Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(3): 6805–6826.
DOI: 10.3934/math.2025312
Received: 31 January 2025
Revised: 12 March 2025
Accepted: 17 March 2025
Published: 26 March 2025

Research article

Existence and stability analysis of a problem of the Caputo fractional derivative with mixed conditions

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Abstract: Recently, initial-boundary-value problems with the Caputo fractional derivative for ordinary differential equations have been intensively studied. This paper studies a nonlinear integro-differential equation of fractional order, containing a composition of fractional derivatives with different origins and mixed conditions. The equation under consideration acts as a model equation of motion in a fractal medium. First, we use three fixed-point theorems to prove the existence and uniqueness results. Then, the Ulam stability criterion of the solution is given. The main results will be illustrated by a proposed example.

Keywords: fractional derivative; existence; Ulam-stability; mixed conditions; iterative methods
Mathematics Subject Classification: 26A33, 34A08, 34B15, 34K20

1. Introduction and problem statement

The analytical apparatus of fractional integro-differentiation and the theory of fractional differential equations demonstrate high efficiency in the description and mathematical modeling of various physical and geophysical processes occurring in fractal environments. Using the concept of the effective rate of the change in the number of parameters of the modeled systems leads to differential equations that contain a composition of fractional differentiation operators of different origins; see [1–3]. More than 300 years ago, scientists used incorrect random orders to generalize ordinary differential equations and integrals using fractional differential equations. The origin of fractional calculus goes to Newton and Leibniz; see [4–6]. Fractional differential equations are useful for many models: physical, biological, genetic, and even economic phenomena. Several recent studies have been carried out by researchers to prove the existence and uniqueness of the solution for fractional differential equations with different conditions (boundary, initial, nonlocal, and integral conditions, etc.). For more details, the reader is referred to [7–9] and the references therein. For a more comprehensive and informative presentation and to illustrate the broad applicability of fractional calculus, we mention the work in [10] for fluid dynamics, mathematical biology [11, 12], image denoising [13, 14], and image super-resolution [15, 16]. In ancient times, the study of the stability of solutions of fractional differential equations was slow, but recently many researchers have done it in different articles in several ways (asymptotically stable, Ulam stable, generalized Ulam stable,...), see [17–19].

In [20], the system is governed by a Caputo fractional with a nonlocal initial condition

$$\begin{cases} {}^{C}D^{\nu}_{0^+}\varpi(z) = h(\varpi(z)) + \psi(z,\varpi(z)) + \int_0^z N(z,s,\varpi(s))ds, \\ \varpi(0) = \sigma \int\limits_0^{\gamma} \varpi(s)ds, \quad 0<\gamma<1, \end{cases} \tag{1.1}$$

is considered. The authors investigated the existence and uniqueness of the solution using the Banach and Krasnoselskii fixed-point theorems.

In [21], using the Banach contraction mapping together with the Burton-Kirk fixed-point theorem, the authors obtained certain results on the existence and uniqueness of a solution for the problem with the Caputo fractional delay

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}\varpi(z) = f(z,\varpi_z,{}^{C}D^{\nu}_{0^+}\varpi),\ 0\le z\le 1, \\ \varpi(0)=0, \qquad \varpi'(0) = aI^{\sigma}_{0^+}\varpi(\eta), \\ b\varpi(1)+c\varpi'(1) = g(\varpi), \\ \varpi(t) = \phi(z),\ z\in[-\tau,0], \end{cases} \tag{1.2}$$

with appropriate conditions on the parameters of nonlocal and integral boundary value. In [22], the problem with the Riemann-Liouville fractional integral in the RHS of

$$\begin{cases} {}^{C}D^{\nu+\kappa}_{0^+}\varpi(z) = \omega(z,\varpi(z)) + I^{\nu}_{0^+}\psi(z,\varpi(z)) + \int_0^z N(z,s,\varpi(s))ds \\ \varpi(0) = b\int_0^{\eta}\varpi(z)dz, \quad 0<\eta<1, \end{cases} \tag{1.3}$$

is considered. Using a combination of the Krasnoselskii and Banach fixed-point theorems, results on the existence and uniqueness are proved. In addition, some suitable conditions are provided that ensure the generalized Ulam stability of the system.

In light of these studies, we shall prove that the unique solution exists with the Ulam stability of the following system:

$$\begin{cases} {}^{C}D^{\nu+\kappa}_{0^+}\varpi(z) = \omega(z,\varpi(z)) + {}^{C}D^{\nu}_{0^+}\psi(z,\varpi(z)) + \displaystyle\int_0^z N(z,s,\varpi(s))ds, \\ \varpi(0) = \varpi_0, \qquad \varpi'(0) = \varpi_1 \displaystyle\int\limits_0^{\gamma}\varpi(s)ds, \quad 0<\gamma<1, \end{cases} \tag{1.4}$$

where ϖ_0,ϖ_1 are real constants, $1<\nu+\kappa\le 2$, the functional ${}^{C}D^{\kappa}_{0^+}$ is the Caputo fractional derivative of order κ, and ω,ψ, and N are appropriate functions given by the following:

$$\begin{array}{ll} \omega & : J\times E \longrightarrow E, \\ \psi & : J\times E \longrightarrow E, \\ N & : J\times J\times E \longrightarrow E, \end{array} \tag{1.5}$$

where E is a Banach space. The existence and stability analysis for the problem (1.4) involving the Caputo fractional derivative with mixed conditions typically refers to studying the behavior and solutions of fractional differential equations (FDEs) under a specific boundary. The Caputo fractional derivative is one of the most commonly used definitions of fractional derivatives, particularly because of its useful properties in physical and engineering applications. Equation $(1.4)_1$ is studied under mixed conditions $(1.4)_2$ to extend the work in [22], where only the integral boundary condition was taken and the problem was with the Riemann-Liouville fractional integral in the RHS of the differential equation and the works [23, 24], in which the authors used fractional integration in the equation with the initial condition, but here we used the fractional derivative instead with mixed conditions.

Our study is based on the three-fixed-point theorem, and the goal is to prove the existence and uniqueness results for the solution. In Section 2, some preliminary and integral equations are given. The generalized stability is shown in Section 3. It should be noted that this representation constitutes a generalization of recent results obtained in this research area. To illustrate our results, we conclude the article with examples.

2. Some preliminary and integral equation

In this section, we present the definitions of fractional integral, fractional Caputo derivative, and some auxiliary lemmas. We refer to [25–28] for essential preliminary concepts regarding fractional calculus and fixed-point theory.

Definition 2.1. *[25] Let $\kappa > 0$ and $\hbar : \mathbb{R}_+ \longrightarrow \mathbb{R}$. The Riemann-Liouville fractional integral of order κ of a function $\hbar$ is given by*

$$I^{\kappa}_{0^+}\hbar(z) = \frac{1}{\Gamma(\kappa)} \int_0^z (z-s)^{\kappa-1}(s)\hbar(s)ds, \qquad z \in \mathbb{R}_+.$$

Definition 2.2. *[28] Let $\kappa > 0$; the order κ Caputo fractional derivative of a function $\hbar : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is defined by*

$${}^{C}D^{\kappa}_{0^+}\hbar(z) = \frac{1}{\Gamma(n-\kappa)} \int_0^z (z-s)^{n-\kappa-1}\hbar^{(n)}(s)ds = I^{n-\kappa}_{0^+}\hbar^{(n)}(z), \qquad z \in \mathbb{R}_+,$$

where $n = [\kappa] + 1$, and the right-hand side is point-wise defined on $\mathbb{R}_+$.

