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Research article

Prioritized Hamy mean operators based on Dombi t-norm and t-conorm for the complex interval-valued Atanassov-Intuitionistic fuzzy sets and their applications in strategic decision-making problems

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Abstract: The complex interval-valued Atanassov intuitionistic fuzzy set theory is an advanced modification of traditional fuzzy information that combines elements of both interval-valued fuzzy information and Atanassov intuitionistic fuzzy set theory, and incorporates complex numbers. Additionally, aggregating a finite number of alternatives into a singleton set is very important, where the Hamy mean operator and prioritized aggregation operator are much more suitable and flexible for depicting such kinds of problems. Our main goal of this manuscript was to analyze the Dombi operational laws based on complex interval-valued Atanassov intuitionistic fuzzy numbers. Further, the prioritized Hamy mean operators based on Dombi operational laws for complex interval-valued Atanassov intuitionistic fuzzy values, called the complex interval-valued Atanassov intuitionistic fuzzy Dombi Hamy mean operator, complex interval-valued Atanassov intuitionistic fuzzy weighted Dombi prioritized Hamy mean operator, complex interval-valued Atanassov intuitionistic fuzzy Dombi Dual Hamy mean operator, and complex interval-valued Atanassov intuitionistic fuzzy weighted Dombi Dual prioritized Hamy mean operator, were proposed. Some dominant and flexible properties of the evaluated operators were also examined. Further, in some multi-attribute decision-making problems, biased conclusions may be produced due to the deficiency of consideration for many relationships between the criteria of decision-making. Therefore, to evaluate the proficiency and reliability of the proposed operators, the multi-attribute decision-making technique based on derived operators for complex interval-valued Atanassov intuitionistic fuzzy values was developed. Finally, the proposed method with some prevailing techniques was compared to show its advantages and benefits.

Keywords: green sustainable chain; complex interval-valued Atanassov-intuitionistic fuzzy sets; prioritized Hamy mean operators; Dombi t-norms; decision-making techniques
Mathematics Subject Classification: 03B52, 68T27, 68T37, 94D05, 03E72

1. Introduction

Decision-making is a valuable and systematic technique or method used for evaluating or analyzing the best decision from a collection of finite alternatives. In this technique, there is a collection of finite alternatives, and according to each alternative, there is a finite number of attributes or criteria, based on assigned criteria, the goal is to decide which alternative is best and which one is worst. In general, it is very complicated to make this decision in some real-life situations, because the attributes cannot be expressed by a crisp set. Therefore, Zadeh [1] proposed the fuzzy set (FS), where the range of FS is a unit interval instead of {0, 1}. FS theory is very famous and reliable for coping with vague and uncertain data because of its features, but it is clear that the technique of FS theory has just a supporting degree, whereas the non-supporting degree or non-supporting function is also very important because it is the major part of every genuine-life problem. Thus, Atanassov [2] extended the model of FS to the model of proposed intuitionistic FS (IFS) by adding the non-support degree or falsity information in the model of FS theory. The information on IFS has received a lot of attention from different scholars, because of their structure, where the truth and falsity functions are the part of IFSs with a valuable and famous characteristic that is the sum of both functions must be contained in unit interval. Additionally, it is also very clear the truth function and falsity function in IFSs are just particular numerical values from the unit interval, but we noticed that the interval-valued information is more beneficial if compared to just a particular numerical value; for example, if we are talking about the temperature of a room, but due to measurement uncertainty, you do not have an exact value, only a range. In genuine-life cases, the temperature might be recorded as 23℃, however, when vagueness is involved in the measurement, the interval-valued data might show the temperature as 22℃, 24℃, indicating a range. To cope with such a problem, Atanassov [3] also investigated the interval-valued IFS (IVIFS), which is superior to IFS, because the truth and falsity grades in IVIFS were computed as interval-valued.

The truth/supporting/membership grade in FS is expressed by a real number, and sometimes it is difficult to express uncertain information; for instance, when purchasing any kind of software, we have two possibilities: The name or version of the software, which are represented by the real and imaginary part of the complex numbers. Thus, the complex FS (CFS) is proposed by Ramot et al. [4]. Moreover, for the model of CFS theory constructed for coping with vague and uncertain data, the information of supporting degree is not enough for coping with some vague and complex problems, because the non-supporting degree or non-supporting function also plays a critical role in the environment of many genuine-life problems. Thus, Alkouri and Salleh [5] developed the complex IFS (CIFS), where the CIFS covered the truth and falsity grades by complex-valued functions. The characteristic of CIFS is that the sum of the real parts (also for imaginary parts) of both functions must be restricted to the unit interval. However, it is better; if we give information in the shape of a sub-interval of unit interval instead of real numbers, because numerous problems can easily be evaluated with the help of interval-valued data instead of real-valued information due to complication and problems. For this, Garg and Rani [6] initiated the complex interval-valued Atanassov intuitionistic fuzzy (CIVAIF) sets, whose truth and falsity grades are computed as complex interval-valued. Moreover, FS, IFSs, and CFS are special cases of the CIFSs.

1.1. Literature review

The truth function in FS theory is defined from any fixed set to the unit interval, where in this case, experts have a lot of space or a wide range of information to make their decision more precisely and accurately. After the construction of FS theory, many scholars have modified or utilized the FS in different scenarios; for instance, fuzzy superior Mandelbrot sets were designed by Mahmood and Ali [7], the generalized fuzzy superior Mandelbrot sets were constructed by Ince and Ersoy [8], the model of interval type-2 fuzzy logic was invented by Castillo and Melin [9], the model of type-2 fuzzy logic was invented by John and Coupland [10], the information of fuzzy differential equation was proposed by Kaleva [11], Buckley and Feuring [12], and Kaleva [13], another form of fuzzy differential equations was designed by Park and Han [14], and the fuzzy implications were evaluated by Pan et al. [15]. Further, from the above assessment of IFS, it is superior and more flexible than the structure of FS theory because of its feature, and both concepts have a lot of advantages; However, the IFS has received massive attraction from different scholars. For instance, distance measures for IVIFSs were designed by Gohain et al. [16], the optimization problems based on IFSs were evaluated by Sharma et al. [17], the fairly operators for IFSs were constructed by Mishra et al. [18], the multi-granulation covering rough IFSs were presented by Xue et al. [19], the decision-making problems for IFSs were discussed by Wieckowski et al. [20], the acceptability analysis for integrated IFSs were derived by Ilbas et al. [21], the model of MAIRCA technique for intuitionistic fuzzy symmetry point were designed by Hezam et al. [22], and a temporal topological structure for IFSs were invented by Atanassov [23].

CFS is the advanced extension of FS, and many scholars have researched it in different fields. For instance, the distance measures and entropy measures for CFS were designed by Liu et al. [24], and the technique of model of neighborhood operators for CFSs was invented by Mahmood et al. [25], the model of ARIMA forecasting for complex neuro-fuzzy information was designed by Li and Chiang [26], the image noise canceling for complex neuro-fuzzy data was constructed by Li et al. [27], and the model of aggregation operators for complex linguistic fuzzy information was designed by Mahmood et al. [28]. Further, from the above assessment of CIFS, it is superior and more flexible than the structure of CFS theory because of its feature, and many scholars have utilized the CIFS in the field of many environments: For instance, Einstein operators for CIFSs were invented by Azeem et al. [29], the prioritized operators for CIFSs were proposed by Ali et al. [30], and the application of software selection based on some new techniques was discussed by Garg et al. [31]. In addition, we know that the algebraic t-norm and t-conorm are very famous and reliable for constructing any kind of aggregation operator. However, Dombi [32] modified the idea of algebraic norms and proposed the novel concept of Dombi t-norm and Dombi t-conorm. Similarly, the HM operators were proposed by Yu [33], which are based on algebraic norms. Further, Dombi Heronian mean operators for IVIFSs were presented by Wu et al. [34], Yu et al. [35] exposed the prioritized operators for IVIFSs, the prioritized operators for CIFSs were presented by Garg and Rani [36], aggregation operators for CIFSs were presented by Garg and Rani [37], Shi et al. [38] derived the power operators for IVIFSs, Chen [39] examined the prioritized operators for IVIFSs, Wang et al. [40] invented the Dombi prioritized operators for CIFSs, and Fang et al. [41] derived the Aczel-Alsina operators for CIVAIF sets (CIVAIFSs). Additionally, Wan et al. [42] designed the intuitionistic fuzzy preference relation, Dong and Wang [43] evaluated the interval-valued intuitionistic fuzzy best-worst technique, Wan et al. [44] proposed the time-series based on a decision-making model for IFSs, Lu et al. [45] derived the intuitionistic fuzzy multiplicative preference relation, and Chen et al. [46] exposed the integrated interval-valued IFS technique with application in the assessment of COVID-19. Gou et al. [47] presented the continuous Pythagorean fuzzy information, Gou et al. [48] determined the consensus-reaching procedure for hesitant fuzzy linguistic information, Gou et al. [49] designed the circular economic and fuzzy set theory, Gou et al. [50] evaluated the improved ORESTE technique for linguistic preference, and Cheng et al. [51] derived the decision-making technique for the large-scale group.

1.2. Research gap, limitation, and motivation of the proposed theory

After a long assessment, we observed that the model of CIVAIFSs is more reliable and more prominent because of their advantages and its features. First, we want to explain the structure of CIVAIFS as it contains the truth function and falsity function in the form of complex numbers, where the real and imaginary parts of both functions are constructed in the form of sub-interval of unit intervals, which provides more space or wide range to experts for making their decision more precisely and accurately. However, we also observed that in every decision-making procedure, much information is lost due to ambiguity and uncertainty. Thus, we observed that every decision-maker faces the following problems during decision-making:

(1) How do we propose new operational laws based on complicated structures?
(2) How do we combine two or more operators?
(3) How do we propose operators?
(4) How do we find the ranking values?

Depicting the above problems, prioritized Hamy mean operators based on Dombi operational laws for CIVAIFSs are very dominant because of their features. First, we talk about the technique of Dombi norms, developed by Jozsef Dombi in 1982, such as

$$D\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2\right) = \begin{cases} 0 & if\ \overline{\overline{\mathbb{A}_f}}^1 = 0\ or\ \overline{\overline{\mathbb{A}_f}}^2 = 0 \\ D_T\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2\right) & if\ \overline{\overline{\mathfrak{I}_s}} = 0 \\ D_{min}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2\right) & if\ \overline{\overline{\mathfrak{I}_s}} = +\infty \\ \dfrac{1}{1+\left(\left(\frac{1-\overline{\overline{\mathbb{A}_f}}^1}{\overline{\overline{\mathbb{A}_f}}^1}\right)^{\overline{\overline{\mathfrak{I}_s}}}+\left(\frac{1-\overline{\overline{\mathbb{A}_f}}^2}{\overline{\overline{\mathbb{A}_f}}^2}\right)^{\overline{\overline{\mathfrak{I}_s}}}\right)^{\frac{1}{\overline{\overline{\mathfrak{I}_s}}}}} & otherwise \end{cases}.$$

Where $0 \leq \overline{\overline{\mathfrak{I}_s}} \leq +\infty$, and the above information is converted for Hamacher norm, if $\overline{\overline{\mathfrak{I}_s}} = 1$, where the Drastic norms "$D_T\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2\right)$" and min norms "$D_{min}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2\right)$" are the special cases of the Dombi norms from the above information. This means that three different types of norms are the special cases of the Dombi norms, which is a very powerful and very constructive technique to cope with vague and uncertain data. Additionally, the Hamy mean (HM) or dual HM (DHM) operators are also very reliable because they is the modified version of the averaging and geometric operators, but in the evaluation of aggregation information, we have unknown weight vectors; in this, we have a high chance of losing information during decision-making techniques; thus, to reduce the inaccuracy and improve the validity of the derived theory, we will use the technique of prioritized technique, which can help us in the construction of weight vector from the original data. After our long analysis and observation, we noticed that no one could derive prioritized operators, Dombi operators, and Hamy operators for CIVAIFSs. Moreover, because of complications and problems, it is also very complex to combine these techniques and develop the model of Dombi prioritized Hamy mean operators and Dombi prioritized dual Hamy mean operators for CIVAIFSs.

1.3. Advantages and main contribution of the proposed theory

No doubt the analysis of the Dombi prioritized Hamy mean operators for CIVAIFSs is very complex, but it is also very clear that they have a lot of benefits, because they can deal with a bundle of problems at one time by fixing the value of parameters. Due to different values of parameters, we can determine numerous techniques and models for FSs to CIVAIFSs. In the scenario of diabetes diagnosis based on multiple factors is a common and valuable practical example for CIVAIFSs. For instance, in medical diagnosis, in early detection of diabetes, patients demonstrate varying functions of test results and symptoms. Often, these results contain uncertainty and ambiguity because of measurement errors, incomplete data, or patient variation. The model of CIVAIFS can help us to depict these problems with the help of complex-valued truth function and falsity function for various factors, such as glucose levels, BMI, and family history, using complex intervals. For each factor, we assign the data in the following ways: According to age, the truth function value is $[0.8,0.9]+i0$ and for falsity value, we have $[0.1,0.2]+\mathrm{i}0$, but according to glucose levels, the truth function value is $[0.7,0.8]+\mathrm{i}0$, and for falsity value, we have $[0.2,0.3]+\mathrm{i}0$. This scenario, CIVAIFS is used to access the numerous ambiguous medical factors affecting diabetes diagnosis, enabling massive well-known and effective early predication. The major advantages (special cases of the proposed operators) of the proposed operators are listed below:

(1) Averaging/geometric operators for FSs (and their extensions).
(2) Dombi averaging/geometric operators for FSs (and their extensions).
(3) Hamacher averaging/geometric operators for FSs (and their extensions).
(4) Prioritized averaging/geometric operators for FSs (and their extensions).
(5) Hamy averaging/geometric operators for FSs (and their extensions).
(6) Dombi prioritized averaging/geometric operators for FSs (and their extensions).
(7) Prioritized Hamy averaging/geometric operators for FSs (and their extensions).
(8) Dombi Hamy averaging/geometric operators for FSs (and their extensions). The geometrical interpretation of the proposed theory is described and explained in Figure 1.

Many operators are the simple cases of the proposed theory. Because HM operators are a general form of operators, prioritized operators, and Dombi t-norm and t-conorm are also general operations of some t-norm and t-conorm. The PHM operators based on Dombi operational laws are not developed. Keeping the advantages of the PHM operators and Dombi norms, we have the following aims:

(1) To initiate Dombi operational laws based on CIVAIF numbers.
(2) To simplify the CIVAIF Dombi HM (CIVAIFDHM) operator, the CIVAIF weighted Dombi prioritized HM (CIVAIFWDPHM) operator, CIVAIF Dombi Dual HM (CIVAIFDDHM) operator, and CIVAIF weighted Dombi dual prioritized HM (CIVAIFWDDPHM) operator.
(3) To evaluate some dominant and flexible properties of the evaluated operators.
(4) To evaluate the proficiency and reliability of the proposed operators, we compute the multi-attribute decision-making (MADM) technique based on derived operators for CIVAIF values.
(5) To compare the proposed method with some prevailing techniques to show its advantages and benefits.

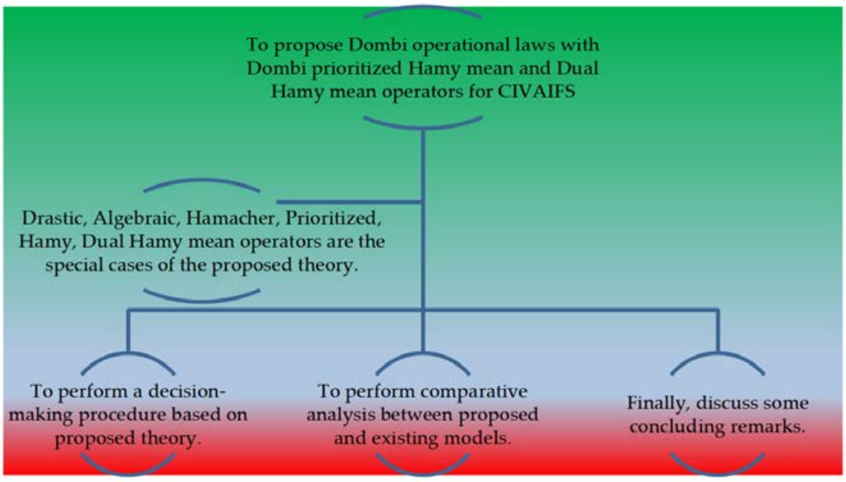

Figure 1. Graphical interpretation of the proposed theory.

1.4. Summary of the proposed manuscript

The structure of this manuscript is the following:

In Section 2, we introduce the CIVAIFSs and their operational laws. Moreover, we state the HM operator, dual Hamy mean operators, Dombi t-norm, and Dombi t-conorm based on a universal set.

