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Research article

The identical approximation regularization method for the inverse problem to a 3D elliptic equation with variable coefficients

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Abstract: In this article, the Cauchy problem for a 3D elliptic equation is considered in a cylindrical domain. To regularize the problem, we propose a regularization method named "identical approximation regularization", which does not require complicated calculations. Two identical approximate regularization solutions are compared in the numerical section. The experimental results show that the Dirichlet reconstruction solution is more effective than the others.

Keywords: Cauchy problem; 3D elliptic equation; identical approximation operator; regularization method; convergence rates
Mathematics Subject Classification: 26D15, 31A25, 31B35

1. Introduction

The Cauchy problem for elliptic equations(CPEE) is one of the emblematic classical inverse problems [1]. As widely known, elliptic equations have been successfully applied in various fields, such as geophysics, wave propagation, vibration, nondestructive testing, etc. [2].

Our research aim is to obtain a stable regularization solution for an inverse 3D elliptic equation problem with variable coefficients.

$$\begin{cases} b_1(x)u_{xx} + b_2(x)u_x + b_3(x)u + u_{yy} + u_{zz} = 0, & (x,y,z) \in \Omega, \\ u(0,y,z) = \varphi(y,z), & (y,z) \in \mathbb{R}^2, \\ u_x(0,y,z) = 0, & (y,z) \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $u = u(x,y,z)$, the domain $\Omega = (0,d) \times \mathbb{R}^2$, variable coefficients $b_1(x)$, $b_2(x)$ and $b_3(x)$ satisfies

$$b_1(x) \in C^2[0,d],\ b_i(x) \in C^1[0,d], i = 2,3;$$

$$0 < M_1 \le b_1(x) \le M_2,\ b_3(x) \le 0,\ x \in [0,d].$$

The exact data $\varphi(y,z)$ and its measurement data $\varphi^{\delta}(y,z)$ satisfy

$$\|\varphi - \varphi^{\delta}\|_{H^s(\mathbb{R}^2)} \leq \delta. \tag{1.2}$$

Here $\delta > 0$ is the noise data, $\|\cdot\|_{H^s(\mathbb{R}^2)}$ represents the norms on Sobolev space $H^s(\mathbb{R}^2)$, *i.e.*

$$\|f\|_{H^s(\mathbb{R}^2)} := (\int_{\mathbb{R}^2} |\hat{f}(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta)^{1/2}. \tag{1.3}$$

Here $f \in L^2(\mathbb{R}^2)$, $\hat{f}$ is the Fourier transform for f. The definitions of Fourier transform and inverse Fourier transform can be found in Ref. [1].

If $b_1(x) = 1$ and $b_i(x) = 0$ for $i = 2, 3$, then problem (1.1) reduces to the 3D Laplace equation, which arises in many physical, engineering fields, for instance, cardiology, geophysics, and so forth [1]. If $b_1(x) = 1$, $b_2(x) = 0$ and $b_3(x) = -k^2$, problem (1.1) simplifies to a 3D modified Cauchy Helmholtz problem, which is also an interesting problem from a physical and mathematical perspective.

There have been some research results on the CPEE. Ref. [3] used a non-local boundary value method to handle the ill-posedness of CPEE; proposing *a posteriori* and *a priori* parameter selection rules methods optimal regularization solution. Ref. [4] proposed the Landweber iteration and steepest descent methods to manage the CPEE. Ref. [5, 6] used the mollification method to solve inverse elliptic problems. The authors in Ref. [7] made a change of variables and removed the non-linearity from the leading term of the equation by studying a non-linear operator elliptic equation, obtaining a convergent iterative procedure. Ref. [8] focused on inverse problems of semilinear elliptic equations with fractional power- type nonlinearities; their arguments are based on the higher-order linearization method. Ref. [9] used the generalized Tikhonov-type regularization method to work out the ill-posedness of a semi-linear elliptic equation. Ref. [10] developed the gradient descent-type methods to handle convex optimization problems in Hilbert space, and applied them in CPEE. Ref. [11] established a variational quasi-reversibility method to solve a prototypical CPEE. In Ref. [12], the inverse source problems of 2D and 3D elliptic-type nonlinear partial differential equations are considered. The authors solved a system of linear algebraic equations that satisfied an over-specified Neumann boundary condition to obtain the unspecified coefficients.

Ref. [13] employed the Fredholm Alternative to address an inverse problem of identifying an unknown source function on the right - hand side of an elliptic equation. Ref. [14] derived the Lipschitz stability of the inverse problem for an admissible class of unknown boundary functions. Their analysis was applicable not only to an interior problem but also provided an extension to the parabolic case. Ref. [15] studied a quasilinear Dirichlet problem dictated by a p-Laplacian-type operator, which was distinguished by an unbounded coefficient. They successfully demonstrated the existence of a bounded weak solution. Nevertheless, articles [13–15] failed to conduct numerical experiments to validate the effectiveness of the proposed method. In Ref. [16], a localized meshless collocation method was put forward to tackle the inverse Cauchy problem related to the fractional heat conduction model within the context of functionally graded materials (FGMs), and in the results and discussions section, three numerical experiments were carried out to demonstrate the efficiency and accuracy of the proposed approach. Similarly, article [17] addresses inverse Cauchy problems in heat conduction for 3D functionally graded materials (FGMs) with heat sources using a semi-analytical boundary collocation solver. The boundary particle method and regularization technique were combined and employed to deal with the ill-posed inverse Cauchy problems. The regularization technique was used to eliminate the effect of the noisy measurement data on the accessible boundary surface of 3D FGMs. However, there is relatively scarce literature available regarding the inverse problems of elliptic equations with variable coefficients in strip - shaped domains.