Lemma 2.1. *[25] For $\kappa > 0$, $i \in \mathbb{R}$, and the appropriate function $\hbar(z) \in C^{n-1}[0, \infty)$, where $\hbar(z)$ exists almost everywhere in any bounded interval of $\mathbb{R}_+$, we have*

$$(I^{\kappa}_{0^+}\,{}^{C}D^{\kappa}_{0^+}\hbar)(z) = \hbar(z) - \sum_{i=0}^{n-1} \frac{\hbar^{(i)}(0)}{i!} z^i.$$

Lemma 2.2. *Let $1 < \nu + \kappa < 2$ and $\varpi_1 \neq \frac{2}{\gamma^2}$. Assume that ω, ψ and N are three continuous functions. If $\varpi \in C(J, E)$, then ϖ is a solution of (1.4) if and only if ϖ satisfies the integral equation*

$$\varpi(z) \;=\; \int_0^z \frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(\omega(s,\varpi(s)) + \int_0^s N(s,\tau,\varpi(\tau))d\tau\Big)ds$$

$$
\begin{aligned}
&+\int_0^z \frac{(z-s)^{\kappa-1}}{\Gamma(\kappa)}\psi(s,\varpi(s))ds+\varpi_0-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^\kappa \\
&+\frac{z\varpi_1}{(1-\frac{\gamma^2}{2}\varpi_1)}\Big[\int_0^\gamma \frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\omega(\tau,\varpi(\tau))+\int_0^\tau N(\tau,\sigma,\varpi(\sigma))d\sigma\Big)d\tau \\
&+\int_0^\gamma \frac{(\gamma-\tau)^\kappa}{\Gamma(\kappa+1)}\psi(\tau,\varpi(\tau))d\tau-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma\varpi_0\Big]. \qquad (2.1)
\end{aligned}
$$

Proof. Assume that $\varpi \in C(J,E)$ is a solution of (1.4), and the goal is to prove that ϖ satisfies (2.1). By Lemma 2.1, we obtain

$$
I_{0^+}^{\nu+\kappa}\, {}^C D_{0^+}^{\nu+\kappa}\varpi(z)=\varpi(z)-\varpi(0)-\varpi'(0)z. \qquad (2.2)
$$

Then, by (1.4), Lemma 2.1, and Definition 2.1, we obtain

$$
\begin{aligned}
I_{0^+}^{\nu+\kappa}\, {}^C D_{0^+}^{\nu+\kappa}\varpi(z) &= I_{0^+}^{\nu+\kappa}\Big(\omega(z,\varpi(z))+\int_0^z N(z,s,\varpi(s))ds+{}^C D_{0^+}^{\nu}\psi(z,\varpi(z))\Big) \\
&= I_{0^+}^{\nu+\kappa}\Big(\omega(z,\varpi(z))+\int_0^z N(z,s,\varpi(s))ds\Big) \\
&\quad +I_{0^+}^{\kappa}\psi(z,\varpi(z))-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^\kappa. \qquad (2.3)
\end{aligned}
$$

Substituting (2.3) in (2.2) with the first condition in (1.4) yields:

$$
\begin{aligned}
\varpi(z) &= I_{0^+}^{\nu+\kappa}\Big(\omega(z,\varpi(z))+\int_0^z N(z,s,\varpi(s))ds\Big) \\
&\quad +I_{0^+}^{\kappa}\psi(z,\varpi(z))-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^\kappa+\varpi_0+\varpi'(0)z. \qquad (2.4)
\end{aligned}
$$

But we have

$$
\begin{aligned}
\frac{\varpi'(0)}{\varpi_1} &= \int_0^\gamma \varpi(s)ds \\
&= \int_0^\gamma \Big[I_{0^+}^{\nu+\kappa}[\omega(s,\varpi(s))+\int_0^s N(s,\tau,\varpi(\tau))d\tau] \\
&\quad +I_{0^+}^{\kappa}\psi(s,\varpi(s))+\varpi'(0)s+\varpi_0-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}s^\kappa\Big]ds \\
&= \int_0^\gamma \Big[I_{0^+}^{\nu+\kappa}[\omega(s,\varpi(s))+\int_0^s N(s,\tau,\varpi(\tau))d\tau]+I_{0^+}^{\kappa}\psi(s,\varpi(s))\Big]ds \\
&\quad -\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma\varpi_0+\frac{\gamma^2}{2}\varpi'(0) \\
&= I_{0^+}^{\nu+\kappa+1}[\omega(\gamma,\varpi(\gamma))+\int_0^\gamma N(\gamma,\tau,\varpi(\tau))d\tau]+I_{0^+}^{\kappa+1}\psi(\gamma,\varpi(\gamma)) \\
&\quad -\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma\varpi_0+\frac{\gamma^2}{2}\varpi'(0).
\end{aligned}
$$

Then, we find

$$\varpi'(0) = \frac{\varpi_1}{(1-\frac{\gamma^2}{2}\varpi_1)}\Big[I_{0^+}^{\nu+\kappa+1}[\omega(\gamma,\varpi(\gamma))+\int_0^\gamma N(\gamma,\tau,\varpi(\tau))d\tau] \\ +I_{0^+}^{\kappa+1}\psi(\gamma,\varpi(\gamma))-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma\varpi_0\Big].$$

Therefore, we obtain

$$\begin{aligned}\varpi(z) &= I_{0^+}^{\nu+\kappa}\Big(\omega(z,\varpi(z))+\int_0^z N(z,s,\varpi(s))ds\Big) \\ &\quad +I_{0^+}^{\kappa}\psi(z,\varpi(z))-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^\kappa+\varpi_0 \\ &\quad +\frac{z\varpi_1}{(1-\frac{\gamma^2}{2}\varpi_1)}\Big[I_{0^+}^{\nu+\kappa+1}[\omega(\gamma,\varpi(\gamma))+\int_0^\gamma N(\gamma,\tau,\varpi(\tau))d\tau] \\ &\quad +I_{0^+}^{\kappa+1}\psi(\gamma,\varpi(\gamma))-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma\varpi_0\Big].\end{aligned}$$

Finally, the integral equivalent Eq (2.1) is obtained. Conversely, using the integral Eq (2.4), which is equivalent to (2.1), we have

$$\varpi(z) = I_{0+}^{\nu+\kappa}\omega(z,\varpi(z))+\int_{z_0}^z N(z,s,\varpi(s))\,ds+I_{0+}^{\kappa}\psi(z,\varpi(z))-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^\kappa+\varpi_0+\varpi'(0)z.$$

Applying the inverse operator of the integral operator on both sides of (2.4) and using the properties of the Caputo derivative (linearity), then

$$\begin{aligned}{}^C D_{0^+}^{\nu+\kappa}\varpi(z) &= {}^C D_{0^+}^{\nu+\kappa}\Big\{I_{0+}^{\nu+\kappa}\omega(z,\varpi(z))+\int_{z_0}^z N(z,s,\varpi(s))\,ds+I_{0+}^{\kappa}\psi(z,\varpi(z))-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^\kappa+\varpi_0+\varpi'(0)z\Big\} \\ &= {}^C D_{0^+}^{\nu+\kappa}I_{0+}^{\nu+\kappa}\omega(z,\varpi(z))+{}^C D_{0^+}^{\nu+\kappa}\int_{z_0}^z N(z,s,\varpi(s))\,ds+{}^C D_{0^+}^{\nu+\kappa}I_{0+}^{\kappa}\psi(z,\varpi(z)) \\ &\quad +{}^C D_{0^+}^{\nu+\kappa}\Big\{-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^\kappa+\varpi_0+\varpi'(0)z\Big\}.\end{aligned}$$

By the properties

$${}^C D_{0^+}^{\nu+\kappa}I_{0+}^{\nu+\kappa}f(t)=f(t),$$

and

$${}^C D_{0^+}^{\nu+\kappa}\Big\{-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^\kappa+\varpi_0+\varpi'(0)z\Big\}=0,$$

will lead to

$$\begin{aligned}{}^C D_{0^+}^{\nu+\kappa}\varpi(z) &= {}^C D_{0^+}^{\nu+\kappa}I_{0^+}^{\nu+\kappa}\omega(s,\varpi(s))+{}^C D_{0^+}^{\nu+\kappa}I_{0^+}^{\kappa+\nu}\int_0^s N(s,\tau,\varpi(\tau))d\tau \\ &\quad +{}^C D_{0^+}^{\nu+\kappa}I_{0^+}^{\kappa}\psi(s,\varpi(s))+{}^C D_{0^+}^{\nu+\kappa}\Big(\varpi(0)+\varpi'(0)z\Big) \\ &= \omega(z,\varpi(z))+{}^C D_{0^+}^{\nu}\psi(z,\varpi(z))+\int_0^z N(z,s,\varpi(s))ds, \qquad (2.5)\end{aligned}$$

because

$$ {}^{C}D_{0^+}^{\nu+\kappa}\Big(\varpi(0)+\varpi'(0)z\Big)=I^{2-\nu-\kappa}\Big[\frac{d^2}{dx^2}(\varpi(0)+\varpi'(0)z)\Big]=0, $$

which means that ϖ verify (1.4).

