Mathematics

Research article

Investigating exact solutions for the (3+1)-dimensional KdV-CBS equation: A non-traveling wave approach

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Abstract: Non-traveling wave solutions are crucial, as they provide deeper insights into the complex dynamics and diverse wave structures of nonlinear systems, expanding the understanding of phenomena beyond traditional traveling wave approaches. This research focuses on deriving explicit non-traveling wave solutions for the (3+1)-dimensional KdV–Calogero–Bogoyavlenskii–Schiff (KdV-CBS) equation. A new method using an improved variable separation technique is applied to find abundant explicit non-traveling wave solutions. This technique integrates elements from both the KdV and CBS equations, extending and unifying previous methodologies. The derived solutions incorporate multiple arbitrary functions, showcasing greater versatility than previous methodologies. By selecting specific forms for these functions, diverse non-traveling exact solutions such as periodic solitary waves and cross soliton-like patterns are constructed. All derived solutions are validated by plugging them into the original equation using Maple software, confirming their correctness. Since non-traveling wave solutions for the (3+1)-dimensional KdV-CBS equation have not been thoroughly explored, this study makes a significant contribution to the field.

1. Introduction

Exact solutions of partial differential equations (PDEs) improve our understanding of the physical phenomena they describe, helping researchers identify key features and behaviors of different nonlinear systems [1,2]. Traveling wave solutions and non-traveling wave solutions represent two distinct classes of solutions to PDEs. Traveling wave solutions are characterized by a wave profile that propagates at a constant speed without changing its shape, often expressed in the form $u(x, t) = f(x - ct)$, where c is the

wave speed. These solutions are widely studied due to their simplicity and applicability to phenomena like solitons and wave propagation in homogeneous media. In contrast, non-traveling wave solutions do not exhibit a fixed propagation pattern and may involve more complex structures, such as spatially localized or time-dependent behaviors. These solutions often arise in systems with inhomogeneous or nonlinear effects, providing deeper insights into the dynamics of more intricate physical systems. While traveling wave solutions are well-established in the literature, non-traveling wave solutions offer a broader perspective, capturing phenomena that cannot be described by traditional traveling wave approaches. In recent years, there has been an increasing focus on discovering non-traveling wave solutions to nonlinear partial differential equations. For example, through symbolic computation and the (G'/G)-expansion method, a range of exact solutions in hyperbolic and trigonometric function forms are derived, leading to various non-traveling wave solutions to the (3+1)-dimensional Boiti–Leon–Manna–Pempinelli equation [3]. By integrating the generalized variable separation method with the extended homoclinic test approach (EHTA), some explicit non-traveling wave solutions of the (3+1)-dimensional potential Yu–Toda–Sasa–Fukuyama (YTSF) equation were constructed in [4]. Using the same methodology, some explicit non-traveling wave solutions for the (2+1)-dimensional breaking soliton equation were derived in [5] using symbolic computation. The research in [6] investigates the variable coefficient Date–Jimbo–Kashiwara–Miwa equation using the EHTA, yielding several diverse non-traveling exact solutions, including kink-like, singular solitary, and periodic solitary wave solutions. Another article [7] investigates non-traveling wave solutions for the time-dependent coefficient Bogoyavlenskii–Kadomtsev–Petviashvili equation.

The celebrated KdV equation is a fundamental nonlinear partial differential equation that models wave propagation in dispersive media, playing a pivotal role in soliton theory and integrable systems [8]. Since its introduction in 1895, the KdV equation has been widely studied, driving significant progress in nonlinear science, fluid dynamics, and mathematical physics. For example, the work of [9] explores the generalized scale-invariant analog of the KdV equation, bridging the KdV and scale-invariant dependent variable (SIdV) equations. Using the tanh-coth method, new traveling wave solutions are derived, revealing bell-shaped solutions consistent with the KdV and SIdV equations. The results highlight the method's effectiveness as a mathematical tool for solving nonlinear problems in cross-disciplinary sciences. In [10], the Whitham modulation equations for the complex modified Korteweg–de Vries (cmKdV) equation with step-like initial data. Using the Lax pair and Krichever's algebro-geometric scheme, the N-genus Whitham equations are derived and solved via Hodograph transformation. Numerical simulations confirm the theory, revealing exotic wave patterns and validating the analytical results. In [11], the author introduced two new (3+1)-dimensional equations: the KdV–Calogero–Bogoyavlenskii–Schiff (KdV-CBS) equation and its negative-order counterpart (nKdV-nCBS). In this work, it has been confirmed that equation satisfies the Painlevé integrability test, and its dispersion relations and multiple soliton solutions are also systematically derived.

In this paper, we study the (3+1)-dimensional KdV-CBS equation [11]

$$u_{xt} + u_{xxxy} + 4u_x u_{xy} + 2u_{xx} u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz} = 0, \tag{1.1}$$

where $u = u(x, y, z, t)$ models the water wave velocity corresponding to the surface of shallow water waves. The symbols λ, μ, and ν denote unspecified coefficients. It is evident that when both ν and μ vanish, Eq (1.1) reduces to the negative-order KdV equation. On the other hand, if $\nu = \lambda = 0$ hold,

Eq (1.1) becomes the negative-order CBS equation.

Developing techniques for the exact wave solutions of the model has attracted the interest of many scholars in the past years. In this direction, the research in [12] explored the problem by applying a traveling wave transformation. In the work of [13], they employed unified and singular manifold methods to derive specific traveling wave solutions for the model in different functional forms. Moreover, [14] derived several categories of rational solutions to a special form of the KdV–Calogero–Bogoyavlenkskii–Schiff equation using symbolic computation. It was also demonstrated that the equation can be reformulated as a trilinear linear equation through truncated Painlevé expansion, leading to explicit representations of the rational solutions. The authors of [15] derived various wave solutions to the generalized form of model (1.1), including lump, soliton, and breather solutions, using symbolic computation and Hirota's method.

This study aimed to employ an innovative generalized variable separation technique to derive abundant explicit non-traveling wave solutions for Eq (1.1). To the best of our knowledge, there have been no analytical solutions of this kind for the equation documented in the existing literature. To achieve this objective, the first step is to explore a new variable transformation that draws on ideas from prior studies. The second section of this article discusses this concept and the findings derived from it. Further, in the last section, a summary of the key conclusions is provided.

2. Main analysis

In this section, we examine the EHTA for finding non-traveling exact solutions to Eq (1.1) in the following generalized form:

$$u = p\,\varphi(\xi, t) + q, \tag{2.1}$$

where $\xi = \alpha x + \theta(y, z, t)$, and $p = p(y, z, t), \varphi(\xi, t), q = q(y, z, t)$, and $\theta(y, z, t)$ are three unknown functions.

Remarks 2.1. *It is important to highlight that the modified variable separation method employed in this research, described by Eq* (2.1)*, provides a more general strategy compared with the related techniques examined in earlier studies [3–7], which mainly focused on* $p(y, z, t) \equiv 1$.

Next, by inserting the symbolic structure Eq (2.1) into Eq (1.1), it is simplified as

$$\delta_1\,\varphi_{\xi\xi\xi\xi} + \delta_2\,\varphi_{\xi\xi\xi} + \delta_3\,\varphi_{\xi\xi} + \delta_4\,\varphi_{\xi\xi}\varphi_\xi + \delta_5\,\varphi_{\xi\xi}\varphi + \delta_6\,\varphi_\xi + \delta_7\,\varphi_\xi^2 + \delta_8\,\varphi_{\xi t} = 0, \tag{2.2}$$

where

$$\delta_1 = \alpha^3\theta_y p, \quad \delta_2 = \alpha^3 p_y, \quad \delta_3 = \alpha p\left(2\alpha q_y + \theta_y\mu + \theta_z\nu + \lambda\alpha + \theta_t\right), \quad \delta_4 = 6\alpha p^2\theta_y,$$
$$\delta_5 = 2\alpha p p_y, \quad \delta_6 = \alpha\left(p_y\mu + p_z\nu + p_t\right), \quad \delta_7 = 4\alpha^2 p p_y, \quad \delta_8 = \alpha p. \tag{2.3}$$

At this point, we are seeking the criteria that would transform Eq (2.2) into a more straightforward and solvable form. To accomplish this, we need to analyze the scenario in which the term $\varphi_{\xi\xi}$ in Eq (2.2) is removed. This leads us to set $\delta_3 = 0$, resulting in

$$q(y, z, t) = -\frac{\alpha\lambda y + \mu\theta + \int (\nu\theta_z + \theta_t)\,dy}{2\alpha} + \Lambda(z, t). \tag{2.4}$$

Considering the remaining coefficients in Eq (2.3) and for the sake of further simplifications, it will be very beneficial if we establish the following two assumptions.