In article [18], we used the identical approximation regularization method to handle the Cauchy problem of the 2D heat conduction equation; the results showed that the identical approximation regularization method is highly efficiency. In this paper, we extend the identical approximation regularization method to consider the problem (1.1), establishing stability error estimates under an *a priori* regularization parameter selection principle. We leave the *a posteriori* selection principle to further research.

The rest of the current article is split into five parts. In Part 2, the ill-posedness of problem (1) is analyzed, and identical approximate operators are introduced. Part 3 focuses on stability estimates of identical approximation operators. Two numerical implementations demonstrate the validity of the method in Part 4. A brief summary is provided in Part 5.

2. Ill-posedness of the problem and identical approximation operator

Let $L = b_1(x)\frac{\partial^2}{\partial x^2} + b_2(x)\frac{\partial}{\partial x} + b_3(x)$. Applying Fourier transform to problem (1.1) for variable (y, z), we have

$$\begin{cases} L\hat{u}(x,\psi,\zeta) = (\psi^2+\zeta^2)\hat{u}(x,\psi,\zeta), & 0<x<d,\ (\psi,\zeta)\in\mathbb{R}^2; \\ \hat{u}(0,\psi,\zeta) = \hat{\varphi}(\psi,\zeta), & (\psi,\zeta)\in\mathbb{R}^2, \\ \hat{u}_x(0,\psi,\zeta) = 0, & (\psi,\zeta)\in\mathbb{R}^2. \end{cases} \tag{2.1}$$

The solution for problem (2.1) is

$$\hat{u}(x,\psi,\zeta) = v(x,\psi,\zeta)\hat{\varphi}(\psi,\zeta). \tag{2.2}$$

The solution $v(x,\psi,\zeta)$ is obtained by solving the following equation.

$$\begin{cases} L\hat{v}(x,\psi,\zeta) = (\psi^2+\zeta^2)\hat{v}(x,\psi,\zeta), & 0<x<d,\ (\psi,\zeta)\in\mathbb{R}^2, \\ \hat{v}(0,\psi,\zeta) = 1, & (\psi,\zeta)\in\mathbb{R}^2, \\ \hat{v}_x(0,\psi,\zeta) = 0, & (\psi,\zeta)\in\mathbb{R}^2. \end{cases} \tag{2.3}$$

The following Lemma will be used in this paper. The proofs of these results can be found in [19].

Lemma 2.1. lemma There exists a unique solution $v(x,\psi,\zeta)$ of (2.3) such that

(1) $v\in W^{2,\infty}(0,d),\ \forall\ (\psi,\zeta)\in\mathbb{R}^2$;

(2) $v(d,\psi,\zeta)\neq 0,\ \forall\ (\psi,\zeta)\in\mathbb{R}^2$;

(3) There exists constants $c_i>0$ $(i=1,2,3,4)$ satisfying

$$|v(x,\psi,\zeta)|\le c_1 e^{\sqrt{\psi^2+\zeta^2}G(x)}, |v(d,\psi,\zeta)|\ge c_2 e^{\sqrt{\psi^2+\zeta^2}G(d)}.$$

Moreover, if $\omega = \sqrt{\psi^2+\zeta^2}\ge\omega_0\ge 0$, then

$$|v_x(x,\psi,\zeta)|\le c_3\omega e^{G(x)\omega}, |v_{xx}(x,\psi,\zeta)|\ge c_4 e^{G(x)\omega},$$

where $G(x)=\int_0^x \frac{ds}{\sqrt{b_1(s)}},\ \forall x\in[0,d]$. If $x=d$, then $\hat{u}(d,\psi,\zeta)=v(d,\psi,\zeta)\hat{\varphi}(\psi,\zeta)$.

It is not difficult to verify that problem (1.1) is severely ill-posed using conclusion (3) of Lemma 2.1. To address it, we employ an identical approximation regularization technique.

In this paper, we exclusively consider the following three operators (referred to as identical approximate operators) (see [20, 21]):

- The Gaussian operator $G_\mu(x) = \frac{1}{(\mu\sqrt{\pi})^n}\prod_{j=1}^{n} e^{-(\frac{x_j}{\mu})^2}$, which has

$$(2\pi)^{n/2}\hat{G}_\mu(\xi) = \prod_{j=1}^{n} e^{-\frac{\mu^2\xi^2}{4}}. \tag{2.4}$$

- The Dirichlet operator $D_\mu(x) = \frac{1}{\pi^n}\prod_{j=1}^{n}\frac{\sin(x_j/\mu)}{x_j}$, satisfies

$$(2\pi)^{n/2}\hat{D}_\mu(\xi) = \begin{cases} 1, & \triangle_\mu = \{\xi_j\ |\xi_j| < 1/\mu\}, \\ 0, & |\xi| \geq 1/\mu. \end{cases} \tag{2.5}$$

- The de la Vallée Poussin operator $V_\mu(x) = (\frac{\mu}{\pi})^n\prod_{j=1}^{n}\frac{\cos(x_j/\mu)-\cos(2x_j/\mu)}{x_j^2}$, and it's Fourier transform fulfills $(2\pi)^{\frac{n}{2}}\hat{V}_\mu = \prod_{j=1}^{n}\lambda(\xi_j)$, where,

$$\lambda(\xi_j) = \begin{cases} 1, & |\xi_j| < 1/\mu, \\ 2-\mu\xi_j, & 1/\mu < |\xi_j| \leq 2/\mu, \\ 0, & |\xi_j| > 2/\mu. \end{cases} \tag{2.6}$$

Here, regularization parameter $\mu > 0$, variable $x = (x_1, x_2, \cdots, x_n)$.

