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Research article

Refinement of Jensen-type inequalities: fractional extensions (global and local)

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Abstract: In this paper, our main objective was to establish new refinements of Jensen's inequality. We focused on the class of convex and harmonic convex functions. In addition, we extended these results to the generalized Caputo-type fractional integral and the generalized local fractional derivative.

Keywords: Jensen's inequality; fractional derivatives and integrals; fractional integral inequalities
Mathematics Subject Classification: 26A33, 26A51, 26D15

1. Introduction

Mathematical inequalities have been fundamental in mathematical research for many years, and their importance has grown even more. In particular, integral inequalities are basic tools in mathematical analysis because they enable us to compare the value of integrals accurately and efficiently. These inequalities facilitate the study of elements in functional spaces and are essential for obtaining bounds in optimization problems.

Many classical inequalities naturally extend to integral operators associated with various fractional derivatives. Due to their extensive applications, these inequalities play a pivotal role in the theory of differential equations and have significant implications in applied mathematics. Some examples of these generalized inequalities are Chebyshev, Gronwall, Ostrowski-type, Hermite-Hadamard-type, Gagliardo-Nirenberg-type, Hardy-type, Grüss-type, Hölder, and Minkowski [12, 16, 22–26, 28, 29].

An important inequality that appears in this context is the well-known Jensen's inequality. Jensen-type inequalities play a crucial role in several fields: economics, statistics, and decision theory, particularly involving concave and convex functions. The inequality asserts that the value of a convex (or concave) function evaluated at the expected value of a random variable is less than (or greater than) the expected value of the function evaluated at that random variable. This principle is fundamental for understanding how nonlinear transformations of random variables affect expected outcomes, and it has significant implications for risk assessment, optimization, and the behavior of economic agents. In economics, for example, Jensen's inequality helps explain why individuals' risk preferences deviate from the expected utility theory, demonstrating how people tend to over- or under-estimate risks depending on the curvature of utility functions. Its applications extend to areas such as portfolio theory, pricing models, and the analysis of income inequality, making it an indispensable tool for both theoretical and applied research. Integral Jensen's inequality [18] says that:

$$\varrho\left(\int_X p\, d\mu\right) \le \int_X (\varrho \circ p)\, d\mu$$

for any probability measure μ defined on any measurable space X, for any μ-integrable function $p : X \to (a,b)$ and any convex function ϱ on the interval (a,b). Equality in Jensen's inequality holds if and only if either ϱ is affine on $p(X)$ or p is constant μ-a.e.

In our previous work [7], we provided several Jensen-type inequalities and used them to obtain novel inequalities in the context of fractional integral operators. In [8], we presented two Jensen-type inequalities involving harmonic convex functions, and applied them to Caputo-type fractional integrals.

Motivated by the results of [7, 8, 27], our contributions in this research move in two directions: In the study of inequalities and in the study of fractional operators. First, we prove some refinements of Jensen's inequality. Moreover, we extend these results to obtain relevant inequalities in the context of fractional integral operators.

Several improvements of Jensen's inequality are known. In particular, in [27], Theorem 1 is shown, which improves the discrete Jensen's inequality with a double inequality. Our main result provides a very general version of the inequalities in Theorem 1, which is the best possible: It holds any general measurable space with any probability measure, and we characterize the case of equalities (Theorem 4). Also, we obtain in Theorem 7 a version of Theorem 4 for harmonic convex functions.

Furthermore, we extend these inequalities to two different fractional integral operators, associated with the generalized Caputo derivative and the generalized local fractional derivative.

2. Jensen-type inequalities

The discrete Jensen's inequality says the following:

Consider an interval $I \subset \mathbb{R}$, $x_i \in I$ for $i = 1, 2, \dots, n$, and w_i positive numbers such that $\sum_{i=1}^{n} w_i = 1$. If $\varrho : I \to \mathbb{R}$ is a convex function, then

$$\varrho\left(\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} w_i \varrho(x_i).$$

Equality in the discrete Jensen's inequality holds if and only if either ϱ is affine on $[\min_i x_i, \max_i x_i]$ or $x_1 = \cdots = x_n$.