2.1. The first assumption: $p(y, z, t) = c$

Without loss of generality, we assume that $p(y, z, t) = 1$ holds. This leads to $\delta_2 = \delta_5 = \delta_6 = \delta_7 = 0$, and Eq (2.2) simplifies to

$$\alpha^2 \theta_y \varphi_{\xi\xi\xi\xi} + 6\theta_y \alpha \varphi_{\xi\xi} \varphi_\xi + \varphi_{\xi t} = 0. \tag{2.5}$$

Now, we integrate Eq (2.5) once with respect to ξ and null the integral constant as

$$\alpha^2 \theta_y \varphi_{\xi\xi\xi} + 3\theta_y \alpha \varphi_\xi^2 + \varphi_t = 0. \tag{2.6}$$

To further simplify Eq (2.6), let us consider $\theta_y(y, z, t) = 1$, which implies

$$\theta(y, z, t) = y + \Theta(z, t). \tag{2.7}$$

If we insert Eq (2.7) into Eq (2.4), the solution becomes

$$q(y, z, t) = -\left(\frac{\lambda\alpha + \mu + \nu\Theta_z(z, t) + \Theta_t(z, t)}{2\alpha} \right) y + \Lambda(z, t). \tag{2.8}$$

Moreover, Eq (2.6) takes the form of

$$\alpha^2 \varphi_{\xi\xi\xi}(\xi, t) + 3\alpha\varphi_\xi^2(\xi, t) + \varphi_t(\xi, t) = 0. \tag{2.9}$$

For solving (2.9), we introduce the new variable $\varphi(\xi, t) = \Phi(\wp)$ along with $\wp = \kappa\xi + \omega t$ in (2.9) and get

$$\alpha^2 \kappa^3 \left(\frac{d^3}{d\wp^3} \Phi(\wp) \right) + 3\alpha\kappa^2 \left(\frac{d}{d\wp} \Phi(\wp) \right)^2 + \omega \left(\frac{d}{d\wp} \Phi(\wp) \right) = 0. \tag{2.10}$$

Now, we apply an analytical technique called the modified generalized exponential rational function method (mGERFM), which was proposed by Ghanbari [16]. On the basis of this method, the solution to Eq (2.10) takes the form of the following structure:

$$\Phi(\wp) = \varepsilon_0 + \sum_{j=1}^{\mathbf{n}} \varepsilon_j \left(\frac{\Gamma'(\wp)}{\Gamma(\wp)} \right)^j + \sum_{j=1}^{\mathbf{n}} \gamma_j \left(\frac{\Gamma(\wp)}{\Gamma'(\wp)} \right)^j, \tag{2.11}$$

where

$$\Gamma(\wp) = \frac{\varsigma_1 e^{\vartheta_1 \wp} + \varsigma_2 e^{\vartheta_2 \wp}}{\varsigma_3 e^{\vartheta_3 \wp} + \varsigma_4 e^{\vartheta_4 \wp}}, \tag{2.12}$$

and $\mathbf{n}$ is the balance number of the given equation. By applying the balance rule given in Eq (2.10), we may infer that $2(\mathbf{n} + 1) = \mathbf{n} + 3$, which gives $\mathbf{n} = 1$. Thus, from Eq (2.11), we obtain

$$\Phi(\wp) = \varepsilon_0 + \varepsilon_1 \left(\frac{\Gamma'(\wp)}{\Gamma(\wp)} \right) + \gamma_1 \left(\frac{\Gamma(\wp)}{\Gamma'(\wp)} \right). \tag{2.13}$$

Upon inserting the expression from Eq (2.13) along with Eq (2.12) in Eq (2.10) and solving the resultant for the unknown parameters, the following solutions are obtained.

Set 1: Inserting $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [2, 0, 1, 1]$ and $[\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4] = [2, 0, 2, 0]$ in Eq (2.12), one has

$$\Gamma(\wp) = \frac{2e^{2\wp}}{e^{2\wp} + 1}. \tag{2.14}$$

Moreover, the rest of the parameters can be obtained as follows:

$$\omega = -4\kappa^3\alpha^2, \varepsilon_1 = -2\kappa\alpha, \gamma_1 = 0, \tag{2.15}$$

where κ, ε_0 are free-chosen parameters.

Taking the obtained results into account in Eqs (2.13) and (2.14), we have

$$\Phi(\wp) = \frac{\varepsilon_0 e^{2\wp} - 4\kappa\alpha + \varepsilon_0}{e^{2\wp} + 1}.$$

Based on the previously mentioned solution, the wave solution to Eq (2.9) is presented below

$$\varphi_1(\xi, t) = \frac{\varepsilon_0 e^{-8\kappa^3 t\alpha^2 + 2\kappa\xi} - 4\kappa\alpha + \varepsilon_0}{e^{-8\kappa^3 t\alpha^2 + 2\kappa\xi} + 1}. \tag{2.16}$$

Consequently, using Eqs (2.16), (2.7), and (2.8) in (2.1), a non-traveling solution for Eq (1.1) is determined as follows:

$$u_1 = \frac{\varepsilon_0 e^{-8\kappa^3 t\alpha^2 + 2\kappa(\alpha x + y + \Theta(z,t))} - 4\kappa\alpha + \varepsilon_0}{e^{-8\kappa^3 t\alpha^2 + 2\kappa(\alpha x + y + \Theta(z,t))} + 1} - \left(\frac{\lambda\alpha + \mu + \nu\Theta_z(z, t) + \Theta_t(z, t)}{2\alpha}\right)y + \Lambda(z, t). \tag{2.17}$$

In Figure 1, we present graphical representations of the solution $u_1(1, 1, z, t)$ described in Eq (2.17). The parameters used are $\lambda = 0.6, \mu = 0.4, \nu = 0.7$, and $\alpha = 1.1$, along with various choices of **(i)** $\Lambda(z, t) = \sin(z + t)$ and $\Theta(z, t) = \cos(z - t)$; **(ii)** $\Lambda(z, t) = \sin(z) + \sin(t)$ and $\Theta(z, t) = zt$; and **(iii)** $\Lambda(z, t) = t\sin(e^z)$ and $\Theta(z, t) = \cos(z)\cos(t)$.
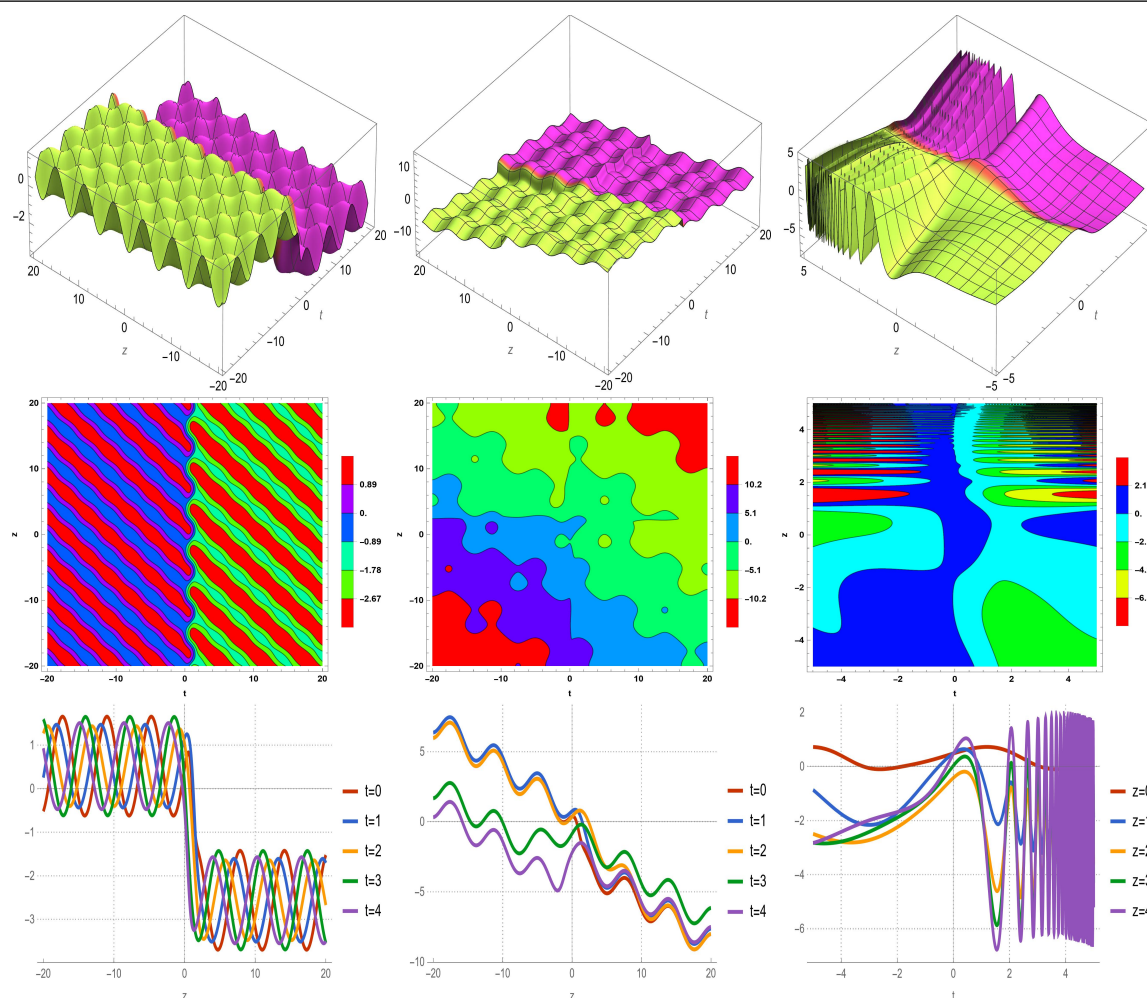
Figure 1. Dynamics of the solution $u_1(1, 1, z, t)$ proposed in Eq (2.17) with $\lambda = 0.6, \mu = 0.4, \nu = 0.7, \kappa = 0.7, \alpha = 1.1$, and $\varepsilon_0 = 1$, corresponding to (first column) $\Lambda(z, t) = \sin(z + t)$ and $\Theta(z, t) = \cos(z - t)$; (second column) $\Lambda(z, t) = \sin(z) + \sin(t)$ and $\Theta(z, t) = zt$; and (third column) $\Lambda(z, t) = t \sin(e^z)$ and $\Theta(z, t) = \cos(z) \cos(t)$.