Modifying the measurement data φ^δ to equation (1.1) by identical approximate operator T_μ, we obtain the modified equation as follows:

$$\begin{cases} Lu^{\mu,\delta} + u^{\mu,\delta}_{yy} + u^{\mu,\delta}_{zz} = 0, \quad (x,y,z) \in \Omega, \\ u^{\mu,\delta}(0,y,z) = (T_\mu * \varphi^\delta)(y,z), \quad (y,z) \in \mathbb{R}^2, \\ u^{\mu,\delta}_x(0,y,z) = 0, \quad (y,z) \in \mathbb{R}^2. \end{cases} \tag{2.7}$$

The solution of problem (2.7) is

$$\hat{u}^{\mu,\delta}(x,\psi,\zeta) = v(x,\psi,\zeta)(\widehat{T_\mu * \varphi^\delta})(\psi,\zeta).$$

Here $T_\mu * \varphi^\delta$ is the 2D convolution defined by [21],

$$(T_\mu * \varphi^\delta)(y,z) = \int\int_{R^2} T_\mu(y',z')\cdot\varphi^\delta(y-y',z-z')dy'dz'.$$

It is well known that [15],

$$(\widehat{T_\mu * \varphi^\delta})(\psi,\zeta) = 2\pi\hat{T}_\mu(\psi,\zeta)\hat{\varphi}^\delta(\psi,\zeta). \tag{2.8}$$

The authors of Ref. [5] applied the Gaussian operator to work out the problem (1.1). In this paper, we will utilize the Dirichlet identical approximation operator and de la Vallée Poussin identical approximation operator to handle problem (1.1), the error estimates and convergence rate for the approximate solutions to exact solutions are obtained. In the numerical experiment part, we compare the three identical approximation solutions, and find that the Dirichlet identical approximation solution is superior to the others.

3. A priori parameter choice and convergence estimate

3.1. Dirichlet identical approximation

For the case in (2.8), taking T_μ as 2D Dirichlet operator

$$D_\mu(y,z) = \frac{\sin(y/\mu)\sin(z/\mu)}{\pi^2 yz}, \ (\mu > 0)$$

we get the regularization solution of problem (1.1),

$$\hat{u}^{\mu,\delta}(x,\psi,\zeta) = v(x,\psi,\zeta)(\widehat{D_\mu * \varphi^\delta})(\psi,\zeta) = 2\pi v(x,\psi,\zeta)\hat{D}_\mu(\psi,\zeta)\hat{\varphi}^\delta(\psi,\zeta). \tag{3.1}$$

We establish the error estimate and approximation effect between the Dirichlet regularized solution and exact solution at $0 < x \le d$.

Theorem 3.1. Let $\varphi(y,z)$ satisfy (1.2). Suppose that problem (1.1) has the exact solution $u(x,y,z)$ and the Dirichlet regularization solution $u^{\mu,\delta}(x,y,z)$.

(i) For any $x \in (0,d)$, if we choose $\mu = \frac{2G(d)}{\ln(M_s/\delta)}$ with *a priori bound* $\|u(d,\cdot,\cdot)\|_{H^s(\mathbb{R}^2)} \le M_s$. Then the following estimate holds

$$\|u - u^{\mu,\delta}\|_{H^s(\mathbb{R}^2)} \le C M_s^{\frac{G(x)}{G(d)}} \delta^{1-\frac{G(x)}{G(d)}}. \tag{3.2}$$

(ii) For $x = d$, if we select $\mu = \frac{2G(d)}{\ln(M_r/\delta)}$ ($r > s \ge 0$) with *a priori bound* $\|u(d,\cdot,\cdot)\|_{H^r(\mathbb{R}^2)} \le M_r$ ($r > s \ge 0$). We obtain

$$\|u(d,\cdot,\cdot) - u^{\mu,\delta}(d,\cdot,\cdot)\|_{H^s(\mathbb{R}^2)} \le 8M_r\Big(\frac{2G(d)}{\ln(M_r/\delta)}\Big)^{r-s} + c_1\delta^{1-\frac{\sqrt{2}}{2}} M_r^{\frac{\sqrt{2}}{2}}. \tag{3.3}$$

Proof. *(i)* We have the following equality

$$\begin{aligned}
\|u - u^{\mu,\delta}\|^2_{H^s(\mathbb{R}^2)} &= \sum_{k=1}^{9} \int\int_{E_k} |\hat{u} - \hat{u}^{\mu,\delta}|^2 (1+\psi^2+\zeta^2)^s d\psi d\zeta \\
&= \sum_{k=1}^{8} \int\int_{E_k} |\hat{u}|^2 (1+\psi^2+\zeta^2)^s d\psi d\zeta + \int\int_{E_9} |\hat{u} - \hat{u}^{\mu,\delta}|^2 (1+\psi^2+\zeta^2)^s d\psi d\zeta \\
&= \sum_{k=1}^{8} \int\int_{E_k} |P(x,\psi,\zeta)\hat{u}(d,\psi,\zeta)|^2 (1+\psi^2+\zeta^2)^s d\psi d\zeta + \\
&\qquad \int\int_{E_9} |v(x,\psi,\zeta)(\hat{\varphi} - \hat{\varphi}^\delta)|^2 (1+\psi^2+\zeta^2)^s d\psi d\zeta.
\end{aligned}$$