In Section 3, we develop the Dombi operational laws for the CIVAIF set. We also propose the CIVAIFDHM operator, CIVAIFWDPHM operator, CIVAIFDDHM operator, and CIVAIFWDDPHM operator. Some basic properties are discussed for the above operators.

In Section 4, we construct the procedure of the MADM technique based on the proposed operators.

In Section 5, we illustrate some numerical examples based on strategic decision-making problems.

In Section 6, we discuss the comparative analysis of the proposed theory.

In Section 7, we briefly discuss some concluding remarks.

2. Preliminaries

In this section, we provide an overview of the prevailing informative and constructive idea of CIVAIFSs, which is the modified version of numerous models. Further, we explain the model of the HM operator and dual HM (DHM) operator with two valuable norms, called the technique of Dombi t-norm (DTN) and Dombi t-conorm (DTCN), which can play an important role in the construction of any types of aggregation operators.

Definition 1. [6] Let $\overline{\overline{\mathbb{u}}}_u$ be the generic element of a fixed set $\overline{\overline{\mathbb{U}}}_u$, then the CIVAIFS $\overline{\overline{\mathbb{A}}_f}$ is defined by:

$$\overline{\overline{\mathbb{A}}_f} = \left\{\left(\overbrace{\text{ҿ}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u), \overbrace{\text{ɳ}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right) : \overline{\overline{\mathbb{u}}}_u \in \overline{\overline{\mathbb{U}}}_u\right\}.$$

Further, we observed that the truth grade is constructed in the shape: $\overbrace{\text{ҿ}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) = \left(\left[\overset{\text{я}-}{\overbrace{\text{ҿ}}}_{\overline{\overline{\ }}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u), \overset{\text{я}+}{\overbrace{\text{ҿ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right], \left[\overset{\text{ʁ}-}{\overbrace{\text{ҿ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u), \overset{\text{ʁ}+}{\overbrace{\text{ҿ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right]\right)$ and the falsity grade is constructed in the shape: $\overbrace{\text{ɳ}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) = \left(\left[\overset{\text{я}-}{\overbrace{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u), \overset{\text{я}+}{\overbrace{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right], \left[\overset{\text{ʁ}-}{\overbrace{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u), \overset{\text{ʁ}+}{\overbrace{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right]\right)$ with conditions, such as

$0 \leq \widetilde{\text{Ꞓ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) + \widetilde{\text{ɳ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) \leq 1$ and $0 \leq \widetilde{\text{Ꞓ}}^{\text{ʁ}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) + \widetilde{\text{ɳ}}^{\text{ʁ}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) \leq 1$. Moreover, we gave the refusal grade, such as $\widetilde{\mathbb{r}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) = \left(\left[\widetilde{\mathbb{r}}^{\text{я}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u), \widetilde{\mathbb{r}}^{\text{я}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right], \left[\widetilde{\mathbb{r}}^{\text{ʁ}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u), \widetilde{\mathbb{r}}^{\text{ʁ}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right]\right) = \left(\left[1 - \left(\widetilde{\text{Ꞓ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) + \widetilde{\text{ɳ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right), 1 - \left(\widetilde{\text{Ꞓ}}^{\text{я}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) + \widetilde{\text{ɳ}}^{\text{я}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right)\right], \left[1 - \left(\widetilde{\text{Ꞓ}}^{\text{ʁ}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) + \widetilde{\text{ɳ}}^{\text{ʁ}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right), 1 - \left(\widetilde{\text{Ꞓ}}^{\text{ʁ}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u) + \widetilde{\text{ɳ}}^{\text{ʁ}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}(\overline{\overline{\mathbb{u}}}_u)\right)\right]\right)$ and the simple form of CIVAIF value (CIVAIFV) is derived from the shape, such as

$\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}} = \left(\left(\left[\widetilde{\text{Ꞓ}}^{\text{я}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}, \widetilde{\text{Ꞓ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right], \left[\widetilde{\text{Ꞓ}}^{\text{ʁ}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}, \widetilde{\text{Ꞓ}}^{\text{ʁ}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right]\right), \left(\left[\widetilde{\text{ɳ}}^{\text{я}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}, \widetilde{\text{ɳ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right], \left[\widetilde{\text{ɳ}}^{\text{ʁ}-}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}, \widetilde{\text{ɳ}}^{\text{ʁ}+}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right]\right)\right), \text{ɯ} = 1, 2, \ldots, p.$

Definition 2. [33] Consider $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}, \text{ɯ} = 1, 2, \ldots, p$ as a finite collection of positive numbers, the HM operator is defined as

$$HM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p}\right) = \frac{\sum_{1 \leq i_1 \leq \cdots \leq i_{\text{ɯ}} \leq p} \left(\prod_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\mathbb{A}_f}}^{i_\emptyset}\right)^{\frac{1}{\text{ɯ}}}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}.$$

Observed that ${}^{\circ}\mathrm{C}_p^{\text{ɯ}} = \frac{p!}{\text{ɯ}!(p-\text{ɯ})!}$, with some characteristics, such as

(1) If $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}} = \overline{\overline{\mathbb{A}_f}}, \text{ɯ} = 1, 2, \ldots, p$, thus $HM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p}\right) = \overline{\overline{\mathbb{A}_f}}$.

(2) If $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}} \leq \overline{\overline{\mathbb{A}_f}}^{\text{ɯ}*}, , \text{ɯ} = 1, 2, \ldots, p$, thus $HM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p}\right) \leq HM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1^*}, \overline{\overline{\mathbb{A}_f}}^{2^*}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p^*}\right)$.

(3) If $\min\left\{\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}\right\} \leq HM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p}\right) \leq \min\left\{\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}\right\}$.

(4) If $\text{ɯ} = 1$, then $HM^{1}\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p}\right) = \frac{1}{p}\prod_{i=1}^{p} \overline{\overline{\mathbb{A}_f}}^{i}$.

Definition 3. [32] Consider $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}, \text{ɯ} = 1, 2$ as two positive numbers, the DTN and DTCN are defined as

$$D\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}\right) = \frac{1}{1 + \left(\left(\frac{1 - \overline{\overline{\mathbb{A}_f}}^{1}}{\overline{\overline{\mathbb{A}_f}}^{1}}\right)^{\overline{\overline{\text{Ɔ}_s}}} + \left(\frac{1 - \overline{\overline{\mathbb{A}_f}}^{2}}{\overline{\overline{\mathbb{A}_f}}^{2}}\right)^{\overline{\overline{\text{Ɔ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ɔ}_s}}}}}, \overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2} \neq 0.$$

$$D^*\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}\right) = 1 - \frac{1}{1 + \left(\left(\frac{\overline{\overline{\mathbb{A}_f}}^{1}}{1 - \overline{\overline{\mathbb{A}_f}}^{1}}\right)^{\overline{\overline{\text{Ɔ}_s}}} + \left(\frac{\overline{\overline{\mathbb{A}_f}}^{2}}{1 - \overline{\overline{\mathbb{A}_f}}^{2}}\right)^{\overline{\overline{\text{Ɔ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ɔ}_s}}}}}, \overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2} \neq 1.$$

3. Dombi prioritized HM operators for CIVAIFSs

In this section, we analyze of construct the model of new operational laws based on the DTN and DTCN for the CIVAIF set. Additionally, we design the HM operators based on Dombi operational laws for the CIVAIF set such as the CIVAIFDHM operator, CIVAIFWDPHM operator, CIVAIFDDHM operator, and CIVAIFDWDPHM operator. Some basic properties are discussed for the above operators.

Definition 4. Consider $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}} = \left(\left(\left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, \ldots, p$ as CIVAIFVs, the Dombi operational laws are given as

$$
\begin{aligned}
&\overline{\overline{\mathbb{A}_f}}^{1} \oplus \overline{\overline{\mathbb{A}_f}}^{2} \\
&= \left(\begin{array}{c}
\left(\begin{array}{c}
\left[1 - \dfrac{1}{1+\left(\left(\dfrac{\overbrace{\varrho_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}-}}{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}}+\left(\dfrac{\overbrace{\varrho_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}-}}{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}}, 1 - \dfrac{1}{1+\left(\left(\dfrac{\overbrace{\varrho_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}+}}{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}_s}}}+\left(\dfrac{\overbrace{\varrho_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}+}}{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}} \right], \\
\left[1 - \dfrac{1}{1+\left(\left(\dfrac{\overbrace{\varrho_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}-}}{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}_s}}}+\left(\dfrac{\overbrace{\varrho_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}-}}{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}}, 1 - \dfrac{1}{1+\left(\left(\dfrac{\overbrace{\varrho_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}+}}{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}_s}}}+\left(\dfrac{\overbrace{\varrho_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}+}}{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}} \right]
\end{array} \right), \\
\left(\begin{array}{c}
\left[\dfrac{1}{1+\left(\left(\dfrac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}-}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}}+\left(\dfrac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}-}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}}, \dfrac{1}{1+\left(\left(\dfrac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}+}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}_s}}}+\left(\dfrac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}+}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}} \right], \\
\left[\dfrac{1}{1+\left(\left(\dfrac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}-}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}_s}}}+\left(\dfrac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}-}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}}, \dfrac{1}{1+\left(\left(\dfrac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}+}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}_s}}}+\left(\dfrac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}+}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}} \right]
\end{array} \right)
\end{array} \right).
\end{aligned}
$$

$$\overline{\overline{\mathbb{A}_f}}^{1} \otimes \overline{\overline{\mathbb{A}_f}}^{2} = \left(\begin{array}{c} \left(\begin{array}{c} \left[\frac{1}{1+\left(\left(\frac{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}-}}{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}}}_s}+\left(\frac{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}-}}{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}}, \frac{1}{1+\left(\left(\frac{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}+}}{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}}}_s}+\left(\frac{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}+}}{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}} \right], \\ \left[\frac{1}{1+\left(\left(\frac{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}-}}{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}}}_s}+\left(\frac{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}-}}{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}}, \frac{1}{1+\left(\left(\frac{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}+}}{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}}}_s}+\left(\frac{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}+}}{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}} \right] \end{array} \right), \\ \left(\begin{array}{c} \left[1-\frac{1}{1+\left(\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}}}_s}+\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}}, 1-\frac{1}{1+\left(\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}}}_s}+\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}} \right], \\ \left[1-\frac{1}{1+\left(\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}}}_s}+\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}}, 1-\frac{1}{1+\left(\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_1}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}}}_s}+\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_2}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}} \right] \end{array} \right) \end{array} \right).$$

$$\overline{\overline{\Psi_s \mathbb{A}_f}}^{\text{ɯ}} = \left(\begin{array}{c} \left(\begin{array}{c} \left[1 - \frac{1}{1 + \left(\overline{\overline{\Psi_s}} \left(\frac{\overbrace{ç_{\overline{\overline{\mathbb{A}_1}}}}^{я-}}{1 - \overbrace{ç_{\overline{\overline{\mathbb{A}_1}}}}^{я-}} \right)^{\overline{\overline{\ni_s}}} \right)^{\frac{1}{\overline{\overline{\ni_s}}}}}, 1 - \frac{1}{1 + \left(\overline{\overline{\Psi_s}} \left(\frac{\overbrace{ç_{\overline{\overline{\mathbb{A}_1}}}}^{я+}}{1 - \overbrace{ç_{\overline{\overline{\mathbb{A}_1}}}}^{я+}} \right)^{\overline{\overline{\ni_s}}} \right)^{\frac{1}{\overline{\overline{\ni_s}}}}} \right], \\ \left[1 - \frac{1}{1 + \left(\overline{\overline{\Psi_s}} \left(\frac{\overbrace{ç_{\overline{\overline{\mathbb{A}_1}}}}^{ʁ-}}{1 - \overbrace{ç_{\overline{\overline{\mathbb{A}_1}}}}^{ʁ-}} \right)^{\overline{\overline{\ni_s}}} \right)^{\frac{1}{\overline{\overline{\ni_s}}}}}, 1 - \frac{1}{1 + \left(\overline{\overline{\Psi_s}} \left(\frac{\overbrace{ç_{\overline{\overline{\mathbb{A}_1}}}}^{ʁ+}}{1 - \overbrace{ç_{\overline{\overline{\mathbb{A}_1}}}}^{ʁ+}} \right)^{\overline{\overline{\ni_s}}} \right)^{\frac{1}{\overline{\overline{\ni_s}}}}} \right] \end{array} \right), \\ \left(\begin{array}{c} \left[\frac{1}{1 + \left(\overline{\overline{\Psi_s}} \left(\frac{1 - \overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{я-}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{я-}} \right)^{\overline{\overline{\ni_s}}} \right)^{\frac{1}{\overline{\overline{\ni_s}}}}}, \frac{1}{1 + \left(\overline{\overline{\Psi_s}} \left(\frac{1 - \overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{я+}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{я+}} \right)^{\overline{\overline{\ni_s}}} \right)^{\frac{1}{\overline{\overline{\ni_s}}}}} \right], \\ \left[\frac{1}{1 + \left(\overline{\overline{\Psi_s}} \left(\frac{1 - \overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{ʁ-}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{ʁ-}} \right)^{\overline{\overline{\ni_s}}} \right)^{\frac{1}{\overline{\overline{\ni_s}}}}}, \frac{1}{1 + \left(\overline{\overline{\Psi_s}} \left(\frac{1 - \overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{ʁ+}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}_1}}}}^{ʁ+}} \right)^{\overline{\overline{\ni_s}}} \right)^{\frac{1}{\overline{\overline{\ni_s}}}}} \right] \end{array} \right) \end{array} \right).$$

$$\left(\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}}\right)^{\overline{\overline{\Psi_s}}} = \left(\begin{array}{c} \left(\left[\frac{1}{1+\left(\overline{\overline{\Psi_s}}\left(\frac{1-\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_1}}^{я-}}{\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_1}}^{я-}}\right)^{\overline{\overline{Э_s}}}\right)^{\frac{1}{\overline{\overline{Э_s}}}}}, \frac{1}{1+\left(\overline{\overline{\Psi_s}}\left(\frac{1-\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_1}}^{я+}}{\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_1}}^{я+}}\right)^{\overline{\overline{Э_s}}}\right)^{\frac{1}{\overline{\overline{Э_s}}}}}\right], \left[\frac{1}{1+\left(\overline{\overline{\Psi_s}}\left(\frac{1-\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_1}}^{ʁ-}}{\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_1}}^{ʁ-}}\right)^{\overline{\overline{Э_s}}}\right)^{\frac{1}{\overline{\overline{Э_s}}}}}, \frac{1}{1+\left(\overline{\overline{\Psi_s}}\left(\frac{1-\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_1}}^{ʁ+}}{\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_1}}^{ʁ+}}\right)^{\overline{\overline{Э_s}}}\right)^{\frac{1}{\overline{\overline{Э_s}}}}}\right]\right), \\ \left(\left[1-\frac{1}{1+\left(\overline{\overline{\Psi_s}}\left(\frac{\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_1}}^{я-}}{1-\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_1}}^{я-}}\right)^{\overline{\overline{Э_s}}}\right)^{\frac{1}{\overline{\overline{Э_s}}}}}, 1-\frac{1}{1+\left(\overline{\overline{\Psi_s}}\left(\frac{\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_1}}^{я+}}{1-\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_1}}^{я+}}\right)^{\overline{\overline{Э_s}}}\right)^{\frac{1}{\overline{\overline{Э_s}}}}}\right], \left[1-\frac{1}{1+\left(\overline{\overline{\Psi_s}}\left(\frac{\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_1}}^{ʁ-}}{1-\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_1}}^{ʁ-}}\right)^{\overline{\overline{Э_s}}}\right)^{\frac{1}{\overline{\overline{Э_s}}}}}, 1-\frac{1}{1+\left(\overline{\overline{\Psi_s}}\left(\frac{\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_1}}^{ʁ+}}{1-\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_1}}^{ʁ+}}\right)^{\overline{\overline{Э_s}}}\right)^{\frac{1}{\overline{\overline{Э_s}}}}}\right]\right) \end{array}\right).$$

Definition 5. Consider $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}} = \left(\left(\left[\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-}, \overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я+}\right], \left[\overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ-}, \overbrace{Ҿ_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ+}\right]\right), \left(\left[\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-}, \overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я+}\right], \left[\overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ-}, \overbrace{ҧ_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ+}\right]\right)\right), \text{ɯ} = 1, 2, .., p$ as a finite collection of CIVAIFVs, the CIVAIFDHM operator is defined as

$$CIVAIFDHM^{\mathfrak{un}}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2, \dots, \overline{\overline{\mathbb{A}_f}}^p\right) = \frac{1}{{}^{\circ}\mathbb{C}_p^{\mathfrak{un}}} \bigoplus_{1\le i_1\le\cdots\le i_{\mathfrak{un}}\le p} \left(\bigotimes_{\emptyset=1}^{\mathfrak{un}} \overline{\overline{\mathbb{A}_f}}^{i_\emptyset}\right)^{\frac{1}{\mathfrak{un}}}.$$

Observed that ${}^{\circ}\mathbb{C}_p^{\mathfrak{un}} = \frac{p!}{\mathfrak{un}!(p-\mathfrak{un})!}$.

