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Research article

Supra soft Omega-open sets and supra soft Omega-regularity

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Abstract: This paper introduces a novel concept of supra-soft topology generated from a specific family of supra-topologies. We define supra-soft ω-open sets as a new class of soft sets that create a finer topology than the original. We explore the properties of these supra-soft ω-open sets and assess the validity of related results from ordinary supra-topological spaces within the framework of supra-soft topological spaces. Additionally, we present two new separation axioms: supra-soft ω-local indiscreteness and supra-soft ω-regularity, demonstrating that both are stronger than traditional supra-soft ω-regularity. We also provide subspace and product theorems for supra-soft ω-regularity and examine the correspondence between our new supra-soft concepts and their classical counterparts in supra-topology.

Keywords: supra-ω-open sets; supra-ω-local indiscrete space; supra-soft regular space; supra-generated soft topology
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1. Introduction and preliminaries

Molodtsov [1] introduced the concept of soft sets in 1999 as an innovative way to deal with uncertain data while modeling real-world situations in several domains such as data science, engineering, economics, and health sciences. Numerous researchers have used the theory of soft sets as a mathematical tool to solve real-world problems (see [2,3]). Shabir and Naz [4] initiated the structure of soft topology and investigated many related topics. After that, several researchers interested in abstract structures attempted to extend topological concepts to include soft topological spaces. For instance, concepts such as soft compactness [5], soft separation axioms [6–10], lower soft separation axioms [11–15], soft mappings [16,17], and soft metrics [18] were introduced. Furthermore, some researchers have investigated the concept of generalized open sets in soft topologies, such as soft semi-open sets [19], soft somewhat open sets [20], soft Q-sets [21], and lower density soft operators [22].

Mashhour et al. [23] defined supra-topological spaces by removing the condition of finite intersections in the traditional definition of topologies. Many topological researchers examined topological notions by using supra-topologies to analyze their properties [24–28]. The authors of [28] used supra-topologies to generate new rough set models for describing information systems. Furthermore, the authors of [29] used supra-topologies in digital image processing.

The concept of supra-soft topological spaces, introduced in 2014 [30], generalizes crisp mathematical structures to include soft ones. It included concepts like continuity [30], compactness [31], separation axioms [32–36], separability [37,38], and generalized open sets [39–41]. Research in the field of supra-soft topologies remains vibrant and active.

This paper proposes new concepts in supra-soft topology that extend traditional supra-topologies through a novel classification known as supra-soft ω-open sets. Some new separation axioms and the development of finer structures using supra-soft ω-open sets connect existing theories to newer aspects of the theory of topological structures. Besides developing our knowledge regarding supra-soft spaces, these results provide fertile ground for future developments of topological methods. In addition, we observe that supra-soft topological structures have not received the attention they deserve, especially since the applications of supra-topological spaces are in many domains [28,29]. Therefore, we expect this paper to offer a new approach to solving practical issues.

For concepts and expressions not described here, we refer the readers to [42,43].

Assume that U is a non-empty set and B is a set of parameters. A soft set over U relative to B is a function $H : B \longrightarrow \mathcal{P}(U)$. $SS(U, B)$ denotes the family of all soft sets over U relative to B. The null soft set and the absolute soft set are denoted by 0_B and 1_B, respectively. Let $H \in SS(U, B)$. If $H(a) = M$ for all $a \in B$, then H is denoted by C_M. If $H(a) = M$ and $H(b) = \emptyset$ for all $b \in B - \{a\}$, then H is denoted by a_M. If $H(a) = \{x\}$ and $H(b) = \emptyset$ for all $b \in B - \{a\}$, then H is called a soft point over U relative to B and denoted by a_x. $SP(U, B)$ denotes the family of all soft points over U relative to B. If $H \in SS(U, B)$ and $a_x \in SP(U, B)$, then a_x is said to belong to H (notation: $a_x \widetilde{\in} H$) if $x \in H(a)$. If $\{H_\alpha : \alpha \in \Delta\} \subseteq SS(U, B)$, then the soft union and soft intersection of $\{H_\alpha : \alpha \in \Delta\}$ are denoted by $\widetilde{\cup}_{\alpha\in\Delta} H_\alpha$ and $\widetilde{\cap}_{\alpha\in\Delta} H_\alpha$, respectively, and are defined by

$\left(\widetilde{\cup}_{\alpha\in\Delta} H_\alpha\right)(a) = \cup_{\alpha\in\Delta} H_\alpha(a)$ and $\left(\widetilde{\cap}_{\alpha\in\Delta} H_\alpha\right)(a) = \cap_{\alpha\in\Delta} H_\alpha(a)$ for all $a \in B$.

The sequel will utilize the following definitions.

Definition 1.1. [18] A soft set $K \in SS(U, B)$ is called countable if $K(b)$ is a countable subset of U for each $b \in B$. $CSS(U, B)$ denotes the family of all countable soft sets from $SS(U, B)$.

Definition 1.2. [23] Let $U \neq \emptyset$ be a set and let $\aleph$ be a family of subsets of U. Then $\aleph$ is a supra-topology on U if

(1) $\{\emptyset, U\} \subseteq \aleph$.

(2) $\aleph$ is closed under an arbitrary union.

We say in this case that $(U, \aleph)$ is a supra-topological space (supra-TS, for short). Members of $\aleph$ are called supra-open sets in $(U, \aleph)$, and their complements are called supra-closed sets in $(U, \aleph)$. $\aleph^c$ denotes the family of all supra-closed sets in $(U, \aleph)$.

Definition 1.3. [23] Let $(U, \aleph)$ be a supra-TS and let $V \subseteq U$. The supra-closure of V in $(U, \aleph)$ is denoted by $Cl_\aleph(V)$ and defined by

$$Cl_\aleph(V) = \cap \{W : W \in \aleph^c \text{ and } V \subseteq W\}.$$

Definition 1.4. [30] A subcollection $\psi \subseteq SS(U,B)$ is called a supra-soft topology on U relative to B if

(1) $\{0_B, 1_B\} \subseteq \psi$.

(2) ψ is closed under arbitrary soft union.

We say in this case (U,ψ,B) is a supra-soft topological space (supra-STS, for short). Members of ψ are called supra-soft open sets in (U,ψ,B), and their soft complements are called supra-soft closed sets in (U,ψ,B). ψ^c will denote the family of all supra-soft closed sets in (U,ψ,B).

Definition 1.5. [30] Let (U,ψ,B) be a Supra-STS and let $K \in SS(U,B)$.

(a) The supra-soft closure of K in (U,ψ,B) is denoted by $Cl_\psi(K)$ and defined by

$$Cl_\psi(K) = \widetilde{\cap}\left\{H : H \in \psi^c \text{ and } K\widetilde{\subseteq}H\right\}.$$

(b) The supra-soft interior of K in (U,ψ,B) is denoted by $Int_\psi(K)$ and defined by

$$Int_\psi(K) = \widetilde{\cup}\left\{T : T \in \psi \text{ and } T\widetilde{\subseteq}K\right\}.$$

Theorem 1.6. [38] For each supra-STS (U,ψ,B) and each $b \in B$, the collection $\{H(b) : H \in \psi\}$ defines a supra-topology on U. This supra-soft topology is denoted by ψ_b.

Definition 1.7. [44] A supra-STS (U,ψ,B) is called a supra-soft compact (resp. supra-soft Lindelof) if for every $\mathcal{M} \subseteq \psi$ with $\widetilde{\cup}_{M\in\mathcal{M}}M = 1_B$, we find a finite (resp. countable) subcollection $\mathcal{M}_1 \subseteq \mathcal{M}$ with $\widetilde{\cup}_{K\in\mathcal{M}_1}M = 1_B$.

Definition 1.8. [45] Let $(U,\aleph)$ be a supra-TS and let $V \subseteq U$. Then, V is called a supra-ω-open in $(U,\aleph)$ if, for each $y \in V$, $S \in \aleph$ and a countable subset $N \subseteq U$ exist such that $y \in S - N \subseteq V$. The collection of all supra-ω-open sets in $(U,\aleph)$ is denoted by $\aleph_\omega$.

Definition 1.9. [46] A supra-STS $(U,\aleph)$ is called supra-Lindelof if, for every $\mathcal{M} \subseteq \aleph$ with $\widetilde{\cup}_{M\in\mathcal{M}}M = U$, we find a countable subcollection $\mathcal{M}_1 \subseteq \mathcal{M}$ with $\widetilde{\cup}_{K\in\mathcal{M}_1}M = U$.

Definition 1.10. A supra-STS (U,ψ,B) is called supra-soft countably compact if, for every countable subcollection $\mathcal{M} \subseteq \psi$ with $\widetilde{\cup}_{M\in\mathcal{M}}M = 1_B$, we find a finite subcollection $\mathcal{M}_1 \subseteq \mathcal{M}$ with $\widetilde{\cup}_{K\in\mathcal{M}_1}M = 1_B$.

Definition 1.11. [31] Let (U,ψ,B) be a supra-STS, $\emptyset \neq V \subseteq U$, and $\psi_V = \left\{G\widetilde{\cap}C_V : G \in \psi\right\}$. Then, (V,ψ_V,B) is called a supra-soft topological subspace of (U,ψ,B).

Definition 1.12. A supra-TS $(U,\aleph)$ is said to be

(1) [45] Supra-regular if, whenever $V \in \aleph^c$ and $y \in U - V$, we find $R, W \in \aleph$ with $y \in R$, $V \subseteq W$, and $R \cap W = \emptyset$;

(2) Supra-ω-regular if, whenever $V \in (\aleph_\omega)^c$ and $y \in U - V$, we find $R \in \aleph$ and $W \in \aleph_\omega$ with $y \in R$, $V \subseteq W$, and $R \cap W = \emptyset$;

(3) Supra-ω-locally indiscrete (supra-ω-L-I, for short) if $\aleph \subseteq (\aleph_\omega)^c$.

Definition 1.13. [36] A supra-STS (U,ψ,B) is called supra-soft regular if, whenever $L \in \psi^c$ and $b_y\widetilde{\in}1_B - L$, we find $G, H \in \psi$ with $b_y\widetilde{\in}G$, $L\widetilde{\subseteq}H$, and $G\widetilde{\cap}H = 0_B$.