For convenience, let $\alpha = 1/\mu$, for each E_k $(1 \le k \le 9)$,

$$E_1 = (-\infty,-\alpha)\times(-\infty,-\alpha),\ E_2 = (-\alpha,\alpha)\times(-\infty,-\alpha), E_3 = (\alpha,+\infty)\times(-\infty,-\alpha),$$

$$E_4 = (\alpha,+\infty)\times(-\alpha,\alpha), E_5 = (\alpha,+\infty)\times(\alpha,+\infty),\ E_6 = (-\alpha,\alpha)\times(\alpha,+\infty),$$

$$E_7 = (-\infty,-\alpha)\times(\alpha,+\infty),\ E_8 = (-\infty,-\alpha)\times(-\alpha,\alpha), E_9 = (-\alpha,\alpha)\times(-\alpha,\alpha).$$

Let

$$P(x,\psi,\zeta) = \frac{v(x,\psi,\zeta)}{v(d,\psi,\zeta)} \quad (\psi,\zeta \in \mathbb{R}).$$

By calculations, we obtain that

$$\|u-u^{\mu,\delta}\|^2_{H^s(\mathbb{R}^2)} \quad \le \sum_{k=1}^{8}(\sup_{E_k}|P(x,\psi,\zeta)|)^2\|\hat{u}(d,\cdot,\cdot)\|^2_{H^s(\mathbb{R}^2)}+\delta^2(\sup_{E_9}|v(x,\psi,\zeta|)^2.$$

Using inequality $\sqrt{\sum_{k=1}^{8} a_k} \le \sum_{k=1}^{8} \sqrt{a_k}$, $(a_k \ge 0)$ and the properties for $v(x,\psi,\zeta)$, we have

$$\begin{aligned}\|u-u^{\mu,\delta}\|_{H^s(\mathbb{R}^2)} &= \sum_{k=1}^{8}\sup_{E_k}|P(x,\psi,\zeta)|\|\hat{u}(d,\cdot,\cdot)\|_2+\delta\sup_{E_9}|v(x,\psi,\zeta)| \\ &\le 8M_s\frac{c_1}{c_2}e^{-(G(d)-G(x))/\mu}+c_1\delta e^{\sqrt{2}/\mu G(x)}.\end{aligned}$$

If we choose $\mu=\frac{2G(d)}{\ln(M_s/\delta)}$, and substitute it into the above inequality, we deduce that

$$\|u-u^{\mu,\delta}\|_{H^s(\mathbb{R}^2)} \le 8\frac{c_1}{c_2}M_s^{\frac{G(x)}{G(d)}}\delta^{1-\frac{G(x)}{G(d)}}+c_1M_s^{\frac{G(x)}{G(d)}}\delta^{1-\frac{G(x)}{G(d)}}. \tag{3.4}$$

Let $c=8\frac{c_1}{c_2}+c_1$, we thereby conclude the error estimate (3.2).
(ii) The following assertion is the straightforward results of (i) by taking $x=d$,

$$\begin{aligned}&\|u(d,\cdot,\cdot)-u^{\mu,\delta}(d,\cdot,\cdot)\|^2_{H^s(\mathbb{R}^2)} \\ &=\sum_{k=1}^{9}\int\int_{E_k}|\hat{u}(d,\psi,\zeta)-\hat{u}^{\mu,\delta}(d,\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta \\ &=\sum_{k=1}^{8}\int\int_{E_k}|\hat{u}(d,\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta+ \\ &\quad \int\int_{E_9}|\hat{u}(d,\psi,\zeta)-\hat{u}^{\mu,\delta}(d,\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta.\end{aligned}$$

Denote $Q(x,\psi,\zeta)=\frac{1}{(1+\psi^2+\zeta^2)^{s/2}} \quad (\psi,\zeta\in\mathbb{R})$, there is

$$\begin{aligned}\|u(d,\cdot,\cdot)-u^{\mu,\delta}(d,\cdot,\cdot)\|_{H^s(\mathbb{R}^2)} &\le \sum_{k=1}^{8}\sup_{E_k}|Q(x,\psi,\zeta)|\|\hat{u}(d,\cdot,\cdot)\|_{H^s(\mathbb{R}^2)}+\delta\sup_{E_9}|v(x,\psi,\zeta))| \\ &\le \frac{8M_r}{(1+1/\mu^2)^{\frac{r-s}{2}}}+c_1\delta e^{\sqrt{2}G(d)/\mu},\ (\ r>s\ge 0).\end{aligned}$$

If we choose $\mu=\frac{2G(d)}{\ln(M_r/\delta)}$, then equation (3.3) holds. □

3.2. De la Vallée Poussin identical approximation

Taking T_μ as a 2D de la Vallée Poussin operator,

$$V_\mu(y,z)=\mu^2\frac{(\cos(y/\mu)-\cos(2y/\mu))((\cos(z/\mu)-\cos(2z/\mu))}{\pi^2y^2z^2}.$$