As seen in Figure 1, for $\Lambda(z, t) = \sin(z + t)$ and $\Theta(z, t) = \cos(z - t)$, the solution u_1 exhibits periodic wave-like behavior due to the trigonometric functions. The interaction of $\sin(z+t)$ and $\cos(z-t)$ creates interference patterns, while the exponential term modulates the amplitude, leading to a combination of periodic oscillations and exponential decay or growth over time. Moreover, with $\Lambda(z, t) = \sin(z)+\sin(t)$ and $\Theta(z, t) = zt$, the solution u_1 shows mixed periodicity in z and t. The linear growth term zt dominates for a large z or t, causing the amplitude of u_1 to increase linearly, while the periodic terms introduce oscillatory behavior, resulting in a modulated wave pattern. Finally, for $\Lambda(z, t) = t \sin(e^z)$ and $\Theta(z, t) = \cos(z) \cos(t)$, the solution u_1 displays high-frequency oscillations in z due to the e^z term. The time-dependent amplitude $t \sin(e^z)$ grows linearly with t, while the periodic modulation from $\cos(z) \cos(t)$ creates complex, nonlinear dynamics, potentially leading to chaotic or intricate wave structures.

Remarks 2.2. *By utilizing a wide range of arbitrary functions within the solution u_1, numerous distinct non-traveling exact solutions to the KdV-CBS equation* (1.1) *can be derived.*

Some special cases of the solution (2.17)

■ For $\Lambda(z,t) = \Lambda_1(z)\,\Lambda_2(t)$ and $\Theta(z,t) = \Theta_1(z)\,\Theta_2(t)$ in Eq (2.17), we obtain

$$u_{1,1} = \frac{\gamma_0 \mathrm{e}^{-8\kappa^3 t\alpha^2 + 2\kappa(\alpha x + y + \Theta_1(z)\Theta_2(t))} - 4\kappa\alpha + \gamma_0}{\mathrm{e}^{-8\kappa^3 t\alpha^2 + 2\kappa(\alpha x + y + \Theta_1(z)\Theta_2(t))} + 1} - \frac{\left(\mu + \Theta_1'(z)\,\Theta_2(t)\,\nu + \lambda\alpha + \Theta_1(z)\,\Theta_2'(t)\right)y}{2\alpha} + \Lambda_1(z)\,\Lambda_2(t)\,.$$

(2.18)

■ For $\Lambda(z,t) = \Lambda_1(z) + \Lambda_2(t)$ and $\Theta(z,t) = \Theta_1(z) + \Theta_2(t)$ in Eq (2.17), we obtain

$$u_{1,2} = \frac{\gamma_0 \mathrm{e}^{-8\kappa^3 t\alpha^2 + 2\kappa(\alpha x + y + \Theta_1(z) + \Theta_2(t))} - 4\kappa\alpha + \gamma_0}{\mathrm{e}^{-8\kappa^3 t\alpha^2 + 2\kappa(\alpha x + y + \Theta_1(z) + \Theta_2(t))} + 1} - \frac{\left(\mu + \Theta_1'(z)\,\nu + \lambda\alpha + \Theta_2'(t)\right)y}{2\alpha} + \Lambda_1(z) + \Lambda_2(t)\,.$$

(2.19)

■ For $\Lambda(z,t) = \mathrm{e}^{z+t}$ and $\Theta(z,t) = \sin(z+t)$ in Eq (2.17), we obtain

$$u_{1,3} = \frac{\gamma_0 \mathrm{e}^{-8\kappa^3 t\alpha^2 + 2\kappa(\alpha x + y + \sin(z+t))} - 4\kappa\alpha + \gamma_0}{\mathrm{e}^{-8\kappa^3 t\alpha^2 + 2\kappa(\alpha x + y + \sin(z+t))} + 1} - \frac{(\mu + \cos(z+t)\,\nu + \lambda\alpha + \cos(z+t))\,y}{2\alpha} + \mathrm{e}^{z+t}\,.$$

(2.20)

Set 2: Inserting $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, -1, 2\mathbf{i}, 0]$ and $[\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4] = [2 + \mathbf{i}, 2 - \mathbf{i}, 0, 0]$ in Eq (2.12), one has

$$\Gamma(\wp) = \sin(\wp)\,\mathrm{e}^{2\wp}\,.$$

(2.21)

Additionally, the remaining parameters can be obtained as follows:

$$\omega = 4\kappa^3\alpha^2,\, \varepsilon_1 = 0,\, \gamma_1 = -10\kappa\alpha,$$

(2.22)

where κ, ε_0 are free-chosen parameters.

Taking the obtained results into account in Eqs (2.13) and (2.14), one achieves

$$\Phi(\wp) = \frac{2(-5\kappa\alpha + \varepsilon_0)\sin(\xi) + \varepsilon_0\cos(\xi)}{2\sin(\xi) + \cos(\xi)}\,.$$

If we account for the solution outlined earlier, the wave solution for Eq (2.9) is provided below:

$$\varphi_2(\xi, t) = \frac{2(-5\kappa\alpha + \varepsilon_0)\sin\left(4\kappa^3 t\alpha^2 + \kappa\xi\right) + \varepsilon_0\cos\left(4\kappa^3 t\alpha^2 + \kappa\xi\right)}{2\sin(4\kappa^3 t\alpha^2 + \kappa\xi) + \cos(4\kappa^3 t\alpha^2 + \kappa\xi)}\,.$$

(2.23)

Hence, if we use Eqs (2.23), (2.7), and (2.8) in (2.1), a non-traveling solution for Eq (1.1) is determined as follows:

$$
\begin{aligned}
u_2 = & \frac{(-10\kappa\alpha + 2\varepsilon_0)\sin\left(4\kappa^3 t\alpha^2 + \kappa\left(\alpha x + y + \Theta(z,t)\right)\right) + \varepsilon_0\cos\left(4\kappa^3 t\alpha^2 + \kappa\left(\alpha x + y + \Theta(z,t)\right)\right)}{2\sin(4\kappa^3 t\alpha^2 + \kappa\left(\alpha x + y + \Theta(z,t)\right)) + \cos(4\kappa^3 t\alpha^2 + \kappa\left(\alpha x + y + \Theta(z,t)\right))} \\
& - \left(\frac{\lambda\alpha + \mu + \nu\Theta_z(z,t) + \Theta_t(z,t)}{2\alpha}\right)y + \Lambda(z,t)\,.
\end{aligned}
$$

(2.24)

Some special cases of the solution (2.24)

■ For $\Lambda(z,t) = \mathrm{e}^t\cos(z)$ and $\Theta(z,t) = t\sin(z)$ in Eq (2.24), we obtain

$$
\begin{aligned}
u_{2,1} = & \frac{(-10\alpha\kappa + 2\gamma_0)\sin\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t\sin(z)\right)\right) + \gamma_0\cos\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t\sin(z)\right)\right)}{2\sin(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t\sin(z)\right)) + \cos(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t\sin(z)\right))} \\
& - \frac{(\mu + t\cos(z)\,\nu + \lambda\alpha + \sin(z))\,y}{2\alpha} + \mathrm{e}^t\cos(z)\,.
\end{aligned}
$$

(2.25)

■ For $\Lambda(z,t) = t - \sin(z)$ and $\Theta(z,t) = t^2 \tan(z)$ in Eq (2.24), we obtain

$$
\begin{aligned}
u_{2,2} = & \frac{(-10\alpha\kappa + 2\gamma_0)\sin\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t^2\tan(z)\right)\right) + \gamma_0\cos\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t^2\tan(z)\right)\right)}{2\sin(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t^2\tan(z)\right)) + \cos(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t^2\tan(z)\right))} \\
& - \frac{\left(\mu + t^2\left(1 + \tan^2(z)\right)\nu + \lambda\alpha + 2t\tan(z)\right)y}{2\alpha} + t - \sin(z).
\end{aligned}
$$

(2.26)

■ For $\Lambda(z,t) = t^2 + z^2$ and $\Theta(z,t) = z^2 - t^2$ in Eq (2.24), we obtain

$$
\begin{aligned}
u_{2,3} = & \frac{(-10\alpha\kappa + 2\gamma_0)\sin\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x - t^2 + z^2 + y\right)\right) + \gamma_0\cos\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x - t^2 + z^2 + y\right)\right)}{2\sin(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x - t^2 + z^2 + y\right)) + \cos(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x - t^2 + z^2 + y\right))} \\
& - \frac{(\lambda\alpha + 2z\nu + \mu - 2t)y}{2\alpha} + t^2 + z^2.
\end{aligned}
$$

(2.27)