Definition 1.14. [37] Let (U,ψ,B) and (V,ϕ,D) be two supra-STSs. Then the supra-soft topology on $U \times V$ relative to $B \times D$ that has $\psi \times \phi$ as a supra-soft base will be denoted by $pr(\psi \times \phi)$.

2. Generated supra-soft topologies via supra-topologies

Theorem 2.1. Let $\{(U,\psi_b) : b \in B\}$ be a family of supra-TSs, and let

$$\psi = \{H \in SS(U,B) : H(b) \in \psi_b \text{ for all } b \in B\}.$$

Then (U,ψ,B) is a supra-STS.
Proof: Since for every $b\in B$, $0_B(b)=\emptyset\in\psi_b$, and $1_B(b)=Y\in\psi_b$, therefore $\{0_B,1_B\}\subseteq\psi$. Let $\{H:H\in\mathcal{H}\}\subseteq\psi$. Then for all $b\in B$ and $H\in\mathcal{H}$, $H(b)\in\psi_b$ and $\cup_{H\in\mathcal{H}}H(b)\in\psi_b$. So, for each $b\in B$, $\left(\widetilde{\cup}_{H\in\mathcal{H}}H\right)(b)=\cup_{H\in\mathcal{H}}H(b)\in\psi_b$. Consequently, $\widetilde{\cup}_{H\in\mathcal{H}}H\in\psi$.
Definition 2.2. Let $\{(U,\psi_b):b\in B\}$ be a family of supra-TSs.

(a) The supra-soft topology $\{H\in SS(U,B):H(b)\in\psi_b \text{ for all } b\in B\}$ is indicated by $\otimes_{b\in B}\psi_b$.

(b) If $\psi_b=\aleph$ for all $b\in B$, then $\otimes_{b\in B}\psi_b$ is indicated by $\mu(\aleph)$.

Theorem 2.3. Let $\{(U,\psi_b):b\in B\}$ be a family of supra-TSs. Then, for each $a\in B$, $\{a_V:V\in\psi_a\}\subseteq\otimes_{b\in B}\psi_b$.
Proof: Let $a\in B$ and let $Z\in\psi_a$. We then have

$$(a_V)(b)=\begin{cases} V & \text{if} \quad b=a,\\ \emptyset & \text{if} \quad b\neq a.\end{cases}$$

Consequently, $(a_V)(b)\in\psi_b$ for all $b\in B$. Hence, $a_V\in\otimes_{b\in B}\psi_b$.
Theorem 2.4. Let $\{(U,\psi_b):b\in B\}$ be a family of supra-TSs and let $H\in SS(U,B)-\{0_B\}$. Then, $H\in\otimes_{b\in B}\psi_b$ if for each $b_y\widetilde{\in}H$, we find $V\in\psi_b$ with $y\in V$ and $b_V\widetilde{\subseteq}H$.
Proof: Necessity. Let $H\in\otimes_{b\in B}\psi_b$ and let $b_y\widetilde{\in}H$. Then, $y\in H(b)\in\psi_b$. Set $V=H(b)$. Thus, we have $V\in\psi_b$, $y\in Z$, and $b_V\widetilde{\subseteq}H$.
Sufficiency. Let $H\in SS(U,B)-\{0_B\}$ such that for each $b_y\widetilde{\in}H$, we find $V\in\psi_b$ with $y\in V$ and $b_V\widetilde{\subseteq}H$. Let $b\in B$. To show that $H(b)\in\psi_b$, let $y\in H(b)$. Then $b_y\widetilde{\in}H$ and, by assumption, we find $V\in\psi_b$ with $b_y\widetilde{\in}b_V\widetilde{\subseteq}H$. Moreover, by Theorem 2.3, $b_V\in\otimes_{b\in B}\psi_b$. Hence, $H\in\otimes_{b\in B}\psi_b$.
Theorem 2.5. Let $\{(U,\psi_b):b\in B\}$ be a family of supra-TSs. Then, $(\otimes_{b\in B}\psi_b)_a=\psi_a$ for all $a\in B$.
Proof: To demonstrate that $(\otimes_{b\in B}\psi_b)_a\subseteq\psi_b$, let $V\in(\otimes_{b\in B}\psi_b)_a$. We then find $H\in\otimes_{b\in B}\psi_b$ with $H(a)=V$. By the definition of $\otimes_{b\in B}\psi_b$, $H(a)\in\psi_b$, and thus $V\in\psi_b$. To demonstrate that $\psi_b\subseteq(\otimes_{b\in B}\psi_b)_a$, let $V\in\psi_b$, then, by Theorem 2.3, $a_V\in\otimes_{b\in B}\psi_b$, and so $a_V(a)=V\in(\otimes_{b\in B}\psi_b)_a$.
Corollary 2.6. If $(U,\aleph)$ is a supra-TS and B is any set of parameters, then $(\mu(\aleph))_b=\aleph$ for all $b\in B$.
Proof: For every $b\in B$, set $\psi_b=\aleph$. Then, $\mu(\aleph)=\otimes_{b\in B}\psi_b$, and, by Theorem 2.5, we get the result.
Theorem 2.7. If (U,ψ,B) is a supra-STS, then $\psi\subseteq\otimes_{b\in B}\psi_b$.
Proof: Let $H\in\psi$. Then, $H(b)\in\psi_b$ for all $b\in B$, and thus, $H\in\otimes_{b\in B}\psi_b$.

The equality in Theorem 2.7 is not often true.

Example 2.8. Let $U=\{1,2,3,4\}$, $B=\{s,t\}$, $H=s_{\{1,2\}}\widetilde{\cup}t_{\{3,4\}}$, and $\psi=\{0_B,1_B,H\}$. Then, $\psi_s=\{\emptyset,U,\{1,2\}\}$, $\psi_t=\{\emptyset,U,\{3,4\}\}$, and $\otimes_{b\in B}\psi_b=\{0_B,1_B,s_{\{1,2\}},t_{\{3,4\}},F\}$. Hence, $\psi\neq\otimes_{b\in B}\psi_b$.
Theorem 2.9. For any supra-STS (U,ψ,B) and any $a\in B$, $(\otimes_{b\in B}\psi_b)_a=\psi_b$.
Proof: The proof is derived from Theorem 2.5.
Theorem 2.10. Let $(U,\aleph)$ be a supra-TS, B be a set of parameters, and $\psi=\{C_V:V\in\aleph\}$. Then, (U,ψ,B) is a supra-STS.
Proof: Since $\emptyset$, $U\in\aleph$, then $0_B=C_\emptyset\in\psi$ and $0_B=C_U\in\psi$. Let $\{C_{V_i}:i\in I\}\subseteq\psi$ where $\{V_i:i\in I\}\subseteq\psi$. We then have $\cup_{i\in I}V_i\in\aleph$ and so $C_{\cup_{i\in I}V_i}\in\psi$. Moreover, it is not difficult to demonstrate that $\widetilde{\cup}_{i\in I}C_{V_i}=C_{\cup_{i\in I}V_i}$. Consequently, $\widetilde{\cup}_{i\in I}C_{V_i}\in\psi$.
Definition 2.11. For every supra-TS $(U,\aleph)$ and every collection of parameters B, the supra-soft topology $\{C_V:V\in\aleph\}$ is indicated by $C(\aleph)$.
Theorem 2.12. For every supra-TS $(U,\aleph)$, and each collection of parameters B, $(C(\aleph))_b=\aleph$ for all $b\in B$.

Proof: Obvious.

Theorem 2.13. Let (U, ψ, B) be a supra-STS with $\psi \subseteq \{C_V : Z \subseteq U\}$, and let $\aleph = \{V \subseteq U : C_V \in \psi\}$. Then $(U, \aleph)$ is a supra-TS.

Proof: Since $0_B = C_\emptyset \in \psi$, and $1_B = C_U \in \psi$, $\emptyset$, $U \in \aleph$. Let $\{V_i : i \in I\} \subseteq \aleph$. Then $\{C_{V_i} : i \in I\} \subseteq \psi$ and so $\widetilde{\cup}_{i\in I} C_{V_i} \in \psi$. Since $\widetilde{\cup}_{i\in I} C_{V_i} = C_{\cup_{i\in I} V_i}$, $\cup_{i\in I} V_i \in \aleph$.

Definition 2.14. Let (U, ψ, B) be a supra-STS with $\psi \subseteq \{C_V : V \subseteq U\}$. Then the supra-topology $\{V \subseteq U : C_V \in \psi\}$ is indicated by $D(\psi)$.

The following two results follow obviously:

Theorem 2.15. For any supra-STS (U, ψ, B) with $\psi \subseteq \{C_V : V \subseteq U\}$, $\psi_b = D(\psi)$ for all $b \in B$.

Theorem 2.16. For every supra-TS $(U, \aleph)$ and every collection of parameters B, $D(C(\aleph)) = \aleph$.

3. Supra-soft omega open sets

Definition 3.1. Let (U, ψ, B) be a supra-STS and let $H \in SS(U, B)$.

(a) A soft point $b_y \in SP(U, B)$ is a supra-soft condensation point of H in (U, ψ, B) if, for each $K \in \psi$ with $b_y \widetilde{\in} K$, $K \widetilde{\cap} H \notin CSS(U, B)$.

(b) The soft set $\widetilde{\cup}\left\{b_y \in SP(U, B) : b_y \text{ is a supra-soft condensation point of } H \text{ in } (U, \psi, B)\right\}$, which is indicated by $Cond(H)$.

(c) H is supra-soft ω-closed in (U, ψ, B) if $Cond(H) \widetilde{\subseteq} H$.

(d) H is supra-soft ω-open in (U, ψ, B) if $1_B - H$ is supra-soft ω-closed in (U, ψ, B).

(e) The collection of all supra-soft ω-open sets in (U, ψ, B) is indicated by ψ_ω.

Theorem 3.2. Let (U, ψ, B) be a supra-STS and let $H \in SS(U, B)$. Then, $H \in \psi_\omega$ iff for each $b_y \widetilde{\in} H$, we find $K \in \psi$ with $b_y \widetilde{\in} K$, and $K - H \in CSS(U, B)$.