Theorem 1. Consider $\overline{\overline{\mathbb{A}_f}}^{\mathfrak{un}} = \left(\left(\left[\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{\mathfrak{un}}}}}}^{\text{я}-}, \overbrace{\varrho_{\overline{\overline{\mathbb{A}_{\mathfrak{un}}}}}}^{\text{я}+}\right], \left[\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{\mathfrak{un}}}}}}^{\text{ʁ}-}, \overbrace{\varrho_{\overline{\overline{\mathbb{A}_{\mathfrak{un}}}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\eta_{\overline{\overline{\mathbb{A}_{\mathfrak{un}}}}}}^{\text{я}-}, \overbrace{\eta_{\overline{\overline{\mathbb{A}_{\mathfrak{un}}}}}}^{\text{я}+}\right], \left[\overbrace{\eta_{\overline{\overline{\mathbb{A}_{\mathfrak{un}}}}}}^{\text{ʁ}-}, \overbrace{\eta_{\overline{\overline{\mathbb{A}_{\mathfrak{un}}}}}}^{\text{ʁ}+}\right]\right)\right), \mathfrak{un} = 1, 2, \dots, p$ as a finite collection of CIVAIFVs. Then, the finalized value of $CIVAIFDHM^{\mathfrak{un}}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2, \dots, \overline{\overline{\mathbb{A}_f}}^p\right)$ is also CIVAIFV, such as

$$CIVAIFDHM^{\mathfrak{un}}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2, \dots, \overline{\overline{\mathbb{A}_f}}^p\right),$$

$$
= \left(\begin{array}{c}
\left(\begin{array}{c}
\left[\begin{array}{c}
1 - \left(1 \Big/ 1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{1 - \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right), \\
1 - \left(1 \Big/ 1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{1 - \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}+}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right)
\end{array} \right], \\
\left[\begin{array}{c}
1 - \left(1 \Big/ 1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{1 - \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}-}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right), \\
1 - \left(1 \Big/ 1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{1 - \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}+}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right)
\end{array} \right]
\end{array} \right), \\
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(1 \Big/ 1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}{1 - \overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right), \\
\left(1 \Big/ 1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}+}}{1 - \overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}+}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right)
\end{array} \right], \\
\left[\begin{array}{c}
\left(1 \Big/ 1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}-}}{1 - \overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}-}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right), \\
\left(1 \Big/ 1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}+}}{1 - \overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}+}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right)
\end{array} \right]
\end{array} \right)
\end{array} \right).
$$

The proof of Theorem 1 is given in Appendix A.

Further, we aim to evaluate the technique of $CIVAIFDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2, \dots, \overline{\overline{\mathbb{A}_f}}^p\right)$ with the help of example, for this, we consider four CIVAIF values, such as: $\overline{\overline{\mathbb{A}_1}}^{\text{ɯ}} = \left(([0.3,0.4],[0.4,0.5]),([0.1,0.2],[0.1,0.2])\right), \overline{\overline{\mathbb{A}_2}}^{\text{ɯ}} = \left(([0.31,0.41],[0.41,0.51]),([0.11,0.21],[0.11,0.21])\right), \overline{\overline{\mathbb{A}_3}}^{\text{ɯ}} = \left(([0.32,0.42],[0.42,0.52]),([0.12,0.22],[0.12,0.22])\right)$ and $\overline{\overline{\mathbb{A}_4}}^{\text{ɯ}} = \left(([0.33,0.43],[0.43,0.53]),([0.13,0.23],[0.13,0.23])\right)$ with $\text{ɯ} = 2$ and $\overline{\overline{\Im_s}} = 2$, thus

$$CIVAIFDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2, \dots, \overline{\overline{\mathbb{A}_f}}^p\right) = \left(([0.3147,0.4147],[0.4147,0.5148]),([0.1149,0.2150],[0.1149,0.2150])\right).$$

Moreover, for this operator, the idempotency, monotonicity, and boundedness can be given as.

Property 1. Consider $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}} = \left(\left(\left[\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, \dots, p$ as a finite collection of CIVAIFVs. If $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}} = \overline{\overline{\mathbb{A}_f}} = \left(\left(\left[\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}}}^{\text{я}-}, \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}}}^{\text{я}+}\right], \left[\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}}}^{\text{ʁ}-}, \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, \dots, p$, then

$$CIVAIFDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2, \dots, \overline{\overline{\mathbb{A}_f}}^p\right) = \overline{\overline{\mathbb{A}_f}}.$$

The proof of Property 1 is given in Appendix B.

Property 2. Consider $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}} = \left(\left(\left[\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, \dots, p$ as a finite collection of CIVAIFVs. If $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}} \leq \overline{\overline{\mathbb{A}_f}}^{\text{ɯ}*}$, thus

$$CIVAIFDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2, \dots, \overline{\overline{\mathbb{A}_f}}^p\right) \leq CIVAIFDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1^*}, \overline{\overline{\mathbb{A}_f}}^{2^*}, \dots, \overline{\overline{\mathbb{A}_f}}^{p^*}\right).$$

The proof of Property 2 is given in Appendix C.

Property 3. Consider $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}} = \left(\left(\left[\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, \dots, p$ as a finite collection of CIVAIFVs. If

$$\overline{\overline{\mathbb{A}_f}}^{-} = \begin{pmatrix} \left(\left[\min_{\text{ɯ}} \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \min_{\text{ɯ}} \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\min_{\text{ɯ}} \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \min_{\text{ɯ}} \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right), \\ \left(\left[\max_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \max_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\max_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \max_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right) \end{pmatrix}$$

and

$$\overline{\overline{\mathbb{A}_f}}^{+} = \begin{pmatrix} \left(\left[\max_{\text{ɯ}} \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \max_{\text{ɯ}} \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\max_{\text{ɯ}} \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \max_{\text{ɯ}} \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right), \\ \left(\left[\min_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-}, \min_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right], \left[\min_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-}, \min_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right) \end{pmatrix}, \text{ɯ} = 1, 2, \dots, p, \text{ thus}$$

$$\overline{\overline{\mathbb{A}}}_f^{\,-} \le CIVAIFDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{\,1}, \overline{\overline{\mathbb{A}}}_f^{\,2}, \dots, \overline{\overline{\mathbb{A}}}_f^{\,p}\right) \le \overline{\overline{\mathbb{A}}}_f^{\,+}.$$

The proof of Property 3 is given in Appendix D.

Definition 6. Consider $\overline{\overline{\mathbb{A}}}_f^{\,\text{ɯ}} = \left(\left(\left[\overbrace{\text{ϱ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ϱ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ϱ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ϱ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, .., p$ as a finite collection of CIVAIFVs, the CIVAIFWDPHM operator is defined as

$$CIVAIFWDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{\,1}, \overline{\overline{\mathbb{A}}}_f^{\,2}, \dots, \overline{\overline{\mathbb{A}}}_f^{\,p}\right) = \begin{cases} \dfrac{\bigoplus_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(\left(\bigotimes_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\mathbb{A}}}_f^{\,i_\emptyset}\right)^{\frac{1}{\text{ɯ}}}\right)}{{}^{\circ}\mathrm{C}_{p-1}^{\text{ɯ}}} & 1 \le \text{ɯ} < p \\ \bigotimes_{\emptyset=1}^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{\,\emptyset}\right)^{\left(\frac{1-\overline{\overline{\partial_{wv}^{\emptyset}}}}{p-1}\right)} & \text{ɯ} = p \end{cases}.$$

Observed that ${}^{\circ}\mathrm{C}_{p-1}^{\text{ɯ}} = \frac{(p-1)!}{\text{ɯ}!(p-1-\text{ɯ})!}$ with a priority degree $\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}$, where $\overline{\overline{\partial_{wv}^{\iota_\emptyset}}} = \frac{\overline{\overline{\pi_{wv}^{\iota_\emptyset}}}}{\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\pi_{wv}^{\iota_J}}}}$, $\overline{\overline{\pi_{wv}^{\iota_1}}} = 1$ and $\overline{\overline{\pi_{wv}^{\iota_\emptyset}}} = \prod_{j=1}^{\emptyset}\overline{\overline{S_{sv}}}\left(\overline{\overline{\mathbb{A}}}_f^{\,j}\right)$.

Theorem 2. Consider $\overline{\overline{\mathbb{A}}}_f^{\,\text{ɯ}} = \left(\left(\left[\overbrace{\text{ϱ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ϱ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ϱ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ϱ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, \dots, p$ as a finite collection of CIVAIFVs. Then, the finalized value of $CIVAIFWDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{\,1}, \overline{\overline{\mathbb{A}}}_f^{\,2}, \dots, \overline{\overline{\mathbb{A}}}_f^{\,p}\right)$ is also a CIVAIFV, such as

$$CIVAIFWDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{\,1}, \overline{\overline{\mathbb{A}}}_f^{\,2}, \dots, \overline{\overline{\mathbb{A}}}_f^{\,p}\right) = \frac{\bigoplus_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(\left(\bigotimes_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\mathbb{A}}}_f^{\,i_\emptyset}\right)^{\frac{1}{\text{ɯ}}}\right)}{{}^{\circ}\mathrm{C}_{p-1}^{\text{ɯ}}}$$

$$
= \left(\begin{array}{c} \left(\left[\begin{array}{c} 1 - \left(1 \Big/ \left(1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 - \sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\partial_{wv}^{\iota_\emptyset}}} \right) \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{1 - \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right) \right), \\ 1 - \left(1 \Big/ \left(1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 - \sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\partial_{wv}^{\iota_\emptyset}}} \right) \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{1 - \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right) \right) \end{array} \right], \left[\begin{array}{c} 1 - \left(1 \Big/ \left(1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 - \sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\partial_{wv}^{\iota_\emptyset}}} \right) \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{1 - \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right) \right), \\ 1 - \left(1 \Big/ \left(1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 - \sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\partial_{wv}^{\iota_\emptyset}}} \right) \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{1 - \overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right) \right) \end{array} \right] \right), \\ \left(\left[\begin{array}{c} \left(1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 - \sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\partial_{wv}^{\iota_\emptyset}}} \right) \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}{1 - \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right), \\ \left(1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 - \sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\partial_{wv}^{\iota_\emptyset}}} \right) \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}{1 - \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right) \end{array} \right], \left[\begin{array}{c} \left(1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 - \sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\partial_{wv}^{\iota_\emptyset}}} \right) \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}{1 - \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right), \\ \left(1 + \left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}} \sum_{1 \le i_1 \le \cdots \le i_{\text{ɯ}} \le p} \left(1 - \sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\partial_{wv}^{\iota_\emptyset}}} \right) \left(1 \Big/ \sum_{\emptyset=1}^{\text{ɯ}} \left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}{1 - \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}} \right)^{\overline{\overline{\text{Э}_s}}} \right) \right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}} \right) \end{array} \right] \right) \end{array} \right).
$$

$$CIVAIFWDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1},\overline{\overline{\mathbb{A}}}_f^{2},\dots,\overline{\overline{\mathbb{A}}}_f^{p}\right)=\otimes_{\emptyset=1}^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{\emptyset}\right)^{\left(\frac{1-\overline{\overline{\partial}}_{wv}^{\emptyset}}{p-1}\right)}$$

$$=\left(\begin{array}{l}\left(\begin{array}{l}\left[1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial}}_{wv}^{\text{ɯ}}}{p-1}\right)\left(\frac{1-\overset{я-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}{\overset{я-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}\right)^{\overline{\overline{\ni}}_s}\right)^{\frac{1}{\overline{\overline{\ni}}_s}},1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial}}_{wv}^{\text{ɯ}}}{p-1}\right)\left(\frac{1-\overset{я+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}{\overset{я+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}\right)^{\overline{\overline{\ni}}_s}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}\right],\\ \left[1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial}}_{wv}^{\text{ɯ}}}{p-1}\right)\left(\frac{1-\overset{ʁ-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}{\overset{ʁ-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}\right)^{\overline{\overline{\ni}}_s}\right)^{\frac{1}{\overline{\overline{\ni}}_s}},1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial}}_{wv}^{\text{ɯ}}}{p-1}\right)\left(\frac{1-\overset{ʁ+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}{\overset{ʁ+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}\right)^{\overline{\overline{\ni}}_s}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}\right]\end{array}\right),\\ \left(\begin{array}{l}\left[1-\left(1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial}}_{wv}^{\text{ɯ}}}{p-1}\right)\left(\frac{\overset{я-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}{1-\overset{я-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}\right)^{\overline{\overline{\ni}}_s}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}\right),1-\left(1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial}}_{wv}^{\text{ɯ}}}{p-1}\right)\left(\frac{\overset{я+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}{1-\overset{я+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}\right)^{\overline{\overline{\ni}}_s}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}\right)\right],\\ \left[1-\left(1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial}}_{wv}^{\text{ɯ}}}{p-1}\right)\left(\frac{\overset{ʁ-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}{1-\overset{ʁ-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}\right)^{\overline{\overline{\ni}}_s}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}\right),1-\left(1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial}}_{wv}^{\text{ɯ}}}{p-1}\right)\left(\frac{\overset{ʁ+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}{1-\overset{ʁ+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}\right)^{\overline{\overline{\ni}}_s}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}\right)\right]\end{array}\right)\end{array}\right).$$

Property 4. Consider $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}}=\left(\left(\left[\overset{я-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{я+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right],\left[\overset{ʁ-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{ʁ+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right]\right),\left(\left[\overset{я-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{я+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right],\left[\overset{ʁ-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{ʁ+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right]\right)\right),\text{ɯ}=1,2,\dots,p$ as a finite collection of CIVAIFVs. If $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}}=\overline{\overline{\mathbb{A}}}_f=\left(\left(\left[\overset{я-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}},\overset{я+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}}\right],\left[\overset{ʁ-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}},\overset{ʁ+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}}\right]\right),\left(\left[\overset{я-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}},\overset{я+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}}\right],\left[\overset{ʁ-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}},\overset{ʁ+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}}\right]\right)\right),\text{ɯ}=1,2,\dots,p$, then

$$CIVAIFWDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1},\overline{\overline{\mathbb{A}}}_f^{2},\dots,\overline{\overline{\mathbb{A}}}_f^{p}\right)=\overline{\overline{\mathbb{A}}}_f\,.$$

Property 5. Consider $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}}=\left(\left(\left[\overset{я-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{я+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right],\left[\overset{ʁ-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{ʁ+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right]\right),\left(\left[\overset{я-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{я+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right],\left[\overset{ʁ-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{ʁ+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right]\right)\right),\text{ɯ}=1,2,\dots,p$ as a finite collection of CIVAIFVs. If $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}}\le\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}*}$, thus

$$CIVAIFWDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1},\overline{\overline{\mathbb{A}}}_f^{2},\dots,\overline{\overline{\mathbb{A}}}_f^{p}\right)\le CIVAIFWDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1^*},\overline{\overline{\mathbb{A}}}_f^{2^*},\dots,\overline{\overline{\mathbb{A}}}_f^{p^*}\right).$$

Property 6. Consider $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}}=\left(\left(\left[\overset{я-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{я+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right],\left[\overset{ʁ-}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{ʁ+}{\widetilde{\text{ȩ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right]\right),\left(\left[\overset{я-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{я+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right],\left[\overset{ʁ-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}},\overset{ʁ+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}\right]\right)\right),\text{ɯ}=1,2,\dots,p$ as a finite collection of CIVAIFVs. If

$$\overline{\overline{\mathbb{A}_f}}^{-} = \left(\begin{array}{c}\left(\left[\min_{ɯ} \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}, \min_{ɯ} \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right], \left[\min_{ɯ} \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}, \min_{ɯ} \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right), \\ \left(\left[\max_{ɯ} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}, \max_{ɯ} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right], \left[\max_{ɯ} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}, \max_{ɯ} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right)\end{array}\right) \quad \text{and}$$

$$\overline{\overline{\mathbb{A}_f}}^{+} = \left(\begin{array}{c}\left(\left[\max_{ɯ} \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}, \max_{ɯ} \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right], \left[\max_{ɯ} \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}, \max_{ɯ} \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right), \\ \left(\left[\min_{ɯ} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}, \min_{ɯ} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right], \left[\min_{ɯ} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}, \min_{ɯ} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right)\end{array}\right), ɯ = 1, 2, \ldots, p, \text{ thus}$$

$$\overline{\overline{\mathbb{A}_f}}^{-} \leq CIVAIFWDPHM^{ɯ}\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p}\right) \leq \overline{\overline{\mathbb{A}_f}}^{+}.$$

Definition 7. Consider $\overline{\overline{\mathbb{A}_f}}^{ɯ} = \left(\left(\left[\overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}, \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right], \left[\overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}, \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right)\right), ɯ = 1, 2, .., p$ as a finite collection of CIVAIFVs, the CIVAIFDDHM operator is defined such as

$$CIVAIFDDHM^{ɯ}\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p}\right) = \left(\bigotimes_{1 \leq i_1 \leq \cdots \leq i_{ɯ} \leq p} \left(\frac{\bigoplus_{\emptyset=1}^{ɯ} \overline{\overline{\mathbb{A}_f}}^{i_{\emptyset}}}{ɯ}\right)^{\frac{1}{ɯ}}\right)^{\frac{1}{{}^{\circ}\mathbb{C}_p^{ɯ}}}.$$

Observed that ${}^{\circ}\mathbb{C}_p^{ɯ} = \frac{p!}{ɯ!(p-ɯ)!}$.