In Figure 2, we present graphical representations of the solution $u_2(1,1,z,t)$ described in Eq (2.24). The parameters used $\lambda = 0.1, \mu = 0.1, \nu = 0.7, \kappa = 0.1, \alpha = 1.1$, and $\varepsilon_0 = 1$, along with various choices of **(i)** $\Lambda(z,t) = z\,\sin(z+t)$ and $\Theta(z,t) = \cos(z+t)$; **(ii)** $\Lambda(z,t) = (t+z)\,\cos(z+t)$ and $\Theta(z,t) = \cos(z-t)$; and **(iii)** $\Lambda(z,t) = \cos(z)\cos(t)$ and $\Theta(z,t) = \cos(z) - \cos(t)$.
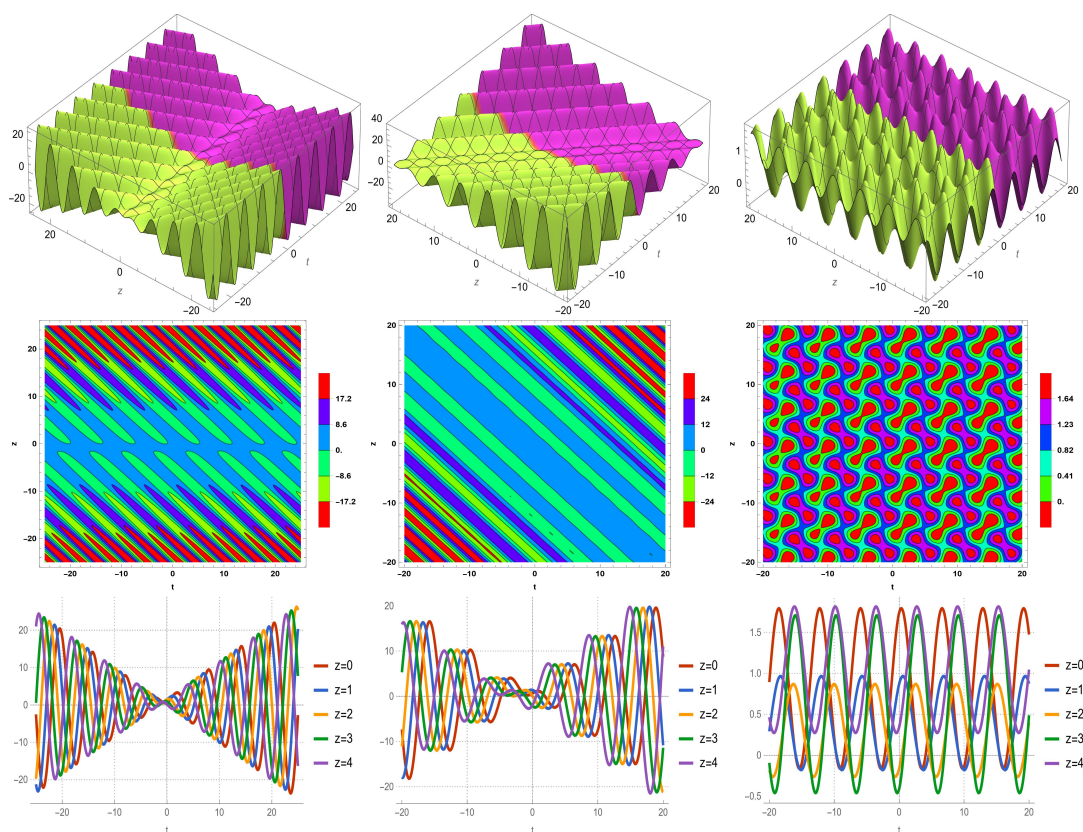
Figure 2. Dynamics of the solution $u_2(1,1,z,t)$ proposed in Eq (2.24) with $\lambda = 0.1, \mu = 0.1, \nu = 0.7, \kappa = 0.7, \alpha = 1.1$, and $\varepsilon_0 = 1$, corresponding to (first column) $\Lambda(z,t) = z\,\sin(z+t)$ and $\Theta(z,t) = \cos(z+t)$; (second column) $\Lambda(z,t) = (t+z)\,\cos(z+t)$ and $\Theta(z,t) = \cos(z-t)$; and (third column) $\Lambda(z,t) = \cos(z)\cos(t)$ and $\Theta(z,t) = \cos(z) - \cos(t)$.

From Figure 2, it is clear that for $\Lambda(z,t) = z\sin(z+t)$ and $\Theta(z,t) = \cos(z+t)$, the solution u_2 combines oscillatory behavior with a spatially growing amplitude due to the $z\sin(z+t)$ term. The trigonometric functions create wave-like patterns, while the linear growth in z amplifies the oscillations, resulting in a solution that grows in magnitude with an increasing z. Moreover, when we take $\Lambda(z,t) = (t+z)\cos(z+t)$ and $\Theta(z,t) = \cos(z-t)$, the solution u_2 exhibits a combination of linear growth and wave interference. The $(t+z)$ term introduces growth in both time and space, while the $\cos(z+t)$ and $\cos(z-t)$ terms create traveling waves, leading to modulated wave patterns with increasing amplitude. Finally, for $\Lambda(z,t) = \cos(z)\cos(t)$ and $\Theta(z,t) = \cos(z) - \cos(t)$, the solution u_2 displays periodic behavior in both z and t. The product $\cos(z)\cos(t)$ introduces standing wave patterns, while the difference $\cos(z) - \cos(t)$ creates phase shifts, resulting in a solution with rich, periodic dynamics and no significant growth or decay.

Set 3: Inserting $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1,1,1,0]$ and $[\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4] = [1,2,0,0]$ in Eq (2.12), one has

$$\Gamma(\wp) = e^{\wp} + e^{2\wp}. \tag{2.28}$$

Moreover, the rest of the parameters can be obtained as follows:

$$\omega = -\kappa^3\alpha^2, \varepsilon_1 = 0, \gamma_1 = -4\kappa\alpha, \tag{2.29}$$

where κ, ε_0 are free-chosen parameters.

Taking the obtained results into account in Eqs (2.13) and (2.14), one gets

$$\Phi(\wp) = \frac{(-4\kappa\alpha + 2\varepsilon_0)\,e^{\xi} - 4\kappa\alpha + \varepsilon_0}{2\,e^{\xi} + 1}.$$

If we take the solution stated beforehand into account, the wave solution to Eq (2.9) is detailed below

$$\varphi_3(\xi,t) = \frac{(-4\kappa\alpha + 2\varepsilon_0)\,e^{-\kappa^3 t\alpha^2 + \kappa\xi} - 4\kappa\alpha + \varepsilon_0}{2\,e^{-\kappa^3 t\alpha^2 + \kappa\xi} + 1}. \tag{2.30}$$

Consequently, if we use Eqs (2.30), (2.7), and (2.8) in (2.1), a non-traveling solution for Eq (1.1) is determined as follows:

$$u_3 = \frac{(-4\kappa\alpha + 2\varepsilon_0)\,e^{-\kappa^3 t\alpha^2 + \kappa(\alpha x + y + \Theta(z,t))} - 4\kappa\alpha + \varepsilon_0}{2\,e^{-\kappa^3 t\alpha^2 + \kappa(\alpha x + y + \Theta(z,t))} + 1} - \left(\frac{\lambda\alpha + \mu + \nu\Theta_z(z,t) + \Theta_t(z,t)}{2\alpha}\right)y + \Lambda(z,t). \tag{2.31}$$

Some special cases of the solution (2.31)

■ For $\Lambda(z,t) = t^2 + z^2$ and $\Theta(z,t) = z^2 - t^2$ in Eq (2.31), we obtain

$$u_{3,1} = \frac{(-4\kappa\alpha + 2\gamma_0)\,e^{-\kappa^3 t\alpha^2 + \kappa(\alpha x - t^2 + z^2 + y)} - 4\kappa\alpha + \gamma_0}{2\,e^{-\kappa^3 t\alpha^2 + \kappa(\alpha x - t^2 + z^2 + y)} + 1} - \frac{(\lambda\alpha + 2z\nu + \mu - 2t)\,y}{2\alpha} + t^2 + z^2. \tag{2.32}$$

■ For $\Lambda(z,t) = z^2 e^t$ and $\Theta(z,t) = t^2\sin(z)$ in Eq (2.31), we obtain

$$u_{3,2} = \frac{(-4\kappa\alpha + 2\gamma_0)\,e^{-\kappa^3 t\alpha^2 + \kappa(\alpha x + y + t^2\sin(z))} - 4\kappa\alpha + \gamma_0}{2\,e^{-\kappa^3 t\alpha^2 + \kappa(\alpha x + y + t^2\sin(z))} + 1} - \frac{\left(\mu + t^2\cos(z)\,\nu + \lambda\alpha + 2t\sin(z)\right)y}{2\alpha} + z^2 e^t. \tag{2.33}$$

■ For $\Lambda(z,t) = z\sin(t)$ and $\Theta(z,t) = t\cos(z)$ in Eq (2.31), we obtain

$$u_{3,3} = \frac{(-4\kappa\alpha + 2\gamma_0)\, e^{-\kappa^3 t\,\alpha^2 + \kappa(\alpha x + y + t\cos(z))} - 4\kappa\alpha + \gamma_0}{2\, e^{-\kappa^3 t\,\alpha^2 + \kappa(\alpha x + y + t\cos(z))} + 1} - \frac{(\mu - t\sin(z)\,\nu + \lambda\alpha + \cos(z))\,y}{2\alpha} + z\sin(t)\,.$$

(2.34)