The de la Vallée Poussin regularization solution for problem (1.1) is,

$$\hat{u}^{\mu,\delta}(x,\psi,\zeta) = v(x,\psi,\zeta)(\widehat{V_\mu * \Phi^\delta})(\psi,\zeta) = 2\pi v(x,\psi,\zeta)\hat{V}_\mu(\psi,\zeta)\hat{\varphi}^\delta(\psi,\zeta). \tag{3.5}$$

Theorem 3.2. Let $\varphi(y,z)$ satisfy (1.2). Suppose that problem (1.1) has the exact solution $u(x,y,z)$ and the De la Vallée Poussin regularization solution $u^{\mu,\delta}(x,y,z)$.
(i) For any $x \in (0,d)$, if we choose $\mu = \frac{4G(d)}{\ln(M_s/\delta)}$ with *a priori* bound $\|u(d,\cdot,\cdot)\|_{H^s(\mathbb{R}^2)} \le M_s$, we have

$$\|u - u^{\mu,\delta}\|_{H^s(\mathbb{R}^2)} \le C M_s^{\frac{G(x)}{G(d)}} \delta^{1-\frac{G(x)}{G(d)}}, \tag{3.6}$$

where $C = 59M_s\frac{c_1}{c_2} + 36c_1$.
(ii) For $x = d$, if we select $\mu = \frac{8G(d)}{\ln(M_r/\delta)}$ $(r > s \ge 0)$ with *a priori* bound $\|u(d,\cdot,\cdot)\|_{H^r(\mathbb{R}^2)} \le M_r$ $(r > s \ge 0)$, the following estimate is valid,

$$\|u(d,\cdot,\cdot) - u^{\mu,\delta}(d,\cdot,\cdot)\|_{H^s(\mathbb{R}^2)} \le 59M_r\left(\frac{8d}{\ln(M_r/\delta)}\right)^{r-s} + 36M_r^{1/2}\delta^{1/2}.$$

Proof. *(i)* Using a similar method with Theorem 3.1 to prove this theorem, we obtain

$$\begin{aligned}
\|u - u^{\mu,\delta}\|^2_{H^s(\mathbb{R}^2)} &= \sum_{k=1}^{25}\int_{D_k} |\hat{u} - \hat{u}^{\mu,\delta}|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta \\
&= \sum_{k=1}^{16}\int_{D_k} |\hat{u}|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta + \sum_{k=17}^{25}\int_{D_k} |\hat{u} - \hat{u}^{\mu,\delta}|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta \\
&\le M_s^2 \sum_{k=1}^{16}(\sup_{D_k}|p(\psi,\zeta)|)^2 + \sum_{k=17}^{25}\int_{D_k} |(\hat{\varphi} - (\widehat{T_\mu * \varphi^\delta}))q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta.
\end{aligned}$$

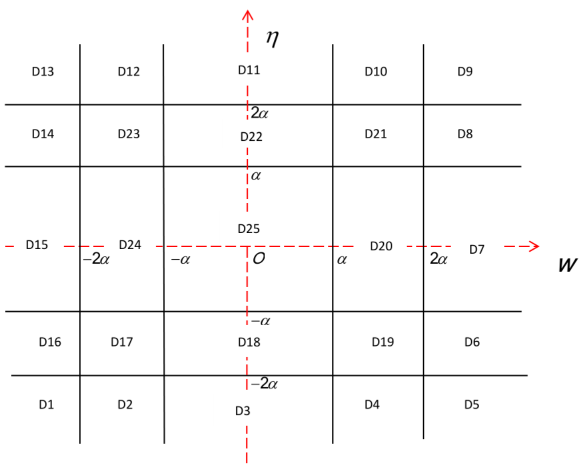


Figure 1. For each D_k $(i = 1, 2, \cdots, 25)$.

Here, $\mathbb{R}^2 = \bigcup_{i=1}^{25} D_i$ (see Fig.1). For simplicity let $\alpha = 1/\mu$, $p(\psi,\zeta) = \frac{v(x,\psi,\zeta)}{v(d,\psi,\zeta)}$, $q(\psi,\zeta) = v(x,\psi,\zeta), (\psi,\zeta \in \mathbb{R})$.

$$\sum_{k=17}^{25}\int_{D_k} |(\hat{\varphi} - (\widehat{T_\mu \varphi^\delta}))q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta =$$
$$\int_{D_{17}} |(\hat{\varphi} - (2-\psi\mu)(2-\zeta\mu)\hat{\varphi}^\delta)q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta +$$
$$\int_{D_{18}} |(\hat{\varphi} - (2-\eta\mu)\hat{\varphi}^\delta)q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta +$$
$$\int_{D_{19}} |(\hat{\varphi} - (2-\psi\mu)(2-\zeta\mu)\hat{\varphi}^\delta)q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta +$$
$$\int_{D_{20}} |(\hat{\varphi} - (2-\psi\mu)\hat{\varphi}^\delta)q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta +$$
$$\int_{D_{21}} |(\hat{\varphi} - (2-\psi\mu)(2-\eta\mu)\hat{\varphi}^\delta)q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta +$$
$$\int_{D_{22}} |(\hat{\varphi} - (2-\eta\mu)\hat{\varphi}^\delta)q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta +$$
$$\int_{D_{23}} |(\hat{\varphi} - (2-\psi\mu)(2-\eta\mu)\hat{\varphi}^\delta)q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta +$$
$$\int_{D_{24}} |(\hat{\varphi} - (2-\psi\mu)\hat{\varphi}^\delta)q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta +$$
$$\int_{D_{25}} |(\hat{\varphi} - \hat{\varphi}^\delta)B(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta.$$

