mathfrak{a}_i-\gamma_i, \quad i=\overline{1,n}, \tag{5.9}$$

$$\sum_{\eta=0}^{i-1}\mathfrak{a}_{i-1-\eta}\Big(\sum_{\nu=1}^{r_\mu} u_{\mu\nu}\,\mathrm{Sp}\,(A_{\mu\nu}A_{00}^{\eta})\Big)=-\sum_{\eta=0}^{i-1}\mathfrak{a}_{i-1-\eta}\,\mathrm{Sp}\,(A_{\mu 0}A_{00}^{\eta}), \quad \mu=\overline{1,\ell}, \quad i=\overline{1,n}, \tag{5.10}$$

$$\begin{gathered}\sum_{\eta=0}^{i-1}\mathfrak{a}_{i-1-\eta}\Big(\sum_{\zeta=1}^{q_\xi} w_{\xi\zeta}(\tau)\,\mathrm{Sp}\,(C_{\xi\zeta}A_{00}^{\eta})\Big)=-\sum_{\eta=0}^{i-1}\mathfrak{a}_{i-1-\eta}\,\mathrm{Sp}\,(C_{\xi 0}(\tau)A_{00}^{\eta})\\ \text{a.e. } \tau\in[-\omega_\xi,-\omega_{\xi-1}], \quad \xi=\overline{1,\ell}, \quad i=\overline{1,n}.\end{gathered} \tag{5.11}$$

The system (5.9) consists of n linear equations with r_0 unknown variables $u_{01},\dots,u_{0r_0}$. For every $\mu=\overline{1,\ell}$, equalities (5.10) represent a linear system of n equations with r_μ unknown variables $u_{\mu 1},\dots,u_{\mu r_\mu}$. For every $\xi=\overline{1,\ell}$, equalities (5.11) represent a linear system of n equations with r_μ unknown functions $w_{\xi 1}(\tau),\dots,w_{\xi r_\mu}(\tau)$. Let us rewrite (5.9)–(5.11) in vector form. Let us construct the matrices (see [32, (50)])

$$G:=\{g_{ij}\}_{i,j=1}^{n}, \; g_{ij}=0 \; (i<j), \quad g_{ij}=\mathfrak{a}_{i-j} \; (i\ge j), \tag{5.12}$$

and (3.8)–(3.12). Denote $d_0:=\mathrm{col}\,(\mathfrak{a}_1-\gamma_1,\dots,\mathfrak{a}_n-\gamma_n)\in\mathbb{K}^n$. Then one can rewrite systems (5.9)–(5.11) in the vector form

$$G\varGamma_0 u_0 = d_0, \tag{5.13}$$
$$G\varGamma_\mu u_\mu = -G\mathcal{X}_\mu, \quad \mu=\overline{1,\ell}, \tag{5.14}$$
$$G\mathcal{Y}_\xi w_\xi(\tau) = -G\mathcal{Z}_\xi(\tau) \quad \text{a.e. } \tau\in[-\omega_\xi,-\omega_{\xi-1}], \quad \xi=\overline{1,\ell}. \tag{5.15}$$

Taking into account that $\det G=1\neq 0$, we see that the system (5.13) is resolvable with respect to u_0, for any given $\gamma_i\in\mathbb{K}$, $i=\overline{1,n}$, if and only if condition $(C1)$ is fulfilled; the systems (5.14) are resolvable with respect to u_μ ($\mu=\overline{1,\ell}$) if and only if condition $(C2)$ is fulfilled; the systems (5.15) are resolvable with respect to $w_\xi(\tau)$ ($\xi = \overline{1,\ell}$) if and only if condition ($C3$) is fulfilled. Since $\mathcal{Z}_\xi(\tau)$, $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$, $\xi = \overline{1,\ell}$, are integrable, it follows that $w_\xi(\tau)$, $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$, $\xi = \overline{1,\ell}$, are integrable as well. Finding u_0, u_μ ($\mu = \overline{1,\ell}$), and $w_\xi(\tau)$ ($\xi = \overline{1,\ell}$) from (5.13)–(5.15), we assign a desirable polynomial (5.1) as the characteristic function for the system (3.1). Theorem 1 is proved.

Remark 1. *Note that the condition $r_0 \geq n$ is necessary for condition* ($C1$).

6. Corollaries

Suppose that the following condition holds:
($C4$) rank $\mathit{\Gamma}_\mu = n$ for all $\mu = \overline{1,\ell}$.
(A necessary condition for ($C4$) is $r_\mu \geq n$, $\mu = \overline{1,\ell}$.) Then, clearly, ($C2$) holds.
Similarly, let the following condition hold:
($C5$) rank $\mathcal{Y}_\xi = n$ for all $\xi = \overline{1,\ell}$.
(A necessary condition for ($C5$) is $q_\xi \geq n$, $\xi = \overline{1,\ell}$.) Then, clearly, ($C3$) holds.
So we have the following corollary.