In Figure 3, we present graphical representations of the solution $u_3(1,1,z,t)$ described in Eq (2.31). The parameters used are $\lambda = 0.6, \mu = 0.4, \nu = 0.7$, and $\alpha = 1.1$, along with various choices of **(i)** $\Lambda(z,t) = e^{\sin(t-z)}$ and $\Theta(z,t) = e^{\sin(t-z)}$; **(ii)** $\Lambda(z,t) = \sin(t)e^{\sin(z)}$ and $\Theta(z,t) = \cos(t)e^{\cos(z)}$; and **(iii)** $\Lambda(z,t) = te^{-\sin(t+z)}$ and $\Theta(z,t) = ze^{\cos(t)}$.
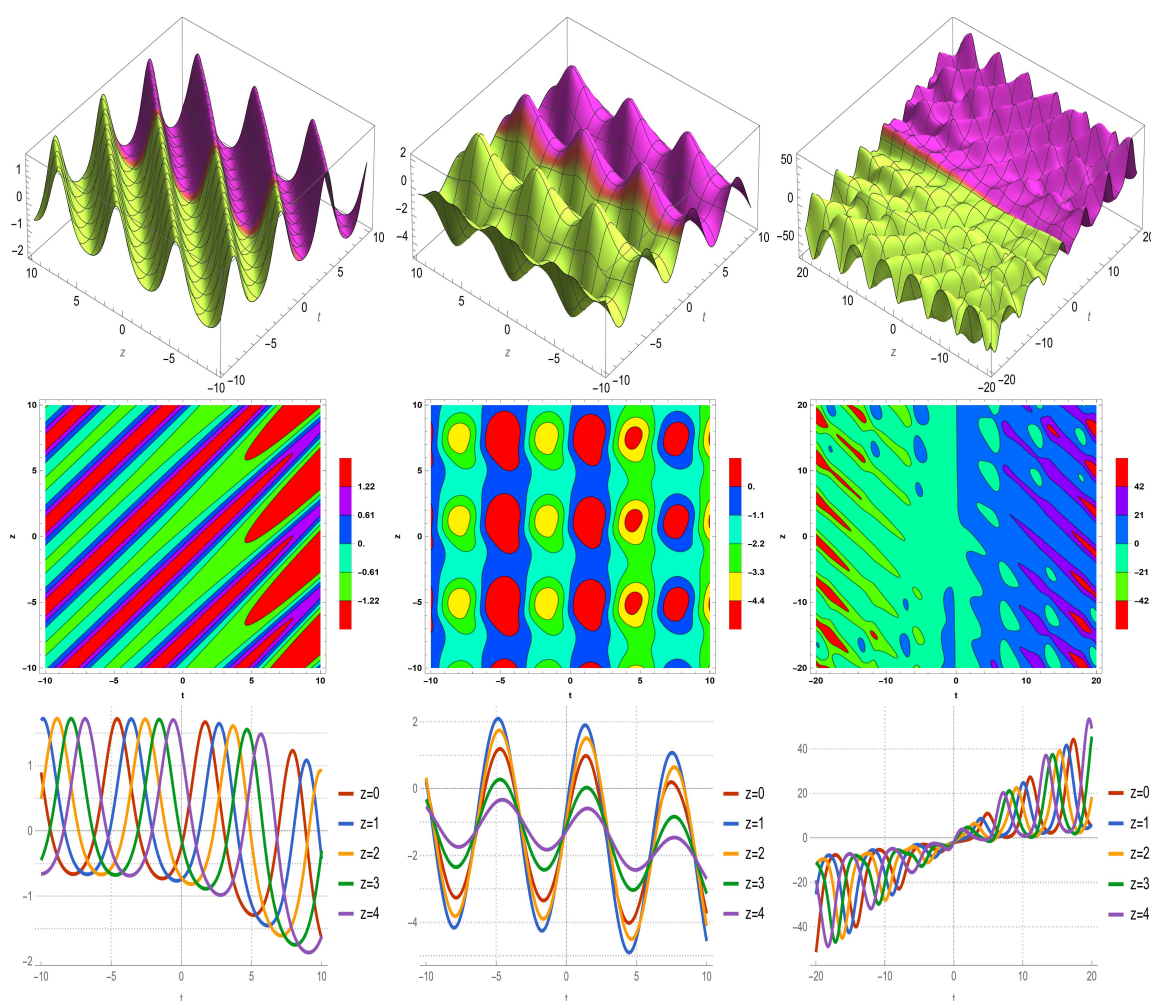
Figure 3. Dynamics of the solution $u_3(1,1,z,t)$ proposed in Eq (2.31) with $\lambda = 0.6, \mu = 0.4, \nu = 0.7, \kappa = 0.7, \alpha = 1.1$, and $\varepsilon_0 = 1$, corresponding to (first column) $\Lambda(z,t) = e^{\sin(t-z)}$ and $\Theta(z,t) = e^{\sin(t-z)}$; (second column) $\Lambda(z,t) = \sin(t)e^{\sin(z)}$ and $\Theta(z,t) = \cos(t)e^{\cos(z)}$; and (third column) $\Lambda(z,t) = te^{-\sin(t+z)}$ and $\Theta(z,t) = ze^{\cos(t)}$.

Figure 3 illustrates that for $\Lambda(z,t) = e^{\sin(t-z)}$ and $\Theta(z,t) = e^{\sin(t-z)}$, the solution u_3 exhibits oscillatory behavior modulated by exponential functions. The term $e^{\sin(t-z)}$ introduces periodic growth and decay in both time and space, resulting in a solution with localized peaks and troughs that propagate in the z–t plane. Moreover, with $\Lambda(z,t) = \sin(t)e^{\sin(z)}$ and $\Theta(z,t) = \cos(t)e^{\cos(z)}$, the solution u_3 combines

time-dependent oscillations with spatially modulated exponential terms. The interplay among $\sin(t)$, $\cos(t)$, and the exponential functions creates a complex pattern of waves with amplitudes that vary periodically in both z and t. Moreover, for $\Lambda(z, t) = te^{-\sin(t+z)}$ and $\Theta(z, t) = ze^{\cos(t)}$, the solution u_3 displays a combination of linear growth in t and z with periodic modulation. The exponential terms $e^{-\sin(t+z)}$ and $e^{\cos(t)}$ introduce localized damping and amplification, leading to a solution that grows linearly over time while exhibiting periodic variations in space.

Set 4: Inserting $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, -1, 1, 0]$ and $[\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4] = [\mathbf{i}, -\mathbf{i}, 0, 0]$ in Eq (2.12), one has

$$\Gamma(\wp) = \cos(\wp). \tag{2.35}$$

Additionally, the remaining parameters can be obtained as follows:

$$\omega = 4\kappa^3\alpha^2, \varepsilon_1 = 0, \gamma_1 = -4\kappa\alpha, \tag{2.36}$$

where κ, ε_0 are free-chosen parameters.

Taking the obtained results into account in Eqs (2.13) and (2.14), we obtain

$$\Phi(\wp) = 2\kappa\alpha \cot(\xi) + \varepsilon_0.$$

If we take the solution stated beforehand into account, the wave solution to Eq (2.9) is presented as follows:

$$\varphi_4(\xi, t) = 2\kappa\alpha \cot\left(4\kappa^3 t\alpha^2 + \kappa\xi\right) + \varepsilon_0. \tag{2.37}$$

Hence, using Eqs (2.37), (2.7), and (2.8) in (2.1), a non-traveling solution for Eq (1.1) is determined as follows:

$$u_4 = 2\kappa\alpha \cot\left(4\kappa^3 t\alpha^2 + \kappa\left(\alpha x + y + \Theta(z, t)\right)\right) + \varepsilon_0 - \left(\frac{\lambda\alpha + \mu + \nu\Theta_z(z, t) + \Theta_t(z, t)}{2\alpha}\right)y + \Lambda(z, t). \tag{2.38}$$

Some special cases of the solution (2.38)

■ For $\Lambda(z, t) = z\sin(t)$ and $\Theta(z, t) = t\cos(z)$ in Eq (2.38), we obtain

$$u_{4,1} = 2\kappa\alpha \cot\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t\cos(z)\right)\right) + \gamma_0 - \frac{\left(\mu - t\sin(z)\nu + \lambda\alpha + \cos(z)\right)y}{2\alpha} + z\sin(t). \tag{2.39}$$

■ For $\Lambda(z, t) = z - \sin(t)$ and $\Theta(z, t) = t + \cos(z)$ in Eq (2.38), we obtain

$$u_{4,2} = 2\kappa\alpha \cot\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + t + \cos(z)\right)\right) + \gamma_0 - \frac{\left(\mu - \sin(z)\nu + \lambda\alpha + 1\right)y}{2\alpha} + z - \sin(t). \tag{2.40}$$

■ For $\Lambda(z, t) = e^z \sin(t)$ and $\Theta(z, t) = e^t \cos(z)$ in Eq (2.38), we obtain

$$u_{4,3} = 2\kappa\alpha \cot\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + e^t \cos(z)\right)\right) + \gamma_0 - \frac{\left(\mu - e^t \sin(z)\nu + \lambda\alpha + e^t \cos(z)\right)y}{2\alpha} + e^z \sin(t). \tag{2.41}$$

Set 5: For $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, -1, 1, 0]$ and $[\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4] = [1 + \mathbf{i}, 1 - \mathbf{i}, 0, 0]$ in Eq (2.12), one has

$$\Gamma(\wp) = e^{\wp} \cos(\wp). \tag{2.42}$$

Moreover, the rest of the parameters can be obtained as follows:

$$\omega = 4\kappa^3\alpha^2, \varepsilon_1 = 0, \gamma_1 = -4\kappa\alpha, \tag{2.43}$$

where κ, ε_0 are free-chosen parameters.