Fix an interval $I \subset \mathbb{R}$. Let $x_i \in I$ and w_i, v_i positive numbers such that $\sum_{i=1}^{n} w_i = 1$ and $0 < v_i < 1$ for $i = 1, 2, \ldots, n$. Also, let J be a proper subset of $\{1, 2, \ldots, n\}$ and $J^c = \{1, 2, \ldots, n\} \setminus J$. If $\varrho : I \to \mathbb{R}$ is a convex function, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, let us define:

$$\mathbb{Z}(\varrho, \mathbf{w}, \mathbf{v}, \mathbf{x}, J) = \varrho\left(\frac{\sum_{i\in J} w_i v_i x_i}{\sum_{i\in J} w_i v_i}\right)\sum_{i\in J} w_i v_i + \varrho\left(\frac{\sum_{i\in J} w_i(1-v_i)x_i}{\sum_{i\in J} w_i(1-v_i)}\right)\sum_{i\in J} w_i(1-v_i)$$
$$+ \varrho\left(\frac{\sum_{i\in J^c} w_i v_i x_i}{\sum_{i\in J^c} w_i v_i}\right)\sum_{i\in J^c} w_i v_i + \varrho\left(\frac{\sum_{i\in J^c} w_i(1-v_i)x_i}{\sum_{i\in J^c} w_i(1-v_i)}\right)\sum_{i\in J^c} w_i(1-v_i).$$

The following refinement of the discrete Jensen's inequality is presented in [27].

Theorem 1. *Consider an interval $I \subset \mathbb{R}$, $x_i \in I$, $0 < v_i < 1$ for $i = 1, 2, \ldots, n$, and w_i positive numbers such that $\sum_{i=1}^{n} w_i = 1$. Also, let J be a proper subset of $\{1, 2, \ldots, n\}$ and $J^c = \{1, 2, \ldots, n\} \setminus J$. If $\varrho : I \to \mathbb{R}$ is a convex function, then*

$$\varrho\left(\sum_{i=1}^{n} w_i x_i\right) \le \mathbb{Z}(\varrho, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{x}, J) \le \sum_{i=1}^{n} w_i \varrho(x_i).$$

If I is an interval in $\mathbb{R} \setminus \{0\}$, a function $\varrho : I \to \mathbb{R}$ is said to be *harmonic convex* on I if

$$\varrho\Big(\frac{xy}{tx + (1-t)y}\Big) \le (1-t)\varrho(x) + t\varrho(y)$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

Dragomir [13] established a Jensen-type inequality for harmonic convex functions as follows.

Theorem 2. *Let I be an interval in $(0, \infty)$ and let ϱ be a harmonic convex function on I. If $w_1, \ldots, w_n \ge 0$ with $\sum_{i=1}^{n} w_i = 1$ and $x_1, \ldots, x_n \in I$, then*

$$\varrho\left(\frac{1}{\sum_{i=1}^{n} \frac{w_i}{x_i}}\right) \le \sum_{i=1}^{n} \varrho(x_i) w_i.$$

In [8], the following generalization of Theorem 2 is found:

Theorem 3. *If μ is a probability measure on space X, and I is an interval in $\mathbb{R} \setminus \{0\}$, let $p : X \to I$ be a measurable function and let ϱ be a harmonic convex function on I such that $\varrho \circ p$ is a μ-integrable function. Then,*

$$\varrho\left(\frac{1}{\int_X \frac{d\mu}{p}}\right) \le \int_X (\varrho \circ p)\, d\mu,$$

where $\varrho(0) = \lim_{t\to 0,\, t\in I} \varrho(t)$ if $\int_X \frac{d\mu}{p} = \pm\infty$.

In this work, we prove several refinements of Jensen's inequality for convex and harmonic convex functions.

3. Major results

If μ is a probability measure on space X, and $I \subset \mathbb{R}$ is an interval, suppose that $p : X \to I$ is a μ-integrable function. Also, let $X_0 \subset X$ be a measurable subset and $X_0^c = X \setminus X_0$.

We remark that the function q is (X_0, μ)-*compatible* if $q : X \to \mathbb{R}$ is a measurable function with $q(x) \in [0, 1]$ for every $x \in X$ and

$$\int_{X_0} q\, d\mu > 0, \qquad \int_{X_0} (1-q)\, d\mu > 0,$$
$$\int_{X_0^c} q\, d\mu > 0, \qquad \int_{X_0^c} (1-q)\, d\mu > 0.$$