Corollary 1. *Suppose that the matrices of the system* (3.1) *have the special form* (3.5)–(3.7). *Suppose that the following conditions hold:* ($C1$) *and* $\big((C2)$ *or* $(C4)\big)$ *and* $\big((C3)$ *or* $(C5)\big)$. *Then the system* (3.1) *is an arbitrary finite spectrum assignable.*

Next, consider a problem of exponential stabilization for the system (3.1): one needs to construct $u_\mu \in \mathbb{K}^{r_\mu}$ ($\mu = \overline{0,\ell}$) and $w_\xi : [-\omega_\xi, -\omega_{\xi-1}] \to \mathbb{K}^{q_\xi}$ ($\xi = \overline{1,\ell}$) such that the system (3.1) is exponentially stable. The system (3.1) is exponentially stable if and only if $\Lambda \subset \mathbb{C}_- := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$. Suppose that the system (3.1) is an arbitrary finite spectrum assignable. Let us choose the polynomial (5.1) such that its roots lie in $\mathbb{C}_-$. Then system (3.1) is exponentially stable. So, we get the following corollary from Corollary 1.

Corollary 2. *Suppose that the matrices of the system* (3.1) *have the special form* (3.5)–(3.7). *Suppose that the following conditions hold:* ($C1$) *and* $\big((C2)$ *or* $(C4)\big)$ *and* $\big((C3)$ *or* $(C5)\big)$. *Then the system* (3.1) *is exponentially stabilizable.*

Remark 2. *Suppose that $\ell = 1$. In this case, Theorem 1 was proved in [33, Theorem 3]. Thus, Theorem 1 is a generalization of [33, Theorem 3] to the case of multiple delays.*

Remark 3. *Suppose that $C_{\xi 0}(\tau) \equiv O$, $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$, $C_{\xi\zeta} = O$, $\xi = \overline{1,\ell}$, $\zeta = \overline{1,q_\xi}$. In this case, Theorem 1 was proved in [34, Theorem 1]. Thus, Theorem 1 is a generalization of [34, Theorem 1] to the case of systems containing, in addition to lumped ones, also distributed delays.*

Remark 4. *Suppose that the system* (2.1) *has the form* (2.7), *i.e., the coefficients of the system* (2.1) *have the form* (2.8)–(2.12). *In this case $\mathit{\Gamma}_0 = \mathit{\Gamma}_\mu = \mathcal{Y}_\xi$ for all $\mu = \overline{1,\ell}$, $\xi = \overline{1,\ell}$. Suppose that, for this system* (2.7), *the conditions of [32, Corollary 3] hold for assigning an arbitrary finite spectrum, i.e., the matrices of the system have the special form and the matrices*

$$\Xi^*\Theta, \quad \Xi^*\Phi_0\Theta, \quad ,\ldots, \quad \Xi^*\Phi_0^{n-1}\Theta \tag{6.1}$$

are linearly independent. One can check that (see, e.g., [34, Remark 1 and Lemma 2]), in this case, the coefficients (2.8)–(2.12) *of the system* (2.1) *have the form* (3.5)–(3.7), *and conditions* ($C1$), ($C4$), *and* $(C5)$ *are fulfilled. Then, by Theorem 1, the system* (2.1) *is an arbitrary finite spectrum assignable. Conversely, if the matrices* (6.1) *are not linearly independent, then condition* $(C1)$ *does not hold and, hence, by Theorem 1, the system* (2.1) *is not an arbitrary finite spectrum assignable. Thus, Theorem 1 extends the results of [32, Corollary 3] from systems* (2.7) *to systems* (2.1). *Similarly, Corollary 2 extends the results of [32, Corollary 4] on stabilization from systems* (2.7) *to systems* (2.1).

7. Case of time-varying $C_{\xi\zeta}$, $\xi = \overline{1,\ell}$, $\zeta = \overline{1,q_\xi}$

Consider the system similar to (3.1) where the matrix coefficients $C_{\xi\zeta}$, $\xi = \overline{1,\ell}$, $\zeta = \overline{1,q_\xi}$, are not constant and are time-varying:

$$\begin{aligned}\dot{x}(t) &= \sum_{\mu=0}^{\ell}\Big(A_{\mu 0}x(t-\omega_\mu) + \sum_{\nu=1}^{r_\mu} u_{\mu\nu}A_{\mu\nu}x(t-\omega_\mu)\Big)\\ &+ \sum_{\xi=1}^{\ell}\Big(\int_{-\omega_\xi}^{-\omega_{\xi-1}}\Big(C_{\xi 0}(\tau) + \sum_{\zeta=1}^{q_\xi} w_{\xi\zeta}(\tau)C_{\xi\zeta}(\tau)\Big)x(t+\tau)\,d\tau\Big).\end{aligned} \tag{7.1}$$