Noting that $9 < (2-\psi\mu)(2-\zeta\mu) < 16$ holds in D_{17}, we get

$$D_{17} = (-2\alpha, -\alpha) \times (-2\alpha, -\alpha).$$

It follows that

$$\begin{aligned}
&(\int_{D_{17}} |(\hat{\varphi} - (2-\psi\mu)(2-\zeta\mu)\hat{\varphi}^\delta)q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta)^{1/2} \\
&= \|(\hat{\varphi} - (2-\psi\mu)(2-\zeta\mu)\hat{\varphi}^\delta)q(\psi,\zeta)(1+\psi^2+\zeta^2)^{s/2}\|_{L^2(D_{17})} \\
&\le \|\hat{\varphi}q(\psi,\zeta)(1+\psi^2+\zeta^2)^{s/2}\|_{L^2(D_{17})} + 16\|(\hat{\varphi}^\delta - \hat{\varphi} + \hat{\varphi})q(\psi,\zeta)(1+\psi^2+\zeta^2)^{s/2}\|_{L^2(D_{17})} \\
&\le 17\|\hat{\varphi}q(\psi,\zeta)(1+\psi^2+\zeta^2)^{s/2}\|_{L^2(D_{17})} + 16\|(\hat{\varphi}^\delta - \hat{\varphi})q(\psi,\zeta)(1+\psi^2+\zeta^2)^{s/2}\|_{L^2(D_{17})} \\
&\le 17M \sup_{D_{17}} |p(\psi,\zeta)| + 16\delta \sup_{D_{17}} |q(\psi,\zeta)|.
\end{aligned}$$

By conducting a series of elaborate calculations, the error estimates on other D_k $(k = 18, \cdots, 24)$ can be obtained. Whereupon we have

$$\begin{aligned}
&\|u - u^{\mu,\delta}\|_{H^s(\mathbb{R}^2)} \\
&\le M \sum_{k=1}^{16} \sup_{D_k} |p(\psi,\zeta)| + \sum_{k=17}^{25} (\int_{D_k} |(\hat{\varphi} - (\widehat{T_\mu * \varphi^\delta}))q(\psi,\zeta)|^2(1+\psi^2+\zeta^2)^s d\psi d\zeta)^{1/2}.
\end{aligned}$$

Combination with the properties of $v(x,\psi,\zeta)$, we deduce that

$$\|u-u^{\mu,\delta}\|_{H^s(\mathbb{R}^2)} \le 59M_s\frac{c_1}{c_2}e^{-(G(d)-G(x))/\mu}+36c_1\delta e^{2\sqrt{2}G(x)/\mu}.$$

Choosing regularization parameter $\mu=\frac{4d}{\ln(M_s/\delta)}$, the error estimate (3.6) is obtained .
(ii) Using similar methods to the above, we can arrive at the statement of (ii). □

Remark 3.1. If $s=0$, it is straightforward to verify that the results of the Theorems 3.1 and 3.2 can be deduced to the convergence estimate in space $L^2(\mathbb{R}^2)$.

4. Numerical tests

This subsection is devoted to the implementation of our theoretical results in Section 3. The works are performed by MATLAB 2020(a).

First, we take $N=101$, discrete interval $[-10,10]\times[-10,10]$, $M=\|u(1,\cdot,\cdot)\|_2$, data $\varphi^\delta(y,z)$ is got through

$$\varphi^\delta(y,z)=\varphi+\delta(2Randn(Size(\varphi))-1),$$

where

$$\varphi=(\varphi(y_i,z_j))^T_{N\times N},\ y_i=\frac{20(i-1)}{N-1},\ z_j=\frac{10(j-1)}{N-1},(i,j=1,2,3,\cdots,N),$$

$$\delta=\|\varphi-\varphi^\delta\|_2=\sqrt{\frac{1}{N\times N}\sum_{i=1}^{N}\sum_{j=1}^{N}(\varphi(y_i,z_j)-\varphi^\delta(y_i,z_j))^2}.$$

Let $u^{\mu,\delta}$ and u denote the regularization solution and exact solution, respectively, and

$$R(u)=\frac{\|u-u^{\mu,\delta}\|_2}{\|u\|_2}$$

signifies the relative error. Let $R_P(u)$, $R_D(u)$ and $R_G(u)$ denote the relative error between the exact solution and de la Vallée Poussin, Dirichlet, and Gaussian regularization solution, respectively. The regularization parameter $\mu=4/\ln(M/\delta)$.

Taking $b_1(x)=x^2+1$, $b_2(x)=x$, $b_3(x)=0$, the solution of equation (2.3) is $\hat{v}(x,\psi,\zeta)=\cosh(G(x)\sqrt{\psi^2+\zeta^2})$, the solution of problem (1.1) is

$$\hat{u}(x,\psi,\zeta)=\cosh(\sqrt{\psi^2+\zeta^2}G(x))\hat{\varphi}(\psi,\zeta).$$

Let $\varphi(y,z)$ be the following functions:
Example 4.1. $\varphi(y,z)=e^{-y^2-z^2}$.
Example 4.2.