Proof: Necessity. Let $H \in \psi_\omega$ and let $b_y \widetilde{\in} H$. Then, $1_B - H$ is soft ω-closed in (U, ψ, B) and $b_y \widetilde{\notin} 1_B - H$. Since $Cond(1_B - H) \widetilde{\subseteq} 1_B - H$, then $b_y \widetilde{\notin} Cond(1_B - H)$, and thus, we find $K \in \psi$ with $b_y \widetilde{\in} K$ and $K \widetilde{\cap} (1_B - H) \in CSS(U, B)$. Since $K \widetilde{\cap} (1_B - H) = K - H$, we are done.

Sufficiency. We show that $H \widetilde{\subseteq} 1_B - Cond(1_B - H)$. Let $b_y \widetilde{\in} H$. Then we find $K \in \psi$ with $b_y \widetilde{\in} K$ and $K - H \in CSS(U, B)$. Thus, we have $K \in \psi$, $b_y \widetilde{\in} K$, and $K \widetilde{\cap} (1_B - H) = K - H \in CSS(U, B)$. Hence, $b_y \widetilde{\in} 1_B - Cond(1_B - H)$.

Theorem 3.3. Let (U, ψ, B) be a supra-STS and let $H \in SS(U, B)$. Then, $H \in \psi_\omega$ iff for each $b_y \widetilde{\in} H$, we find $K \in \psi$ and $F \in CSS(U, B)$ with $b_y \widetilde{\in} K - F \widetilde{\subseteq} H$.

Proof: Necessity. Let $H \in \psi_\omega$ and let $b_y \widetilde{\in} H$. By Theorem 3.2, we find $K \in \psi$ with $b_y \widetilde{\in} K$ and $K - H \in CSS(U, B)$. Set $F = K - H$. We then have $F \in CSS(U, B)$ with $b_y \widetilde{\in} K - F = K - (K - H) = H \widetilde{\subseteq} H$.

Sufficiency. Let $b_y \widetilde{\in} H$. Then, by assumption, we find $K \in \psi$ and $F \in CSS(U, B)$ with $b_y \widetilde{\in} K - F \widetilde{\subseteq} H$. Since $K - H \widetilde{\subseteq} F$, $K - H \in CSS(U, B)$. Consequently, by Theorem 3.2, $H \in \psi_\omega$.

Theorem 3.4. For any supra-STS (U, ψ, B), $\psi \subseteq \psi_\omega$.

Proof: Let $H \in \psi$ and let $b_y \widetilde{\in} H$. Set $K = H$ and $F = 0_B$. We then have $K \in \psi$ and $F \in CSS(U, B)$ with $b_y \widetilde{\in} K - F = K \widetilde{\subseteq} K = H$. Consequently, by Theorem 3.2, $H \in \psi_\omega$.

Theorem 3.5. For any supra-STS (U, ψ, B), (U, ψ_ω, B) is a supra-STS.

Proof: Since (U, ψ, B) is a supra-STS, $\{0_B, 1_B\} \subseteq \psi$. So, by Theorem 3.4, $\{0_B, 1_B\} \subseteq \psi_\omega$. Let $\mathcal{H} \subseteq \psi_\omega$ and let $b_y \widetilde{\in} \widetilde{\cup}_{H\in\mathcal{H}} H$. Choose $H_\circ \in \mathcal{H}$ with $b_y \widetilde{\in} H_\circ$. Since $H_\circ \in \psi_\omega$, by Theorem 3.3, we find $K \in \psi$ and $F \in CSS(U, B)$ with $b_y \widetilde{\in} K - F \widetilde{\subseteq} H_\circ \widetilde{\subseteq} \widetilde{\cup}_{H\in\mathcal{H}} H$. Again, by Theorem 3.3, $\widetilde{\cup}_{H\in\mathcal{H}} H \in \psi_\omega$.

The example that follows demonstrates that equality in general cannot take the place of inclusion in

Theorem 3.4.

Example 3.6. Let $Y = \mathbb{R}$, $B = \mathbb{N}$, and $\psi = \{0_B, 1_B, C_{(-\infty,0]}, C_{[0,\infty)}\}$. Then (U, ψ, B) is a supra-STS and $C_{(0,\infty)} \in \psi_\omega - \psi$.

Theorem 3.7. Let (U, ψ, B) be a supra-STS. Then, $\psi = \psi_\omega$ iff $\{K - F : K \in \psi$ and $F \in CSS(U, B)\} \subseteq \psi$.

Proof: Necessity. Let $\psi = \psi_\omega$. Then, by Theorem 3.3, $\{K - F : K \in \psi$ and $F \in CSS(U, B)\} \subseteq \psi_\omega = \psi$.
Sufficiency. Let $\{K - F : K \in \psi$ and $F \in CSS(U, B)\} \subseteq \psi$. By Theorem 3.4, it is enough to demonstrate that $\psi_\omega \subseteq \psi$. Let $H \in \psi_\omega - \{0_B\}$ and let $b_y \widetilde{\in} H$. Then, by Theorem 3.3, we find $K \in \psi$ and $F \in CSS(U, B)$ with $b_y \widetilde{\in} K - F \widetilde{\subseteq} H$. Since $\{K - F : K \in \psi$ and $F \in CSS(U, B)\} \subseteq \psi$, $K - F \in \psi$. Consequently, $H \in \psi$.

Theorem 3.8. Let (U, ψ, B) be a supra-STS. Then for all $b \in B$, $(\psi_b)_\omega = (\psi_\omega)_b$.

Proof: Let $b \in B$. To demonstrate that $(\psi_b)_\omega \subseteq (\psi_\omega)_b$, let $S \in (\psi_b)_\omega$ and let $y \in S$. We then find $M \in \psi_b$ and a countable set $N \subseteq U$ with $y \in M - N \subseteq S$. Since $M \in \psi_b$, we find $G \in \psi$ with $G(b) = M$. Since $b_N \in CSS(U, B)$, $G - b_N \in \psi_\omega$ and $(G - b_N)(b) = G(b) - N = M - N \in (\psi_\omega)_b$. Consequently, $S \in (\psi_\omega)_b$. To demonstrate that $(\psi_\omega)_b \subseteq (\psi_b)_\omega$, let $S \in (\psi_\omega)_b$ and let $y \in S$. Choose $H \in \psi_\omega$ with $H(b) = S$. Since $b_y \widetilde{\in} H \in \psi_\omega$, by Theorem 3.3, we find $K \in \psi$ and $F \in CSS(U, B)$ with $b_y \widetilde{\in} K - F \widetilde{\subseteq} H$. Consequently, we have $K(b) \in \psi_b$, $F(b)$ is a countable subset of U, and $y \in K(b) - F(b) \subseteq H(b) = S$. Hence, $S \in (\psi_b)_\omega$.

Corollary 3.9. Let (U, ψ, B) be a supra-STS. If $H \in \psi_\omega$, then for each $b \in B$, $H(b) \in (\psi_b)_\omega$.

*Proof:*Let $H \in \psi_\omega$ and let $b \in B$. Then, $H(b) \in (\psi_\omega)_b$, and, by Theorem 3.8, $H(b) \in (\psi_b)_\omega$.

Theorem 3.10. For any family of supra-TSs $\{(U, \psi_b) : b \in B\}$, $(\otimes_{b \in B} \psi_b)_\omega = \otimes_{b \in B} (\psi_b)_\omega$.

Proof: Let $H \in (\otimes_{b \in B} \psi_b)_\omega$. To demonstrate that $H \in \otimes_{b \in B} (\psi_b)_\omega$, we show that $H(a) \in (\psi_a)_\omega$ for all $a \in B$. Let $a \in B$ and let $y \in H(a)$. We then have $a_y \widetilde{\in} H \in (\otimes_{b \in B} \psi_b)_\omega$ and, by Theorem 3.3, we find $K \in \otimes_{b \in B} \psi_b$ and $F \in CSS(U, B)$ with $a_y \widetilde{\in} K - F \widetilde{\subseteq} H$. Consequently, we have $K(a) \in \psi_a$, $F(a)$ is a countable subset of U, and $y \in K(a) - F(a) \subseteq H(a)$. This implies that $H(a) \in (\psi_a)_\omega$. Conversely, let $H \in \otimes_{b \in B} (\psi_b)_\omega$. To demonstrate that $H \in (\otimes_{b \in B} \psi_b)_\omega$, let $a_y \widetilde{\in} H$. Then, $y \in H(a)$. Since $H \in \otimes_{b \in B} (\psi_b)_\omega$, $H(a) \in (\psi_a)_\omega$. Since $y \in H(a) \in (\psi_a)_\omega$, we find $M \in \psi_a$ and a countable set $N \subseteq U$ with $y \in M - N \subseteq H(a)$. Thus, we have $a_M \in \otimes_{b \in B} \psi_b$, $a_N \in CSS(U, B)$, and $a_y \widetilde{\in} a_M - a_N \widetilde{\subseteq} H$. Consequently, by Theorem 3.3, $H \in (\otimes_{b \in B} \psi_b)_\omega$.

Corollary 3.11. For every supra-TS $(U, \aleph)$ and every collection of parameters B, $(\mu(\aleph))_\omega = \mu(\aleph_\omega)$ for every $b \in B$.

Proof: For each $b \in B$, set $\psi_b = \aleph$. Then $\mu(\aleph) = \otimes_{b \in B} \psi_b$ and, by Theorem 3.10,

$$\begin{aligned} (\mu(\aleph))_\omega &= (\otimes_{b \in B} \psi_b)_\omega \\ &= \otimes_{b \in B} (\psi_b)_\omega \\ &= \mu(\aleph_\omega). \end{aligned}$$

Definition 3.12. A supra-STS (U, ψ, B) is called supra-soft locally countable (supra-soft L-C, for short) if, for each $b_y \in SP(U, B)$, we find $K \in \psi \cap CSS(U, B)$ with $b_y \widetilde{\in} K$.

Theorem 3.13. If (U, ψ, B) is supra-soft L-C, then $\psi_\omega = SS(U, B)$.

Proof: It is sufficient to show that $SP(U, B) \subseteq \psi_\omega$. Let $b_y \in SP(U, B)$. Since (U, ψ, B) is supra-soft L-C, then we find $K \in \psi \cap CSS(U, B)$ with $b_y \widetilde{\in} K$. Since $K \in CSS(U, B)$, then $K - b_y \widetilde{\in} CSS(U, B)$. Thus, by Theorem 3.3, $K - \left(K - b_y\right) = b_y \in \psi_\omega$.