$t > 0$, with initial conditions $x(\tau) = x_0(\tau)$, $\tau \in [-\omega_\ell, 0]$; here $0 = \omega_0 < \omega_1 < \ldots < \omega_\ell$ are constant delays; $x_0\colon [-\omega_\ell, 0] \to \mathbb{K}^n$ is a continuous function; $x \in \mathbb{K}^n$ is a state vector; $u_\mu = \operatorname{col}(u_{\mu 1}, \ldots, u_{\mu r_\mu}) \in \mathbb{K}^{r_\mu}$, $\mu = \overline{0,\ell}$, are control vectors; $w_\xi(\tau) = \operatorname{col}(w_{\xi 1}(\tau), \ldots, w_{\xi q_\xi}(\tau)) \in \mathbb{K}^{q_\xi}$, $\xi = \overline{1,\ell}$, $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$, are control vector functions; $A_{\mu\nu} \in M_n(\mathbb{K})$, $\mu = \overline{0,\ell}$, $\nu = \overline{0,r_\mu}$; $C_{\xi 0}\colon [-\omega_\xi, -\omega_{\xi-1}] \to M_n(\mathbb{K})$ are integrable functions, $C_{\xi\zeta}\colon [-\omega_\xi, -\omega_{\xi-1}] \to M_n(\mathbb{K})$ are continuous functions, $\xi = \overline{1,\ell}$, $\zeta = \overline{1,q_\xi}$.

Suppose that the matrices of the system (7.1) have the special form (3.5), (3.6), and

$$C_{\xi\zeta}(\tau) = \begin{bmatrix} 0 & 0 \\ \widehat{C}_{\xi\zeta}(\tau) & 0\end{bmatrix}, \quad \widehat{C}_{\xi\zeta}(\tau) \in M_{n-p+1,p}(\mathbb{K}), \quad \xi = \overline{0,\ell},\ \zeta = \overline{1,q_\xi},\ \tau \in [-\omega_\xi, -\omega_{\xi-1}]. \tag{7.2}$$

Construct the matrices (3.8)–(3.10), the matrix

$$\mathcal{W}_\xi(\tau) = \begin{bmatrix} \operatorname{Sp}(C_{\xi 1}(\tau)) & \operatorname{Sp}(C_{\xi 2}(\tau)) & \ldots & \operatorname{Sp}(C_{\xi q_\xi}(\tau)) \\ \operatorname{Sp}(C_{\xi 1}(\tau)A_{00}) & \operatorname{Sp}(C_{\xi 2}(\tau)A_{00}) & \ldots & \operatorname{Sp}(C_{\xi q_\xi}(\tau)A_{00}) \\ \ldots & \ldots & \ldots & \ldots \\ \operatorname{Sp}(C_{\xi 1}(\tau)A_{00}^{n-1}) & \operatorname{Sp}(C_{\xi 2}(\tau)A_{00}^{n-1}) & \ldots & \operatorname{Sp}(C_{\xi q_\xi}(\tau)A_{00}^{n-1})\end{bmatrix}, \tag{7.3}$$

and the matrix (3.12).

Theorem 2. *Suppose that the matrices of the system* (7.1) *have the special form* (3.5), (3.6)*, and* (7.2)*. Then the system* (7.1) *is an arbitrary finite spectrum assignable if and only if the following conditions hold:* $(C1)$ *and* $(C2)$ *and*

$(C6)$ *For every* $\xi = \overline{1,\ell}$*, for almost every* $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$*, the system of linear equations*

$$\mathcal{W}_\xi(\tau)\mathfrak{X}_\xi(\tau) = \mathcal{Z}_\xi(\tau)$$

is resolvable with respect to $\mathfrak{X}_\xi(\tau) \in \mathbb{K}^{q_\xi}$ *and the solution* $\mathfrak{X}_\xi(\tau)$*,* $\tau \in [-\omega_\xi, -\omega_{\xi-1}]$*, is an integrable function on* $[-\omega_\xi, -\omega_{\xi-1}]$.

The proof of Theorem 2 repeats the proof of Theorem 1 up to formula (5.15), with replacing $C_{\xi\zeta}$ by $C_{\xi\zeta}(\tau)$, $\xi = \overline{1,\ell}$, $\zeta = \overline{1,q_\xi}$. Instead of (5.15), we have the equality

$$G\mathcal{W}_\xi w_\xi(\tau) = -G\mathcal{Z}_\xi(\tau) \quad \text{a.e. } \tau \in [-\omega_\xi, -\omega_{\xi-1}], \quad \xi = \overline{1,\ell}. \tag{7.4}$$

Resolvability of system (7.4) is equivalent to condition ($C6$), which completes the proof.

Corollary 3. *Conditions of Theorem 2 are sufficient for exponential stabilization of the system* (7.1).