Theorem 3. Consider $\overline{\overline{\mathbb{A}_f}}^{ɯ} = \left(\left(\left[\overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}, \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right], \left[\overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}, \overbrace{\text{ɐ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right)\right), ɯ = 1, 2, \ldots, p$ as a finite collection of CIVAIFVs. Then the finalized value of $CIVAIFDDHM^{ɯ}\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p}\right)$ is also a CIVAIFV, such as

$$CIVAIFDDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1},\overline{\overline{\mathbb{A}_f}}^{2},\dots,\overline{\overline{\mathbb{A}_f}}^{p}\right)$$

$$=\left(\begin{matrix}\left(\begin{matrix}\left[1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}{1-\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}}\right)\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}},1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}+}}{1-\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}_s}}}\right)\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}\right],\\ \left[1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}-}}{1-\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}_s}}}\right)\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}},1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}+}}{1-\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}_s}}}\right)\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}\right]\end{matrix}\right),\\ \left(\begin{matrix}\left[\begin{matrix}1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}}\right)\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}\right),\\ 1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}+}}{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}_s}}}\right)\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}\right)\end{matrix}\right],\\ \left[\begin{matrix}1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}-}}{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Э}_s}}}\right)\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}\right),\\ 1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}+}}{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Э}_s}}}\right)\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}\right)\end{matrix}\right]\end{matrix}\right)\end{matrix}\right).$$

Further, we aim to evaluate the technique of $CIVAIFDDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1},\overline{\overline{\mathbb{A}_f}}^{2},\dots,\overline{\overline{\mathbb{A}_f}}^{p}\right)$ with the help of example, for this, we consider four CIVAIF values, such as: $\overline{\overline{\mathbb{A}_1}}^{\text{ɯ}}=\left(([0.3,0.4],[0.4,0.5]),([0.1,0.2],[0.1,0.2])\right)$, $\overline{\overline{\mathbb{A}_2}}^{\text{ɯ}}=\left(([0.31,0.41],[0.41,0.51]),([0.11,0.21],[0.11,0.21])\right)$, $\overline{\overline{\mathbb{A}_3}}^{\text{ɯ}}=\left(([0.32,0.42],[0.42,0.52]),([0.12,0.22],[0.12,0.22])\right)$ and $\overline{\overline{\mathbb{A}_4}}^{\text{ɯ}}=\left(([0.33,0.43],[0.43,0.53]),([0.13,0.23],[0.13,0.23])\right)$ with $\text{ɯ}=2$ and $\overline{\overline{\text{Э}_s}}=2$, thus

$$CIVAIFDDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1},\overline{\overline{\mathbb{A}_f}}^{2},\dots,\overline{\overline{\mathbb{A}_f}}^{p}\right)=\left(([0.3150,0.4151],[0.4151,0.5151]),([0.1141,0.2145],[0.1141,0.2145])\right).$$

Property 7. Consider $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}=\left(\left(\left[\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-},\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right],\left[\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-},\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right),\left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-},\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}+}\right],\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}-},\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{ʁ}+}\right]\right)\right)$, $\text{ɯ}=1,2,\dots,p$ as a finite collection of CIVAIFVs. If $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}=\overline{\overline{\mathbb{A}_f}}=\left(\left(\left[\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}}}}}^{\text{я}-},\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}}}}}^{\text{я}+}\right],\left[\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}}}}}^{\text{ʁ}-},\overbrace{\text{Ꞓ}_{\overline{\overline{\mathbb{A}}}}}^{\text{ʁ}+}\right]\right),\left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}}}^{\text{я}-},\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}}}^{\text{я}+}\right],\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}}}^{\text{ʁ}-},\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}}}^{\text{ʁ}+}\right]\right)\right)$, $\text{ɯ}=1,2,\dots,p$, then

$$CIVAIFDDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1}, \overline{\overline{\mathbb{A}}}_f^{2}, \dots, \overline{\overline{\mathbb{A}}}_f^{p}\right) = \overline{\overline{\mathbb{A}}}_f .$$

Property 8. Consider $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}} = \left(\left(\left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, \dots, p$ as a finite collection of CIVAIFVs. If $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}} \leq \overline{\overline{\mathbb{A}}}_f^{\text{ɯ}*}$, thus

$$CIVAIFDDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1}, \overline{\overline{\mathbb{A}}}_f^{2}, \dots, \overline{\overline{\mathbb{A}}}_f^{p}\right) \leq CIVAIFDDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1^*}, \overline{\overline{\mathbb{A}}}_f^{2^*}, \dots, \overline{\overline{\mathbb{A}}}_f^{p^*}\right).$$

Property 9. Consider $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}} = \left(\left(\left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, \dots, p$ as a finite collection of CIVAIFVs. If

$$\overline{\overline{\mathbb{A}}}_f^{-} = \begin{pmatrix} \left(\left[\min_{\text{ɯ}} \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \min_{\text{ɯ}} \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\min_{\text{ɯ}} \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \min_{\text{ɯ}} \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right), \\ \left(\left[\max_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \max_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\max_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \max_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right) \end{pmatrix}$$

and

$$\overline{\overline{\mathbb{A}}}_f^{+} = \begin{pmatrix} \left(\left[\max_{\text{ɯ}} \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \max_{\text{ɯ}} \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\max_{\text{ɯ}} \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \max_{\text{ɯ}} \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right), \\ \left(\left[\min_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \min_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\min_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \min_{\text{ɯ}} \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right) \end{pmatrix}, \text{ɯ} = 1, 2, \dots, p, \text{ thus}$$

$$\overline{\overline{\mathbb{A}}}_f^{-} \leq CIVAIFDDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1}, \overline{\overline{\mathbb{A}}}_f^{2}, \dots, \overline{\overline{\mathbb{A}}}_f^{p}\right) \leq \overline{\overline{\mathbb{A}}}_f^{+}.$$

Definition 8. Consider $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}} = \left(\left(\left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, .., p$ as a finite collection of CIVAIFVs, the CIVAIFWDDPHM operator is defined as

$$CIVAIFWDDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1}, \overline{\overline{\mathbb{A}}}_f^{2}, \dots, \overline{\overline{\mathbb{A}}}_f^{p}\right) = \begin{cases} \dfrac{\bigotimes_{1 \leq i_1 \leq \cdots \leq i_{\text{ɯ}} \leq p} \left(\dfrac{\bigoplus_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\mathbb{A}}}_f^{i_\emptyset}}{\text{ɯ}}\right)^{\left(1 - \sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)}}{{}^{\circ}\mathrm{C}_{p-1}^{\text{ɯ}}} & 1 \leq \text{ɯ} < p \\ \bigoplus_{\emptyset=1}^{\text{ɯ}} \left(\dfrac{1 - \overline{\overline{\partial_{wv}^{\emptyset}}}}{p-1}\right)\left(\overline{\overline{\mathbb{A}}}_f^{\emptyset}\right) & \text{ɯ} = p \end{cases}.$$

Observed that ${}^{\circ}\mathrm{C}_{p-1}^{\text{ɯ}} = \frac{(p-1)!}{\text{ɯ}!(p-1-\text{ɯ})!}$ with a priority degree $\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}$, where $\overline{\overline{\partial_{wv}^{\iota_\emptyset}}} = \frac{\overline{\overline{\pi_{wv}^{\iota_\emptyset}}}}{\sum_{\emptyset=1}^{\text{ɯ}} \overline{\overline{\pi_{wv}^{\iota_J}}}}$, $\overline{\overline{\pi_{wv}^{\iota_1}}} = 1$ and $\overline{\overline{\pi_{wv}^{\iota_\emptyset}}} = \prod_{j=1}^{\emptyset} \overline{\overline{S_{sv}}}\left(\overline{\overline{\mathbb{A}}}_f^{j}\right)$.

Theorem 4. Consider $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}} = \left(\left(\left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right), \left(\left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{я}+}\right], \left[\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}-}, \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{\text{ʁ}+}\right]\right)\right), \text{ɯ} = 1, 2, \dots, p$ as a finite collection of CIVAIFVs. Then, the finalized value of

$CIVAIFWDDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1}, \overline{\overline{\mathbb{A}}}_f^{2}, \ldots, \overline{\overline{\mathbb{A}}}_f^{p}\right)$ is also a CIVAIFV, such as

$$CIVAIFWDDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}}}_f^{1}, \overline{\overline{\mathbb{A}}}_f^{2}, \ldots, \overline{\overline{\mathbb{A}}}_f^{p}\right) = \frac{\bigotimes_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(\frac{\bigoplus_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\mathbb{A}}}_f^{i_\emptyset}}{\text{ɯ}}\right)^{\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)}}{{}^{\circ}C_{p-1}^{\text{ɯ}}}$$

$$= \left(\begin{array}{c}
\left(\begin{array}{c}
\left[\begin{array}{c}
1\Big/\left(1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}{1-\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}, \\
1\Big/\left(1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}{1-\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}
\end{array}\right], \\
\left[\begin{array}{c}
1\Big/\left(1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ᴚ}-}}{1-\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ᴚ}-}}\right)^{\overline{\overline{\text{Э}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}, \\
1\Big/\left(1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ᴚ}+}}{1-\overbrace{\text{ȩ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ᴚ}+}}\right)^{\overline{\overline{\text{Э}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}
\end{array}\right]
\end{array}\right), \\
\left(\begin{array}{c}
\left[\begin{array}{c}
1-\left(1\Big/\left(1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right), \\
1-\left(1\Big/\left(1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Э}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right)
\end{array}\right], \\
\left[\begin{array}{c}
1-\left(1\Big/\left(1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ᴚ}-}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ᴚ}-}}\right)^{\overline{\overline{\text{Э}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right), \\
1-\left(1\Big/\left(1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1-\sum_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\partial_{wv}^{\iota_\emptyset}}}\right)\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ᴚ}+}}{\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ᴚ}+}}\right)^{\overline{\overline{\text{Э}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right)
\end{array}\right]
\end{array}\right)
\end{array}\right).$$

$$CIVAIFWDDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1},\overline{\overline{\mathbb{A}_f}}^{2},\dots,\overline{\overline{\mathbb{A}_f}}^{p}\right)=\bigoplus_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overline{\overline{\partial_{wv}^{\emptyset}}}}{p-1}\right)\left(\overline{\overline{\mathbb{A}_f}}^{\emptyset}\right)$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}\left[1-\left(1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial_{wv}^{\text{ɯ}}}}}{p-1}\right)\left(\frac{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-}}{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-}}\right)^{\overline{\overline{\ni_s}}}\right)^{\frac{1}{\overline{\overline{\ni_s}}}}\right),1-\left(1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial_{wv}^{\text{ɯ}}}}}{p-1}\right)\left(\frac{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}}{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}}\right)^{\overline{\overline{\ni_s}}}\right)^{\frac{1}{\overline{\overline{\ni_s}}}}\right)\right],\\ \left[1-\left(1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial_{wv}^{\text{ɯ}}}}}{p-1}\right)\left(\frac{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-}}{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-}}\right)^{\overline{\overline{\ni_s}}}\right)^{\frac{1}{\overline{\overline{\ni_s}}}}\right),1-\left(1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial_{wv}^{\text{ɯ}}}}}{p-1}\right)\left(\frac{\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}}{1-\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}}\right)^{\overline{\overline{\ni_s}}}\right)^{\frac{1}{\overline{\overline{\ni_s}}}}\right)\right]\end{array}\right),\\ \left(\begin{array}{c}\left[1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial_{wv}^{\text{ɯ}}}}}{p-1}\right)\left(\frac{1-\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-}}{\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-}}\right)^{\overline{\overline{\ni_s}}}\right)^{\frac{1}{\overline{\overline{\ni_s}}}},1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial_{wv}^{\text{ɯ}}}}}{p-1}\right)\left(\frac{1-\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}}{\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}}\right)^{\overline{\overline{\ni_s}}}\right)^{\frac{1}{\overline{\overline{\ni_s}}}}\right],\\ \left[1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial_{wv}^{\text{ɯ}}}}}{p-1}\right)\left(\frac{1-\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-}}{\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-}}\right)^{\overline{\overline{\ni_s}}}\right)^{\frac{1}{\overline{\overline{\ni_s}}}},1\Big/1+\left(\sum_{\text{ɯ}=1}^{p}\left(\frac{1-\overline{\overline{\partial_{wv}^{\text{ɯ}}}}}{p-1}\right)\left(\frac{1-\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}}{\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}}\right)^{\overline{\overline{\ni_s}}}\right)^{\frac{1}{\overline{\overline{\ni_s}}}}\right]\end{array}\right)\end{array}\right).$$

Property 10. Consider $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}=\left(\left(\left[\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-},\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}\right],\left[\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-},\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}\right]\right),\left(\left[\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-},\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}\right],\left[\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-},\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}\right]\right)\right),\text{ɯ}=1,2,\dots,p$ as a finite collection of CIVAIFVs. If $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}=\overline{\overline{\mathbb{A}_f}}=\left(\left(\left[\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}}}}}^{я-},\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}}}}}^{я+}\right],\left[\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}}}}}^{ʁ-},\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}}}}}^{ʁ+}\right]\right),\left(\left[\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}}}}}^{я-},\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}}}}}^{я+}\right],\left[\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}}}}}^{ʁ-},\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}}}}}^{ʁ+}\right]\right)\right),\text{ɯ}=1,2,\dots,p$, then

$$CIVAIFWDDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1},\overline{\overline{\mathbb{A}_f}}^{2},\dots,\overline{\overline{\mathbb{A}_f}}^{p}\right)=\overline{\overline{\mathbb{A}_f}}.$$

Property 11. Consider $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}=\left(\left(\left[\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-},\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}\right],\left[\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-},\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}\right]\right),\left(\left[\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-},\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}\right],\left[\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-},\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}\right]\right)\right),\text{ɯ}=1,2,\dots,p$ as a finite collection of CIVAIFVs. If $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}\leq\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}*}$, thus

$$CIVAIFWDDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1},\overline{\overline{\mathbb{A}_f}}^{2},\dots,\overline{\overline{\mathbb{A}_f}}^{p}\right)\leq CIVAIFWDDPHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1^*},\overline{\overline{\mathbb{A}_f}}^{2^*},\dots,\overline{\overline{\mathbb{A}_f}}^{p^*}\right).$$

Property 12. Consider $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}}=\left(\left(\left[\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-},\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}\right],\left[\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-},\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}\right]\right),\left(\left[\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-},\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}\right],\left[\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-},\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}\right]\right)\right),\text{ɯ}=1,2,\dots,p$ as a finite collection of CIVAIFVs. If $\overline{\overline{\mathbb{A}_f}}^{-}=\left(\begin{array}{c}\left(\left[\min_{\text{ɯ}}\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-},\min_{\text{ɯ}}\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}\right],\left[\min_{\text{ɯ}}\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-},\min_{\text{ɯ}}\overbrace{\text{Ҿ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}\right]\right),\\ \left(\left[\max_{\text{ɯ}}\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я-},\max_{\text{ɯ}}\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{я+}\right],\left[\max_{\text{ɯ}}\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ-},\max_{\text{ɯ}}\overbrace{\text{Ҧ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{ʁ+}\right]\right)\end{array}\right)$ and

$$\overline{\overline{\mathbb{A}_f}}^{+}=\left(\begin{array}{c}\left(\left[\max_{ɯ}\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-},\max_{ɯ}\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right],\left[\max_{ɯ}\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-},\max_{ɯ}\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right),\\ \left(\left[\min_{ɯ}\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-},\min_{ɯ}\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}\right],\left[\min_{ɯ}\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-},\min_{ɯ}\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}\right]\right)\end{array}\right), ɯ=1,2,\dots,p, \text{ thus}$$

$$\overline{\overline{\mathbb{A}_f}}^{-}\leq CIVAIFWDDPHM^{ɯ}\left(\overline{\overline{\mathbb{A}_f}}^{1},\overline{\overline{\mathbb{A}_f}}^{2},\dots,\overline{\overline{\mathbb{A}_f}}^{p}\right)\leq\overline{\overline{\mathbb{A}_f}}^{+}.$$

4. Strategic decision-making problems based on proposed operators

The decision-making technique contains many types, and the multi-attribute decision-making technique is one of them, which is for accessing the most preferable decision among the collection of finite number of alternatives. For this, in this section, we perform the strategic decision-making framework under the consideration of the designed aggregation operators, called the CIVAIFDHM operator, CIVAIFDDHM operator, CIVAIFWDPHM operator, and CIVAIFWDDPHM operator to designate the hegemony and rationality of the invented models.