For the other side of the proof, substituting z by 0 in (2.1) will make clear that the non-local condition in (1.4) matches. Thus, ϖ is a solution of (1.4). This concludes the proof. □

3. Existence of solution with different approaches

We discuss the existence of a globally unique solution by using three fixed-point theorems in different directions. System (1.4) is transformed into a related fixed-point system as

$$ \varpi=\mathcal{F}\varpi, $$

here the operator

$$ \mathcal{F}:C(J,E)\longrightarrow C(J,E), $$

is given by:

$$ \begin{aligned} \mathcal{F}\varpi(z) &= \int_0^z\frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(\omega(s,\varpi(s))+\int_0^s N(s,\tau,\varpi(\tau))d\tau\Big)ds \\ &+\int_0^z\frac{(z-s)^{\kappa-1}}{\Gamma(\kappa)}\psi(s,\varpi(s))ds+\varpi_0-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^{\kappa} \\ &+\frac{\varpi_1 z}{(1-\frac{\gamma^2}{2}\varpi_1)}\Big[\gamma\varpi_0-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+2)}\gamma^{\kappa+1}\Big] \\ &+\frac{\varpi_1 z}{(1-\frac{\gamma^2}{2}\varpi_1)}\Big[\int_0^{\gamma}\frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\omega(\tau,\varpi(\tau))+\int_0^{\tau}N(\tau,\sigma,\varpi(\sigma))d\sigma\Big)d\tau \\ &+\int_0^{\gamma}\frac{(\gamma-\tau)^{\kappa}}{\Gamma(\kappa+1)}\psi(\tau,\varpi(\tau))d\tau\Big]. \end{aligned} $$

Noting by

$$ \begin{aligned} \varLambda &= \frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{2+\kappa+\nu}\Big(\frac{1+\kappa+\nu}{2+\kappa+\nu}+\frac{|\varpi_1|\gamma^{\nu+\kappa+1}}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big) \\ &+\frac{\|\mu_2\|_{L^\infty}}{2+\kappa}\Big(\frac{2+\kappa}{\kappa+1}+\frac{|\varpi_1|\gamma^{\kappa+1}}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big), \end{aligned} \tag{3.1} $$

$$ \varLambda_1 = \frac{|\varpi_1|L}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\frac{2\gamma^{\nu+\kappa+1}}{2+\nu+\kappa}+\frac{\gamma^{\kappa+1}}{2+\kappa}\Big]. \tag{3.2} $$

$$ \begin{aligned} \delta &= \frac{2}{\Gamma(\nu+\kappa+1)}+\frac{1}{\Gamma(\kappa+1)} \\ &+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{2\gamma^{\nu+\kappa+1}}{\Gamma(\nu+\kappa+2)}+\frac{\gamma^{\kappa+1}}{\Gamma(\kappa+2)}\Big), \end{aligned} \tag{3.3} $$

$$\delta_1 = \frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)} + |\varpi_0| + \frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1} + \gamma|\varpi_0|\Big), \tag{3.4}$$

$$\begin{aligned}\delta_2 &= \frac{1}{\nu+\kappa+1} + \frac{2}{\nu+\kappa+2} + \frac{1}{\kappa+1}\\ &+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{\gamma^{\nu+\kappa+1}}{\nu+\kappa+2} + \frac{\gamma^{\kappa+1}}{\kappa+2} + \frac{2\gamma^{\nu+\kappa+2}}{\nu+\kappa+3}\Big),\end{aligned} \tag{3.5}$$

where μ_1, μ_2, and μ_3 will be defined in (H2).

3.1. The existence results using Leray-Schauder nonlinear alternative

Theorem 3.1. *Assume $\omega, \psi \in C(J\times\mathbb{R},\mathbb{R})$ and $N \in C(J\times J\times\mathbb{R},\mathbb{R})$ are continuous functions. Assume that*

(H0) There exist functions $f_1, f_2 \in C(J,\mathbb{R}^+)$, $f_3 \in C(J\times J,\mathbb{R}^+)$, with $f = \max\{f_1, f_2, f_3\}$ and nondecreasing functions

$$g_1, g_2, g_3 : \mathbb{R}^+ \to \mathbb{R}^+,$$

with $g = \max\{g_1, g_2, g_3\}$ such that

$$|\omega(z,\varpi(z))| \le f_1(z)g_1(\|\varpi\|),$$

$$|\psi(z,\varpi(z))| \le f_2(z)g_2(\|\varpi\|),$$

and

$$|N(z,s,\varpi(s))| \le f_3(s)g_3(\|\varpi\|),$$

for all $z\in[0,1]$, $s\in[0,1]$, $\varpi\in\mathbb{R}$. Also, assume that there exists a constant $M>0$ such that

$$\frac{M}{\|f\|g(M)\delta_2+\delta_1} > 1.$$

Then problem (1.4) has at least one solution in J.

Proof. For $r>0$, let

$$B_r = \{z \in C([0,1],\mathbb{R}) : \|\varpi\| \le r\},$$

where $r>0$, be a bounded set in $C([0,1],\mathbb{R})$. We shall prove that F maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. By using the assumption (H1) and some computations (The beta function and its properties together with the gamma function)

$$(\kappa,\nu) = \int_0^1 \tau^{\kappa-1}(1-\tau)^{\nu-1}d\tau,$$

$$\int_0^1 (1-s)^{\kappa+\nu}s^{\kappa}\,ds = \nu(\kappa+1,\kappa+\nu).$$

The well-known Beta function and Gamma function are related as

$$(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},$$

and

$$\int_0^{\gamma}(\gamma-\tau)^{\kappa+\nu}\tau^{\kappa}\,d\tau=\gamma^{2\kappa+\nu+1}\,\nu(\kappa+1,\kappa+\nu+1),$$