8. Examples

Example 1. Let us present an example illustrating Theorem 2. Suppose $\mathbb{K} = \mathbb{R}$, $n = 3$, $\ell = 2$, $p = 2$, $r_0 = 3$, $r_1 = 3$, $r_2 = 2$, $q_1 = 3$, $q_2 = 2$, and the matrices of the system (7.1) have the following form:

$$A_{00} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 2 & 1 \\ 2 & -1 & -1 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 0 \end{bmatrix}, \quad A_{02} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 1 & 0 \end{bmatrix}, \quad A_{03} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$A_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$
$$A_{20} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8.1}$$
$$C_{10}(\tau) = \begin{bmatrix} 0 & 0 & 0 \\ \cos\tau & \sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \end{bmatrix}, \; C_{11}(\tau) \equiv \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \; C_{12}(\tau) \equiv \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 0 \end{bmatrix}, \; C_{13}(\tau) \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix},$$
$$C_{20}(\tau) = \begin{bmatrix} 0 & 0 & 0 \\ \cos\tau & 0 & 0 \\ 0 & \sin\tau & 0 \end{bmatrix}, \quad C_{21}(\tau) = \begin{bmatrix} 0 & 0 & 0 \\ \sin\tau & 0 & 0 \\ -\cos\tau & 0 & 0 \end{bmatrix}, \quad C_{22}(\tau) = \begin{bmatrix} 0 & 0 & 0 \\ 2\cos\tau & 0 & 0 \\ \sin\tau & \cos\tau & 0 \end{bmatrix}.$$

The matrices (8.1) of the system (7.1) have the special forms (3.5), (3.6), and (7.2). We have $\chi(A_{00};\lambda) = \lambda^3 - 1$. Hence, $\mathfrak{a}_1 = 0$, $\mathfrak{a}_2 = 0$, $\mathfrak{a}_3 = -1$. Let's calculate the matrices (5.12), (3.8)–(3.10), (7.3), and (3.12): we obtain $G = I$,

$$\varGamma_0 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \quad \varGamma_1 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \varGamma_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{X}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathcal{X}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$
$$\mathcal{W}_1(\tau) \equiv \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{W}_2(\tau) = \begin{bmatrix} 0 & 0 \\ \sin\tau & 3\cos\tau \\ \sin\tau - \cos\tau & 3\cos\tau + \sin\tau \end{bmatrix}, \tag{8.2}$$
$$\mathcal{Z}_1(\tau) = \begin{bmatrix} \sin\tau \\ 2\cos\tau + 2\sin\tau \\ 2\cos\tau + 2\sin\tau \end{bmatrix}, \quad \mathcal{Z}_2(\tau) = \begin{bmatrix} 0 \\ \cos\tau + \sin\tau \\ \cos\tau + \sin\tau \end{bmatrix}.$$

One can see that conditions $(C1)$, $(C2)$, and $(C6)$ hold. Hence, by Theorem 2, the system (7.1) with the matrices (8.1) is an arbitrary finite spectrum assignable. Let us construct that control $u_0 \in \mathbb{R}^3$, $u_1 \in \mathbb{R}^3$, $u_2 \in \mathbb{R}^2$, $w_1(\tau) \in \mathbb{R}^3$ $(\tau \in [-\omega_1, 0])$, $w_2(\tau) \in \mathbb{R}^2$ $(\tau \in [-\omega_2, -\omega_1])$. Suppose, for example, that $\mathfrak{p}(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda + 3)$ in (5.1). We have $\gamma_1 = 6$, $\gamma_2 = 11$, $\gamma_3 = 6$. Hence,

$$d_0 = \operatorname{col}(\mathfrak{a}_1 - \gamma_1, \mathfrak{a}_2 - \gamma_2, \mathfrak{a}_3 - \gamma_3) = (-6, -11, -7). \tag{8.3}$$

Resolving the systems (5.13), (5.14), and (7.4) with coefficients (8.2), (8.3), we obtain

$$\begin{gathered} u_0 = \operatorname{col}\left(-\frac{37}{3}, -\frac{35}{3}, -\frac{19}{3}\right), \quad u_1 = \operatorname{col}\left(\frac{1}{3}, -\frac{4}{3}, -\frac{1}{3}\right), \quad u_2 = \operatorname{col}(-2, -1), \\ w_1(\tau) = \operatorname{col}(-2\cos\tau - 2\sin\tau, -2\cos\tau - 3\sin\tau, 2\cos\tau + 3\sin\tau), \\ w_2(\tau) = \operatorname{col}\left(-\frac{\sin\tau(\cos\tau + \sin\tau)}{1 + 2\cos^2\tau}, -\frac{\cos\tau(\cos\tau + \sin\tau)}{1 + 2\cos^2\tau}\right). \end{gathered} \tag{8.4}$$