For a MADM problem, the collection of alternatives is $\overline{\overline{\mathbb{A}_f}}^{1},\overline{\overline{\mathbb{A}_f}}^{2},\dots,\overline{\overline{\mathbb{A}_f}}^{p}$ and the collection of finite attributes is $\overline{\overline{\mathbb{A}_{AT}}}^{1},\overline{\overline{\mathbb{A}_{AT}}}^{2},\dots,\overline{\overline{\mathbb{A}_{AT}}}^{q}$. For each attribute, its weight is w_j and $\sum_{j=1}^{q}w_j=1$. Moreover, the evaluation value of alternative $\overline{\overline{\mathbb{A}_f}}^{\emptyset}$ under the attribute $\overline{\overline{\mathbb{A}_{AT}}}^{j}$ is expressed by the CIVAIFSs with the truth grade in the shape: $\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}(\overline{\overline{ɯ}}_{\emptyset})=\left(\left[\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}(\overline{\overline{ɯ}}_{\emptyset}),\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}(\overline{\overline{ɯ}}_{\emptyset})\right],\left[\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}(\overline{\overline{ɯ}}_{\emptyset}),\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}(\overline{\overline{ɯ}}_{\emptyset})\right]\right)$ and the falsity grade in the shape: $\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}(\overline{\overline{ɯ}}_{\emptyset})=\left(\left[\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}(\overline{\overline{ɯ}}_{\emptyset}),\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}(\overline{\overline{ɯ}}_{\emptyset})\right],\left[\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}(\overline{\overline{ɯ}}_{\emptyset}),\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}(\overline{\overline{ɯ}}_{\emptyset})\right]\right)$ with two valuable and valid conditions, such as $0\leq\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}(\overline{\overline{ɯ}}_{\emptyset})+\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}(\overline{\overline{ɯ}}_{\emptyset})\leq 1$ and $0\leq\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}(\overline{\overline{ɯ}}_{\emptyset})+\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}(\overline{\overline{ɯ}}_{\emptyset})\leq 1$. Moreover, we revised the basic condition of the refusal grade, such as $\overbrace{ɼ_{\overline{\overline{\mathbb{A}_{ɯ}}}}}(\overline{\overline{ɯ}}_{\emptyset})=\left(\left[\overbrace{ɼ_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}(\overline{\overline{ɯ}}_{\emptyset}),\overbrace{ɼ_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}(\overline{\overline{ɯ}}_{\emptyset})\right],\left[\overbrace{ɼ_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}(\overline{\overline{ɯ}}_{\emptyset}),\overbrace{ɼ_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}(\overline{\overline{ɯ}}_{\emptyset})\right]\right)=\left(\left[1-\left(\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}(\overline{\overline{ɯ}}_{\emptyset})+\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я+}(\overline{\overline{ɯ}}_{\emptyset})\right),1-\left(\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}(\overline{\overline{ɯ}}_{\emptyset})+\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{я-}(\overline{\overline{ɯ}}_{\emptyset})\right)\right],\left[1-\left(\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}(\overline{\overline{ɯ}}_{\emptyset})+\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ+}(\overline{\overline{ɯ}}_{\emptyset})\right),1-\left(\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}(\overline{\overline{ɯ}}_{\emptyset})+\overbrace{\eta_{\overline{\overline{\mathbb{A}_{ɯ}}}}}^{ʁ-}(\overline{\overline{ɯ}}_{\emptyset})\right)\right]\right)$ and the simple form of CIVAIFV is expressed by $\overline{\overline{\mathbb{A}_f}}^{\emptyset}=\left(\left(\left[\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{\emptyset}}}}}^{я-},\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{\emptyset}}}}}^{я+}\right],\left[\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{\emptyset}}}}}^{ʁ-},\overbrace{\varrho_{\overline{\overline{\mathbb{A}_{\emptyset}}}}}^{ʁ+}\right]\right),\left(\left[\overbrace{\eta_{\overline{\overline{\mathbb{A}_{\emptyset}}}}}^{я-},\overbrace{\eta_{\overline{\overline{\mathbb{A}_{\emptyset}}}}}^{я+}\right],\left[\overbrace{\eta_{\overline{\overline{\mathbb{A}_{\emptyset}}}}}^{ʁ-},\overbrace{\eta_{\overline{\overline{\mathbb{A}_{\emptyset}}}}}^{ʁ+}\right]\right)\right),\emptyset=1,2,\dots,p$. For this decision problem, we give the following decision steps.

Step 1. Construct the decision matrix and normalize it by

$$N=\begin{cases}\left(\left(\left[\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{я-},\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{я+}\right],\left[\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{ꓤ-},\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{ꓤ+}\right]\right),\left(\left[\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{я-},\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{я+}\right],\left[\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{ꓤ-},\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{ꓤ+}\right]\right)\right) & benefit\ type\\ \left(\left(\left[\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{я-},\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{я+}\right],\left[\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{ꓤ-},\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{ꓤ+}\right]\right),\left(\left[\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{я-},\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{я+}\right],\left[\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{ꓤ-},\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\emptyset}}}^{ꓤ+}\right]\right)\right) & cost\ type\end{cases}.$$

Step 2. Aggregate the normalized information based on the CIVAIFDHM operator, CIVAIFDDHM operator, CIVAIFWDPHM operator, and CIVAIFWDDPHM operator.
Step 3. Analyze the numerical values from CIVAIF values with the help of Score values or Accuracy values, such as, where the score value and accuracy value are given as

$$\overline{\overline{S_{sv}}}\left(\overline{\overline{\mathbb{A}_f}}^{ɯ}\right)=\frac{1}{4}\left(\left(\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{я-}+\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{ꓤ-}\right)-\left(\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{я-}+\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{ꓤ-}\right)+\left(\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{я+}+\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{ꓤ+}\right)-\left(\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{я+}+\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{ꓤ+}\right)\right)\in[-1,1].$$

$$\overline{\overline{A_{av}}}\left(\overline{\overline{\mathbb{A}_f}}^{ɯ}\right)=\frac{1}{4}\left(\left(\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{я-}+\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{ꓤ-}\right)+\left(\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{я-}+\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{ꓤ-}\right)+\left(\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{я+}+\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{ꓤ+}\right)+\left(\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{я+}+\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{ɯ}}}^{ꓤ+}\right)\right)\in[0,1].$$

Additionally, for comparing any two CIVAIFVs, we have the following characteristics, such as
(1) If $\overline{\overline{S_{sv}}}\left(\overline{\overline{\mathbb{A}_f}}^{1}\right)>\overline{\overline{S_{sv}}}\left(\overline{\overline{\mathbb{A}_f}}^{2}\right)$, thus $\overline{\overline{\mathbb{A}_f}}^{1}>\overline{\overline{\mathbb{A}_f}}^{2}$.
(2) If $\overline{\overline{S_{sv}}}\left(\overline{\overline{\mathbb{A}_f}}^{1}\right)<\overline{\overline{S_{sv}}}\left(\overline{\overline{\mathbb{A}_f}}^{2}\right)$, thus $\overline{\overline{\mathbb{A}_f}}^{1}<\overline{\overline{\mathbb{A}_f}}^{2}$.
(3) If $\overline{\overline{S_{sv}}}\left(\overline{\overline{\mathbb{A}_f}}^{1}\right)=\overline{\overline{S_{sv}}}\left(\overline{\overline{\mathbb{A}_f}}^{2}\right)$, thus
  i) If $\overline{\overline{A_{av}}}\left(\overline{\overline{\mathbb{A}_f}}^{1}\right)>\overline{\overline{A_{av}}}\left(\overline{\overline{\mathbb{A}_f}}^{2}\right)$, thus $\overline{\overline{\mathbb{A}_f}}^{1}>\overline{\overline{\mathbb{A}_f}}^{2}$.
  ii) If $\overline{\overline{A_{av}}}\left(\overline{\overline{\mathbb{A}_f}}^{1}\right)<\overline{\overline{A_{av}}}\left(\overline{\overline{\mathbb{A}_f}}^{2}\right)$, thus $\overline{\overline{\mathbb{A}_f}}^{1}<\overline{\overline{\mathbb{A}_f}}^{2}$.

Step 4. Analyze the rank of all alternatives according to their Score values for addressing the best preference among the collection of the finite number of values.

The geometrical abstract of the decision-making technique is given in Figure 2. Further, we aim to perform the application of the assessment of the best green sustainable chain by using the model of invented models. This application can show the supremacy and validity of the designed models.

5. Selection of the best green sustainable chain

Green sustainability refers to procedures or techniques that have less impact on plants, human activities, energy production, and many others. To fulfill this demand, green sustainable methods must be used. They place a high priority on preserving biodiversity, reducing waste and pollution, and safeguarding natural resources. Several kinds of environmentally friendly and sustainable techniques may be applied in various spheres of life and businesses. Here are five instances that the collection of alternatives:

(1) Renewable Energy $\left(\overline{\overline{\mathbb{A}_f}}^{1}\right)$: Using renewable energy can help reduce dependency on fossil fuels and greenhouse gas emissions. Examples of renewable energy sources include solar, wind, hydropower, and geothermal energy.

(2) Energy Efficiency $\left(\overline{\overline{\mathbb{A}}}_f^{\,2}\right)$: Using energy-efficient technology and procedures, such as smart thermostats, LED lighting, and energy-efficient appliances, can help cut down on energy usage and waste.

(3) Sustainable Transportation $\left(\overline{\overline{\mathbb{A}}}_f^{\,3}\right)$: Promoting electric automobiles, cycling, walking, and public transit can help reduce carbon emissions and reliance on fossil fuels.

(4) Sustainable Agriculture $\left(\overline{\overline{\mathbb{A}}}_f^{\,4}\right)$: Techniques like organic farming, permaculture, crop rotation, and agroforestry can improve soil health, use less chemical input, and have a less negative impact on the environment.

(5) Water Reduction and Recycling $\left(\overline{\overline{\mathbb{A}}}_f^{\,5}\right)$: Resource conservation and waste reduction may be achieved by implementing recycling programs, composting organic waste, and promoting the use of recycled products.

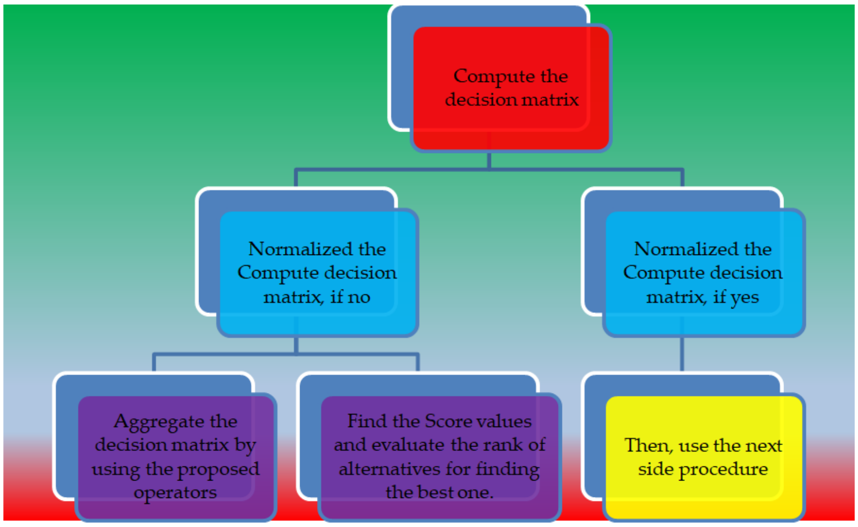

Figure 2. Graphical abstract of the decision-making algorithm.

For the selection of the best one among the above five alternatives, we have the following four attributes such as: $\overline{\overline{\mathbb{A}_{AT}}}^{1}$: Social impact, $\overline{\overline{\mathbb{A}_{AT}}}^{2}$: Environmental impact, $\overline{\overline{\mathbb{A}_{AT}}}^{3}$: Political impact, and $\overline{\overline{\mathbb{A}_{AT}}}^{4}$: Other internal and external resources. For each attribute, we have the following weight vector, such as *0.4, 0.3, 0.2, and 0.1*. Finally, to solve this problem, we have the following procedure:

Step 1. Construct the matrix. Because all attributes are beneficial, there is no need to normalize them. See Table 1.

Table 1. Original CIVAIF decision matrix.

	$\overline{\overline{\mathbb{A}_{AT}}}^{1}$	$\overline{\overline{\mathbb{A}_{AT}}}^{2}$	$\overline{\overline{\mathbb{A}_{AT}}}^{3}$	$\overline{\overline{\mathbb{A}_{AT}}}^{4}$
$\overline{\overline{\mathbb{A}_{f}}}^{1}$	$\left(\begin{pmatrix}[0.3,0.4],\\ [0.4,0.5]\end{pmatrix},\begin{pmatrix}[0.1,0.2],\\ [0.1,0.2]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.31,0.41],\\ [0.41,0.51]\end{pmatrix},\begin{pmatrix}[0.11,0.21],\\ [0.11,0.21]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.32,0.42],\\ [0.42,0.52]\end{pmatrix},\begin{pmatrix}[0.12,0.22],\\ [0.12,0.22]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.33,0.43],\\ [0.43,0.53]\end{pmatrix},\begin{pmatrix}[0.13,0.23],\\ [0.13,0.23]\end{pmatrix}\right)$

Continued on next page

	$\overline{\overline{\mathbb{A}}}_{AT}^{1}$	$\overline{\overline{\mathbb{A}}}_{AT}^{2}$	$\overline{\overline{\mathbb{A}}}_{AT}^{3}$	$\overline{\overline{\mathbb{A}}}_{AT}^{4}$
$\overline{\overline{\mathbb{A}}}_{f}^{2}$	$\left(\begin{pmatrix}[0.5,0.6],\\ [0.4,0.5]\end{pmatrix}, \begin{pmatrix}[0.2,0.3],\\ [0.1,0.3]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.51,0.61],\\ [0.41,0.51]\end{pmatrix}, \begin{pmatrix}[0.21,0.31],\\ [0.11,0.31]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.52,0.62],\\ [0.42,0.52]\end{pmatrix}, \begin{pmatrix}[0.22,0.32],\\ [0.12,0.32]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.53,0.63],\\ [0.43,0.53]\end{pmatrix}, \begin{pmatrix}[0.23,0.33],\\ [0.13,0.33]\end{pmatrix}\right)$
$\overline{\overline{\mathbb{A}}}_{f}^{3}$	$\left(\begin{pmatrix}[0.7,0.8],\\ [0.6,0.7]\end{pmatrix}, \begin{pmatrix}[0.1,0.2],\\ [0.1,0.2]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.71,0.81],\\ [0.61,0.71]\end{pmatrix}, \begin{pmatrix}[0.11,0.21],\\ [0.11,0.21]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.72,0.82],\\ [0.62,0.72]\end{pmatrix}, \begin{pmatrix}[0.12,0.22],\\ [0.12,0.22]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.73,0.83],\\ [0.63,0.73]\end{pmatrix}, \begin{pmatrix}[0.13,0.23],\\ [0.13,0.23]\end{pmatrix}\right)$
$\overline{\overline{\mathbb{A}}}_{f}^{4}$	$\left(\begin{pmatrix}[0.6,0.7],\\ [0.6,0.7]\end{pmatrix}, \begin{pmatrix}[0.1,0.2],\\ [0.1,0.2]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.61,0.71],\\ [0.61,0.71]\end{pmatrix}, \begin{pmatrix}[0.11,0.21],\\ [0.11,0.21]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.62,0.72],\\ [0.62,0.72]\end{pmatrix}, \begin{pmatrix}[0.12,0.22],\\ [0.12,0.22]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.63,0.73],\\ [0.63,0.73]\end{pmatrix}, \begin{pmatrix}[0.13,0.23],\\ [0.13,0.23]\end{pmatrix}\right)$
$\overline{\overline{\mathbb{A}}}_{f}^{5}$	$\left(\begin{pmatrix}[0.3,0.4],\\ [0.4,0.5]\end{pmatrix}, \begin{pmatrix}[0.2,0.3],\\ [0.1,0.2]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.31,0.41],\\ [0.41,0.51]\end{pmatrix}, \begin{pmatrix}[0.21,0.31],\\ [0.11,0.21]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.32,0.42],\\ [0.42,0.52]\end{pmatrix}, \begin{pmatrix}[0.22,0.32],\\ [0.12,0.22]\end{pmatrix}\right)$	$\left(\begin{pmatrix}[0.33,0.43],\\ [0.43,0.53]\end{pmatrix}, \begin{pmatrix}[0.23,0.33],\\ [0.13,0.23]\end{pmatrix}\right)$

Step 2. Aggregate the normalized information based on the CIVAIFDHM operator, CIVAIFDDHM operator, CIVAIFWDPHM operator, and CIVAIFWDDPHM operator, and the results are shown in Table 2.