we obtain

$$\begin{aligned}
\|\mathcal{F}\varpi(z)\| &\leq \int_0^z \frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(f_1(s)g_1(\|\varpi(s)\|)+\int_0^s f_3(s)g_3(\|\varpi(\tau)\|)d\tau\Big)ds \\
&\quad+\int_0^z \frac{(z-s)^{\kappa-1}}{\Gamma(\kappa)}f_2(s)g_2(\|\varpi(s)\|)ds+\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+|\varpi_0| \\
&\quad+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big] \\
&\quad+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\int_0^{\gamma}\frac{(\gamma-\tau)^{\kappa+1}}{\Gamma(\kappa+2)}f_2(\tau)g_2(\|\varpi(\tau)\|)d\tau \\
&\quad+\int_0^{\gamma}\frac{(\gamma-\tau)^{\nu+\kappa+1}}{\Gamma(\nu+\kappa+2)}\Big(f_1(\tau)g_1(\|\varpi(\tau)\|)+\int_0^{\tau}f_3(\tau)g_3(\|\varpi(\sigma)\|)d\sigma\Big)d\tau\Big] \\
&\leq \|f\|g(\|\varpi\|)\Big[\int_0^z\frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}(1+s)ds+\int_0^z\frac{(z-s)^{\kappa-1}}{\Gamma(\kappa)}ds \\
&\quad+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\int_0^{\gamma}\frac{(\gamma-\tau)^{\kappa+1}}{\Gamma(\kappa+2)}d\tau+\int_0^{\gamma}\frac{(\gamma-\tau)^{\nu+\kappa+1}}{\Gamma(\nu+\kappa+2)}(1+\tau)\Big)d\tau\Big] \\
&\quad+\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+|\varpi_0|+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big] \\
&\leq \|f\|g(\|\varpi\|)\Big[\frac{1}{\nu+\kappa+1}+\frac{2}{\nu+\kappa+2}+\frac{1}{\kappa+1} \\
&\quad+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{\gamma^{\nu+\kappa+1}}{\nu+\kappa+2}+\frac{\gamma^{\kappa+1}}{\kappa+2}+\frac{2\gamma^{\nu+\kappa+2}}{\nu+\kappa+3}\Big)\Big] \\
&\quad+\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+|\varpi_0|+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big) \\
&= \|f\|g(\|\varpi\|)\delta_2+\delta_1<+\infty.
\end{aligned} \tag{3.6}$$

Let $z_1, z_2 \in [0,1]$, with $z_1 < z_2$, and $\varpi \in B_r$, in which B_r is a bounded set of $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
&\|(\mathcal{F}\varpi)(z_2)-(\mathcal{F}\varpi)(z_1)\| \\
\leq\ & \int_{z_1}^{z_2}\frac{(z_2-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(\|\omega(s,\varpi(s))\|+\int_0^s\|N(s,\tau,\varpi(\tau))\|d\tau\Big)ds \\
&+\int_{z_1}^{z_2}\frac{(z_2-s)^{\kappa-1}}{\Gamma(\kappa)}\|\psi(s,\varpi(s))\|ds \\
&+\int_0^{z_1}\frac{(z_1-s)^{\nu+\kappa-1}-(z_2-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(\|\psi(s,\varpi(s))\|+\int_0^s\|N(s,\tau,\varpi(\tau))\|d\tau\Big)ds \\
&+\int_0^{z_1}\frac{(z_1-s)^{\kappa-1}-(z_2-s)^{\kappa-1}}{\Gamma(\kappa)}\|\psi(s,\varpi(s))\|ds+\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}(z_2^{\kappa}-z_1^{\kappa})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\varpi_1|(z_2-z_1)}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|+\int_0^\gamma\frac{(\gamma-\tau)^\kappa}{\Gamma(\kappa+1)}\psi(\tau,\varpi(\tau))d\tau \\
&+\int_0^\gamma\frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\omega(\tau,\varpi(\tau))+\int_0^\tau N(\tau,\sigma,\varpi(\sigma))d\sigma\Big)d\tau\Big]. \\
\leq\ & \|f\|g(\|\varpi\|)\Big[\frac{2(z_2-z_1)^{\nu+\kappa}+\left|z_1^{\nu+\kappa}-z_2^{\nu+\kappa}\right|}{\Gamma(\nu+\kappa+1)}+\frac{2(z_2-z_1)^\kappa+\left|z_1^\kappa-z_2^\kappa\right|}{\Gamma(\kappa+1)} \\
&+(z_2-z_1)\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\frac{\gamma^{\nu+\kappa+1}}{\nu+\kappa+2}+\frac{\gamma^{\kappa+1}}{\kappa+2}+\frac{2\gamma^{\nu+\kappa+2}}{\nu+\kappa+3}\Big]\Big] \\
&+(z_2^\kappa-z_1^\kappa)\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+(z_2-z_1)\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|)}\Big[\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big].
\end{aligned} \tag{3.7}
$$

If $(z_2 - z_1) \longrightarrow 0$, then the RHS of (3.7) tends to zero independently of $\varpi \in B_r$. Therefore,

$$\|\mathcal{F}\varpi(z_2) - \mathcal{F}\varpi(z_1)\| \longrightarrow 0,$$

so F maps bounded sets into equi-continuous sets of C.

Owing to the Arzela-Ascoli theorem, we find

$$\mathcal{F} : C([0, 1], \mathbb{R}) \longrightarrow C([0, 1], \mathbb{R}),$$

is completely continuous.

Applying the Leray-Schauder nonlinear alternative when we show the boundedness of the set of all solutions to

$$\varpi = \epsilon\mathcal{F}\varpi \quad for \quad 0 < \epsilon < 1.$$

Let ϖ be a solution of (1.4). Thus, by (3.6) we obtain

$$|\varpi(z)| \leq \|f\|g(\|\varpi\|)\delta_2 + \delta_1,$$

which implies

$$\frac{\|\varpi\|}{\|f\|g(\|\varpi\|)\delta_2 + \delta_1} \leq 1.$$

Then by (H2), $\exists M > 0$ where $M \neq \|\varpi\|$. Define the set

$$Y = \{\varpi \in C([0, 1], \mathbb{R})/\ \|\varpi\| < M\},$$

so

$$\mathcal{F} : \bar{Y} \longrightarrow C([0, 1], \mathbb{R}),$$

is completely continuous. By the choice of Y, there is no $z \in \partial Y$ where

$$\varpi = \epsilon\mathcal{F}\varpi \qquad \text{for } \epsilon \in (0, 1),$$

then owing to the nonlinear Leray-Schauder type, we find that $\mathcal{F}$ has a fixed point $\varpi \in \bar{Y}$ which is exactly the solution of the IVP (1.4). □

Example 3.1. *Let the next fractional integro-differential system*

$$\begin{aligned} {}^{C}D_{0^+}^{\frac{12}{7}}\varpi(z) &= \frac{\sin z}{22+e^z}\Big(\frac{|u(z)|}{1+\|\varpi\|}+\cos(\varpi(z))\Big)+{}^{C}D_{0^+}^{\frac{8}{7}}\Big(\frac{\cos z}{18}+\frac{e^{-z}}{z^2+18e^{-z}}\Big)\varpi(z) \\ &+ \int_0^z \frac{e^{s-z-1}}{15}\Big(\varpi(s)+2e^{-|\varpi(s)|}\Big)ds, \end{aligned}$$

$$\varpi(0)=1, \qquad \varpi'(0)=6\int_0^{\frac{2}{3}}\varpi(s)ds,$$

where

$$\nu=\frac{4}{7}, \qquad \kappa=\frac{8}{7}, \quad \varpi_0=1, \quad \varpi_1=6, \quad \gamma=\frac{2}{3}.$$

Clearly

$$\delta_1=2.7868, \qquad \delta_2=4.6248,$$

$$|\psi(z,\varpi(z))|\le\Big(\frac{|\cos z|}{18}+\frac{e^{-z}}{z^2+18e^{-z}}\Big)\|\varpi\|\le\Big(\frac{1}{18}+\frac{e^{-z}}{z^2+18e^{-z}}\Big)\|\varpi\|,$$

$$|\omega(z,\varpi(z))|\le\frac{|\sin z|}{22+e^z}\Big(\frac{\|\varpi\|}{1+\|\varpi\|}+|\cos\varpi|\Big)\le\frac{1}{22+e^z}\Big(1+\|\varpi\|\Big),$$

$$|N(z,s,\varpi(x))|\le\frac{e^{s-z-1}}{15}\Big(\|\varpi\|+2e^{-\|\varpi\|}\Big)\le\frac{e^{s-z-1}}{15}\Big(\|\varpi\|+2\Big),$$

with

$$f_1(z)=\frac{1}{18}+\frac{e^{-z}}{z^2+18e^{-z}}, \qquad \|f_1\|=\frac{1}{9},$$

$$f_2(z)=\frac{1}{22+e^z}, \qquad \|f_2\|=\frac{1}{23},$$

$$f_3(z,s)=\frac{e^{s-z-1}}{15}, \qquad \|f_3\|=\frac{1}{15},$$

$$g_1(\|\varpi\|)=\|\varpi\|, \qquad g_2(\|\varpi\|)=\|\varpi\|+2, \qquad g_3(\|\varpi\|)=2+\|\varpi\|.$$

$$f=\max\Big\{\frac{1}{9},\frac{1}{23},\frac{1}{15}\Big\}=\frac{1}{9},$$

$$g=\max\Big\{\|\varpi\|,\ \|\varpi\|+1,\ \|\varpi\|+2\Big\}=\|\varpi\|+2.$$

Since all the conditions of Theorem3.1 are satisfied, the problem (3.8) has at least one solution on $[0,1]$.