Corollary 3.14. If (U, ψ, B) is a supra-STS with U being countable, then $\psi_\omega = SS(U, B)$.

Theorem 3.15. Let (U, ψ, B) be a supra-STS. Then (U, ψ_ω, B) is supra-soft countably compact iff $SP(U, B)$ is finite.

Proof: Necessity. Let (U, ψ_ω, B) be supra-soft countably compact and assume, however, that $SP(U, B)$

is infinite. Choose a denumerable subset $\{a_n : n \in \mathbb{N}\} \subseteq SP(U,B)$ with $a_i \neq a_j$ when $i \neq j$. For each $n \in \mathbb{N}$, set $H_n = 1_B - \widetilde{\cup}_{k\geq n} a_k$. We then have $\widetilde{\cup}_{n\in\mathbb{N}} H_n = 1_B$ and $\{H_n : n \in \mathbb{N}\} \subseteq \psi_\omega$. Since (U,ψ_ω,B) is supra-soft countably compact, we find $\{H_{n_1}, H_{n_2}, ..., H_{n_k}\} \subseteq \{H_n : n \in \mathbb{N}\}$ with $n_1 < n_2 < ... < n_k$ and $\widetilde{\cup}_{i\in\{n_1,n_2,...,n_k\}} H_i = H_{n_k} = 1_B$, which is a contradiction.
Sufficiency. Suppose that $SP(U,B)$ is finite. Then $SS(U,B)$ is finite. Thus, (U,ψ_ω,B) is supra-soft compact, and hence (U,ψ_ω,B) is supra-soft countably compact.
Corollary 3.16. Let (U,ψ,B) be a supra-STS. Then (U,ψ_ω,B) is supra-soft compact iff $SP(U,B)$ is finite.
Lemma 3.17. Let (U,ψ,B) be a supra-STS, and let $\mathcal{K}$ be a supra-soft base of (U,ψ,B). Then, (U,ψ,B) is supra-soft Lindelof iff for every $\mathcal{K}_1 \subseteq \mathcal{K}$ with $\widetilde{\cup}_{K\in\mathcal{K}_1} K = 1_B$, we find a countable subcollection $\mathcal{K}_2 \subseteq \mathcal{K}_1$ with $\widetilde{\cup}_{K\in\mathcal{K}_2} K = 1_B$.
Proof: Necessity. Let (U,ψ,B) be supra-soft Lindelof. Let $\mathcal{K}_1 \subseteq \mathcal{K}$ with $\widetilde{\cup}_{K\in\mathcal{K}_1} K = 1_B$. Then, $\mathcal{K}_1 \subseteq \psi$ with $\widetilde{\cup}_{K\in\mathcal{K}_1} K = 1_B$, and so, we find a countable subcollection $\mathcal{K}_2 \subseteq \mathcal{K}_1$ with $\widetilde{\cup}_{K\in\mathcal{K}_2} K = 1_B$.
Sufficiency. Let $\mathcal{H} \subseteq \psi$ with $\widetilde{\cup}_{H\in\mathcal{H}} H = 1_B$. For each $b_y \in SP(U,B)$, choose $H_{b_y} \in \mathcal{H}$ with $b_y \widetilde{\in} H_{b_y}$. Since $\mathcal{K}$ is a supra-soft base of (U,ψ,B), for each $b_y \in SP(U,B)$, we find $K_{b_y} \in \mathcal{K}$ with $b_y \widetilde{\in} K_{b_y} \widetilde{\subseteq} H_{b_y}$. Let $\mathcal{K}_1 = \left\{K_{b_y} : b_y \in SP(U,B)\right\}$. We then have $\mathcal{K}_1 \subseteq \mathcal{K}$ with $\widetilde{\cup}_{K\in\mathcal{K}_1} K = 1_B$, and, by assumption, we find a countable subcollection $\mathcal{K}_2 \subseteq \mathcal{K}_1$ with $\widetilde{\cup}_{K\in\mathcal{K}_2} K = 1_B$. Choose a countable subset $\gamma \subseteq SP(U,B)$ with $\mathcal{K}_2 = \left\{K_{b_y} : b_y \in \gamma\right\}$. Let $\mathcal{H}_1 = \left\{H_{b_y} : b_y \in \gamma\right\}$. Then, $\mathcal{H}_1$ is a countable subcollection of $\mathcal{H}$ with $\widetilde{\cup}_{K\in\mathcal{H}_1} H = 1_B$. Consequently, (U,ψ,B) is supra-soft Lindelof.
Theorem 3.18. Let (U,ψ,B) be a supra-STS with B being countable. Then (U,ψ,B) is supra-soft Lindelof iff (U,ψ_ω,B) is supra-soft Lindelof.
Proof: Necessity. Let (U,ψ,B) be supra-soft Lindelof. Set $\mathcal{R} = \{K - F : K \in \psi \text{ and } F \in CSS(U,B)\}$. Then, by Theorem 3.3, $\mathcal{R}$ is a supra-soft base of (U,ψ_ω,B). We apply Lemma 3.17. Let $\mathcal{R}_1 \subseteq \mathcal{R}$ with $\widetilde{\cup}_{R\in\mathcal{R}_1} R = 1_B$, say $\mathcal{R}_1 = \left\{K_j - F_j : \text{where } K_j \in \psi \text{ and } F_j \in CSS(U,B) : j \in J\right\}$. Since $\widetilde{\cup}_{j\in J} K_j = 1_B$ and (U,ψ,B) is supra-soft Lindelof, then there is a countable subset $J_1 \subseteq J$ with $\widetilde{\cup}_{j\in J_1} K_j = 1_B$. Set $F = \widetilde{\cup}_{j\in J_1} F_j$. Then $F \in CSS(U,B)$. For each $b_y \widetilde{\in} F$, choose $j_{b_y} \in J$ with $b_y \widetilde{\in} K_{j_{b_y}} - F_{j_{b_y}}$. Let

$$\mathcal{R}_2 = \left\{K_j - F_j : j \in J_1\right\} \cup \left\{K_{j_{b_y}} - F_{j_{b_y}} : b_y \widetilde{\in} F\right\}.$$

Then, $\mathcal{R}_2 \subseteq \mathcal{R}_1$, $\mathcal{R}_2$ is countable, and $\widetilde{\cup}_{R\in\mathcal{R}_2} R = 1_B$.
Sufficiency. Let (U,ψ_ω,B) be supra-soft Lindelof. By Theorem 3.4, $\psi \subseteq \psi_\omega$. Thus, (U,ψ,B) is supra-soft Lindelof.
Theorem 3.19. Let (U,ψ,B) be a supra-STS and let $\emptyset \neq V \subseteq U$. Then, $(\psi_V)_\omega = (\psi_\omega)_V$.
Proof: To show that $(\psi_V)_\omega \subseteq (\psi_\omega)_V$, let $S \in (\psi_V)_\omega$ and let $b_y \widetilde{\in} S$. By Theorem 3.3, we find $M \in \psi_V$ and $L \in CSS(V,B)$ with $b_y \widetilde{\in} M - L \widetilde{\subseteq} S$. Choose $K \in \psi$ with $M = K \widetilde{\cap} C_V$. Then, $K - L \in \psi_\omega$, $b_y \widetilde{\in} K - L$, and $(K - L) \widetilde{\cap} C_V = M - L \widetilde{\subseteq} S$. Consequently, $S \in (\psi_\omega)_V$. Conversely, to show that $(\psi_\omega)_V \subseteq (\psi_V)_\omega$, let $S \in (\psi_\omega)_V$ and let $b_y \widetilde{\in} S$. Choose $H \in \psi_\omega$ with $S = H \cap C_V$. Since $b_y \widetilde{\in} H \in \psi_\omega$, by Theorem 3.3, we find $K \in \psi$ and $F \in CSS(U,B)$ with $b_y \widetilde{\in} K - F \widetilde{\subseteq} H$. Set $T = K \widetilde{\cap} C_V$. We then have $T \in \psi_V$, $F \widetilde{\cap} C_V \in CSS(V,B)$, and $b_y \widetilde{\in} T - \left(F \widetilde{\cap} C_V\right) \widetilde{\subseteq} S$. Again, by Theorem 3.3, $S \in (\psi_V)_\omega$.
Theorem 3.20. Let $\{(U,\psi_b) : b \in B\}$ be a family of supra-TSs. Then $(U, \otimes_{b\in B}\psi_b, B)$ is supra-soft Lindelof iff B is countable and (U,ψ_b) is supra-Lindelof for all $b \in B$.
Proof: Necessity. Let $(U, \otimes_{b\in B}\psi_b, B)$ be supra-soft Lindelof. Since $\{b_U : b \in B\} \subseteq \otimes_{b\in B}\psi_b$ with $\widetilde{\cup}_{b\in B} b_U = 1_B$, we find a countable subset $B_1 \subseteq B$ with $\widetilde{\cup}_{b\in B_1} b_U = 1_B$. We must have $B_1 = B$, and

hence B is countable. Let $a \in B$. To show that (U, ψ_b) is supra-Lindelof, let $\mathcal{Y} \subseteq \psi_a$ with $\cup_{Y \in \mathcal{Y}} Y = U$. Let $\mathcal{K} = \{a_Y : Y \in \mathcal{Y}\} \cup \{b_U : b \in B - \{a\}\}$. Then, $\mathcal{K} \subseteq \otimes_{b \in B} \psi_b$ and $\widetilde{\cup}_{K \in \mathcal{K}} K = 1_B$. Since $(U, \otimes_{b \in B} \psi_b, B)$ is supra-soft Lindelof, we find a countable subcollection $\mathcal{K}_1 \subseteq \mathcal{K}$ with $\widetilde{\cup}_{K \in \mathcal{K}_1} K = 1_B$. Consequently, we find a countable subcollection $\mathcal{Y}_1 \subseteq \mathcal{Y}$ with $\mathcal{K}_1 = \{a_Y : Y \in \mathcal{Y}_1\} \cup \{b_U : b \in B - \{a\}\}$. Moreover, we must have $\cup_{Y \in \mathcal{Y}_1} Y = U$. This shows that (U, ψ_b) is supra-Lindelof.