If we take the obtained results in Eqs (2.13) and (2.14) into account, it reads

$$\Phi(\wp) = -2\kappa\alpha \tan(\xi) + 2\kappa\alpha + \varepsilon_0.$$

If we take the solution stated beforehand into account, the wave solution to Eq (2.9) is detailed below

$$\varphi_5(\xi, t) = -2\kappa\alpha \tan\left(4\kappa^3 t\alpha^2 + \kappa\xi\right) + 2\kappa\alpha + \varepsilon_0. \tag{2.44}$$

Therefore, by substituting Eqs (2.44), (2.7), and (2.8) into (2.1), we derive a non-traveling solution for Eq (1.1) as follows

$$u_5 = -2\kappa\alpha \tan\left(4\kappa^3 t\alpha^2 + \kappa\left(\alpha x + y + \Theta(z, t)\right)\right) + 2\kappa\alpha + \varepsilon_0 - \left(\frac{\lambda\alpha + \mu + \nu\Theta_z(z, t) + \Theta_t(z, t)}{2\alpha}\right) y + \Lambda(z, t). \tag{2.45}$$

Some special cases of the solution (2.45)

■ For $\Lambda(z, t) = e^z \sin(t)$ and $\Theta(z, t) = e^t \cos(z)$ in Eq (2.45), we obtain

$$u_{5,1} = -2\kappa\alpha \tan(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + e^t \cos(z)\right)) + 2\kappa\alpha + \gamma_0 \\ - \frac{(\mu - e^t \sin(z)\,\nu + \lambda\alpha + e^t \cos(z))\,y}{2\alpha} + e^z \sin(t). \tag{2.46}$$

■ For $\Lambda(z, t) = t\,e^{z-t}$ and $\Theta(z, t) = z\,e^{z+t}$ in Eq (2.45), we obtain
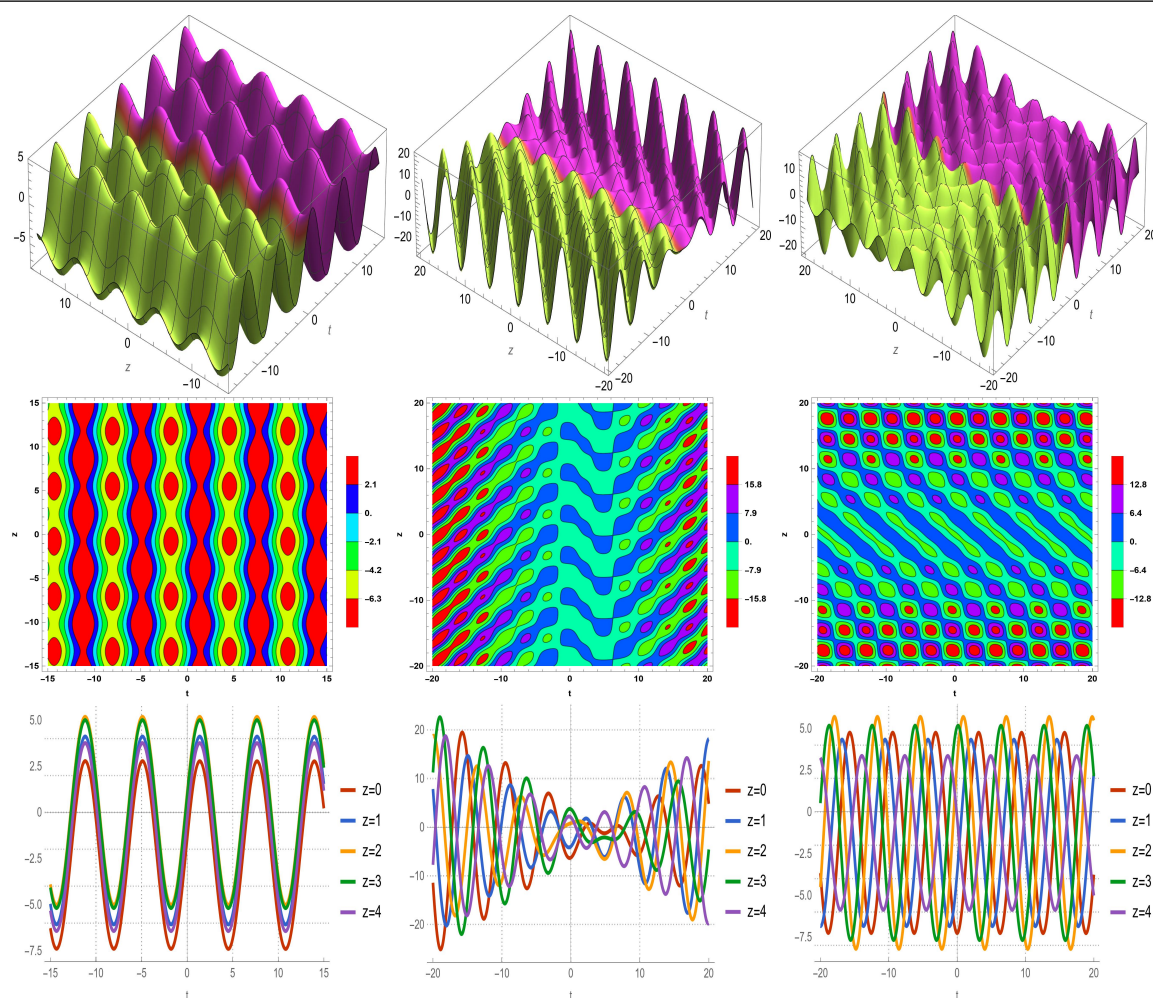
$$u_{5,2} = -2\kappa\alpha \tan(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + y + z\,e^{z+t}\right)) + 2\kappa\alpha + \gamma_0 \\ - \frac{(\mu + (e^{z+t} + z\,e^{z+t})\,\nu + \lambda\alpha + z\,e^{z+t})\,y}{2\alpha} + t\,e^{z-t}. \tag{2.47}$$

■ For $\Lambda(z, t) = z^2 - t^2$ and $\Theta(z, t) = t^2 + z^2$ in Eq (2.45), we obtain

$$u_{5,3} = -2\kappa\alpha \tan\left(4\kappa^3 t\,\alpha^2 + \kappa\left(\alpha x + t^2 + z^2 + y\right)\right) + 2\kappa\alpha + \gamma_0 - \frac{(\lambda\alpha + 2z\nu + \mu + 2t)\,y}{2\alpha} + z^2 - t^2. \tag{2.48}$$

In Figure 4, we present graphical representations of the solution $u_5(1, 1, z, t)$ described in Eq (2.45). The parameters used are $\lambda = 0.6, \mu = 0.4, \nu = 0.2, \kappa = 0.4, \alpha = 0.1$, and $\varepsilon_0 = 1$, along with various choices of **(i)** $\Lambda(z, t) = t \cos(t - z)$ and $\Theta(z, t) = \sin(t) \cos(z)$; **(ii)** $\Lambda(z, t) = \sin(t) - e^z$ and $\Theta(z, t) = \cos(z) - e^t$; and **(iii)** $\Lambda(z, t) = z \cos(t) \sin(z)$ and $\Theta(z, t) = \sin(t + z)$.

Figure 4. Dynamics of the solution $u_5(1, 1, z, t)$ proposed in Eq (2.45) with $\lambda = 0.6, \mu = 0.4, \nu = 0.2, \kappa = 0.4, \alpha = 0.1$, and $\varepsilon_0 = 1$, corresponding to (first column) $\Lambda(z, t) = \cos(t) - \cos(z)$ and $\Theta(z, t) = \cos(t) + \cos(z)$; (second column) $\Lambda(z, t) = t\cos(t - z)$ and $\Theta(z, t) = \sin(t)\cos(z)$; and (third column) $\Lambda(z, t) = z\cos(t)\sin(z)$ and $\Theta(z, t) = \sin(t + z)$.

As seen in Figure 4, if one takes $\Lambda(z, t) = \cos(t) - \cos(z)$ and $\Theta(z, t) = \cos(t) + \cos(z)$, the solution u_5 exhibits periodic behavior in both z and t. The trigonometric terms create standing wave patterns, while the tan function introduces phase shifts, resulting in a solution with smooth oscillations and no significant growth or decay over time. Moreover, with $\Lambda(z, t) = t\cos(t - z)$ and $\Theta(z, t) = \sin(t)\cos(z)$, the solution u_5 combines linear growth in t with wave-like behavior. The term $t\cos(t - z)$ introduces a time-dependent amplitude, while $\sin(t)\cos(z)$ creates traveling waves, leading to a solution that grows linearly in time with modulated oscillations. Finally, if we have $\Lambda(z, t) = z\cos(t)\sin(z)$ and $\Theta(z, t) = \sin(t + z)$, the solution u_5 displays spatially growing oscillations due to the $z\cos(t)\sin(z)$ term. The $\sin(t+z)$ term introduces phase shifts, resulting in a solution with complex wave patterns and amplitudes that grow linearly with z.

2.2. The second assumption: $p_y = 0$

Here, we consider another way of simplifying Eq (2.2) as $p_y = 0$, or equivalently $p(y, z, t) = p(z, t)$. Then Eq (2.2) reduces to

$$\theta_y \left(\alpha^2 p \varphi_{\xi\xi\xi\xi} + 6 p^2 \varphi_{\xi\xi} \varphi_\xi \right) + (p_z v + p_t) \varphi_\xi + p \varphi_{\xi t} = 0. \tag{2.49}$$

Now, considering $\theta_y = 0$ in Eq (2.49) yields

$$(p_z v + p_t) \varphi_\xi(\xi, t) + p \varphi_{\xi t}(\xi, t) = 0. \tag{2.50}$$

This condition also implies that $\theta(y, z, t) = \Omega(z, t)$. Taking this result into account in (2.4), one gets

$$q(y, z, t) = -\left(\frac{\lambda \alpha + v \Omega_z(z, t) + \Omega_t(z, t)}{2\alpha} \right) y + \Lambda(z, t). \tag{2.51}$$

Further, we integrate Eq (2.50) once with respect to ξ and null the integral constant as

$$(p_z v + p_t) \varphi(\xi, t) + p \varphi_t(\xi, t) = 0. \tag{2.52}$$

If we assume that $p_z v + p_t = p$ holds, than we have

$$p(z, t) = \hbar_1 \left(\frac{tv - z}{v} \right) e^{\frac{z}{v}}, \tag{2.53}$$

where $\hbar_1$ is an arbitrary non–zero function.

Taking Eq (2.53) into account in (2.52), we derive the following equation:

$$\varphi(\xi, t) + \varphi_t(\xi, t) = 0, \tag{2.54}$$

which possesses the following solution:

$$\varphi(\xi, t) = \hbar_2(\xi) e^{-t}, \tag{2.55}$$

where $\hbar_2$ is another arbitrary non–zero function.