3.2. Owing to the Krasnoselskii's fixed point

Theorem 3.2. *Let*

$$\omega,\psi:[0,1]\times E\longrightarrow E,$$

and

$$N:[0,1]\times[0,1]\times E\longrightarrow E,$$

be continuous functions satisfying

(H1)

$$\|\omega(z,\varpi(z))-\omega(z,v^*(z))\|\leq L_1\|\varpi(z)-v^*(z)\|,\quad z\in[0,1],\ \ \varpi,v^*\in X,$$

$$\|\psi(z,\varpi(z))-\psi(z,v^*(z))\|\leq L_2\|\varpi(z)-v^*(z)\|,\quad z\in[0,1],\ \ \varpi,v^*\in E,$$

$$\|N(z,s,\varpi(s))-N(z,s,v^*(s))\|\leq L_3\|\varpi(s)-v^*(s)\|,\quad (z,s)\in G,\ \ \varpi,v^*\in X,$$

hold where $L_1,L_2,L_3\geq 0$ *with*

$$L=\max\{L_1,L_2,L_3\},$$

and

$$G=\{(z,s):0\leq s\leq x\leq 1\}.$$

(H2) $\exists\mu_1^*,\mu_2,\mu_3\in L^\infty([0,1],\mathbb{R}^+)$ *such that*

$$\|\omega(z,\varpi(z))\|\leq\mu_1^*(z)\|\varpi(z)\|,\quad 0\leq z\leq 1,\ \ \varpi\in E,$$

$$\|\psi(z,\varpi(z))\|\leq\mu_2(z)\|\varpi(z)\|,\quad 0\leq z\leq 1,\ \ \varpi\in E,$$

$$\|N(z,s,\varpi(s))\|\leq\mu_3(z)\|\varpi(s)\|,\quad (z,s)\in G,\ \ \varpi\in E.$$

If $\Lambda\leq 1$ *and* $L\Lambda_1\leq 1$, *then problem (1.4) has at least one solution on* $[0,1]$.

Proof. Let $\varpi\in C(J,E)$, we define

$$\|\varpi\|_1=\max\{e^{-x}\|\varpi(z)\|:\ \ z\in[0,1]\},$$

and consider the closed ball

$$B_r=\{\varpi\in C(J,E):\|\varpi\|_1\leq r\}.$$

Then, the operators $\mathcal{F}_1,\mathcal{F}_2$ are defined on B_r as

$$\begin{aligned}\mathcal{F}_1\varpi(z) &= \int_0^z\frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(\omega(s,\varpi(s))+\int_0^s N(s,\tau,\varpi(\tau))d\tau\Big)ds\\ &\quad+\int_0^z\frac{(z-s)^{\kappa-1}}{\Gamma(\kappa)}\psi(s,\varpi(s))ds+\varpi_0-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+1)}z^\kappa\\ &\quad+\frac{\varpi_1 x}{(1-\frac{\gamma^2}{2}\varpi_1)}\Big[\gamma\varpi_0-\frac{\psi(0,\varpi_0)}{\Gamma(\kappa+2)}\gamma^{\kappa+1}\Big],\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}\mathcal{F}_2\varpi(z) &= \frac{\varpi_1 x}{(1-\frac{\gamma^2}{2}\varpi_1)}\Big[\int_0^\gamma\frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\omega(\tau,\varpi(\tau))+\int_0^\tau N(\tau,\sigma,\varpi(\sigma))d\sigma\Big)d\tau\\ &\quad+\int_0^\gamma\frac{(\gamma-\tau)^\kappa}{\Gamma(\kappa+1)}\psi(\tau,\varpi(\tau))d\tau\Big].\end{aligned}\tag{3.9}$$

For $\varpi,v^*\in B_r$, $z\in[0,1]$, by fixed $r\geq\frac{\delta_1}{1-\Lambda}$ and by the assumption (H2), we find:

$$e^z\|\mathcal{F}_1\varpi(z)+\mathcal{F}_2v^*(z)\|_1 \leq \int_0^z\frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(\mu_1(s)\|\varpi(s)\|+\int_0^s\mu_3(s)\|\varpi(\tau)\|d\tau\Big)ds$$

$$
\begin{aligned}
&+\int_0^z \frac{(z-s)^{\kappa-1}}{\Gamma(\kappa)}\mu_2(s)\|\varpi(s)\|ds+\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+|\varpi_0| \\
&+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big] \\
&+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\int_0^\gamma \frac{(\gamma-\tau)^{\kappa}}{\Gamma(\kappa+1)}\mu_2(\tau)\|v^*(\tau)\|d\tau \\
&+\int_0^\gamma \frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\mu_1(\tau)\|v^*(\tau)\|+\int_0^\tau \mu_3(\tau)\|v^*(\sigma)\|d\sigma\Big)d\tau\Big] \\
\leq\ &\int_0^z \frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(\|\mu_1\|_{L^\infty}\|\varpi\|_1 e^s+\|\mu_3\|_{L^\infty}\|\varpi\|_1(e^s-1)\Big)ds \\
&+\|\mu_2\|_{L^\infty}\|\varpi\|_1\int_0^z \frac{(z-s)^{\kappa-1}}{\Gamma(\kappa)}e^s ds+\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+|\varpi_0| \\
&+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big] \\
&+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\int_0^\gamma \frac{(\gamma-\tau)^{\kappa}}{\Gamma(\kappa+1)}\|\mu_2\|_{L^\infty}\|v^*\|_2 e^\tau d\tau \\
&+\int_0^\gamma \frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\|\mu_1\|_{L^\infty}\|v^*\|_1 e^\tau+\|\mu_3\|_{L^\infty}\|v^*\|_3(e^\tau-1)\Big)d\tau\Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\mathcal{F}_1\varpi\ &+\ \mathcal{F}_2 v^*\|_1 \\
\leq\ &\frac{z^{\nu+\kappa}}{\nu+\kappa+1}(\|\mu_1\|_{L^\infty}\|\varpi\|_1+\|\mu_3\|_{L^\infty}\|\varpi\|_1)+\frac{z^\kappa}{\kappa+1}\|\mu_2\|_{L^\infty}\|\varpi\|_1 \\
&+e^{-z}\Big[\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+|\varpi_0|+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big)\Big] \\
&+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\frac{\gamma^{\kappa+1}}{\kappa+2}\|\mu_2\|_{L^\infty}\|v^*\|_1+\frac{\gamma^{\nu+\kappa+1}}{\nu+\kappa+2}\Big(\|\mu_1\|_{L^\infty}\|v^*\|_1+\|\mu_3\|_{L^\infty}\|v^*\|_3\Big)\Big] \\
\leq\ &r\Big[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{2+\kappa+\nu}\Big(\frac{1+\kappa+\nu}{2+\kappa+\nu}+\frac{|\varpi_1|\gamma^{\nu+\kappa+1}}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big)+\frac{\|\mu_2\|_{L^\infty}}{2+\kappa}\Big(\frac{2+\kappa}{\kappa+1}+\frac{|\varpi_1|\gamma^{\kappa+1}}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big)\Big] \\
&+\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+|\varpi_0|+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big) \\
=\ &r\Lambda+\delta_1\leq r.
\end{aligned}
$$