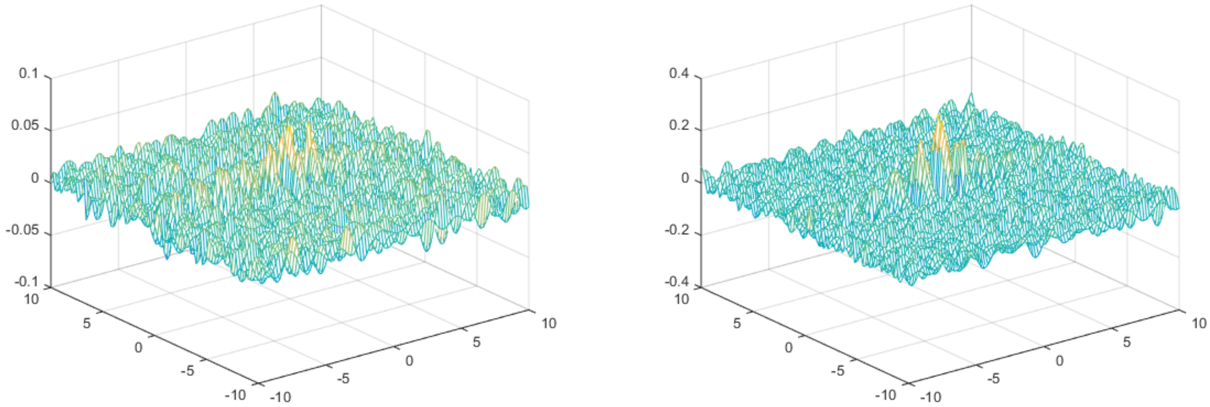
$$\varphi(y,z)=\begin{cases}9-y^2-z^2, & y^2+z^2\le 9,\\ 0, & y^2+z^2>9.\end{cases}$$

Table 1 presents the relative L^2-errors of the three kinds of reconstructions in Example 4.1 for noise levels $\delta=0.001$. Fig.2 shows the error of $R_D(u)$ to Example 4.1 at $x=0.4$ and $x=0.7$ with

$\delta = 10^{-3}$. Fig.3 displays the Dirichlet regularization solution at $x = 0.4$, $x = 0.7$ under conditions $\delta = 10^{-3}$, $E = 200$.

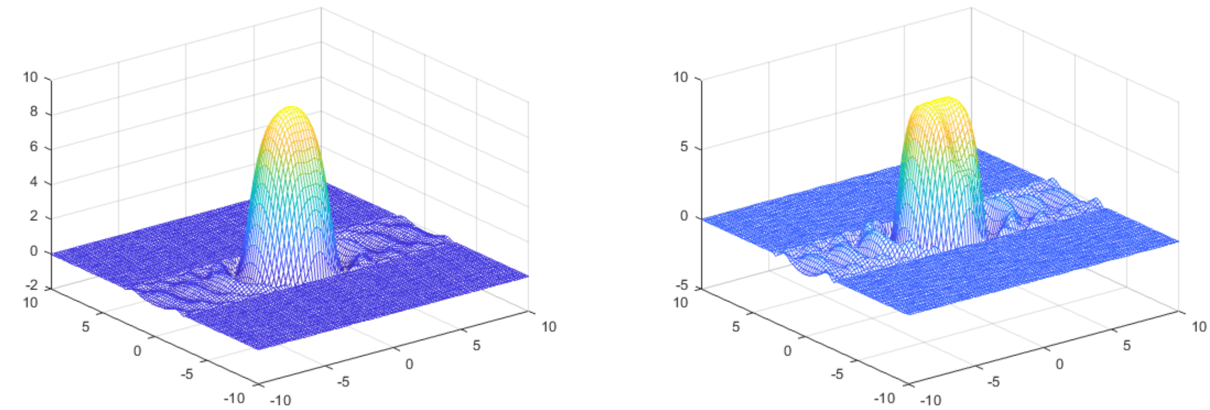
Table 1. The error for $\delta = 0.001$ of Example 4.1.

x	$x = 0.1$	$x = 0.5$	$x = 0.9$
$R_P(u)$	0.2133	5.5791	87.8083
$R_D(u)$	0.0726	0.1025	0.3104
$R_G(u)$	0.0779	0.1751	21.4179



(a) The error at $x = 0.4$ (b) The error at $x = 0.7$

Figure 2. Example 4.1: The error between the Dirichlet regularization solution and exact solution (a) at $x = 0.4$ (b) at $x = 0.7$.



(a) The Dirichlet regularization solution at $x = 0.4$ (b) The Dirichlet regularization solution at $x = 0.7$

Figure 3. Example 4.2: The Dirichlet approximation solution (a) at $x = 0.4$ (b) at $x = 0.7$.

It is evident from Table 1 and Figures 2-3 that the method proposed in this paper exhibits both stability and efficiency. Notably, the reconstruction accuracy is satisfactory, and the results in Table 1 indicate that the Dirichlet operator achieves a lower relative error compared to other operators.

5. Conclusions

This article presents a regularization method called the identical approximation regularization method to address the Cauchy problem for a 3D elliptic equation. The Cauchy problem, which involves reconstructing an unknown function from noisy data on its boundary or derivatives, is known to be ill-posed and highly sensitive to data perturbations. Two numerical tests are presented. The numerical results verify the stability of the regularization method, and the accuracy of the procedure is quite acceptable. Stable approximate errors are obtained under an *a priori* parameter choice rule. However, this study only focuses on *a priori* estimates and a-posteriori estimates are not considered. We leave the development of an *a posteriori* parameter selection rule for future research.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there are no conflicts of interest.