Thus, a general non-traveling solution for Eq (1.1) is obtained as follows:

$$u_6 = e^{\frac{z}{v} - t} \hbar_1 \left(\frac{tv - z}{v} \right) \hbar_2(\alpha x + \Omega(z, t)) - \left(\frac{\lambda \alpha + v \Omega_z(z, t) + \Omega_t(z, t)}{2\alpha} \right) y + \Lambda(z, t), \tag{2.56}$$

where $\hbar_1, \hbar_2$ are arbitrary one-variable functions, and Λ, Ω are arbitrary functions dependent on z, t.

Some special cases of the solution (2.56)

■ For $\Lambda(z, t) = -t^2 + z^2, \Omega(z, t) = t^2 + z^2$, and $\hbar_1(\xi) = \sin(\xi), \hbar_2(\xi) = e^\xi$ in Eq (2.56), we obtain

$$u_{6,1} = \sin\left(\frac{tv - z}{v} \right) e^{\alpha x + t^2 + z^2 + \frac{z}{v} - t} - \frac{(\lambda \alpha + 2zv + 2t)}{2\alpha} y - t^2 + z^2. \tag{2.57}$$

■ For $\Lambda(z, t) = t \cos(z), \Omega(z, t) = z \cos(t)$, and $\hbar_1(\xi) = \xi, \hbar_2(\xi) = \xi e^\xi$ in Eq (2.56), we obtain

$$u_{6,2} = \frac{(tv - z)(\alpha x + z \cos(t)) e^{\alpha x + z \cos(t) + \frac{z}{v} - t}}{v} - \frac{(\cos(t) v + \lambda \alpha - z \sin(t))}{2\alpha} y + t \cos(z). \tag{2.58}$$

■ For $\Lambda(z,t) = e^z - \cos(t)$, $\Omega(z,t) = -\sin(-z+t)$, and $\hbar_1(\xi) = \cos(\xi)$, $\hbar_2(\xi) = \xi\sin(\xi)$ in Eq (2.56), we obtain

$$u_{6,3} = \cos\left(\frac{t\nu - z}{\nu}\right) e^{\frac{z}{\nu}-t}(\alpha x - \sin(-z+t))\sin(\alpha x - \sin(-z+t))$$
$$- \frac{(\cos(-z+t)\nu + \lambda\alpha - \cos(-z+t))}{2\alpha}y + e^z - \cos(t). \tag{2.59}$$

In Figure 5, we present graphical representations of the solution $u_6(1,1,z,t)$ described in Eq (2.56). The parameters used are $\lambda = 0.6, \mu = 0.4, \nu = 0.7$, and $\alpha = 1.1$, along with various choices of (i) $\Lambda(z,t) = \sin(z-t), \Omega(z,t) = \sin(z+t)$, and $\hbar_1(\xi) = e^\xi, \hbar_2(\xi) = \cos(\xi)$; (ii) $\Lambda(z,t) = \sin(z) - \cos(t)$, $\Theta(z,t) = \sin(z) + \cos(t)$, and $\hbar_1(\xi) = \hbar_2(\xi) = e^\xi$; and (iii) $\Lambda(z,t) = t\cos(z), \Theta(z,t) = t\sin(z)$, and $\hbar_1(\xi) = \hbar_2(\xi) = \xi e^\xi$.
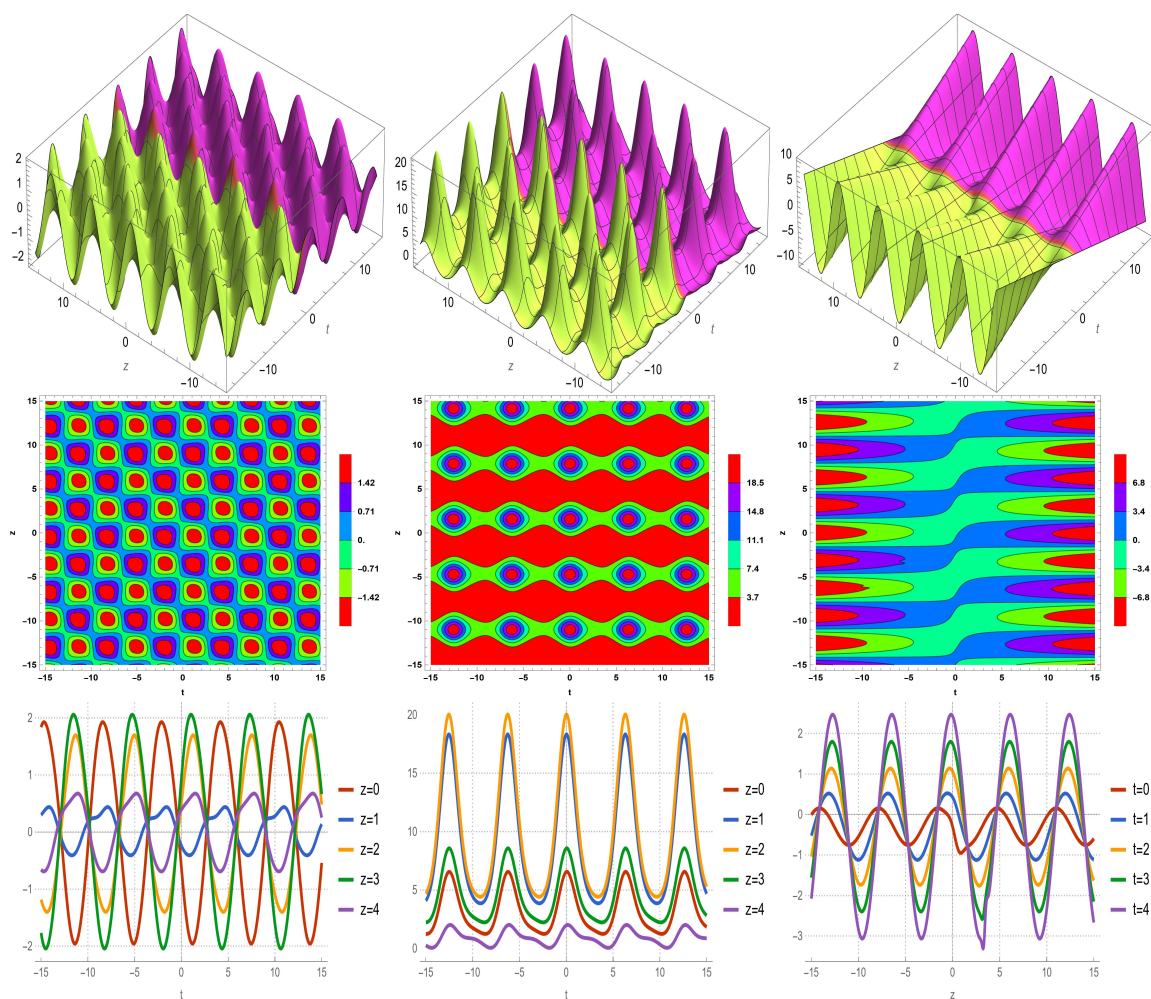
Figure 5. Dynamics of the solution $u_6(1,1,z,t)$ proposed in Eq (2.56) with $\lambda = 0.6, \mu = 0.4, \nu = 0.7$, and $\alpha = 1.1$, corresponding to (first column) $\Lambda(z,t) = \sin(z-t), \Omega(z,t) = \sin(z+t), \hbar_1(\xi) = e^\xi$, and $\hbar_2(\xi) = \cos(\xi)$; (second column) $\Lambda(z,t) = \sin(z) - \cos(t), \Theta(z,t) = \sin(z) + \cos(t)$, and $\hbar_1(\xi) = \hbar_2(\xi) = e^\xi$; and (third column) $\Lambda(z,t) = t\cos(z), \Theta(z,t) = t\sin(z)$, and $\hbar_1(\xi) = \hbar_2(\xi) = \xi e^\xi$.

As seen in Figure 5, if we take $\Lambda(z,t) = \sin(z-t)$, $\Omega(z,t) = \sin(z+t)$, $\hbar_1(\xi) = e^\xi$, and $\hbar_2(\xi) = \cos(\xi)$

into account, the solution u_6 exhibits wave-like behavior with exponential growth and trigonometric modulation. The terms $\sin(z - t)$ and $\sin(z + t)$ create traveling waves, while e^ξ and $\cos(\xi)$ introduce exponential growth and periodic oscillations, respectively. Moreover, if we take $\Lambda(z, t) = \sin(z) - \cos(t)$, $\Omega(z, t) = \sin(z) + \cos(t)$, and $\hbar_1(\xi) = \hbar_2(\xi) = e^\xi$ into account, the solution u_6 combines periodic oscillations with exponential growth. The trigonometric terms create standing waves, while the exponential functions amplify the solution over time, leading to a solution with growing amplitude and a periodic structure. Finally, for $\Lambda(z, t) = t\cos(z)$, $\Omega(z, t) = t\sin(z)$, and $\hbar_1(\xi) = \hbar_2(\xi) = \xi e^\xi$, the solution u_6 displays time-dependent growth with spatially periodic behavior. The terms $t\cos(z)$ and $t\sin(z)$ introduce linear growth in time, while ξe^ξ adds exponential growth, resulting in a solution that grows both linearly and exponentially with time.