This implies that $(\mathcal{F}_1 u+\mathcal{F}_2 v)\in B_r$. We use the estimations:

$$
\frac{e^s}{e^z}\leq 1,\quad \frac{e^\tau}{e^z}\leq 1,\quad \frac{e^\tau-1}{e^z}\leq 1,\quad \frac{e^s-1}{e^z}\leq 1.
$$

Now, we have to show that $\mathcal{F}_2$ is a contraction mapping. For $\varpi, v^*\in E$ and $0\leq z\leq 1$, we have

$$
e^x\|\mathcal{F}_2\varpi(z)-\mathcal{F}_2 v^*(z)\|_1 \ \leq\ \frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\int_0^\gamma \frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\|\omega(\tau,\varpi(\tau))-\omega(\tau,v^*(\tau))\|
$$

$$
\begin{aligned}
& + \int_0^{\tau} \|N(\tau,\sigma,\varpi(\sigma)) - N(\tau,\sigma,v^*(\sigma))\|d\sigma\Big)d\tau \\
& + \int_0^{\gamma} \frac{(\gamma-\tau)^{\kappa}}{\Gamma(\kappa+1)}\|\psi(\tau,\varpi(\tau)) - \psi(\tau,v^*(\tau))\|d\tau\Big] \\
\leq\ & \frac{|\varpi_1|L}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\int_0^{\gamma} \frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\|\varpi - v^*\|_1 e^{\tau} + \|\varpi - v^*\|_1(e^{\tau}-1)\Big) \\
& + \int_0^{\gamma} \frac{(\gamma-\tau)^{\kappa}}{\Gamma(\kappa+1)}\|\varpi - v^*\|_1 e^{\tau} d\tau\Big].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\mathcal{F}_2\varpi - \mathcal{F}_2 v^*\|_1 &\leq \frac{|\varpi_1|L}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\frac{2\gamma^{\nu+\kappa+1}}{2+\nu+\kappa} + \frac{\gamma^{\kappa+1}}{2+\kappa}\Big]\|\varpi - v^*\|_1. \\
&= L\Lambda_1\|\varpi - v^*\|_1.
\end{aligned}
$$

As long as $L\Lambda_1 \leq 1$, then $\mathcal{F}_2$ is a contraction. Since the functions h, f, and K are continuous, we have the operator $\mathcal{F}_1$ is continuous. Also, $\mathcal{F}_1 B_r \subset B_r$, for $\varpi \in B_r$, i.e., $\mathcal{F}_1$ is uniformly bounded on B_r as

$$
\begin{aligned}
e^x\|\mathcal{F}_1\varpi(z)\|_1 \leq\ & \int_0^z \frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(\|\mu_1\|_{L^\infty}\|\varpi\|_1 e^s + \|\mu_3\|_{L^\infty}\|\varpi\|_1(e^s-1)\Big)ds \\
& + \|\mu_2\|_{L^\infty}\|\varpi\|_1\int_0^z \frac{(z-s)^{\kappa}}{\Gamma(\kappa)}e^s ds + \frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}z^{\kappa} + |\varpi_0| \\
& + \frac{|\varpi_1|x}{(1-\frac{\gamma^2}{2}|\varpi_1|)}\Big[\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1} + \gamma|\varpi_0|\Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\mathcal{F}_1\varpi\|_1 \leq\ & r\Big[\frac{\|\mu_1\|_{L^\infty} + \|\mu_3\|_{L^\infty}}{\nu+\kappa+1} + \frac{\|\mu_2\|_{L^\infty}}{1+\kappa}\Big] \\
& + \frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)} + |\varpi_0| + \frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1} + \gamma|\varpi_0|\Big) \\
=\ & r\Lambda + \delta_1 \leq r.
\end{aligned}
$$

We will now show that $\overline{(\mathcal{F}_1 B_r)}$ is equicontinuous. So, let us define

$$
\begin{aligned}
\overline{\omega} &= \sup_{(s,\varpi)\in[0,1]\times B_r} \|\omega(s,u)\|, \\
\overline{\psi} &= \sup_{(s,\varpi)\in[0,1]\times B_r} \|\psi(s,\varpi)\|, \\
\overline{N} &= \sup_{(s,\tau,\varpi)\in G\times B_r} \int_0^s \|N(z,s,\varpi)\|dz. \qquad (3.10)
\end{aligned}
$$

For any $\varpi \in B_r$ and for each $z_1, z_2 \in [0,1]$ with $z_1 \leq z_2$, we have

$$
\|(\mathcal{F}_1\varpi)(z_2) - (\mathcal{F}_1\varpi)(z_1)\| \qquad (3.11)
$$

$$\leq \frac{\overline{\varpi}+\overline{N}}{\Gamma(\nu+\kappa+1)}\Big[2\big|z_2-z_1\big|^{\nu+\kappa}+\big|z_1^{\nu+\kappa}-z_2^{\nu+\kappa}\big|\Big]+\frac{\overline{\psi}}{\Gamma(\kappa+1)}\Big[2\big|z_2-z_1\big|^{\kappa}+\big|z_1^{\kappa}-z_2^{\nu}\big|\Big]$$
$$+(z_2^{\kappa}-z_1^{\kappa})\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+(z_2-z_1)\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|)}\Big[\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big].$$

The right hand side of (3.11) is independent of ϖ and tends to 0 when $|z_2-z_1|\longrightarrow 0$, which means that

$$|\mathcal{F}_1\varpi(z_2)-\mathcal{F}_1\varpi(z_1)|\longrightarrow 0,$$

this implies that $\overline{\mathcal{F}_1 B_r}$ is equicontinuous, then $\mathcal{F}_1$ is relatively compact on B_r. So, owing to the Arzela-Ascoli theorem, $\mathcal{F}_1$ is compact on B_r, and then, all conditions of Theorem 3.2 are fulfilled. Then, the operator $\mathcal{F}_1+\mathcal{F}_2$ has a fixed point on B_r. Therefore, problem (1.4) has at least one solution on [0, 1]. □

Example 3.2. *Consider the following fractional integro-differential problem.*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{9}{5}}\varpi(z)=\dfrac{\varpi(z)(7-z^2)}{245}+{}^{C}D_{0^+}^{\frac{8}{5}}\varpi(z)-\dfrac{2-\ln(z+1)}{86}\varpi(z)+\displaystyle\int_0^z\frac{2+e^{-(s^2+z^2)}}{93}\varpi(s)ds \\ \varpi(0)=1,\quad \varpi'(0)=6\displaystyle\int_0^{\frac{2}{3}}\varpi(s)ds, \end{cases} \tag{3.12}$$

Where

$$\nu=\frac{1}{5},\qquad \kappa=\frac{8}{5},\quad \varpi_0=1,\quad \varpi_1=6,\quad \gamma=\frac{2}{3}.$$

For $\varpi, v^*\in E=\mathbb{R}^+$ *and* $z\in[0,1]$, *we obtain*

$$L_1=\frac{8}{245},\ L_2=\frac{2+\ln(2)}{86},\ L_3=\frac{1}{31},\ \psi(0,\varpi_0)=\psi(0,1)=\frac{1}{35},$$

$$\mu_1(z)=\frac{7-z^2}{245},\quad \mu_2(z)=\frac{2-\ln(z+1)}{86},\quad \mu_3(z)=\frac{2+e^{-(s^2+z^2)}}{93},$$

yields

$$\|\mu_1\|_{L_\infty}=\frac{1}{35},\quad \|\mu_2\|_{L_\infty}=\frac{1}{43},\quad \|\mu_3\|_{L_\infty}=\frac{1}{31},$$

and

$$L=\frac{1}{31}.$$

Using the above data, we obtain

$$\Lambda=0.2285,\quad \Lambda_1=7.6401,\quad L\Lambda_1=0.2465<1,\quad \delta_1=13.1677.$$

fixing $r\geq\frac{\delta_1}{1-\Lambda}=17.0673$ *then, problem (3.12) has at least one solution on* $[0,1]$ *by Theorem3.2*

3.3. The existence and uniqueness

Theorem 3.3. *Let (H1) hold. If* $L\Lambda < 1$*, then (1.4) has a unique solution in* $[0,1]$.