References

1. A. Kirsch, *An introduction to the mathematical theory of inverse problems*, 2 Eds., New York: Springer, 1971. http://doi.org/10.1007/978-1-4419-8474-6
2. C. Wang, J. Wang, Primal-Dual Weak Galerkin Finite Element Methods for Elliptic Cauchy Problems, *Comput. Math. Appl.*, **79** (2020), 746–763. https://doi.org/10.1016/j.camwa.2019.07.031
3. D.N. Hào, N.V. Duc, D. Lesnic, A non-local boundary value problem method for the Cauchy problem for elliptic equations, *Inverse Probl.*, **25** (2009), 055002. http://doi.org/10.1088/0266-5611/25/5/055002
4. V. I. Vasil'Ev, A. M. Kardashevsky, V. V. Popov, G. A. Prokopev, Iterative solution of the inverse Cauchy problem for an elliptic equation by the conjugate gradient method, *AIP Conf. Proc.*, **1895** (2017), 110009. http://doi.org/10.1063/1.5007415

5. H. Cheng, X. L. Feng, F. L. Fu, A mollification regularization method for the Cauchy problem of an elliptic equation in a multi-dimensional case, *Inverse Probl. Sci. Eng.*, **18** (2010), 977–982. http://doi.org/10.1080/17415977.2010.492519

6. D. N. Hào, A mollification method for ill-posed problems, *Numer. Math.*, **68** (1994), 469–506. http://doi.org/10.1007/s002110050073

7. F. Berntsson, V. A. Kozlov, D. Wokiyi, Solvability of a non-linear Cauchy problem for an elliptic equation, *Int. J. Comput. Math.*, **96** (2019), 2317–2333. http://doi.org/10.1080/00207160.2019.1569640

8. M. Lassas, T. Liimatainen, Y. H. Lin, M. Salo, Inverse problems for elliptic equations with power type nonlinearities, *J. Math. Pures Appl.*, **53** (2021), 23–28. http://doi.org/10.1016/J.MATPUR.2020.11.006

9. H. Zhang, Modified Tikhonov Method for Cauchy Problem of Elliptic Equation with Variable Coefficients, *Amer. J. Comput. Math.*, **4** (2014), 213–222. http://doi.org/10.4236/ajcm.2014.43018

10. M. Vladislav, S. Kabanikhin, M. Shishlenin, N. S. Novikov, A. Vasin, A. V. Gasnikov, Convex optimization with inexact gradients in Hilbert space and applications to elliptic inverse problems, *Comput. Optim. Appl.*, 2021. http://doi.org/10.13140/RG.2.2.16851.91681

11. V. A. Khoa, P. T. Hoangnhan, Constructing a variational quasi-reversibility method for a Cauchy problem for elliptic equations, *Math. Methods Appl. Sci.*, **44** (2021), 3334–3355. http://doi.org/10.1002/mma.6945

12. C. S. Liu, L. Qiu, Solving the 2D and 3D nonlinear inverse source problems of elliptic type partial differential equations by a homogenization function method, *Numer. Methods Partial Differ. Eq.: Int. J.*, (2023), 1287–1298. http://doi.org/10.1002/num.22934

13. Z. Kaytmaz, M. Yldz, Solvability of an Inverse Problem for an Elliptic-Type Equation, *Cumhuriyet Sci. J.*, **41** (2024), 130–134. http://doi.org/10.17776/csj.1359651

14. M. Choulli, H. Takase, Lipschitz stability for an elliptic inverse problem with two measurements, *Cumhuriyet Sci. J.*, **41** (2025), 130–134. http://doi.org/10.1007/s40687-025-00504-y

15. D. Motreanu, E. Tornatore, Elliptic equations with unbounded coefficient, convection term and intrinsic operator, *Math. Z.*, **308** (2024), 38. http://doi.org/10.1007/s00209-024-03591-9

16. Q. Xi, Z. Fu, C. Alves, A. Ji, A semi-analytical boundary collocation solver for the inverse Cauchy problems in heat conduction under 3D FGMs with heat source, *Numer. Heat Trans., Part B: Fundam.*, **76** (2019), 311–327. http://doi.org/10.1080/10407790.2019.1665386

17. H. Wen, J. F. Zhuo, C. T. Zhuo, G. Yan, A meshless collocation method for solving the inverse Cauchy problem associated with the variable-order fractional heat conduction model under functionally graded materials, *Eng. Anal. Boundary Elem.*, **140** (2022), 132–144. http://doi.org/10.1016/j.enganbound.2022.04.007

18. S. Q. He, X. F. Feng, Identical approximation operator and regularization method for the Cauchy problem of 2-D heat conduction equation, *Math. Methods Appl. Sci.*, **4** (2021), 12931–12944. http://doi.org/10.1002/mma.7596

19. D. N. Hào, P. M. Hien, H. Sahli, Stability results for a Cauchy problem for an elliptic equation, *Inverse Probl.*, **23** (2007), 421–461. http://doi.org/10.1088/0266-5611/23/1/024

20. S. Rajnikant, *Real and complex analysis*, Singapore: Springer, 2013. http://doi.org/10.1007/978-981-13-0938-0
21. S. M. Nikol'skii, *Approximation of functions of several variables and imbedding theorems*, Berlin: Springer, 1975. http://doi.org/10.1007/978-3-642-65711-5

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