Table 2. Aggregated CIVAIF decision matrix.

	CIVAIFDHM	CIVAIFDDHM
$\overline{\overline{\mathbb{A}}}_{f}^{1}$	$\begin{pmatrix}([0.3147,0.4147],[0.4147,0.5148]),\\ ([0.1149,0.2150],[0.1149,0.2150])\end{pmatrix}$	$\begin{pmatrix}([0.3150,0.4151],[0.4151,0.5151]),\\ ([0.1141,0.2145],[0.1141,0.2145])\end{pmatrix}$
$\overline{\overline{\mathbb{A}}}_{f}^{2}$	$\begin{pmatrix}([0.5148,0.6148],[0.4147,0.5148]),\\ ([0.2150,0.3150],[0.1149,0.2150])\end{pmatrix}$	$\begin{pmatrix}([0.5151,0.6152],[0.4151,0.5151]),\\ ([0.2145,0.3147],[0.1141,0.2145])\end{pmatrix}$
$\overline{\overline{\mathbb{A}}}_{f}^{3}$	$\begin{pmatrix}([0.7149,0.8149],[0.6148,0.7149]),\\ ([0.1149,0.2150],[0.1149,0.2150])\end{pmatrix}$	$\begin{pmatrix}([0.7153,0.8155],[0.6152,0.7153]),\\ ([0.1141,0.2145],[0.1141,0.2145])\end{pmatrix}$
$\overline{\overline{\mathbb{A}}}_{f}^{4}$	$\begin{pmatrix}([0.6148,0.7149],[0.6148,0.7149]),\\ ([0.1149,0.2150],[0.1149,0.2150])\end{pmatrix}$	$\begin{pmatrix}([0.6152,0.7153],[0.6152,0.7153]),\\ ([0.1141,0.2145],[0.1141,0.2145])\end{pmatrix}$
$\overline{\overline{\mathbb{A}}}_{f}^{5}$	$\begin{pmatrix}([0.3147,0.4147],[0.4147,0.5148]),\\ ([0.2150,0.3150],[0.1149,0.2150])\end{pmatrix}$	$\begin{pmatrix}([0.3150,0.4151],[0.4151,0.5151]),\\ ([0.2145,0.3147],[0.1141,0.4145])\end{pmatrix}$
	CIVAIFWDPHM	CIVAIFWDDPHM
$\overline{\overline{\mathbb{A}}}_{f}^{1}$	$\begin{pmatrix}([0.3186,0.4187],[0.4187,0.5187]),\\ ([0.1603,0.2839],[0.1603,0.2839])\end{pmatrix}$	$\begin{pmatrix}([0.3985,0.505],[0.505,0.6042]),\\ ([0.1182,0.2185],[0.1182,0.2185])\end{pmatrix}$
$\overline{\overline{\mathbb{A}}}_{f}^{2}$	$\begin{pmatrix}([0.5185,0.6185],[0.4184,0.5185]),\\ ([0.2836,0.3982],[0.16,0.2836])\end{pmatrix}$	$\begin{pmatrix}([0.6039,0.6966],[0.5047,0.6039]),\\ ([0.2182,0.3184],[0.1179,0.2182])\end{pmatrix}$
$\overline{\overline{\mathbb{A}}}_{f}^{3}$	$\begin{pmatrix}([0.7171,0.8172],[0.6171,0.7171]),\\ ([0.1581,0.2819],[0.1581,0.2819])\end{pmatrix}$	$\begin{pmatrix}([0.7823,0.8638],[0.6954,0.7823]),\\ ([0.1164,0.2168],[0.1164,0.2168])\end{pmatrix}$
$\overline{\overline{\mathbb{A}}}_{f}^{4}$	$\begin{pmatrix}([0.6174,0.7174],[0.6174,0.7174]),\\ ([0.1585,0.2823],[0.1585,0.2823])\end{pmatrix}$	$\begin{pmatrix}([0.6956,0.7825],[0.6956,0.7825]),\\ ([0.1167,0.2171],[0.1167,0.2171])\end{pmatrix}$
$\overline{\overline{\mathbb{A}}}_{f}^{5}$	$\begin{pmatrix}([0.3189,0.419],[0.419,0.5190]),\\ ([0.2842,0.3988],[0.1607,0.2842])\end{pmatrix}$	$\begin{pmatrix}([0.3988,0.5052],[0.5052,0.6044]),\\ ([0.2188,0.3189],[0.1185,0.2188])\end{pmatrix}$

Step 3. Analyze the numerical values from CIVAIF values with the help of Score values or Accuracy values, see Table 3.

Table 3. The score values.

	CIVAIFDHM	CIVAIFDDHM	CIVAIFWDPHM	CIVAIFWDDPHM
$\overline{\overline{\mathbb{A}}}_f^{1}$	0.2498	0.2507	0.1965	0.3347
$\overline{\overline{\mathbb{A}}}_f^{2}$	0.2998	0.3007	0.2371	0.3840
$\overline{\overline{\mathbb{A}}}_f^{3}$	0.5499	0.5510	0.4971	0.6143
$\overline{\overline{\mathbb{A}}}_f^{4}$	0.4999	0.5009	0.4470	0.5721
$\overline{\overline{\mathbb{A}}}_f^{5}$	0.1997	0.2006	0.1369	0.2846

Step 4. Analyze the rank of all alternatives according to their Score values to address the best preference among the collection of finite number of values, see Table 4.

Table 4. Ranking values of the proposed work.

Methods	Ranking values
CIVAIFDHM	$\overline{\overline{\mathbb{A}}}_f^{3} > \overline{\overline{\mathbb{A}}}_f^{4} > \overline{\overline{\mathbb{A}}}_f^{2} > \overline{\overline{\mathbb{A}}}_f^{1} > \overline{\overline{\mathbb{A}}}_f^{5}$
CIVAIFDDHM	$\overline{\overline{\mathbb{A}}}_f^{3} > \overline{\overline{\mathbb{A}}}_f^{4} > \overline{\overline{\mathbb{A}}}_f^{2} > \overline{\overline{\mathbb{A}}}_f^{1} > \overline{\overline{\mathbb{A}}}_f^{5}$
CIVAIFWDPHM	$\overline{\overline{\mathbb{A}}}_f^{3} > \overline{\overline{\mathbb{A}}}_f^{4} > \overline{\overline{\mathbb{A}}}_f^{2} > \overline{\overline{\mathbb{A}}}_f^{1} > \overline{\overline{\mathbb{A}}}_f^{5}$
CIVAIFWDDPHM	$\overline{\overline{\mathbb{A}}}_f^{3} > \overline{\overline{\mathbb{A}}}_f^{4} > \overline{\overline{\mathbb{A}}}_f^{2} > \overline{\overline{\mathbb{A}}}_f^{1} > \overline{\overline{\mathbb{A}}}_f^{5}$

From Table 4, we obtained the best choice is $\overline{\overline{\mathbb{A}}}_f^{3}$, which is Sustainable transportation, and all operators such as CIVAIFDHM, CIVAIFDDHM, CIVAIFWDPHM, and CIVAIFWDDPHM operators produced the same ranking results. The geometrical form of the data in Table 3 is described in Figure 3.

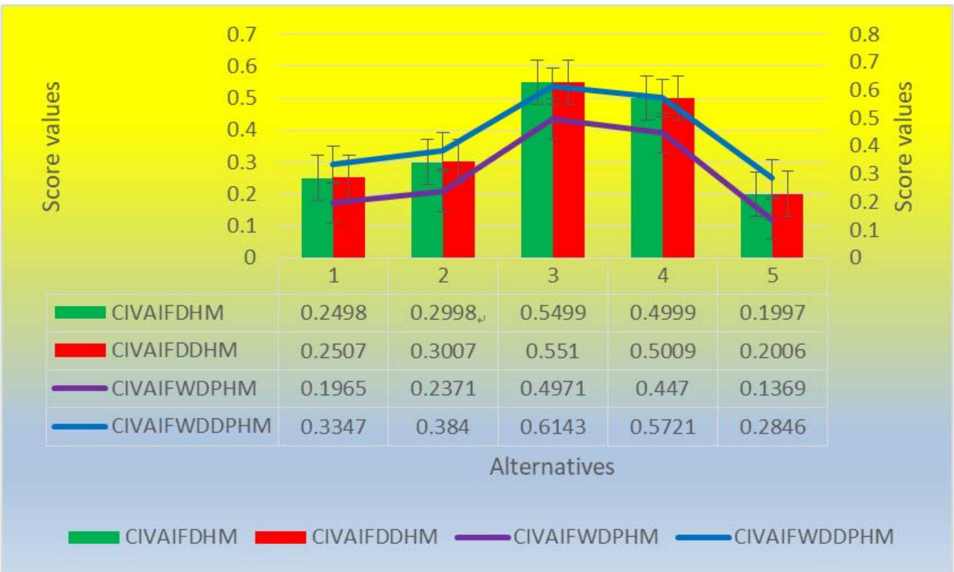

Figure 3. Graphical interpretation of the data in Table 3.

The different colors in Figure 3 represent the ranking behavior of the alternatives according to their Score values, and from Figure 3, we can determine the best and worst optimal among the collection of five alternatives.

6. Comparative analysis

Describing or analyzing the supremacy or validity of any type of technique are very problematic and challenging tasks for individuals. The goal of comparative analysis is to analyze the supremacy and validity of the derived theory by comparing its ranking values or structure with the structure or ranking values of numerous existing models. For this, we aim to arrange information based on fuzzy sets and related work to the proposed theory; we arranged the following models: Dombi Heronian mean operators for IVIFSs were presented by Wu et al. [34], Yu et al. [35] exposed the prioritized operators for IVIFSs, the prioritized operators for CIFSs were presented by Garg and Rani [36], aggregation operators for CIFSs was presented by Garg and Rani [37], Shi et al. [38] derived the power operators for IVIFSs, Chen [39] examined the prioritized operators for IVIFSs, Wang et al. [40] invented the Dombi prioritized operators for CIFSs, and Fang et al. [41] derived the Aczel-Alsina operators for CIVAIFSs. Considering the data in Table 1, the comparative models or ranking values are listed in Table 5.

Table 5. The mathematical comparative analysis.

Methods	Score values	Ranking values
Wu et al. [34]	0.0,0.0,0.0,0.0,0.0	No
Yu et al. [35]	0.0,0.0,0.0,0.0,0.0	No
Garg and Rani [36]	0.0,0.0,0.0,0.0,0.0	No
Garg and Rani [37]	0.0,0.0,0.0,0.0,0.0	No
Shi et al. [38]	0.0,0.0,0.0,0.0,0.0	No
Chen [39]	0.0,0.0,0.0,0.0,0.0	No
Wang et al. [40]	0.0,0.0,0.0,0.0,0.0	No
Fang et al. [41]	$-0.1334, -0.1150, 0.061, 0.028, -0.19$	$\overline{\overline{\mathbb{A}_f}}^3 > \overline{\overline{\mathbb{A}_f}}^4 > \overline{\overline{\mathbb{A}_f}}^2 > \overline{\overline{\mathbb{A}_f}}^1 > \overline{\overline{\mathbb{A}_f}}^5$
	$0.4456, 0.4618, 0.5791, 0.5466, 0.4377$	$\overline{\overline{\mathbb{A}_f}}^3 > \overline{\overline{\mathbb{A}_f}}^4 > \overline{\overline{\mathbb{A}_f}}^2 > \overline{\overline{\mathbb{A}_f}}^1 > \overline{\overline{\mathbb{A}_f}}^5$
CIVAIFDHM	$0.2498, 0.2998, 0.5499, 0.4999, 0.1997$	$\overline{\overline{\mathbb{A}_f}}^3 > \overline{\overline{\mathbb{A}_f}}^4 > \overline{\overline{\mathbb{A}_f}}^2 > \overline{\overline{\mathbb{A}_f}}^1 > \overline{\overline{\mathbb{A}_f}}^5$
CIVAIFDDHM	$0.2507, 0.3007, 0.5510, 0.5009, 0.2006$	$\overline{\overline{\mathbb{A}_f}}^3 > \overline{\overline{\mathbb{A}_f}}^4 > \overline{\overline{\mathbb{A}_f}}^2 > \overline{\overline{\mathbb{A}_f}}^1 > \overline{\overline{\mathbb{A}_f}}^5$
CIVAIFWDPHM	$0.1965, 0.2371, 0.4971, 0.4470, 0.1369$	$\overline{\overline{\mathbb{A}_f}}^3 > \overline{\overline{\mathbb{A}_f}}^4 > \overline{\overline{\mathbb{A}_f}}^2 > \overline{\overline{\mathbb{A}_f}}^1 > \overline{\overline{\mathbb{A}_f}}^5$
CIVAIFWDDPHM	$0.3347, 0.3840, 0.6143, 0.5721, 0.2846$	$\overline{\overline{\mathbb{A}_f}}^3 > \overline{\overline{\mathbb{A}_f}}^4 > \overline{\overline{\mathbb{A}_f}}^2 > \overline{\overline{\mathbb{A}_f}}^1 > \overline{\overline{\mathbb{A}_f}}^5$

From the Table 5, we see the best choice is $\overline{\overline{\mathbb{A}_f}}^3$, which is Sustainable transportation, and all operators such as CIVAIFDHM, CIVAIFDDHM, CIVAIFWDPHM, and CIVAIFWDDPHM operators produced the same ranking results. Further, we also observed that only the work of Fang et al. [41] gave the same ranking results for both averaging and geometric operators based on CIVAIFSs, but the other techniques have failed due to their limitations and problems. The limitations and problems of the models are described in Table 6.

Table 6. The theoretical and logical comparative analysis.

Methods	Truth function	Falsiy function	Interval-valued information	Two-dimension data	Algebraic operators	Drastic operators	Max-Min operators	Hamacher operators	Dombi Operators	Hamy operators	Prioritized operators
Wu et al. [34]	yes	Yes	Yes	no	yes	yes	yes	yes	yes	no	no
Yu et al. [35]	yes	Yes	Yes	no	yes	no	no	no	no	no	yes
Garg and Rani [36]	yes	Yes	No	yes	yes	no	no	no	no	no	yes
Garg and Rani [37]	yes	Yes	No	yes	yes	no	no	no	no	no	no
Shi et al. [38]	yes	Yes	Yes	no	yes	no	no	no	no	no	no
Chen [39]	yes	Yes	Yes	no	yes	no	no	no	no	no	yes
Wang et al. [40]	yes	Yes	No	yes	yes	yes	yes	yes	yes	no	yes
Fang et al. [41]	yes	Yes	Yes	yes	yes	yes	yes	no	no	no	no
Proposed theory	yes	Yes	Yes	yes	yes	yes	yes	yes	yes	yes	Yes

From the data in Table 6, we noticed the advantages and limitations or weaknesses of the techniques. Because of these, they failed to cope with vague and uncertain data. The limitation of the existing models is described below:

(1) Dombi Heronian mean operators for IVIFSs were presented by Wu et al. [34] and Yu et al. [35] exposed the prioritized operators for IVIFSs. These operators are computed based on IVIFSs, where the definition of IVIFSs is described in the following: For instance, let $\overline{\overline{\mathbb{u}}}_u$ be the generic element of a fixed set $\overline{\overline{\mathbb{U}}}_u$, then the IVIFS $\overline{\overline{\mathbb{A}_f}}$ is defined by: $\overline{\overline{\mathbb{A}_f}} = \left\{\left(\overbrace{\text{ȩ}}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}(\overline{\overline{\mathbb{u}}}_u), \overbrace{\text{ɳ}}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}(\overline{\overline{\mathbb{u}}}_u)\right) : \overline{\overline{\mathbb{u}}}_u \in \overline{\overline{\mathbb{U}}}_u\right\}$. Further, we observed that the truth grade is constructed in the shape: $\overbrace{\text{ȩ}}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}(\overline{\overline{\mathbb{u}}}_u) = \left(\left[\overbrace{\text{ȩ}}^{\text{я}-}_{\overline{\overline{\text{un}}}}(\overline{\overline{\mathbb{u}}}_u), \overbrace{\text{ȩ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}(\overline{\overline{\mathbb{u}}}_u)\right]\right)$ and the falsity grade is constructed in the shape: $\overbrace{\text{ɳ}}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}(\overline{\overline{\mathbb{u}}}_u) = \left(\left[\overbrace{\text{ɳ}}^{\text{я}-}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}(\overline{\overline{\mathbb{u}}}_u), \overbrace{\text{ɳ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}(\overline{\overline{\mathbb{u}}}_u)\right]\right)$ with conditions, such as $0 \leq \overbrace{\text{ȩ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}(\overline{\overline{\mathbb{u}}}_u) + \overbrace{\text{ɳ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}(\overline{\overline{\mathbb{u}}}_u) \leq 1$, and the simple form of IVIF value is derived from the shape, such as $\overline{\overline{\mathbb{A}_f}}^{\text{un}} = \left(\left(\left[\overbrace{\text{ȩ}}^{\text{я}-}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}, \overbrace{\text{ȩ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}\right]\right), \left(\left[\overbrace{\text{ɳ}}^{\text{я}-}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}, \overbrace{\text{ɳ}}^{\text{я}+}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}\right]\right)\right), \text{un} = 1, 2, \ldots, p$, meaning that the IVIFSs did not contain the term $\left[\overbrace{\text{ȩ}}^{\text{ʁ}-}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}, \overbrace{\text{ȩ}}^{\text{ʁ}+}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}\right]$ and $\left[\overbrace{\text{ɳ}}^{\text{ʁ}-}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}, \overbrace{\text{ɳ}}^{\text{ʁ}+}_{\overline{\overline{\mathbb{A}_{\text{un}}}}}\right]$. Thus, due to this extra term in the proposed theory, the technique of Wu et al. [34] and the model of Yu et al. [35] have failed.