Proof. Let

$$M = \max\{M_1, M_2, M_3\}, \tag{3.13}$$

where $M_1, M_2, M_3 > 0$ and

$$M_1 = \sup_{z\in[0,1]} \|\omega(z,0)\|, \tag{3.14}$$

$$M_2 = \sup_{z\in[0,1]} \|\psi(z,0)\|, \tag{3.15}$$

$$M_3 = \sup_{(z,s)\in G} \|\int_0^z N(z,s,0)ds\|. \tag{3.16}$$

For

$$r_1 \geq \frac{M\delta + \delta_1}{1 - L\delta},$$

fixed, let

$$D_{r_1} = \{z \in C([0,1],E) : \quad \|\varpi\| \leq r_1\}.$$

By (H1), we have

$$\begin{aligned}\|\omega(z,\varpi(z))\| &= \|\omega(z,\varpi(z)) - \omega(z,0) + \omega(z,0)\|\\ &\leq L_1\|\varpi\| + M_1,\end{aligned}$$

and

$$\|\psi(z,\varpi(z))\| \leq L_2\|\varpi\| + M_2, \tag{3.17}$$

and

$$\|\int_0^z N(z,s,\varpi(s))\| \leq L_3\|\varpi\| + M_3.$$

First step: We shall prove that $TD_r \subset D_r$. Let $t \in [0,1]$ and $\varpi \in D_r$, we have

$$\begin{aligned}\|(\mathcal{F}\varpi)(z)\| \leq\ & \int_0^z \frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big(\|\omega(s,\varpi(s))\| + \int_0^s \|N(s,\tau,\varpi(\tau))\|d\tau\Big)ds\\ &+ \int_0^z \frac{(z-s)^{\kappa-1}}{\Gamma(\kappa)}\|\psi(s,\varpi(s))\|ds + \frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)} + |\varpi_0|\\ &+ \frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\int_0^\gamma \frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\|\omega(\tau,\varpi(\tau))\| + \int_0^\tau \|N(\tau,\lambda,\varpi(\lambda))\|d\lambda\Big)d\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\gamma \frac{(\gamma-\tau)^\kappa}{\Gamma(\kappa+1)}\|\psi(\tau,\varpi(\tau))\|d\tau+\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big]\\
\le\ & (Lr+M)\Big[\frac{1}{\nu+\kappa+1}+\frac{2}{\nu+\kappa+2}+\frac{1}{\kappa+1}\\
&+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{\gamma^{\nu+\kappa+1}}{\nu+\kappa+2}+\frac{\gamma^{\kappa+1}}{\kappa+2}+\frac{2\gamma^{\nu+\kappa+2}}{\nu+\kappa+3}\Big)\Big]\\
&+\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+1)}+|\varpi_0|+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{|\psi(0,\varpi_0)|}{\Gamma(\kappa+2)}\gamma^{\kappa+1}+\gamma|\varpi_0|\Big)\\
=\ & (Lr_1+M)\delta_2+\delta_1\le r_1.
\end{aligned}
$$

Hence, $\mathcal{F}D_{r_1}\subset D_{r_1}$.

Second step: We prove now that $\mathcal{F}:D_{r_1}\longrightarrow D_r$ is a contraction. By $(H1)$, let $z\in[0,1]$ and $\forall\varpi,v^*\in D_{r_1}$, we have

$$
\begin{aligned}
&\|(\mathcal{F}\varpi)(z)-(\mathcal{F}v^*)(z)\|\\
\le\ & \int_0^z \frac{(z-s)^{\nu+\kappa-1}}{\Gamma(\nu+\kappa)}\Big[\|\omega(s,\varpi(s))-\omega(s,v^*(s))\|\\
&+\int_0^s \|N(s,\tau,\varpi(\tau))-N(s,\tau,v^*(\tau))\|d\tau\Big]ds\\
&+\int_0^z \frac{(z-s)^{\kappa-1}}{\Gamma(\kappa)}\|\psi(s,\varpi(s))-\psi(s,v^*(s))\|ds\\
&+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big[\int_0^\gamma \frac{(\gamma-\tau)^{\nu+\kappa}}{\Gamma(\nu+\kappa+1)}\Big(\|\omega(\tau,\varpi(\tau))-\omega(\tau,v^*(\tau))\|\\
&+\int_0^\tau \|N(\tau,\sigma,\varpi(\sigma))-N(\tau,\sigma,v^*(\sigma))\|d\sigma\Big)d\tau\\
&+\int_0^\gamma \frac{(\gamma-\tau)^\kappa}{\Gamma(\kappa+1)}\|\psi(\tau,\varpi(\tau))-\psi(\tau,v^*(\tau))\|d\tau\Big]d\tau\\
\le\ & L\Big[\frac{1}{\nu+\kappa+1}+\frac{2}{\nu+\kappa+2}+\frac{1}{\kappa+1}\\
&+\frac{|\varpi_1|}{|1-\frac{\gamma^2}{2}\varpi_1|}\Big(\frac{\gamma^{\nu+\kappa+1}}{\nu+\kappa+2}+\frac{\gamma^{\kappa+1}}{\kappa+2}+\frac{2\gamma^{\nu+\kappa+2}}{\nu+\kappa+3}\Big)\Big]\|\varpi-v^*\|\\
=\ & L\delta_2\|\varpi-v^*\|.
\end{aligned}
$$

Since $L\delta_2<1$, we have $\mathcal{F}$ is a contraction. All conditions of Banach's fixed point theorem are satisfied, then $\exists\varpi\in C(J,E)$ such that $\mathcal{F}\varpi=\varpi$, which is considered as a unique solution of (1.4) in $C(J,E)$. □

Example 3.3. *Consider the following fractional integro-differential problem:*

$$
\begin{cases}
{}^CD_{0^+}^{\frac{12}{7}}\varpi(z)=\omega(z,\varpi(z))+{}^CD_{0^+}^{\frac{6}{7}}\psi(z,\varpi(z))+\displaystyle\int_0^z N(z,s,\varpi(s))ds,\\
\varpi(0)=1,\quad \varpi'(0)=6\displaystyle\int_0^{\frac{2}{3}}\varpi(s)ds,\quad 0<\gamma<1,
\end{cases}
\tag{3.18}
$$

where

$$\nu = \kappa = \frac{6}{7}, \quad \varpi_0 = 1, \quad \varpi_1 = 6, \quad \gamma = \frac{2}{3}.$$

Taking $\varpi, \nu^* \in E = \mathbb{R}^+$ *and* $z \in [0, 1]$*, and*

$$\omega(z, \varpi(z)) = \frac{\varpi(z) - (1 - x^2)}{13} + \frac{14}{39}|\varpi|, \tag{3.19}$$

$$M_1 = \frac{1}{13}, \ L_1 = \frac{4}{39},$$

$$\psi(z, \varpi(z)) = \frac{|\varpi| + 5(z + 1)}{17} + \frac{1}{34}\varpi(z), \tag{3.20}$$

$$M_2 = \frac{5}{17}, \qquad L_2 = \frac{3}{34}, \tag{3.21}$$

$$N(z, s, \varpi(s)) = \frac{4e^{-(s+z)}}{19} + \frac{2}{57}|\varpi|, \tag{3.22}$$

$$M_3 = \frac{4}{19}, \qquad L_3 = \frac{2}{57}.$$

Thus,

$$\psi(0, \varpi_0) = \psi(0, 1) = \frac{42}{17}, \quad \text{and} \qquad L = \max\{L_1, L_2, L_3\} = \frac{4}{34}.$$

$$\delta_2 = 13.1677, \quad L\delta_2 = 0.4743 < 1, \quad \delta_1 = 9.0338.$$

Then, there exists a unique solution for the problem (3.18) on $[0, 1]$ *by application of Theorem 3.3*