The system (7.1) with the matrices (8.1) closed-loop by the control (8.4) takes the form

$$\begin{aligned} \dot{x}(t) = &\begin{bmatrix} -1 & 1 & 0 \\ 14/3 & -4 & 1 \\ 0 & -20/3 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ -2/3 & 0 & 0 \\ 0 & 2/3 & 0 \end{bmatrix} x(t - \omega_1) + \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} x(t - \omega_2) \\ &+ \int_{-\omega_1}^{0} \begin{bmatrix} 0 & 0 & 0 \\ \alpha(\tau) & 0 & 0 \\ 0 & -\alpha(\tau) & 0 \end{bmatrix} x(t + \tau)\, d\tau + \int_{-\omega_2}^{-\omega_1} \begin{bmatrix} 0 & 0 & 0 \\ \beta(\tau) & 0 & 0 \\ 0 & -\beta(\tau) & 0 \end{bmatrix} x(t + \tau)\, d\tau, \end{aligned} \tag{8.5}$$

where $\alpha(\tau) = -\cos\tau - \sin\tau, \beta(\tau) = \dfrac{\cos^3\tau - \sin\tau\cos^2\tau - \sin\tau}{1 + 2\cos^2\tau}$. Calculating the characteristic function for the system (8.5), we obtain that $\psi(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda + 3)$. In particular, the system (8.5) is exponentially stable.

Example 2. Let us consider the mathematical model of an automatic rheostat voltage regulator [36,37]. The dynamics are described by a system of differential equations with delay [36, 37]

$$\begin{aligned} &\theta_1 \ddot{\alpha}(t) + \varepsilon\alpha(t) + B\alpha(t - \tau) - \varepsilon\beta(t) = A, \\ &\theta_2 \ddot{\beta}(t) + k\dot{\beta}(t) + \varepsilon\beta(t) - \varepsilon\alpha(t) = 0. \end{aligned} \tag{8.6}$$

Here θ_1 is a moment of inertia of the regulator anchor, θ_2 is a moment of inertia of the damper sector and the regulator washer, α is the angle of rotation of the regulator anchor, β is the angle of rotation of the damper sector and the regulator washer, k is a damping coefficient, $A = 2CU_nU_0 - 2CU_n\gamma\psi_0 - 2CU_n^2$; $B, \varepsilon, C, U_n, \gamma, \psi_0$ are constants, U_0 is a voltage at the anchor terminals, $\tau > 0$ is a constant delay. Let us assume that the voltage U_0 can be chosen from the class of piecewise continuous functions. Then, the value A in (8.6) acquires the meaning of control $A = v(t) \in \mathbb{R}$. Let us introduce a change of variables $x = \operatorname{col}(\alpha, \dot{\alpha}, \beta, \dot{\beta}) \in \mathbb{R}^4$. Then, system (8.6) will be written in the form

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}x(t - \tau) + \mathcal{G}v(t), \tag{8.7}$$

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\varkappa & 0 & \varkappa & 0 \\ 0 & 0 & 0 & 1 \\ \rho & 0 & -\rho & -\sigma \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{8.8}$$

$$\varkappa = \varepsilon/\theta_1, \quad \rho = \varepsilon/\theta_2, \quad \sigma = k/\theta_2, \quad \delta = B/\theta_1. \tag{8.9}$$

Suppose that $\varepsilon \neq 0$. Then, $\varkappa \neq 0, \rho \neq 0$. The free system (8.7)–(8.9) (i.e., a system with $v(t) \equiv 0$) is not asymptotically stable for some values of parameters. For example, let $k = 2$, $\theta_1 = \theta_2 = \varepsilon = 1$, $B = 1/2$, $\tau = \pi/2$. Then, $\varkappa = 1, \rho = 1, \sigma = 2, \delta = 1/2$. Then, the characteristic equation has the root $\lambda = i$ [36]. In [36], the linear *state* feedback regulator with delay is constructed in the form

$$u(t) = Q_0 x(t) + Q_1 x(t - \tau), \quad Q_0, Q_1 \in M_{1,4}(\mathbb{R}), \tag{8.10}$$

such that the characteristic function $\psi(\lambda)$ of the closed-loop system (8.7)–(8.10) is equal to $(\lambda + 2)^4$ (in particular, the closed-loop system is asymptotically stable).