Sufficiency. Let B be countable, and (U, ψ_b) be supra-Lindelof for all $b \in B$. Let $\mathcal{H} = \{b_V : b \in B \text{ and } V \in \psi_b\}$. By Theorem 2.4, $\mathcal{H}$ is a supra-soft base of $(U, \otimes_{b \in B} \psi_b, B)$. We apply Lemma 3.17. Let $\mathcal{T} \subseteq \mathcal{H}$ with $\widetilde{\cup}_{T \in \mathcal{T}} T = 1_B$. For each $b \in B$, let $\mathcal{T}_b = \{V \subseteq U : b_V \in \mathcal{T}\}$. For each $b \in B$, we have $\mathcal{T}_b \subseteq \psi_b$ with $\cup_{Y \in \mathcal{T}_b} Y = U$, and, we find a countable subcollection $\mathcal{L}_b \subseteq \mathcal{T}_b$ with $\cup_{Y \in \mathcal{L}_b} Y = U$. Let $\mathcal{T}_1 = \{b_V : b \in B \text{ and } V \in \mathcal{L}_b\}$. Since B is countable, $\mathcal{T}_1$ is countable. Consequently, we have a $\mathcal{T}_1$ that is a countable subcollection of $\mathcal{T}$ with $\widetilde{\cup}_{T \in \mathcal{T}_1} T = 1_B$. It follows that $(U, \otimes_{b \in B} \psi_b, B)$ is supra-soft Lindelof.

Definition 3.21. A supra-STS (U, ψ, B) is called supra-soft anti locally countable (supra-soft A-L-C, for short) if for any $G, H \in \psi$, either $G \widetilde{\cap} H = 0_B$ or $G \widetilde{\cap} H \notin CSS\,(U, B)$.

Theorem 3.22. A supra-STS (U, ψ, B) is supra-soft A-L-C iff (U, ψ_ω, B) is supra-soft A-L-C.

Proof: Necessity. Let (U, ψ, B) be supra-soft A-L-C. Assume, however, we have $G, H \in \psi_\omega$ with $G \widetilde{\cap} H \in CSS(U, B) - \{0_B\}$. Choose $b_y \widetilde{\in} G \widetilde{\cap} H$. By Theorem 3.3, we find $M, N \in \psi$ and $F, L \in CSS(U, B)$ with $b_y \widetilde{\in} M - F \widetilde{\subseteq} G$ and $b_y \widetilde{\in} N - L \widetilde{\subseteq} H$. Consequently, $M \widetilde{\cap} N \widetilde{\subseteq} \left(G \widetilde{\cap} H\right) \widetilde{\cup} \left(F \widetilde{\cup} L\right)$. This implies that $M \widetilde{\cap} N \in CSS(U, B) - \{0_B\}$. Consequently, (U, ψ, B) is not supra-soft A-L-C, which is a contradiction.

Sufficiency. Obvious.

Theorem 3.23. Let (U, ψ, B) be supra-soft A-L-C. Then for all $H \in \psi_\omega$, $Cl_\psi(H) = Cl_{\psi_\omega}(H)$.

Proof: Let (U, ψ, B) be supra-soft A-L-C and let $H \in \psi_\omega$. Since, by Theorem 3.4, $\psi \subseteq \psi_\omega$, $Cl_{\psi_\omega}(H) \widetilde{\subseteq} Cl_\psi(H)$. To demonstrate that $Cl_\psi(H) \widetilde{\subseteq} Cl_{\psi_\omega}(H)$, let $b_y \widetilde{\in} Cl_\psi(H)$, and let $K \in \psi_\omega$ with $b_y \widetilde{\in} K$. By Theorem 3.3, we find $M \in \psi$ and $F \in CSS(U, B)$ with $b_y \widetilde{\in} M - F \widetilde{\subseteq} K$. Since $b_y \widetilde{\in} M \in \psi$ and $b_y \widetilde{\in} Cl_\psi(H)$, $M \widetilde{\cap} H \neq 0_B$. Choose $a_x \widetilde{\in} M \widetilde{\cap} H$. Since $H \in \psi_\omega$, by Theorem 3.3, we find $N \in \psi$ and $L \in CSS(U, B)$ with $a_x \widetilde{\in} N - L \widetilde{\subseteq} H$. Since $a_x \widetilde{\in} M \widetilde{\cap} N$ and (U, ψ, B) is supra-soft A-L-C, $M \widetilde{\cap} N \notin CSS\,(U, B)$. Thus, $(M - F) \widetilde{\cap} (N - L) \neq 0_B$, and hence, $K \widetilde{\cap} H \neq 0_B$. Consequently, $b_y \widetilde{\in} Cl_{\psi_\omega}(H)$.

The following example demonstrates that Theorem 3.23 is no longer true when the assumption of being "supra-soft A-L-C" is removed.

Example 3.24. Let $U = \mathbb{Z}$, $B = \{a, b\}$, and $\psi = \{0_B, 1_B, C_\mathbb{N}\}$. Then, $C_\mathbb{N} \in \psi \subseteq \psi_\omega$. We have $Cl_\psi(C_\mathbb{N}) = 1_B$, but $Cl_{\psi_\omega}(C_\mathbb{N}) = C_\mathbb{N} \neq 1_B$.

In Theorem 3.23, the assumption "$H \in \psi_\omega$" cannot be eliminated.

Example 3.25. Let $\aleph$ be the usual topology on $\mathbb{R}$. Consider $(\mathbb{R}, \mu(\aleph), \mathbb{N})$. Let $H \in SS\,(\mathbb{R}, \mathbb{N})$ be defined by $H(b) = \mathbb{Q} - \{b\}$ for all $b \in \mathbb{N}$. Since $H \in CSS\,(\mathbb{R}, \mathbb{N})$, $Cl_{(\mu(\aleph))_\omega}(H) = H$. Moreover, $Cl_{\mu(\aleph)}(H) = 1_B$.

Theorem 3.26. Let (U, ψ, B) be supra-soft A-L-C. Then for all $H \in (\psi_\omega)^c$, $Int_\psi(H) = Int_{\psi_\omega}(H)$.

Proof: Let (U, ψ, B) be supra-soft A-L-C and let $H \in (\psi_\omega)^c$. Then, $1_B - H \in \psi_\omega$ and, by Theorem 3.23, $Cl_\psi(1_B - H) = Cl_{\psi_\omega}(1_B - H)$. Thus,

$$
\begin{aligned}
Int_\psi(H) &= 1_B - Cl_\psi(1_B - H) \\
&= 1_B - Cl_{\psi_\omega}(1_B - H) \\
&= Int_{\psi_\omega}(H).
\end{aligned}
$$

Theorem 3.27. Let (U, ψ, B) be a supra-soft Lindelof space. If $V \subseteq U$ with $C_V \in (\psi_\omega)^c - \{0_B\}$, then (V, ψ_V, B) is supra-soft Lindelof.

Proof: Let (U,ψ,B) be a supra-soft Lindelof space and let $V \subseteq U$ with $C_V \in (\psi_\omega)^c - \{0_B\}$. By Theorem 3.18, (U,ψ_ω,B) is supra-soft Lindelof. Since $C_V \in (\psi_\omega)^c$, by Theorem 3.6 of [45], $(V,(\psi_\omega)_V,B)$ is supra-soft Lindelof. By Theorem 3.19, $(V,(\psi_V)_\omega,B)$ is supra-soft Lindelof. Again, by Theorem 3.18, we must have (V,ψ_V,B) is supra-soft Lindelof.

4. Supra-soft ω-local indiscreetness

Theorem 4.1. If (U,ψ,B) is a supra-STS with $\psi \subseteq \psi^c$, then, (U,ψ,B) is a soft topological space.
Proof: Let $G,H \in \psi$. Then, $G,H \in \psi^c$ and $1_B - G, 1_B - H \in \psi$. Therefore, $1_B - \left(G\widetilde{\cap}H\right) = (1_B - G)\widetilde{\cup}(1_B - H) \in \psi$. Thus, $1_B - \left(G\widetilde{\cap}H\right) \in \psi^c$. Hence, $G\widetilde{\cap}H \in \psi$.
Definition 4.2. A supra-STS (U,ψ,B) is said to be
(a) Supra-soft locally indiscrete (supra-soft L-I, for short) if $\psi \subseteq \psi^c$;
(b) Supra-soft ω-locally indiscrete (supra-soft ω-L-I, for short) if $\psi \subseteq (\psi_\omega)^c$.
Theorem 4.3. A supra-STS (U,ψ,B) is supra-soft L-I iff (U,ψ,B) is supra-soft L-I as a soft topological space.
Proof: This follows from Theorem 4.1.
Theorem 4.4. Supra-soft L-C supra-STSs are supra-soft ω-L-I.
Proof: Let (U,ψ,B) be a supra-soft L-C. Then, by Theorem 3.13, $\psi_\omega = SS(U,B)$. Thus, $\psi \subseteq \psi_\omega = (\psi_\omega)^c = SS(U,B)$, and hence (U,ψ,B) is supra-soft ω-L-I.

Theorem 4.4's implication is not reversible in general.

Example 4.5. Let $U = \mathbb{R}$, $B = \{a,b\}$, and $\psi = \{0_B, 1_B, C_{\mathbb{N}\cup\{-1\}}, C_{\mathbb{Z}-\mathbb{N}}, C_{\mathbb{Z}}\}$. Consider the supra-STS (U,ψ,B). Since $\{C_{\mathbb{N}\cup\{-1\}}, C_{\mathbb{Z}-\mathbb{N}}, C_{\mathbb{Z}}\} \subseteq CSS(U,B)$, then $\{C_{\mathbb{N}\cup\{-1\}}, C_{\mathbb{Z}-\mathbb{N}}, C_{\mathbb{Z}}\} \subseteq (\psi_\omega)^c$. Consequently, we have $\psi \subseteq (\psi_\omega)^c$, and hence, (U,ψ,B) is supra-soft ω-L-I. Moreover, it is clear that (U,ψ,B) is not supra-soft L-C.
Theorem 4.6. Every supra-soft L-I supra-STS is supra-soft ω-L-I.
Proof: Let (U,ψ,B) be supra-soft L-I, and thus $\psi \subseteq \psi^c$. Since $\psi \subseteq \psi_\omega$, $\psi^c \subseteq (\psi_\omega)^c$. Consequently, $\psi \subseteq (\psi_\omega)^c$. Hence, (U,ψ,B) is supra-soft ω-L-I.