(2) The prioritized operators for CIFSs were presented by Garg and Rani [36], aggregation operators for CIFSs were presented by Garg and Rani [37], and Wang et al. [40] invented the Dombi prioritized operators for CIFSs. These operators are computed based on IVIFSs, where the

definition of IVIFSs is described in the following: For instance, let $\overline{\overline{\mathbb{u}}}_u$ be the generic element of a fixed set $\overline{\overline{\mathbb{U}}}_u$, then the IVIFS $\overline{\overline{\mathbb{A}_f}}$ is defined by: $\overline{\overline{\mathbb{A}_f}} = \left\{\left(\widetilde{\text{Ҿ}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u), \widetilde{\text{ɳ}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u)\right) : \overline{\overline{\mathbb{u}}}_u \in \overline{\overline{\mathbb{U}}}_u\right\}$. Further, we observed that the truth grade is constructed in the shape: $\widetilde{\text{Ҿ}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) = \left(\overset{\text{я}-}{\widetilde{\text{Ҿ}}}_{\overline{\overline{un}}}(\overline{\overline{\mathbb{u}}}_u), \overset{\text{ʁ}-}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u)\right)$ and the falsity grade is constructed in the shape: $\widetilde{\text{ɳ}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) = \left(\overset{\text{я}-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u), \overset{\text{ʁ}-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u)\right)$ with conditions, such as $0 \leq \overset{\text{я}+}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) + \overset{\text{я}+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) \leq 1$ and $0 \leq \overset{\text{ʁ}+}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) + \overset{\text{ʁ}+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) \leq 1$. Moreover, the simple form of CIF value is derived from the shape, such as $\overline{\overline{\mathbb{A}_f}}^{un} = \left(\left(\overset{\text{я}-}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}, \overset{\text{ʁ}-}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}\right), \left(\overset{\text{я}-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}, \overset{\text{ʁ}-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}\right)\right), un = 1, 2, \ldots, p.$, meaning that the CIFSs did not contain the term $\left(\overset{\text{я}+}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}, \overset{\text{ʁ}+}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}\right)$ and $\left(\overset{\text{я}+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}, \overset{\text{ʁ}+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}\right)$. Thus, due to this extra term in the proposed theory, the technique of Garg and Rani [36, 37] and the model of Wang et al. [40] has been failed.

(3) Shi et al. [38] derived the power operators for IVIFSs, and Chen [39] examined the prioritized operators for IVIFSs, These operators are computed based on IVIFSs, where the definition of IVIFSs is described in the following: For instance, let $\overline{\overline{\mathbb{u}}}_u$ be the generic element of a fixed set $\overline{\overline{\mathbb{U}}}_u$, then the IVIFS $\overline{\overline{\mathbb{A}_f}}$ is defined by: $\overline{\overline{\mathbb{A}_f}} = \left\{\left(\widetilde{\text{Ҿ}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u), \widetilde{\text{ɳ}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u)\right) : \overline{\overline{\mathbb{u}}}_u \in \overline{\overline{\mathbb{U}}}_u\right\}$. Further, we observed that the truth grade is constructed in the shape: $\widetilde{\text{Ҿ}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) = \left(\left[\overset{\text{я}-}{\widetilde{\text{Ҿ}}}_{\overline{\overline{un}}}(\overline{\overline{\mathbb{u}}}_u), \overset{\text{я}+}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u)\right]\right)$ and the falsity grade is constructed in the shape: $\widetilde{\text{ɳ}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) = \left(\left[\overset{\text{я}-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u), \overset{\text{я}+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u)\right]\right)$ with conditions, such as $0 \leq \overset{\text{я}+}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) + \overset{\text{я}+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}(\overline{\overline{\mathbb{u}}}_u) \leq 1$, and the simple form of IVIF value is derived from the shape, such as $\overline{\overline{\mathbb{A}_f}}^{un} = \left(\left(\left[\overset{\text{я}-}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}, \overset{\text{я}+}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}\right]\right), \left(\left[\overset{\text{я}-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}, \overset{\text{я}+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}\right]\right)\right), un = 1, 2, \ldots, p$, meaning that the IVIFSs did not contain the term $\left[\overset{\text{ʁ}-}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}, \overset{\text{ʁ}+}{\widetilde{\text{Ҿ}}}_{\overline{\overline{\mathbb{A}_{un}}}}\right]$ and $\left[\overset{\text{ʁ}-}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}, \overset{\text{ʁ}+}{\widetilde{\text{ɳ}}}_{\overline{\overline{\mathbb{A}_{un}}}}\right]$. Thus, due to this extra term in the proposed theory, the technique of Shi et al. [38] and the model of Chen [39] have failed.

(4) Fang et al. [41] derived the Aczel-Alsina operators for CIVAIFSs, but the structure of proposed operators and existing operators [41] are defined based on the same models, called the CIVAIFSs, but their defined operators are different. From the data in Table 5, we noticed that the proposed operators and existing operators provided the ranking results, but from the data in Table 6, we also noticed the limitations of the existing technique [41] and the advantages of the proposed theory.

The major advantages (special cases of the proposed operators) of the proposed operators are listed below:

(1) Averaging/geometric operators for FSs.
(2) Dombi averaging/geometric operators for FSs.
(3) Hamacher averaging/geometric operators for FSs.

(4) Prioritized averaging/geometric operators for FSs.
(5) Hamy averaging/geometric operators for FSs.
(6) Dombi prioritized averaging/geometric operators for FSs.
(7) Prioritized Hamy averaging/geometric operators for FSs.
(8) Dombi Hamy averaging/geometric operators for FSs.

After our long assessment, we concluded that the proposed theory is a modified version of many existing models and techniques to cope with vague and uncertain data. Hence, the proposed techniques are novel and better than others [34–40] according to the analysis in Tables 6 and 7.

7. Conclusions

The major contributions of this manuscript are listed below:

(1) Proposed the Dombi operational laws based on the Dombi t-norm and Dombi t-conorm for the CIVAIF set.
(2) Developed the CIVAIFDHM operator, CIVAIFWDPHM operator, CIVAIFDDHM operator, and CIVAIFWDDPHM operator. Some basic properties are discussed for the above operators.
(3) Developed the MADM method based on the proposed operators.
(4) Compared the proposed method with prevailing methods.

CIVAIFSs are very flexible techniques, but in the presence of the truth grade, abstinence, and falsity grades, they are not working feasibly, CIVAIFSs deal only with truth and falsity grades. Thus, we are required to develop the idea of complex interval-valued picture fuzzy sets and their extensions.

In the future, the PHM operators based on Dombi operational laws will be utilized for complex Pythagorean fuzzy sets, complex q-rung orthopair fuzzy sets, complex picture fuzzy sets, artificial intelligence, machine learning, and decision-making. Further, we also aim to read the following articles: Fuzzy-model-based lateral control [52], analysis of hybrid machine learning [53], the fuzzy model for predication severity [54], analysis of neuro-fuzzy model [55], and precise neuro-fuzzy model [56].

Appendix section

Appendix A. Consider

$$\otimes_{\emptyset=1}^{\mathrm{un}} \overline{\overline{\mathbb{A}}}_f^{\,i_\emptyset} =$$

$$\left(\begin{array}{c}\left(\begin{array}{c}\left[1\Big/1+\left(\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}},1\Big/1+\left(\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right],\\ \left[1\Big/1+\left(\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}},1\Big/1+\left(\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right]\end{array}\right),\\ \left(\begin{array}{c}\left[1-\left(1\Big/1+\left(\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right),1-\left(1\Big/1+\left(\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right)\right],\\ \left[1-\left(1\Big/1+\left(\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right),1-\left(1\Big/1+\left(\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right)\right]\end{array}\right)\end{array}\right).$$

$$\left(\bigotimes_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\mathbb{A}}_f}^{i_{\emptyset}}\right)^{\frac{1}{\text{ɯ}}}=$$

$$\left(\begin{array}{c}\left(\begin{array}{c}\left[1\Big/1+\left(\frac{1}{\text{ɯ}}\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}},1\Big/1+\left(\frac{1}{\text{ɯ}}\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right],\\ \left[1\Big/1+\left(\frac{1}{\text{ɯ}}\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}},1\Big/1+\left(\frac{1}{\text{ɯ}}\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right]\end{array}\right),\\ \left(\begin{array}{c}\left[1-\left(1\Big/1+\left(\frac{1}{\text{ɯ}}\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right),1-\left(1\Big/1+\left(\frac{1}{\text{ɯ}}\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{я}+}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right)\right],\\ \left[1-\left(1\Big/1+\left(\frac{1}{\text{ɯ}}\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right),1-\left(1\Big/1+\left(\frac{1}{\text{ɯ}}\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{Ͽ}_s}}}\right)^{\frac{1}{\overline{\overline{\text{Ͽ}_s}}}}\right)\right]\end{array}\right)\end{array}\right).$$

$$\bigoplus_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(\bigotimes_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\mathbb{A}}}_f^{\,i_\emptyset}\right)^{\frac{1}{\text{ɯ}}}$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}\left[1-\left(1\Big/1+\left(\sum_{\binom{1\le i_1\le}{\ldots\le i_{\text{ɯ}}\le p}}\left(\text{ɯ}\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}},1-\left(1\Big/1+\left(\sum_{\binom{1\le i_1\le}{\ldots\le i_{\text{ɯ}}\le p}}\left(\text{ɯ}\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}}\right],\\ \left[1-\left(1\Big/1+\left(\sum_{\binom{1\le i_1\le}{\ldots\le i_{\text{ɯ}}\le p}}\left(\text{ɯ}\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}},1-\left(1\Big/1+\left(\sum_{\binom{1\le i_1\le}{\ldots\le i_{\text{ɯ}}\le p}}\left(\text{ɯ}\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}{\overbrace{\text{ҿ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}}\right]\end{array}\right),\\ \left(\begin{array}{c}\left[1\Big/1+\left(\sum_{\binom{1\le i_1\le}{\ldots\le i_{\text{ɯ}}\le p}}\left(\text{ɯ}\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}},1\Big/1+\left(\sum_{\binom{1\le i_1\le}{\ldots\le i_{\text{ɯ}}\le p}}\left(\text{ɯ}\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}}\right],\\ \left[1\Big/1+\left(\sum_{\binom{1\le i_1\le}{\ldots\le i_{\text{ɯ}}\le p}}\left(\text{ɯ}\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}},1\Big/1+\left(\sum_{\binom{1\le i_1\le}{\ldots\le i_{\text{ɯ}}\le p}}\left(\text{ɯ}\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}}\right]\end{array}\right)\end{array}\right)$$

$$CIVAIFDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1},\overline{\overline{\mathbb{A}_f}}^{2},\ldots,\overline{\overline{\mathbb{A}_f}}^{p}\right)=\frac{1}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\oplus_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\left(\otimes_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\mathbb{A}_f}}^{i_\emptyset}\right)^{\frac{1}{\text{ɯ}}}$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}\left[\begin{array}{c}1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}{\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}},\\ 1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}{\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}}\end{array}\right],\\ \left[\begin{array}{c}1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}{\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}},\\ 1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}{\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}}\end{array}\right]\end{array}\right),\\ \left(\begin{array}{c}\left[\begin{array}{c}\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}{1-\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}},\\ \left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}{1-\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}}\end{array}\right],\\ \left[\begin{array}{c}\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}{1-\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}},\\ \left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}{1-\overbrace{\text{ƞ}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\text{϶}}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\text{϶}}}_s}}\end{array}\right]\end{array}\right)\end{array}\right).$$

Hence, the result is proven.

Appendix B. Let $\overline{\overline{\mathbb{A}_f}}^{\text{ɯ}} = \overline{\overline{\mathbb{A}_f}} = \left(\left(\left[\overbrace{\varrho}^{я-}_{\overline{\overline{\mathbb{A}}}}, \overbrace{\varrho}^{я+}_{\overline{\overline{\mathbb{A}}}}\right], \left[\overbrace{\varrho}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}, \overbrace{\varrho}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}\right]\right), \left(\left[\overbrace{\eta}^{я-}_{\overline{\overline{\mathbb{A}}}}, \overbrace{\eta}^{я+}_{\overline{\overline{\mathbb{A}}}}\right], \left[\overbrace{\eta}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}, \overbrace{\eta}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}\right]\right)\right), \text{ɯ} = 1, 2, \ldots, p$, the

$$CIVAIFDHM^{\text{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^1, \overline{\overline{\mathbb{A}_f}}^2, \ldots, \overline{\overline{\mathbb{A}_f}}^p\right) = \frac{1}{{}^{\circ}C_p^{\text{ɯ}}} \oplus_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(\otimes_{\emptyset=1}^{\text{ɯ}}\overline{\overline{\mathbb{A}_f}}^{i_\emptyset}\right)^{\frac{1}{\text{ɯ}}}$$

$$= \left(\begin{array}{l}
\left(\begin{array}{l}
\left[\begin{array}{l}
1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\varrho}^{я-}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}{\overbrace{\varrho}^{я-}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}\right)^{\overline{\overline{\ni}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}, \\
1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\varrho}^{я+}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}{\overbrace{\varrho}^{я+}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}\right)^{\overline{\overline{\ni}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}
\end{array}\right], \\
\left[\begin{array}{l}
1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\varrho}^{ʁ-}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}{\overbrace{\varrho}^{ʁ-}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}\right)^{\overline{\overline{\ni}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}, \\
1-\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\varrho}^{ʁ+}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}{\overbrace{\varrho}^{ʁ+}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}\right)^{\overline{\overline{\ni}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}
\end{array}\right]
\end{array}\right), \\
\left(\begin{array}{l}
\left[\begin{array}{l}
\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\eta}^{я-}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}{1-\overbrace{\eta}^{я-}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}\right)^{\overline{\overline{\ni}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}, \\
\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\eta}^{я+}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}{1-\overbrace{\eta}^{я+}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}\right)^{\overline{\overline{\ni}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}
\end{array}\right], \\
\left[\begin{array}{l}
\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\eta}^{ʁ-}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}{1-\overbrace{\eta}^{ʁ-}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}\right)^{\overline{\overline{\ni}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}, \\
\left(1\Big/1+\left(\frac{\text{ɯ}}{{}^{\circ}C_p^{\text{ɯ}}}\sum_{1\le i_1\le\cdots\le i_{\text{ɯ}}\le p}\left(1\Big/\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\eta}^{ʁ+}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}{1-\overbrace{\eta}^{ʁ+}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}\right)^{\overline{\overline{\ni}}_s}\right)\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}
\end{array}\right]
\end{array}\right)
\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\left[1-\left(1\Big/1+\left(1\Big/\left(\frac{1-\overbrace{\text{ҿ}}^{я-}_{\overline{\overline{\mathbb{A}}}}}{\overbrace{\text{ҿ}}^{я-}_{\overline{\overline{\mathbb{A}}}}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right),1-\left(1\Big/1+\left(1\Big/\left(\frac{1-\overbrace{\text{ҿ}}^{я+}_{\overline{\overline{\mathbb{A}}}}}{\overbrace{\text{ҿ}}^{я+}_{\overline{\overline{\mathbb{A}}}}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right)\right],\left[1-\left(1\Big/1+\left(1\Big/\left(\frac{1-\overbrace{\text{ҿ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}}{\overbrace{\text{ҿ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right),1-\left(1\Big/1+\left(1\Big/\left(\frac{1-\overbrace{\text{ҿ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}}{\overbrace{\text{ҿ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right)\right]\right),\\ \left(\left[1\Big/1+\left(1\Big/\left(\frac{\overbrace{\text{ϥ}}^{я-}_{\overline{\overline{\mathbb{A}}}}}{1-\overbrace{\text{ϥ}}^{я-}_{\overline{\overline{\mathbb{A}}}}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}},1\Big/1+\left(1\Big/\left(\frac{\overbrace{\text{ϥ}}^{я+}_{\overline{\overline{\mathbb{A}}}}}{1-\overbrace{\text{ϥ}}^{я+}_{\overline{\overline{\mathbb{A}}}}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right],\left[1\Big/1+\left(1\Big/\left(\frac{\overbrace{\text{ϥ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}}{1-\overbrace{\text{ϥ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}},1\Big/1+\left(1\Big/\left(\frac{\overbrace{\text{ϥ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}}{1-\overbrace{\text{ϥ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}}\right)^{\overline{\overline{\text{Э}}}_s}\right)^{\frac{1}{\overline{\overline{\text{Э}}}_s}}\right]\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\left[1-\frac{1}{1+\frac{1}{\frac{1-\overbrace{\text{ҿ}}^{я-}_{\overline{\overline{\mathbb{A}}}}}{\overbrace{\text{ҿ}}^{я-}_{\overline{\overline{\mathbb{A}}}}}}},1-\frac{1}{1+\frac{1}{\frac{1-\overbrace{\text{ҿ}}^{я+}_{\overline{\overline{\mathbb{A}}}}}{\overbrace{\text{ҿ}}^{я+}_{\overline{\overline{\mathbb{A}}}}}}}\right],\left[1-\frac{1}{1+\frac{1}{\frac{1-\overbrace{\text{ҿ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}}{\overbrace{\text{ҿ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}}}},1-\frac{1}{1+\frac{1}{\frac{1-\overbrace{\text{ҿ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}}{\overbrace{\text{ҿ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}}}}\right]\right),\\ \left(\left[\frac{1}{1+\frac{1}{\frac{\overbrace{\text{ϥ}}^{я-}_{\overline{\overline{\mathbb{A}}}}}{1-\overbrace{\text{ϥ}}^{я-}_{\overline{\overline{\mathbb{A}}}}}}},\frac{1}{1+\frac{1}{\frac{\overbrace{\text{ϥ}}^{я+}_{\overline{\overline{\mathbb{A}}}}}{1-\overbrace{\text{ϥ}}^{я+}_{\overline{\overline{\mathbb{A}}}}}}}\right],\left[\frac{1}{1+\frac{1}{\frac{\overbrace{\text{ϥ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}}{1-\overbrace{\text{ϥ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}}}}},\frac{1}{1+\frac{1}{\frac{\overbrace{\text{ϥ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}}{1-\overbrace{\text{ϥ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}}}}\right]\right)\end{array}\right)$$