Let us consider a slightly modified system (8.7)–(8.9), namely: We will assume that the system has not one but two control inputs and has the form

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}x(t - \tau) + \mathcal{H}v(t). \tag{8.11}$$

Here $x \in \mathbb{R}^4$, $v = \mathrm{col}(v_1, v_2) \in \mathbb{R}^2$, the matrices $\mathcal{A}$ and $\mathcal{B}$ are taken from (8.8), and the matrix $\mathcal{H}$ has the form

$$\mathcal{H} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Suppose that the output is not the entire state vector but only part of its coordinates:

$$y_1 = x_1, \quad y_2 = x_2, \quad y = (y_1, y_2) \in \mathbb{R}^2. \tag{8.12}$$

Suppose that the controller $v(t)$ in the system (8.11) is constructed as linear static *output* feedback with delay:

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} q_1^0 & q_2^0 \\ q_3^0 & q_4^0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} q_1^1 & q_2^1 \\ q_3^1 & q_4^1 \end{bmatrix} \begin{bmatrix} y_1(t - \tau) \\ y_2(t - \tau) \end{bmatrix}. \tag{8.13}$$

Then, the closed-loop system (8.11), (8.13) takes the form

$$\begin{gathered} \dot{x}(t) = A_0 x(t) + q_1^0 A_1 x(t) + q_2^0 A_2 x(t) + q_3^0 A_3 x(t) + q_4^0 A_4 x(t) \\ + B_0 x(t - \tau) + q_1^1 B_1 x(t - \tau) + q_2^1 B_2 x(t - \tau) + q_3^1 B_3 x(t - \tau) + q_4^1 B_4 x(t - \tau), \end{gathered} \tag{8.14}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{8.15}$$

$$A_0 = \mathcal{A}, \quad B_0 = \mathcal{B}, \quad B_1 = A_1, \quad B_2 = A_2, \quad B_3 = A_3, \quad B_4 = A_4. \tag{8.16}$$

Thus, system (8.14)–(8.16) has the form (2.1) where $\ell = 1$, $r_0 = r_1 = 4$, $\omega_1 = \tau$, and all $C_{\xi\zeta} = O \in M_n(\mathbb{K})$.

One can see that the matrices (8.15) and (8.16) of the system (8.14) have the special form (3.5)–(3.7). Let us show that the conditions of Theorem 1 are satisfied and construct a control q_i^0, q_i^1, $i = 1, \ldots, 4$, that makes the characteristic function $\psi(\lambda)$ equal to the given polynomial $\mathfrak{p}(\lambda)$. We have

$\chi(A_0;\lambda) = \lambda^4 + \sigma\lambda^3 + (\rho+\varkappa)\lambda^2 + \sigma\varkappa\lambda$. Hence, $\mathfrak{a}_1 = \sigma$, $\mathfrak{a}_2 = \rho+\varkappa$, $\mathfrak{a}_3 = \sigma\varkappa$, and $\mathfrak{a}_4 = 0$. Let us calculate the matrices (3.8)–(3.10) and (5.12) (the matrices (3.11) and (3.12) are zero, so $(C3)$ holds): we obtain

$$\varGamma_0 = \varGamma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\varkappa & 0 & \varkappa \\ -\varkappa & 0 & \varkappa & -\sigma\varkappa \end{bmatrix}, \quad X_1 = \begin{bmatrix} 0 \\ -\delta \\ 0 \\ \delta\varkappa \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \sigma & 1 & 0 & 0 \\ \rho+\varkappa & \sigma & 1 & 0 \\ \sigma\varkappa & \rho+\varkappa & \sigma & 1 \end{bmatrix}. \tag{8.17}$$

We have $\det\varGamma_0 = \varkappa^2$. Since $\varepsilon \neq 0$, it follows that $\varkappa \neq 0$. So, $(C1)$ and $(C2)$ hold. Hence, by Theorem 1, the system (8.14)–(8.16) is an arbitrary finite spectrum assignable. Suppose, for example, that $\mathfrak{p}(\lambda) = (\lambda + 1)^4$ in (5.1). We have $\gamma_1 = 4$, $\gamma_2 = 6$, $\gamma_3 = 4$, $\gamma_4 = 1$. Hence,

$$d_0 = \operatorname{col}(\mathfrak{a}_1 - \gamma_1, \mathfrak{a}_2 - \gamma_2, \mathfrak{a}_3 - \gamma_3, \mathfrak{a}_4 - \gamma_4) = (\sigma - 4, \rho + \varkappa - 6, \sigma\varkappa - 4, -1). \tag{8.18}$$

Resolving the systems (5.13) and (5.14) with coefficients (8.17) and (8.18), we obtain

$$\begin{aligned} &q_1^0 = -\sigma^2 + \varkappa + \rho + 4\sigma - 6, \quad q_2^0 = \sigma - 4, \\ &q_3^0 = (\sigma^2\rho - \varkappa\rho - 4\sigma\rho + 6\rho - \rho^2 - 1)/\varkappa, \\ &q_4^0 = (\sigma - 2)(\sigma^2 - 2\rho - 2\sigma + 2)/\varkappa, \\ &q_1^1 = \delta, \quad q_2^1 = q_3^1 = q_4^1 = 0. \end{aligned} \tag{8.19}$$

Calculating the characteristic function for the system (8.14)–(8.16) with coefficients (8.19), we obtain that $\psi(\lambda) = (\lambda + 1)^4$. In particular, this system is exponentially stable.