Theorem 4.6's implication is not reversible in general.

Example 4.7. Let $U = \mathbb{Q}$, $B = \mathbb{N}$, and $\psi = \{0_B, 1_B, C_{\mathbb{N}\cup\{-1\}}, C_{\mathbb{Z}-\mathbb{N}}, C_{\mathbb{Z}}\}$. Consider the supra-STS (U,ψ,B). Then, (U,ψ,B) is supra-soft L-C, and, by Theorem 3.13, $\psi_\omega = SS(U,B)$. Thus, $(\psi_\omega)^c = \psi_\omega = SS(U,B)$, and hence (U,ψ,B) is supra-soft ω-L-I. Moreover, since $C_{\mathbb{Z}-\mathbb{N}} \in \psi - \psi^c$, (U,ψ,B) is not supra-soft L-I.
Theorem 4.8. If (U,ψ,B) is supra-soft A-L-C, and supra-soft ω-L-I, then (U,ψ,B) is supra-soft L-I.
Proof: Let $H \in \psi$. Since (U,ψ,B) is supra-soft ω-L-I, then $H \in (\psi_\omega)^c$, and hence, $Cl_{\psi_\omega}(H) = H$. Since (U,ψ,B) is supra-soft A-L-C, then, by Theorem 3.23, $Cl_\psi(H) = Cl_{\psi_\omega}(H)$. Thus, $Cl_\psi(H) = H$, and hence, $H \in \psi^c$. Consequently, (U,ψ,B) is supra-soft L-I.

Example 4.7 is an example of a supra-soft L-C supra-STS that is not supra-soft L-I. An example of a supra-soft L-I supra-STS that is not supra-soft L-C is as follows:

Example 4.9. Let $U = [0,1] \cup [2,3]$, $B = \{a,b\}$, and $\psi = \{0_B, 1_B, C_{[0,1]}, C_{[2,3]}\}$. Consider the supra-STS (U,ψ,B). Then, (U,ψ,B) is not supra-soft L-C. Since $\psi = \psi^c$, (U,ψ,B) is supra-soft L-I.
Theorem 4.10. If (U,ψ,B) is supra-soft ω-L-I, then (U,ψ_b) is supra ω-L-I for all $b \in B$.
Proof: Since (U,ψ,B) is supra-soft ω-L-I, then $\psi \subseteq (\psi_\omega)^c$. Let $V \in \psi_b$. Choose $K \in \psi$ with $K(b) = V$. Therefore, we have $K \in (\psi_\omega)^c$, and hence, $V = K(b) \in ((\psi_\omega)_b)^c$. But, by Theorem 3.8, $(\psi_\omega)_b = (\psi_b)_\omega$.

Then, $V \in ((\psi_b)_\omega)^c$. This proves that $\psi_b \subseteq ((\psi_b)_\omega)^c$, and hence, (U, ψ_b) is supra ω-L-I.

Theorem 4.11. Let $\{(U, \psi_b) : b \in B\}$ be a family of supra-TSs. Then $(U, \otimes_{b\in B}\psi_b, B)$ is supra-soft ω-L-I iff (U, ψ_b) is supra ω-L-I for all $b \in B$.

Proof: Necessity. Let $(U, \otimes_{b\in B}\psi_b, B)$ be supra-soft ω-L-I. Then, by Theorem 4.10, $(U, (\otimes_{b\in B}\psi_b)_b)$ is supra ω-L-I for all $b \in B$. But, by Theorem 2.5, $(\otimes_{b\in B}\psi_b)_b = \psi_b$ for all $b \in B$. This ends the proof.

Sufficiency. Let (U, ψ_b) be supra ω-L-I for all $b \in B$. Let $K \in \otimes_{b\in B}\psi_b$. Then, $K(b) \in \psi_b$ for all $b \in B$. Since (U, ψ_b) is supra ω-L-I for all $b \in B$, $K(b) \in ((\psi_b)_\omega)^c$ for all $b \in B$. Therefore, $K \in (\otimes_{b\in B}(\psi_b)_\omega)^c$. Now, by Theorem 3.10, $K \in ((\otimes_{b\in B}\psi_b)_\omega)^c$. It follows that $(U, \otimes_{b\in B}\psi_b, B)$ is supra-soft ω-L-I.

Corollary 4.12. Let $(U, \aleph)$ be a supra-TS and B be a set of parameters. Then, $(U, \mu(\aleph), B)$ is supra-soft ω-L-I iff $(U, \aleph)$ is supra ω-L-I.

Proof: For every $b \in B$, set $\aleph_b = \aleph$. Then, $\mu(\aleph) = \otimes_{b\in B}\psi_b$. Theorem 4.11 ends the proof.

The example that follows demonstrates that, generally, the conclusion in Theorem 4.10 is not true in reverse.

Example 4.13. Let $U = \mathbb{R}$ and $B = \{s, t\}$. Let

$T = \{(s, (-\infty, 0)), (t, (-\infty, 1))\}$,
$S = \{(s, [0, 1)), (t, (-\infty, 1))\}$,
$W = \{(s, [1, 2)), (t, [1, \infty))\}$,
$L = \{(s, [2, \infty)), (t, [1, \infty))\}$,
$N = \{(s, \emptyset), (t, (-\infty, 1))\}$,
$M = \{(s, \emptyset), (t, [1, \infty))\}$.

Consider the supra-STS (U, ψ, B), where ψ is the supra-soft topology having $\{T, S, W, L, N, M\}$ as a supra-soft base. Then, ψ_a is the supra-topology on U having $\{(-\infty, 0), [0, 1), [1, 2), [2, \infty)\}$ as a supra base, and ψ_b is the supra-topology on U having $\{(-\infty, 1), [1, \infty)\}$ as a supra base. Hence, (U, ψ_a) and (U, ψ_b) are both supra L-I. Since (U, ψ, B) is supra-soft A-L-C and $1_B - T \notin \psi$, by Theorem 4.8, $1_B - T \notin \psi_\omega$. This implies that (U, ψ, B) is not supra-soft ω-L-I.

5. Supra-soft ω-regularity

Definition 5.1. A supra-STS (U, ψ, B) is called supra-soft ω-regular (supra-soft ω-r, for short) if whenever $L \in \psi^c$ and $b_y \widetilde{\in} 1_B - L$, we find $G \in \psi$ and $H \in \psi_\omega$ with $b_y \widetilde{\in} G$, $L \widetilde{\subseteq} H$, and $G \widetilde{\cap} H = 0_B$.

Theorem 5.2. A supra-STS (U, ψ, B) is supra-soft ω-r iff whenever $T \in \psi$ and $b_y \widetilde{\in} T$, we find $G \in \psi$ with $b_y \widetilde{\in} G \widetilde{\subseteq} Cl_{\psi_\omega}(G) \widetilde{\subseteq} T$.

Proof: Necessity. Let (U, ψ, B) be supra-soft ω-r. Let $T \in \psi$ and $b_y \widetilde{\in} T$. Then, we have $1_B - T \in \psi^c$ and $b_y \widetilde{\in} 1_B - (1_B - T)$. We then find $G \in \psi$ and $H \in \psi_\omega$ with $b_y \widetilde{\in} G$, $1_B - T \widetilde{\subseteq} H$, and $G \widetilde{\cap} H = 0_B$. Since $1_B - T \widetilde{\subseteq} H$, $1_B - H \widetilde{\subseteq} T$. Since $G \widetilde{\cap} H = 0_B$, $G \widetilde{\subseteq} 1_B - H$, and so, $b_y \widetilde{\in} G \widetilde{\subseteq} Cl_{\psi_\omega}(G) \widetilde{\subseteq} Cl_{\psi_\omega}(1_B - H) = 1_B - H \widetilde{\subseteq} T$.

Sufficiency. Let $L \in \psi^c$ and $b_y \widetilde{\in} 1_B - L$. By assumption, we find $G \in \psi$ with $b_y \widetilde{\in} G \widetilde{\subseteq} Cl_{\psi_\omega}(G) \widetilde{\subseteq} 1_B - L$. Set $H = 1_B - Cl_{\psi_\omega}(G)$. Then, $H \in \psi_\omega$, $L \widetilde{\subseteq} H$, and $G \widetilde{\cap} H = 0_B$. Consequently, (U, ψ, B) is supra-soft ω-r.

Theorem 5.3. If (U, ψ, B) is supra-soft ω-L-I, then (U, ψ, B) is supra-soft ω-r.

Proof: Let $T \in \psi$ and $b_y \widetilde{\in} T$. Since (U, ψ, B) is supra-soft ω-L-I, $T \in (\psi_\omega)^c$, and so, $T = Cl_{\psi_\omega}(T)$. Hence, we have $T \in \psi$ and $b_y \widetilde{\in} T \widetilde{\subseteq} Cl_{\psi_\omega}(T) \widetilde{\subseteq} T$. Thus, by Theorem 5.2, (U, ψ, B) is supra-soft ω-r.

Corollary 5.4. If (U, ψ, B) is supra-soft L-C, then (U, ψ, B) is supra-soft ω-r.

Proof: This follows from Theorems 4.4 and 5.3.

Theorem 5.5. Supra-soft regularity implies supra-soft ω-regularity.

Proof: Let (U, ψ, B) be supra-soft regular. Let $T \in \psi$ and $b_y \widetilde{\in} T$. By the supra-soft regularity of (U, ψ, B), we find $G \in \psi$ with $b_y \widetilde{\in} G \widetilde{\subseteq} Cl_{\psi}(G) \widetilde{\subseteq} T$. Since $Cl_{\psi_\omega}(G) \widetilde{\subseteq} Cl_{\psi}(G)$, we have $b_y \widetilde{\in} G \widetilde{\subseteq} Cl_{\psi_\omega}(G) \widetilde{\subseteq} Cl_{\psi}(G) \widetilde{\subseteq} T$. Consequently, (U, ψ, B) is supra-soft ω-r.