$$=\left(\left(\left[\overbrace{\text{ҿ}}^{я-}_{\overline{\overline{\mathbb{A}}}},\overbrace{\text{ҿ}}^{я+}_{\overline{\overline{\mathbb{A}}}}\right],\left[\overbrace{\text{ҿ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}},\overbrace{\text{ҿ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}\right]\right),\left(\left[\overbrace{\text{ϥ}}^{я-}_{\overline{\overline{\mathbb{A}}}},\overbrace{\text{ϥ}}^{я+}_{\overline{\overline{\mathbb{A}}}}\right],\left[\overbrace{\text{ϥ}}^{ʁ-}_{\overline{\overline{\mathbb{A}}}},\overbrace{\text{ϥ}}^{ʁ+}_{\overline{\overline{\mathbb{A}}}}\right]\right)\right)=\overline{\overline{\mathbb{A}}}_f.$$

Appendix C. If $\overline{\overline{\mathbb{A}}}_f^{\text{ɯ}} \leq \overline{\overline{\mathbb{A}}}_f^{\text{ɯ}*}$, thus $\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-} \leq \overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}$, $\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ-} \leq \overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ-\ *}$, $\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я+} \leq \overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я+\ *}$, $\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ+} \leq \overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ+\ *}$ and $\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-} \geq \overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}$, $\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ-} \geq \overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ-\ *}$, $\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я+} \geq \overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я+\ *}$, $\overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ+} \geq \overbrace{\eta_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{ʁ+\ *}$, thus

$$\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-} \leq \overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *} \Rightarrow 1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-} \geq 1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *} \Rightarrow \frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}} \geq \frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}$$

$$\Rightarrow \sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}}\right)^{\overline{\overline{\mathfrak{I}}}_s} \geq \sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}\right)^{\overline{\overline{\mathfrak{I}}}_s} \Rightarrow \frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}}\right)^{\overline{\overline{\mathfrak{I}}}_s}} \geq \frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}\right)^{\overline{\overline{\mathfrak{I}}}_s}}$$

$$\Rightarrow \sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}}\right)^{\overline{\overline{\mathfrak{I}}}_s}} \geq \sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}\right)^{\overline{\overline{\mathfrak{I}}}_s}}$$

$$\Rightarrow \frac{1}{1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\iota_{\emptyset}}}}^{я-}}\right)^{\overline{\overline{\mathfrak{I}}}_s}}\right)^{\frac{1}{\overline{\overline{\mathfrak{I}}}_s}}}$$

$$\geq \frac{1}{1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{1-\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}{\overbrace{\text{ҽ}_{\overline{\overline{\mathbb{A}}}_{\text{ɯ}}}}^{я-\ *}}\right)^{\overline{\overline{\mathfrak{I}}}_s}}\right)^{\frac{1}{\overline{\overline{\mathfrak{I}}}_s}}}$$

$$\Rightarrow 1-\frac{1}{\left(1+\left(\frac{\text{ш}}{{}^{\circ}\mathrm{C}_p^{\text{ш}}}\sum_{1\le i_1\le\cdots\le i_{\text{ш}}\le p}\frac{1}{\sum_{\emptyset=1}^{\text{ш}}\left(\frac{1-\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}{\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}-}}\right)^{\overline{\overline{\ni}}_s}}\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}}$$

$$\le 1-\frac{1}{\left(1+\left(\frac{\text{ш}}{{}^{\circ}\mathrm{C}_p^{\text{ш}}}\sum_{1\le i_1\le\cdots\le i_{\text{ш}}\le p}\frac{1}{\sum_{\emptyset=1}^{\text{ш}}\left(\frac{1-\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\text{ш}}}}^{\text{я}-\ *}}{\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\text{ш}}}}^{\text{я}-\ *}}\right)^{\overline{\overline{\ni}}_s}}\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}}$$

and

$$\Rightarrow 1-\frac{1}{\left(1+\left(\frac{\text{ш}}{{}^{\circ}\mathrm{C}_p^{\text{ш}}}\sum_{1\le i_1\le\cdots\le i_{\text{ш}}\le p}\frac{1}{\sum_{\emptyset=1}^{\text{ш}}\left(\frac{1-\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}{\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{я}+}}\right)^{\overline{\overline{\ni}}_s}}\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}}$$

$$\le 1-\frac{1}{\left(1+\left(\frac{\text{ш}}{{}^{\circ}\mathrm{C}_p^{\text{ш}}}\sum_{1\le i_1\le\cdots\le i_{\text{ш}}\le p}\frac{1}{\sum_{\emptyset=1}^{\text{ш}}\left(\frac{1-\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\text{ш}}}}^{\text{я}+\ *}}{\overbrace{\text{ę}_{\overline{\overline{\mathbb{A}}}_{\text{ш}}}}^{\text{я}+\ *}}\right)^{\overline{\overline{\ni}}_s}}\right)\right)^{\frac{1}{\overline{\overline{\ni}}_s}}}$$

Similarly, for the imaginary part of the truth grade, we have

$$\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\mathrm{un}}}}^{\text{ʁ}-} \leq \overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\mathrm{un}}}}^{\text{ʁ}-\;*} \Rightarrow 1-\frac{1}{\left(1+\left(\frac{\mathrm{un}}{{}^{\circ}\mathrm{C}_p^{\mathrm{un}}}\sum_{1\leq i_1\leq\cdots\leq i_{\mathrm{un}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\mathrm{un}}\left(\frac{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}{\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}-}}\right)^{\overline{\overline{\overline{\text{Ͽ}}}}_s}}\right)\right)^{\frac{1}{\overline{\overline{\overline{\text{Ͽ}}}}_s}}}$$

$$\leq 1-\frac{1}{\left(1+\left(\frac{\mathrm{un}}{{}^{\circ}\mathrm{C}_p^{\mathrm{un}}}\sum_{1\leq i_1\leq\cdots\leq i_{\mathrm{un}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\mathrm{un}}\left(\frac{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\mathrm{un}}}}^{\text{ʁ}-\;*}}{\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\mathrm{un}}}}^{\text{ʁ}-\;*}}\right)^{\overline{\overline{\overline{\text{Ͽ}}}}_s}}\right)\right)^{\frac{1}{\overline{\overline{\overline{\text{Ͽ}}}}_s}}}$$

and

$$\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\mathrm{un}}}}^{\text{ʁ}+} \leq \overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\mathrm{un}}}}^{\text{ʁ}+\;*} \Rightarrow 1-\frac{1}{\left(1+\left(\frac{\mathrm{un}}{{}^{\circ}\mathrm{C}_p^{\mathrm{un}}}\sum_{1\leq i_1\leq\cdots\leq i_{\mathrm{un}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\mathrm{un}}\left(\frac{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}{\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{\text{ʁ}+}}\right)^{\overline{\overline{\overline{\text{Ͽ}}}}_s}}\right)\right)^{\frac{1}{\overline{\overline{\overline{\text{Ͽ}}}}_s}}}$$

$$\leq 1-\frac{1}{\left(1+\left(\frac{\mathrm{un}}{{}^{\circ}\mathrm{C}_p^{\mathrm{un}}}\sum_{1\leq i_1\leq\cdots\leq i_{\mathrm{un}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\mathrm{un}}\left(\frac{1-\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\mathrm{un}}}}^{\text{ʁ}+\;*}}{\overbrace{\varrho_{\overline{\overline{\mathbb{A}}}_{\mathrm{un}}}}^{\text{ʁ}+\;*}}\right)^{\overline{\overline{\overline{\text{Ͽ}}}}_s}}\right)\right)^{\frac{1}{\overline{\overline{\overline{\text{Ͽ}}}}_s}}}$$

Furthermore, we have falsity information, such as

$$\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-} \geq \overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\text{ɯ}}}}}}^{\text{я}-\ *} \Rightarrow \frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}} \geq \frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}} \Rightarrow \sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}} \geq \sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}}\right)^{\overline{\overline{\text{Э}_s}}}$$

$$\Rightarrow \sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}}} \leq \sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}}\right)^{\overline{\overline{\text{Э}_s}}}}$$

$$\Rightarrow 1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}$$

$$\leq 1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}}\right)^{\overline{\overline{\text{Э}_s}}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}$$

$$\Rightarrow \frac{1}{1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-}}\right)^{\overline{\overline{\text{Э}_s}}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}} \geq \frac{1}{1+\left(\frac{\text{ɯ}}{{}^{\circ}\mathrm{C}_p^{\text{ɯ}}}\sum_{1\leq i_1\leq\cdots\leq i_{\text{ɯ}}\leq p}\frac{1}{\sum_{\emptyset=1}^{\text{ɯ}}\left(\frac{\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}}{1-\overbrace{\text{ɳ}_{\overline{\overline{\mathbb{A}_{\iota_\emptyset}}}}}^{\text{я}-\ *}}\right)^{\overline{\overline{\text{Э}_s}}}}\right)^{\frac{1}{\overline{\overline{\text{Э}_s}}}}}$$

and

$$\Rightarrow \frac{1}{1+\left(\frac{\mathfrak{m}}{{}^{\circ}C_p^{\mathfrak{m}}}\sum_{1\le i_1\le\cdots\le i_{\mathfrak{m}}\le p}\frac{1}{\sum_{\emptyset=1}^{\mathfrak{m}}\left(\frac{\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я+}}{1-\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я+}}\right)^{\overline{\overline{\ni}}_s}}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}} \ge \frac{1}{1+\left(\frac{\mathfrak{m}}{{}^{\circ}C_p^{\mathfrak{m}}}\sum_{1\le i_1\le\cdots\le i_{\mathfrak{m}}\le p}\frac{1}{\sum_{\emptyset=1}^{\mathfrak{m}}\left(\frac{\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я+}{}^{*}}{1-\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я+}{}^{*}}\right)^{\overline{\overline{\ni}}_s}}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}}$$

Similarly, for the imaginary part of the truth grade, we have

$$\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\mathfrak{m}}}}^{я-} \le \overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\mathfrak{m}}}}^{я-}{}^{*} \Rightarrow \frac{1}{1+\left(\frac{\mathfrak{m}}{{}^{\circ}C_p^{\mathfrak{m}}}\sum_{1\le i_1\le\cdots\le i_{\mathfrak{m}}\le p}\frac{1}{\sum_{\emptyset=1}^{\mathfrak{m}}\left(\frac{\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я-}}{1-\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я-}}\right)^{\overline{\overline{\ni}}_s}}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}} \ge \frac{1}{1+\left(\frac{\mathfrak{m}}{{}^{\circ}C_p^{\mathfrak{m}}}\sum_{1\le i_1\le\cdots\le i_{\mathfrak{m}}\le p}\frac{1}{\sum_{\emptyset=1}^{\mathfrak{m}}\left(\frac{\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я-}{}^{*}}{1-\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я-}{}^{*}}\right)^{\overline{\overline{\ni}}_s}}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}}$$

and

$$\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\mathfrak{m}}}}^{я+} \le \overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\mathfrak{m}}}}^{я+}{}^{*} \Rightarrow \frac{1}{1+\left(\frac{\mathfrak{m}}{{}^{\circ}C_p^{\mathfrak{m}}}\sum_{1\le i_1\le\cdots\le i_{\mathfrak{m}}\le p}\frac{1}{\sum_{\emptyset=1}^{\mathfrak{m}}\left(\frac{\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я+}}{1-\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я+}}\right)^{\overline{\overline{\ni}}_s}}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}} \ge \frac{1}{1+\left(\frac{\mathfrak{m}}{{}^{\circ}C_p^{\mathfrak{m}}}\sum_{1\le i_1\le\cdots\le i_{\mathfrak{m}}\le p}\frac{1}{\sum_{\emptyset=1}^{\mathfrak{m}}\left(\frac{\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я+}{}^{*}}{1-\overbrace{\mathfrak{n}_{\overline{\overline{\mathbb{A}}}_{\iota_\emptyset}}}^{я+}{}^{*}}\right)^{\overline{\overline{\ni}}_s}}\right)^{\frac{1}{\overline{\overline{\ni}}_s}}}$$

Thus, by the information in the Score value and Accuracy value, we can get

$$CIVAIFDHM^{\mathfrak{m}}\left(\overline{\overline{\mathbb{A}}}_f^{1},\overline{\overline{\mathbb{A}}}_f^{2},\dots,\overline{\overline{\mathbb{A}}}_f^{p}\right) \le CIVAIFDHM^{\mathfrak{m}}\left(\overline{\overline{\mathbb{A}}}_f^{1^*},\overline{\overline{\mathbb{A}}}_f^{2^*},\dots,\overline{\overline{\mathbb{A}}}_f^{p^*}\right).$$

Appendix D. Based on the Propertys 1 and 2, we have

$$CIVAIFDHM^{\mathfrak{m}}\left(\overline{\overline{\mathbb{A}_f a}}^{1},\overline{\overline{\mathbb{A}}}_f^{2},\dots,\overline{\overline{\mathbb{A}}}_f^{p}\right) \le CIVAIFDHM^{\mathfrak{m}}\left(\overline{\overline{\mathbb{A}}}_f^{1^+},\overline{\overline{\mathbb{A}}}_f^{2^+},\dots,\overline{\overline{\mathbb{A}}}_f^{p^+}\right) = \overline{\overline{\mathbb{A}}}_f^{+}$$

$$CIVAIFDHM^{\mathfrak{m}}\left(\overline{\overline{\mathbb{A}}}_f^{1},\overline{\overline{\mathbb{A}}}_f^{2},\dots,\overline{\overline{\mathbb{A}}}_f^{p}\right) \ge CIVAIFDHM^{\mathfrak{m}}\left(\overline{\overline{\mathbb{A}}}_f^{1^-},\overline{\overline{\mathbb{A}}}_f^{2^-},\dots,\overline{\overline{\mathbb{A}}}_f^{p^-}\right) = \overline{\overline{\mathbb{A}}}_f^{-}$$

Thus,

$$\overline{\overline{\mathbb{A}_f}}^{-} \leq CIVAIFDHM^{\mathrm{ɯ}}\left(\overline{\overline{\mathbb{A}_f}}^{1}, \overline{\overline{\mathbb{A}_f}}^{2}, \ldots, \overline{\overline{\mathbb{A}_f}}^{p}\right) \leq \overline{\overline{\mathbb{A}_f}}^{+}.$$

Author contributions

Shichao Li: Conceptualization, Methodology, Formal analysis, Writing–review, Editing, Visualization; Zeeshan Ali: Conceptualization, Software, Validation, Formal analysis, Investigation, Data Curation, Writing–review, Editing*;* Peide Liu: Software, Validation, Investigation, Resources, Data Curation, Writing–review, Editing, Visualization Supervision, Project administration, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

About the publication of this manuscript the authors declare that they have no conflict of interest.

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