9. Conclusions

We have studied a problem of arbitrary finite spectrum assignment for a bilinear control system defined by a linear time-invariant system of differential equations with multiple lumped and distributed delays in the state variable. We have obtained the criterion for solvability of this problem when the coefficients of the system have a special form. This criterion is expressed in terms of rank conditions for matrices constructed from the matrix coefficients of the system. Corollaries on stabilization of a bilinear system with delays have been obtained. The results extend the previously obtained results for less general cases. Modeling examples have been presented.

Author contributions

All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was funded by the Ministry of Science and Higher Education of the Russian Federation in the framework of the state assignment, project FEWS-2024-0009.

Conflict of interest

The authors declare no conflicts of interest.

References

1. R. R. Mohler, *Bilinear control processes: with applications to engineering, ecology, and medicine*, Elsevier, 1973. https://doi.org/10.1016/s0076-5392(08)x6126-0
2. V. Jurdjevic, J. P. Quinn, Controllability and stability, *J. Differ. Equations*, **28** (1978), 381–389. https://doi.org/10.1016/0022-0396(78)90135-3
3. M. Slemrod, Stabilization of bilinear control systems with applications to nonconservative problems in elasticity, *SIAM J. Control Optim.*, **16** (1978), 131–141. https://doi.org/10.1137/0316010
4. P. O. Gutman, Stabilizing controllers for bilinear systems, *IEEE Trans. Automat. Control*, **26** (1981), 917–922. https://doi.org/10.1109/tac.1981.1102742
5. J. Tsinias, N. Kalouptsidis, A. Bacciotti, Lyapunov functions and stability of dynamical polysystems, *Math. Syst. Theory*, **19** (1986), 333–354. https://doi.org/10.1007/bf01704919
6. F. Amato, C. Cosentino, A. S. Fiorillo, A. Merola, Stabilization of bilinear systems via linear state-feedback control, *IEEE Trans. Circuits Syst. II: Express Briefs*, **56** (2009), 76–80. https://doi.org/10.1109/tcsii.2008.2008528
7. V. Andrieu, S. Tarbouriech, Global asymptotic stabilization for a class of bilinear systems by hybrid output feedback, *IEEE Trans. Automat. Control*, **58** (2013), 1602–1608. https://doi.org/10.1109/tac.2012.2229842
8. S. Lin, B. De Schutter, D. Li, Stabilizing controller design for state-based switching bilinear systems, *2018 IEEE Conference on Decision and Control (CDC)*, 2018, 6452–6457. https://doi.org/10.1109/cdc.2018.8618956
9. M. N. Monfared, Y. Kawano, J. E. Machado, D. Astolfi, M. Cucuzzella, Stabilization for a class of bilinear systems: a unified approach, *IEEE Control Syst. Lett.*, **7** (2023), 2791–2796. https://doi.org/10.1109/lcsys.2023.3289703
10. Y. Kawano, M. Cucuzzella, Stabilization for a class of positive bilinear systems, *IEEE Control Syst. Lett.*, **7** (2023), 3519–3524. https://doi.org/10.1109/lcsys.2023.3334725
11. V. A. Zaitsev, Uniform global asymptotic stabilisation of bilinear non-homogeneous periodic systems with stable free dynamics, *Int. J. Syst. Sci.*, **48** (2017), 3403–3410. https://doi.org/10.1080/00207721.2017.1385875
12. V. A. Zaitsev, Sufficient conditions for uniform global asymptotic stabilization of discrete-time periodic bilinear systems, *IFAC-PapersOnLine*, **50** (2017), 11529–11534. https://doi.org/10.1016/j.ifacol.2017.08.1623
13. V. A. Zaitsev, Uniform global asymptotic stabilization of bilinear non-homogeneous periodic discrete-time systems, *2018 14th International Conference "Stability and Oscillations of Nonlinear Control Systems" (Pyatnitskiy's Conference) (STAB), IEEE*, 2018. https://doi.org/10.1109/stab.2018.8408412

14. M. Ouzahra, Stabilisation of infinite-dimensional bilinear systems using a quadratic feedback control, *Int. J. Control*, **82** (2009), 1657–1664. https://doi.org/10.1080/00207170802657348

15. M. Ouzahra, Exponential and weak stabilization of constrained bilinear systems, *SIAM J. Control Optim.*, **48** (2010), 3962–3974. https://doi.org/10.1137/080739161

16. A. Benzaza, M. Ouzahra, Weak and strong stabilisation of bilinear systems in a Banach space, *Int. J. Control*, **92** (2019), 2784–2790. https://doi.org/10.1080/00207179.2018.1459858

17. D. W. C. Ho, G. Lu, Y. Zheng, Global stabilisation for bilinear systems with time delay, *IEE Proceedings-Control Theory Appl.*, **149** (2002), 89–94. https://doi.org/10.1049/ip-cta:20020114