Lemma 5.6. Let (U, ψ, B) be a supra-STS and let $K \in SS(U, B)$. Then, for every $b \in B$, $Cl_{\psi_b}(K(b)) \subseteq \left(Cl_{\psi}(K)\right)(b)$.

Proof: Let $y \in Cl_{\psi_b}(K(b))$. We show that $b_y \widetilde{\in} Cl_{\psi}(K)$. Let $G \in \psi$ with $b_y \widetilde{\in} G$. We then have $y \in G(b) \in \psi_b$. Since $y \in Cl_{\psi_b}(K(b))$, $K(b) \cap G(b) \neq \emptyset$. Thus, $\left(K \widetilde{\cap} G\right)(b) = K(b) \cap G(b) \neq \emptyset$, and hence, $K \widetilde{\cap} G \neq 0_B$. It follows that $b_y \widetilde{\in} Cl_{\psi}(K)$.

Theorem 5.7. If (U, ψ, B) is supra-soft regular, then (U, ψ_b) is supra-regular for all $b \in B$.

Proof: Let (U, ψ, B) be supra-soft regular and let $b \in B$. Let $V \in \psi_b$ and $y \in V$. Choose $T \in \psi$ with $T(b) = V$. Then, $b_y \widetilde{\in} T \in \psi$, and, by the supra-soft regularity of (U, ψ, B), we find $G \in \psi$ with $b_y \widetilde{\in} G \widetilde{\subseteq} Cl_{\psi}(G) \widetilde{\subseteq} T$. Thus, we have $G(b) \in \psi_b$, and, by Lemma 5.6, $y \in G(b) \subseteq Cl_{\psi_b}(G(b)) \subseteq \left(Cl_{\psi}(G)\right)(b) \subseteq T(b) = V$. Consequently, (U, ψ_b) is supra-regular.

Lemma 5.8. Let $\{(U, \psi_b) : b \in B\}$ be a family of supra-TSs and let $K \in SS(U, B)$. Then, $Cl_{\psi_a}(K(a)) = \left(Cl_{\otimes_{b \in B}\psi_b} K\right)(a)$ for every $a \in B$.

Proof: Let $a \in B$. Then, by Lemma 5.6, $Cl_{(\otimes_{b \in B}\psi_b)_a}(K(a)) \subseteq \left(Cl_{\otimes_{b \in B}\psi_b}(K)\right)(a)$. Moreover, by Theorem 2.5, $(\otimes_{b \in B}\psi_b)_a = \psi_a$. Hence, $Cl_{\psi_a}(K(a)) \subseteq \left(Cl_{\otimes_{b \in B}\psi_b} K\right)(a)$. To demonstrate that $\left(Cl_{\otimes_{b \in B}\psi_b} K\right)(a) \subseteq Cl_{\psi_a}(K(a))$, let $y \in \left(Cl_{\otimes_{b \in B}\psi_b} K\right)(a)$, and let $V \in \psi_a$ with $y \in V$. We then have $a_y \widetilde{\in} a_V \in \otimes_{b \in B}\psi_b$. Since $y \in \left(Cl_{\otimes_{b \in B}\psi_b} K\right)(a)$, $a_y \widetilde{\in} Cl_{\otimes_{b \in B}\psi_b} K$, and so, $a_V \widetilde{\cap} K \neq 0_B$. Consequently, $V \cap K(a) \neq \emptyset$. This shows that $y \in Cl_{\psi_a}(K(a))$.

Lemma 5.9. Let $\{(U, \psi_b) : b \in B\}$ be a family of supra-TSs. Then, for any $a \in B$ and $V \subseteq U$, $Cl_{\otimes_{b \in B}\psi_b}(a_V) = a_{Cl_{\psi_a}(V)}$.

Proof: Let $a \in B$ and $V \subseteq U$. Let $b \in B$. Then, by Lemma 5.8, $\left(Cl_{\otimes_{b \in B}\psi_b} a_V\right)(b) = Cl_{\psi_a}(a_V(b)) =$

$$\begin{cases} Cl_{\psi_a}(V) & \text{if} \quad b = a, \\ \emptyset & \text{if} \quad b \neq a. \end{cases}$$

Consequently, $Cl_{\otimes_{b \in B}\psi_b}(a_V) = a_{Cl_{\psi_a}(V)}$.

Theorem 5.10. Let $\{(U, \psi_b) : b \in B\}$ be a family of supra-TSs. Then, $(U, \otimes_{b \in B}\psi_b, B)$ is supra-soft regular if (U, ψ_b) is supra-regular for all $b \in B$.

Proof: Necessity. Let $(U, \otimes_{b \in B}\psi_b, B)$ be supra-soft regular. Then, by Theorem 5.7, $(U, (\otimes_{b \in B}\psi_b)_b, B)$ is supra-regular for all $b \in B$. But, by Theorem 2.5, $(\otimes_{b \in B}\psi_b)_b = \psi_b$ for all $b \in B$. This completes the proof.

Sufficiency. Let (U, ψ_b) be supra-regular for all $b \in B$. Let $T \in \otimes_{b \in B}\psi_b$ and let $a_y \widetilde{\in} T$. Then, $y \in T(a) \in \psi_a$ and, by the supra-regularity of (U, ψ_a), we find $V \in \psi_a$ with $y \in V \subseteq Cl_{\psi_a}(V) \subseteq T(a)$. Consequently, we have $a_y \widetilde{\in} a_V \in \otimes_{b \in B}\psi_b$, and, by Lemma 5.9, $Cl_{\otimes_{b \in B}\psi_b}(a_V) = a_{Cl_{\psi_a}(V)} \widetilde{\subseteq} T$. Therefore, $(U, \otimes_{b \in B}\psi_b, B)$ is supra-soft regular.

Corollary 5.11. Let $(U, \aleph)$ be a supra-TS and B be a set of parameters. Then $(U, \mu(\aleph), B)$ is supra-soft regular iff $(U, \aleph)$ is supra-regular.

Proof: For every $b \in B$, set $\aleph_b = \aleph$. Then, $\mu(\aleph) = \otimes_{b \in B}\psi_b$. Theorem 5.10 completes the proof.

Theorem 5.12 If (U, ψ, B) is supra-soft ω-r, then (U, ψ_a) is supra-ω-regular for all $a \in B$.

Proof: Let (U, ψ, B) be supra-soft ω-r and let $a \in B$. Let $V \in \psi_a$ and $y \in V$. Pick $T \in \psi$ with $T(a) = T$. We then have $a_y \widetilde{\in} T \in \psi$, and, by the supra-soft ω-regularity of (U, ψ, B) and Theorem 5.2, we find $G \in \psi$ with $a_y \widetilde{\in} G \widetilde{\subseteq} Cl_{\psi_\omega}(G) \widetilde{\subseteq} T$. Thus, we have $G(a) \in \psi_a$, and, by Lemma 5.6, $y \in G(a) \subseteq Cl_{\psi_\omega}(G(a)) \subseteq \left(Cl_{\psi_\omega}(G)\right)(a) \subseteq T(a) = V$. Consequently, (U, ψ_a) is supra-ω-regular.

Theorem 5.13. Let $\{(U, \psi_b) : b \in B\}$ be a family of supra-TSs. Then, $(U, \otimes_{b\in B}\psi_b, B)$ is supra-soft ω-r iff (U, ψ_b) is supra-ω-regular for all $a \in B$.

Proof: Necessity. Let $(U, \otimes_{b\in B}\psi_b, B)$ be supra-soft ω-r. Then, by Theorem 5.12, $(U, (\otimes_{b\in B}\psi_b)_b, B)$ is supra-ω-regular for all $b \in B$. But, by Theorem 2.5, $(\otimes_{b\in B}\psi_b)_b = \psi_b$ for all $b \in B$. This completes the proof.

Sufficiency. Let (U, ψ_b) be supra-ω-regular for all $b \in B$. Let $T \in \otimes_{b\in B}\psi_b$, and let $a_y \widetilde{\in} T$. Then, $y \in T(a) \in \psi_a$. Since (U, ψ_a) is supra-ω-regular, we find $V \in \psi_a$ with $y \in V \subseteq Cl_{(\psi_a)_\omega}(V) \subseteq T(a)$. Consequently, we have $a_y \widetilde{\in} a_V \in \otimes_{b\in B}\psi_b$ and, by Lemma 5.9, $Cl_{\otimes_{b\in B}(\psi_b)_\omega}(a_V) = a_{Cl_{(\psi_a)_\omega}(V)} \widetilde{\subseteq} T$. Moreover, by Theorem 3.10, $(\otimes_{b\in B}\psi_b)_\omega = \otimes_{b\in B}(\psi_b)_\omega$. Consequently, $Cl_{\otimes_{b\in B}(\psi_b)_\omega}(a_V) = Cl_{(\otimes_{b\in B}\psi_b)_\omega}(a_V)$. This shows that $(U, \otimes_{b\in B}\psi_b, B)$ is supra-soft ω-r.

Corollary 5.14. Let $(U, \aleph)$ be a supra-TS and B be a set of parameters. Then $(U, \mu(\aleph), B)$ is supra-soft ω-r iff $(U, \aleph)$ is supra-ω-regular.

Proof: For every $b \in B$, set $\aleph_b = \aleph$. Then, $\mu(\aleph) = \otimes_{b\in B}\psi_b$. Theorem 5.13 completes the proof.

The opposites of Theorem 5.3 and Corollary 5.4 are false.

Example 5.15. Let $B = \{s, t\}$. Let ψ_s and ψ_t be the usual and the discrete topologies on $\mathbb{R}$. Consider the supra-STS $(\mathbb{R}, \otimes_{b\in B}\psi_b, B)$. Then, the supra-TSs $(\mathbb{R}, \psi_s)$ and $(\mathbb{R}, \psi_t)$ are supra-regular. Thus, by Theorem 5.10, $(\mathbb{R}, \otimes_{b\in B}\psi_b, B)$ is supra-soft regular. Hence, by Theorem 5.5, $(\mathbb{R}, \otimes_{b\in B}\psi_b, B)$ is supra-soft ω-r. Conversely, since $(-\infty, 0) \in \psi_s - ((\psi_s)_\omega)^c$, $(\mathbb{R}, \psi_s)$ is not supra-ω-L-I. So, by Theorem 4.11, $(\mathbb{R}, \otimes_{b\in B}\psi_b, B)$ is not supra-soft ω-L-I. Moreover, clearly, $(\mathbb{R}, \otimes_{b\in B}\psi_b, B)$ is not supra-soft L-C.