Remarks 2.3. *All the proposed solutions in this work have been validated using Maple by substituting them back into the original equation.*

3. The phase portraits of Eq (2.10)

In this part, we aim to investigate the dynamical properties of the given third-order nonlinear differential equation presented in Eq (2.10). To this end, let us rewrite the Eq (2.10) as a system of first-order ordinary differential equations

$$\begin{cases} \frac{d\mathcal{U}}{d\wp} = \mathcal{V}, \\ \frac{d\mathcal{V}}{d\wp} = \mathcal{W}, \\ \frac{d\mathcal{W}}{d\wp} = -\frac{3\alpha\kappa^2\mathcal{V}^2 + \omega\mathcal{V}}{\alpha^2\kappa^3}. \end{cases} \tag{3.1}$$

This is now a system of three first-order ODEs in the variables $\mathcal{U}(\wp)$, $\mathcal{V}(\wp)$, and $\mathcal{W}(\wp)$, and we analyze its dynamic system properties. The system (3.1) possesses a Hamiltonian system in the following form:

$$\mathcal{H}(\mathcal{U}, \mathcal{V}, \mathcal{W}) = \frac{1}{2}\mathcal{U}^2 + \frac{1}{2}\mathcal{W}^2 - \frac{1}{\alpha\kappa}\mathcal{V}^3 - \frac{\omega}{2\alpha^2\kappa^3}\mathcal{V}^2 = \mathbf{h}. \tag{3.2}$$

The equilibrium points of the system occur where the derivatives are zero, i.e., where

$$\mathcal{V} = 0, \quad \mathcal{W} = 0, \quad -\frac{3\alpha\kappa^2\mathcal{V}^2 + \omega\mathcal{V}}{\alpha^2\kappa^3} = 0.$$

From this equation, we see that both $\mathcal{V} = 0$ and $\mathcal{W} = 0$ must hold. Thus, the equilibrium points are of the form $(\mathcal{U}, 0, 0)$, where $\mathcal{U}$ is arbitrary. This indicates that the system has a line of equilibrium points along the $\mathcal{U}$-axis.

Moreover, to analyze the stability of the equilibrium points, we linearize the system around an equilibrium point $(\mathcal{U}_0, 0, 0)$. The Jacobian matrix of the system is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{6\alpha\kappa^2\mathcal{V} + \omega}{\alpha^2\kappa^3} & 0 \end{pmatrix}.$$

At the equilibrium point $(u^*, 0, 0)$, the Jacobian matrix is simplified to

$$J(u^*, 0, 0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{\omega}{\alpha^2\kappa^3} & 0 \end{pmatrix}.$$

The eigenvalues of this matrix determine the stability of the equilibrium points. The characteristic equation is

$$\det(J - \lambda I) = -\lambda^3 - \frac{\omega}{\alpha^2 \kappa^3} \lambda = 0.$$

This gives the eigenvalues

$$\lambda = 0, \quad \lambda = \pm \sqrt{-\frac{\omega}{\alpha^2 \kappa^3}}.$$

In this part, phase portraits of the system are studied. These plots are studied by analyzing the behavior of the trajectories in the $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ space. The system exhibits different dynamic behaviors depending on the parameters α, κ, and ω, as plotted in Figures 6 and 7.
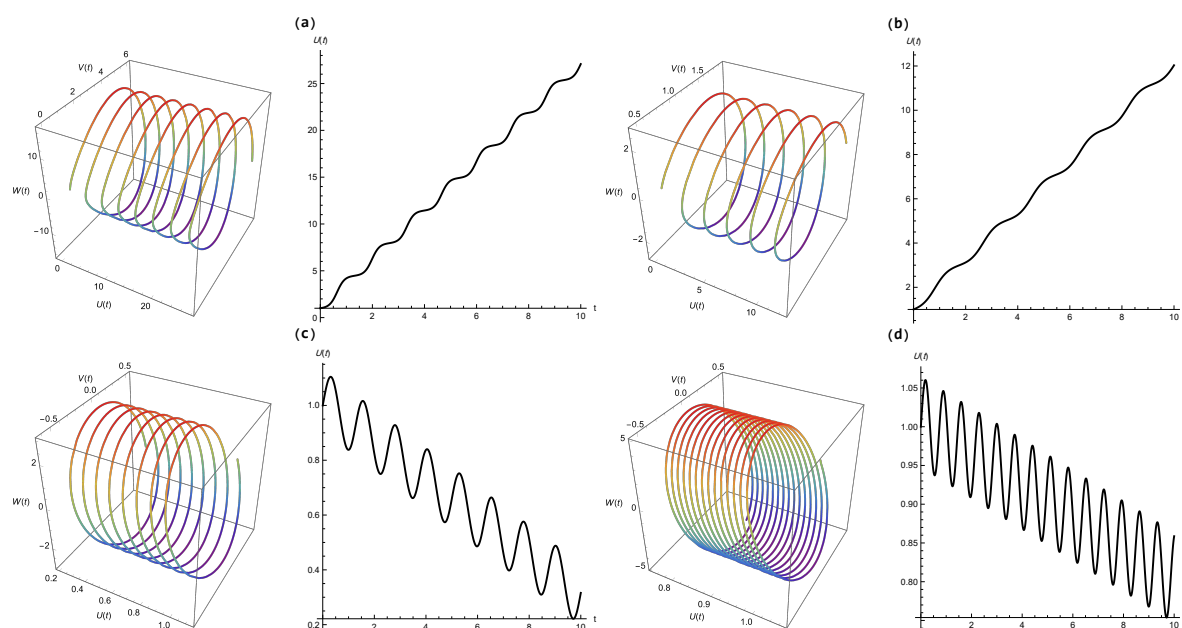
Figure 6. Phase portraits of the system (3.1) corresponding to $\kappa = \alpha = 0.5$ and (a) $\omega = -1.5$, (b) $\omega = -0.1$, (c) $\omega = 1$, and (d) $\omega = 2.5$.
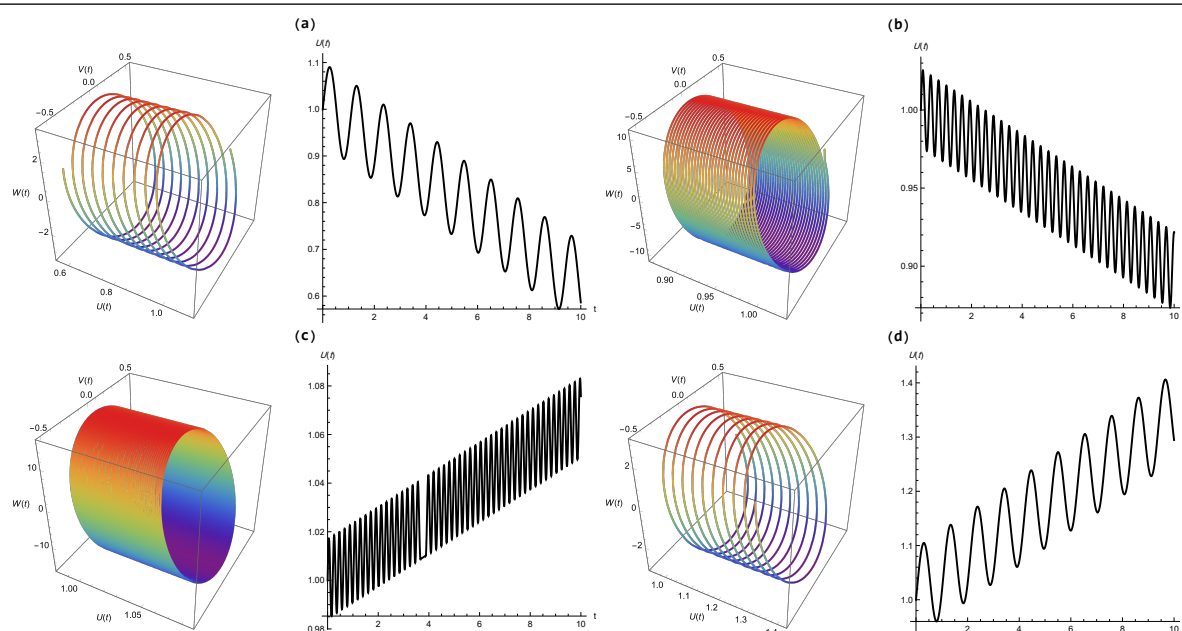
Figure 7. Phase portraits of the system (3.1) corresponding to $\omega = 1, \kappa = 0.3$ and (a) $\alpha = 1$, (b) $\alpha = 0.3$, (c) $\alpha = -0.2$, and (d) $\alpha = -1$.

4. Conclusions

In this study, we have successfully derived a wide range of explicit non-traveling wave solutions for the (3+1)-dimensional KdV–Calogero–Bogoyavlenskii–Schiff (KdV-CBS) equation using a modified generalized variable separation technique. This innovative approach combines the features of the KdV and CBS equations, allowing for the generation of diverse and complex solutions that were not previously documented in the literature. The solutions obtained include periodic solitary waves, cross soliton-like waves, and other non-traveling wave structures, demonstrating the versatility and effectiveness of the proposed method. The modified generalized variable separation technique introduced in this paper generalizes several existing frameworks and provides a broader scope for solving nonlinear partial differential equations. By incorporating arbitrary functions into the solutions, we were able to explore a wide variety of wave behaviors and patterns. All solutions were rigorously validated using symbolic computation with Maple, ensuring their accuracy and consistency with the original equation. The results of this study highlight the importance of advanced separation methods in tackling complex nonlinear PDEs, offering valuable insights into the mathematical modeling of wave phenomena. The non-traveling wave solutions presented here not only enrich the theoretical understanding of the KdV-CBS equation but also have potential applications in various physical contexts, such as fluid dynamics and wave propagation. Future research could focus on investigating the stability and dynamics of these solutions, as well as exploring their implications in real-world physical systems. Additionally, the methodology developed in this study could be extended to other nonlinear PDEs, opening new avenues for the discovery of exact solutions in mathematical physics.

Use of Generative AI tools declaration

The author declares they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that they have no conflicts of interest.

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