18. Z. Hamidi, R. El Ayadi, M. Ouzahra, Feedback stabilization of non-homogeneous bilinear systems with a finite time delay, *Math. Methods Appl. Sci.*, **46** (2023), 1317–1334. https://doi.org/10.1002/mma.8582

19. Z. Hamidi, A. Elazzouzi, M. Ouzahra, Feedback stabilization of delayed bilinear systems, *Differ. Equ. Dyn. Syst.*, **32** (2024), 87–100. https://doi.org/10.1007/s12591-021-00560-0

20. V. Zaitsev, Global asymptotic stabilization of autonomous bilinear complex systems, *Eur. J. Control*, **65** (2022), 100644. https://doi.org/10.1016/j.ejcon.2022.100644

21. V. Zaitsev, E. Zaitsev, Global asymptotic stabilization of single-input bilinear discrete time-invariant complex systems, *2022 IEEE 61st Conference on Decision and Control (CDC)*, 2022, 5769–5774. https://doi.org/10.1109/cdc51059.2022.9993411

22. V. Zaitsev, Global asymptotic stabilization of time-invariant bilinear non-homogeneous complex systems, *2023 62nd IEEE Conference on Decision and Control (CDC)*, 2023, 6659–6664. https://doi.org/10.1109/cdc49753.2023.10383400

23. D. Chalishajar, D. Kasinathan, R. Kasinathan, R. Kasinathan, Exponential stability, T-controllability and optimal controllability of higher-order fractional neutral stochastic differential equation via integral contractor, *Chaos, Soliton. Fract.*, **186** (2024), 115278. https://doi.org/10.1016/j.chaos.2024.115278

24. D. Kasinathan, R. Kasinathan, R. Kasinathan, D. Chalishajar, Exponential stability for higher-order impulsive fractional neutral stochastic integro-delay differential equations with mixed brownian motions and non-local conditions, *Int. J. Optim. Control: Theories Appl. (IJOCTA)*, **15** (2025), 101. https://doi.org/10.36922/ijocta.1524

25. C. I. Byrnes, X. Wang, The additive inverse eigenvalue problem for Lie perturbations, *SIAM J. Matrix Anal. Appl.*, **14** (1993), 113–117. https://doi.org/10.1137/0614009

26. W. Helton, J. Rosenthal, X. Wang, Matrix extensions and eigenvalue completions, the generic case, *T. Am. Math. Soc.*, **349** (1997), 3401–3408. https://doi.org/10.1090/s0002-9947-97-01975-2

27. M. T. Chu, Inverse eigenvalue problems, *SIAM Rev.*, **40** (1998), 1–39. https://doi.org/10.1137/S0036144596303984

28. M. Kim, J. Rosenthal, X. Wang, Pole placement and matrix extension problems: a common point of view, *SIAM J. Control Optim.*, **42** (2004), 2078–2093. https://doi.org/10.1137/s0363012999354429

29. V. A. Zaitsev, Necessary and sufficient conditions in a spectrum control problem, *Diff. Equat.*, **46** (2010), 1789–1793. https://doi.org/10.1134/S0012266110120128

30. W. Michiels, S. I. Niculescu, *Stability and stabilization of time-delay systems: an eigenvalue-based approach,* Philadelphia: SIAM, 2007. https://doi.org/10.1137/1.9780898718645

31. V. Zaitsev, I. Kim, Arbitrary coefficient assignment by static output feedback for linear differential equations with non-commensurate lumped and distributed delays, *Mathematics*, **9** (2021), 2158. https://doi.org/10.3390/math9172158

32. V. Zaitsev, I. Kim, Arbitrary coefficient assignment to the characteristic function of the scalar type for linear systems with state delays by linear static delayed output feedback, *Eur. J. Control*, **76** (2024), 100968. https://doi.org/10.1016/j.ejcon.2024.100968

33. V. Zaitsev, I. Kim, Finite spectrum assignment problem and stabilization of bilinear systems with both lumped and distributed delay, *IFAC-PapersOnLine*, **58** (2024), 25–30. https://doi.org/10.1016/j.ifacol.2024.10.137

34. V. A. Zaitsev, I. G. Kim, V. E. Khartovskii, Finite spectrum assignment problem for bilinear systems with several delays, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, **29** (2019), 319–331. https://doi.org/10.20537/vm190303

35. I. G. Kim, V. A. Zaitsev, Spectrum assignment by static output feedback for linear systems with time delays in states, *2018 14th International Conference "Stability and Oscillations of Nonlinear Control Systems" (Pyatnitskiy's Conference) STAB, IEEE*, 2018. https://doi.org/10.1109/stab.2018.8408365

36. I. Asmykovich, V. M. Marchenko, Spectrum control in systems with delay, *Automat. Remote Control*, **37** (1976), 975–984.

37. D. A. Vikker, Dynamics of automatic rheostat voltage regulators, *Elektrichestvo*, **9** (1934), 26–30.