The contrary of Theorem 5.5 is generally untrue.

Example 5.16. Let $U = \mathbb{Z}$, $B = \mathbb{R}$, and $\aleph$ be the cofinite topology on U. Then, $(U, \aleph)$ is not supra-regular. So, by Corollary 5.11, $(U, \mu(\aleph), B)$ is not supra-soft regular. Since $(U, \mu(\aleph), B)$ is supra-soft L-C, by Corollary 5.4, $(U, \mu(\aleph), B)$ is supra-soft ω-r.

Example 5.17. Consider (U, ψ, B) as shown in Example 4.13. In Example 4.13, we showed that both (U, ψ_s) and (U, ψ_t) are supra-L-I, which means they are supra-regular and thus are supra-ω-regular. Assume that (U, ψ, B) is supra-soft ω-r. If we let $y = -1$, then $t_y \widetilde{\in} 1_B - (1_B - T)$ with $1_B - T \in \psi^c$. Therefore, we find $G \in \psi$ and $H \in \psi_\omega$ with $t_y \widetilde{\in} G$, $1_B - T \widetilde{\subseteq} H$, and $G \widetilde{\cap} H = 0_B$. One can easily check that we must have $t_y \widetilde{\in} T \widetilde{\subseteq} G$, and so $T \widetilde{\cap} H = 0_B$. Thus, $H \widetilde{\subseteq} 1_B - T$, which implies that $H = 1_B - T$. But we have shown in Example 4.13 that $1_B - T \notin (\psi_\omega)^c$. Consequently, (U, ψ, B) is not supra-soft ω-r, and, by Theorem 5.5, (U, ψ, B) is not supra-soft regular.

Theorem 5.18. If (U, ψ, B) is supra-soft A-L-C and supra-soft ω-r, then (U, ψ, B) is supra-soft regular.

Proof: This follows from the definitions and Theorem 3.23.

Theorem 5.19. Let (U, ψ, B) and (V, ϕ, D) be two supra-STSs. In this case:

(a) $(pr(\psi \times \phi))_\omega \subseteq pr(\psi_\omega \times \phi_\omega)$;

(b) For any $S \in SS(U, B)$ and $K \in SS(V, D)$, $Cl_{\psi_\omega}(S) \times Cl_{\phi_\omega}(K) \widetilde{\subseteq} Cl_{(pr(\psi\times\phi))_\omega}(S \times K)$.

Proof: (a) Let $T \in (pr(\psi \times \phi))_\omega$ and let $(s, t)_{(x,y)} \widetilde{\in} T$. We then find $L \in pr(\psi \times \phi)$ and $H \in CSS(U \times V, B \times D)$ with $(s, t)_{(x,y)} \widetilde{\in} L - H \widetilde{\subseteq} T$. Choose $F \in \psi$ and $G \in \phi$ with $(s, t)_{(x,y)} \widetilde{\in} F \times G \widetilde{\subseteq} L$. Set $M = \left(\widetilde{\cup}\left\{c_z : (c, d)_{(z,w)} \widetilde{\in} H \text{ for some } d_w \widetilde{\in} SP(V, D)\right\}\right) - s_x$ and $N = \left(\widetilde{\cup}\left\{d_w : (c, d)_{(z,w)} \widetilde{\in} H \text{ for some } c_z \widetilde{\in} SP(U, B)\right\}\right) - t_y$. Then, $M \in CSS(U, B)$ and $N \in CSS(V, D)$. Therefore, we have $F - M \in \psi_\omega$, $G - N \in \phi_\omega$, and $(s, t)_{(x,y)} \widetilde{\in} (F - M) \times (G - N) \widetilde{\subseteq} (F \times G) - (M \times N) \widetilde{\subseteq} L - H \widetilde{\subseteq} T$. Consequently, $T \in pr(\psi_\omega \times \phi_\omega)$.

(b) Let $(s,t)_{(x,y)}\widetilde{\in}Cl_{\psi_\omega}(S)\times Cl_{\phi_\omega}(K)$, and let $T\in(pr(\psi\times\phi))_\omega$ with $(s,t)_{(x,y)}\widetilde{\in}T$. By (a), $T\in pr(\psi_\omega\times\phi_\omega)$, and thus, we find $W\in\psi_\omega$ and $E\in\phi_\omega$ with $(s,t)_{(x,y)}\widetilde{\in}W\times E\widetilde{\subseteq}T$. Since $s_x\widetilde{\in}W\widetilde{\cap}Cl_{\psi_\omega}(S)$ and $t_y\widetilde{\in}E\widetilde{\cap}Cl_{\phi_\omega}(K)$, $W\widetilde{\cap}S\neq 0_B$ and $E\widetilde{\cap}K\neq 0_D$. Consequently, $(W\times E)\widetilde{\cap}(S\times K)\neq 0_{B\times D}$, and hence, $T\widetilde{\cap}(S\times K)\neq 0_{B\times D}$. This implies that $(s,t)_{(x,y)}\widetilde{\in}Cl_{(pr(\psi\times\phi))_\omega}(S\times K)$.

Theorem 5.20. Let (U,ψ,B) and (V,ϕ,D) be two supra-STSs. If $(U\times V, pr(\psi\times\phi), B\times D)$ is supra-soft ω-r, then (U,ψ,B) and (V,ϕ,D) are supra-soft ω-r.

Proof: Let $F\in\psi$, $G\in\phi$, $s_x\widetilde{\in}F$, and $t_y\widetilde{\in}G$. Then, $(s,t)_{(x,y)}\widetilde{\in}F\times G\in pr(\psi\times\phi)$, and, by the supra-soft ω-regularity of $(U\times V, pr(\psi\times\phi), B\times D)$, we find $K\in pr(\psi\times\phi)$ with $(s,t)_{(x,y)}\widetilde{\in}K\widetilde{\subseteq}Cl_{(pr(\psi\times\phi))_\omega}(K)\widetilde{\subseteq}F\times G$. Choose $S\in\psi$ and $T\in\phi$ with $(s,t)_{(x,y)}\widetilde{\in}S\times T\widetilde{\subseteq}K$. Then, by Theorem 5.19 (b), $(s,t)_{(x,y)}\widetilde{\in}S\times T\widetilde{\subseteq}Cl_{\psi_\omega}(S)\times Cl_{\phi_\omega}(T)\widetilde{\subseteq}Cl_{(pr(\psi\times\phi))_\omega}(T\times S)\widetilde{\subseteq}Cl_{(pr(\psi\times\phi))_\omega}(K)\widetilde{\subseteq}F\times G$. Consequently, we have $s_x\widetilde{\in}S\widetilde{\subseteq}Cl_{\psi_\omega}(S)\widetilde{\subseteq}F$ and $t_y\widetilde{\in}T\widetilde{\subseteq}Cl_{\psi_\omega}(T)\widetilde{\subseteq}G$. It follows that (U,ψ,B) and (V,ϕ,D) are supra-soft ω-r.

Question 5.21. Let (U,ψ,B) and (V,ϕ,D) be two supra-soft ω-r supra-STSs. Is $(U\times V, pr(\psi\times\phi), B\times D)$ supra-soft ω-r?

Theorem 5.22. If (U,ψ,B) is a supra-soft ω-r supra-STS, then for any $\emptyset\neq V\subseteq U$, (V,ψ_V,B) is supra-soft ω-r.

Proof: Let $M\in(\psi_V)^c$ and $b_y\widetilde{\in}C_V-M$. Choose $N\in\psi^c$ with $M=N\widetilde{\cap}C_V$. Since (U,ψ,B) is a supra-soft ω-r, and we have $N\in\psi^c$ and $b_y\widetilde{\in}1_B-N$, we find $F\in\psi$ and $G\in\psi_\omega$ with $a_y\widetilde{\in}F$, $N\widetilde{\subseteq}G$, and $F\widetilde{\cap}G=0_B$. Then, $b_y\widetilde{\in}F\widetilde{\cap}C_V\in\psi_V$, $M=N\widetilde{\cap}C_V\widetilde{\subseteq}G\widetilde{\cap}C_V$ with $G\widetilde{\cap}C_V\in(\psi_\omega)_V$, and $\left(F\widetilde{\cap}C_V\right)\widetilde{\cap}(G\widetilde{\cap}C_V)=\left(F\widetilde{\cap}G\right)\widetilde{\cap}C_V=0_B\widetilde{\cap}C_V=0_B$. Moreover, by Theorem 3.19, $G\widetilde{\cap}C_V\in(\psi_V)_\omega$. This completes the proof.

6. Conclusion and future directions

Soft set theory demonstrates its effectiveness as a mathematical strategy for addressing uncertainty, which is crucial for cognitive analysis and artificial intelligence. Based on soft set theory, many mathematical structures have emerged, including soft topologies and some of their extensions, such as supra-soft topologies.

In this paper, we first defined and investigated a new supra-soft topology using a collection of classical supra-topologies. We then defined supra-soft ω-open sets, a new generalization of supra-soft open sets, using the supra-soft open sets and the countable soft sets. We also showed that supra-soft ω-open sets form a new supra-soft set that is finer than the given supra-soft topology. Finally, we defined and investigated two new classes of supra-topological spaces: supra-soft ω-local indiscrete and supra-soft ω-regular spaces. Specifically, we obtained subspace and product results of supra-soft ω-regular spaces. Finally, we explored the connections between our new concepts and their counterparts in supra-topology.

We intend to do the following in the future papers:

(i) Define new continuity concepts between supra-soft topological spaces via supra-soft ω-open sets.

(ii) Define supra-soft semi ω-open sets in supra-soft topological spaces.

(iii) Define soft ω-Hausdorff spaces in supra-soft topological spaces.

(iv) Explore how our new notions and results can be applied in digital and approximation spaces, as well as decision-making problems.

(v) Define supra-fuzzy ω-open sets in supra-fuzzy topological spaces.

Author contributions

Dina Abuzaid and Samer Al-Ghour: Conceptualization, methodology, formal analysis, writing–original draft, writing–review and editing, and funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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