Example 3.4. *Consider the following fractional integro-differential problem:*

$$\begin{cases} {}^C D_{0^+}^{\frac{17}{8}} \varpi(z) = \omega(z, \varpi(z)) + {}^C D_{0^+}^{\frac{5}{8}} \psi(z, \varpi(z)) + \int_0^z N(z, s, \varpi(s)) ds, \\ \varpi(0) = 16, \quad \varpi'(0) = 7.6 \int_0^{\frac{4}{5}} \varpi(s) ds, \quad 0 < \gamma < 1, \end{cases} \tag{3.23}$$

where

$$\nu = \frac{12}{8}, \ \kappa = \frac{5}{8}, \quad \varpi_0 = 16, \quad \varpi_1 = 77.6, \quad \gamma = \frac{4}{5}.$$

To illustrate our result in Theorem 3.3, let $\varpi, \nu^* \in E = \mathbb{R}^+$*,* $0 \le z \le 1$*, and*

$$\omega(z, \varpi(z)) = \frac{3\varpi(z) + (|\cos(z\pi)| + x^2)}{22} + \frac{14}{66}|\varpi|, \qquad M_1 = \frac{1}{11}, \qquad L_1 = \frac{10}{33},$$

$$\psi(z,\varpi(z)) = \frac{5|\varpi| + (4z-1)}{23} - z + \frac{2}{46}\varpi(z), \qquad M_2 = \frac{3}{23}, \qquad L_2 = \frac{6}{23},$$
$$N(z,s,\varpi(s)) = \frac{7(s+z) + e^{-14(s+z)}}{37} + \frac{9}{74}|\varpi|, \qquad M_3 = \frac{14}{23}, \qquad L_3 = \frac{9}{74}.$$

Thus,

$$\psi(0,\varpi_0) = \psi(0,1) = \frac{10}{46}, \quad \textit{and} \qquad L = \max\{L_1, L_2, L_3\} = \frac{10}{33}.$$
$$\delta_2 = 2.3621, \quad L\delta_2 = 0.757 < 1, \quad \delta_1 = 60.5888.$$

then, there exists a unique solution for the problem (3.23) on $[0,1]$ *by application of Theorem 3.3.*

4. Generalized Ulam stability criterion

We show the Ulam stability of (1.4), using integration $v^*(z) = Fv^*(z)$.
Let us define the non-linear continuous operator

$$Q : C([0,1],E) \longrightarrow C([0,1],E),$$

as follows

$$Qv^*(z) = ^C D^{\nu+\kappa}v^*(z) - ^C D^{\nu}_{0^+}\omega(z,v^*(z)) - \psi(z,v^*(z)) - \int_0^z N(z,s,v^*(s))ds.$$

Definition 4.1. *Let* $\epsilon > 0$ *and* v *be a solution of (1.4), where*

$$\|Qv^*\| \le \epsilon, \tag{4.1}$$

We say that problem (1.4) is Ulam-Hyers stable if there exists $\beta > 0$ *and solution* $\varpi \in C([0,1],E)$ *of (1.4) satisfying*

$$\|\varpi - v^*\| \le \beta\epsilon^*, \tag{4.2}$$

here $\epsilon^* > 0$ *depending on* ϵ.

Definition 4.2. *Let* $g \in C(\mathbb{R}^+,\mathbb{R}^+)$ *and solution* v *of (1.4), there exists a solution* $\varpi \in C([0,1],E)$ *of (1.4) where*

$$\|\varpi(z) - v^*(z)\| \le g(\epsilon), \quad 0 \le x \le 1. \tag{4.3}$$

Then (1.4) is said to be stable in the generalized Ulam-Hyers sense.

Definition 4.3. *Let* $\epsilon > 0$ *and* $\theta \in C([0,1],\mathbb{R}^+)$. *Let* v *be a solution of (1.4); we say that the system (1.4) is stable in the Ulam-Hyers-Rassias sense with respect to* θ *if*

$$\|Qv^*(z)\| \le \epsilon\theta(z), \quad 0 \le x \le 1, \tag{4.4}$$

and $\exists\beta > 0$ *and a solution* $v \in C([0,1],E)$ *of (1.4) where*

$$\|\varpi(z) - v^*(z)\| \le \beta\epsilon_*\theta(z), \quad 0 \le x \le 1, \tag{4.5}$$

here $\epsilon_* > 0$ *depending on* ϵ.

Theorem 4.1. *Let (H1) in Theorem 3.1 hold with $L\delta < 1$. The (1.4) is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $\varpi \in C([0,1], E)$ be a solution of (1.4), satisfying (2.1) in the sense of Theorem 3.3, and any solution v^* satisfying (4.1). Then, we obtain:

$$\begin{aligned} \|v^*(z) - \varpi(z)\| &= \|v^*(z) - \mathcal{F}v^*(z) + \mathcal{F}v^*(z) - \varpi(z)\| \\ &= \|v^*(z) - \mathcal{F}v^*(z) + \mathcal{F}v^*(z) - \mathcal{F}\varpi(z)\| \\ &\leq \|\mathcal{F}v^*(z) - \mathcal{F}\varpi(z)\| + \|Fv^*(z) - Id(v^*(z))\| \\ &\leq \|\mathcal{F}v^*(z) - \mathcal{F}\varpi(z)\| + \|Qv^*\| \\ &\leq L\delta\|\varpi - v^*\| + \epsilon, \end{aligned} \tag{4.6}$$

because $L\delta < 1$ and $\epsilon > 0$, we obtain

$$\|\varpi - v^*\| \leq \frac{\epsilon}{1 - L\delta}.$$

Let $\epsilon_* = \frac{\epsilon}{1-L\delta}$ be fixed and $\beta = 1$, we obtain the Ulam-Hyers stability requirements. Besides, the generalized Ulam-Hyers stability is followed by taking

$$g(\epsilon) = \frac{\epsilon}{1 - L\delta}.$$

□

Theorem 4.2. *Let (H1) holds with $L < \delta^{-1}$, and $\exists\theta \in C([0,1], \mathbb{R}^+)$ satisfy (4.4). Then (1.4) is Ulam-Hyers-Rassias stable with respect to θ.*

Proof. As in the proof of Theorem 4.1, we have

$$\|\varpi(z) - v^*(z\| \leq \epsilon_*\theta(z), \quad 0 \leq z \leq 1,$$

where $\epsilon_* = \frac{\epsilon}{1-L\delta}$. □

5. Conclusions

This article delves into the study of a fractional-order non-linear integro-differential equation, featuring a composition of fractional derivatives with distinct origins and mixed conditions. This equation serves as a model for the motion of a system within a fractal medium. The paper utilizes three fixed-point theorems to establish the existence and uniqueness of a solution. Subsequently, it presents the Ulam stability criterion for the obtained solution. Finally, a proposed example is provided to illustrate the main results. Complete studies are made in which our contributions extend the work in [22]. The problem of the fractional Caputo derivative (in the RHS and LHS of the differential equation) with mixed conditions is considered in (1.4) to improve the model (1.3), where only the integral boundary condition was taken, and the problem was with the Riemann-Liouville fractional integral in the RHS of the differential equation. This problem can be studied by a generalized Caputo derivative with changing conditions, especially when the derivative comes in the psi Caputo derivative; then study it with different origins and suitable mixed conditions.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

Conflict of interest

The authors declare that they have no conflict of interest.

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