Suppose that $\varrho : I \to \mathbb{R}$ is a convex function and q is (X_0, μ)-compatible. Since $p(x) \in I$ for every $x \in X$, we establish

$$\begin{aligned}\mathbb{Z}(\varrho, p, q, X_0, \mu) = {} & \varrho\left(\frac{\int_{X_0} pq\, d\mu}{\int_{X_0} q\, d\mu}\right)\int_{X_0} q\, d\mu + \varrho\left(\frac{\int_{X_0} p(1-q)\, d\mu}{\int_{X_0}(1-q)\, d\mu}\right)\int_{X_0}(1-q)\, d\mu \\ & + \varrho\left(\frac{\int_{X_0^c} pq\, d\mu}{\int_{X_0^c} q\, d\mu}\right)\int_{X_0^c} q\, d\mu + \varrho\left(\frac{\int_{X_0^c} p(1-q)\, d\mu}{\int_{X_0^c}(1-q)\, d\mu}\right)\int_{X_0^c}(1-q)\, d\mu.\end{aligned}$$

Theorem 1 is an interesting improvement of the discrete Jensen's inequality. We have the following refinement of Jensen's inequality, which generalizes Theorem 1 and an inequality in [21], for any probability measure μ and any measurable space X.

Theorem 4. *Let μ be a probability measure on space X, and let $X_0 \subset X$ be a measurable subset and $I \subset \mathbb{R}$ be an interval. Also, let $p : X \to I$ be a μ-integrable function, and let q be a (X_0, μ)-compatible function. If $\varrho : I \to \mathbb{R}$ is a convex function, then pq and $p(1-q)$ are μ-integrable functions and*

$$\varrho\left(\int_X p\, d\mu\right) \leq \mathbb{Z}(\varrho, p, q, X_0, \mu) \leq \int_X (\varrho \circ p)\, d\mu. \tag{3.1}$$

These inequalities reverse if ϱ is a concave function.

Let us define the set

$$A = \left\{\frac{\int_{X_0} pq\, d\mu}{\int_{X_0} q\, d\mu}, \frac{\int_{X_0} p(1-q)\, d\mu}{\int_{X_0}(1-q)\, d\mu}, \frac{\int_{X_0^c} pq\, d\mu}{\int_{X_0^c} q\, d\mu}, \frac{\int_{X_0^c} p(1-q)\, d\mu}{\int_{X_0^c}(1-q)\, d\mu}\right\}.$$

The equality in the first inequality holds if and only if either ϱ is affine on $[\min A, \max A]$ *or the set A has a single point.*

The equality in the second inequality holds if and only if the two following statements hold:

(a) ϱ is affine on $p(X_0)$ or p is constant μ-a.e. on X_0,
(b) ϱ is affine on $p(X_0^c)$ or p is constant μ-a.e. on X_0^c.

Proof. Note that since $p \in L^1(X,\mu)$ and $0 \le q \le 1$, then pq and $p(1-q)$ are μ-integrable functions.

The strategy in the proof of the first inequality is to apply the discrete Jensen's inequality in an appropriate way. Since p takes values on I and q is a (X_0,μ)-compatible function, one can check that all elements of set A belong to I, i.e., $A \subset I$. Since q is a (X_0,μ)-compatible function, we have a convex linear combination with four addends, and the discrete Jensen's inequality implies

$$
\begin{aligned}
\varrho\Big(\int_X p\,d\mu\Big) &= \varrho\left(\frac{\int_{X_0} pq\,d\mu}{\int_{X_0} q\,d\mu}\int_{X_0} q\,d\mu + \frac{\int_{X_0} p(1-q)\,d\mu}{\int_{X_0}(1-q)\,d\mu}\int_{X_0}(1-q)\,d\mu\right.\\
&\quad\left. + \frac{\int_{X_0^c} pq\,d\mu}{\int_{X_0^c} q\,d\mu}\int_{X_0^c} q\,d\mu + \frac{\int_{X_0^c} p(1-q)\,d\mu}{\int_{X_0^c}(1-q)\,d\mu}\int_{X_0^c}(1-q)\,d\mu\right)\\
&\le \varrho\left(\frac{\int_{X_0} pq\,d\mu}{\int_{X_0} q\,d\mu}\right)\int_{X_0} q\,d\mu + \varrho\left(\frac{\int_{X_0} p(1-q)\,d\mu}{\int_{X_0}(1-q)\,d\mu}\right)\int_{X_0}(1-q)\,d\mu\\
&\quad + \varrho\left(\frac{\int_{X_0^c} pq\,d\mu}{\int_{X_0^c} q\,d\mu}\right)\int_{X_0^c} q\,d\mu + \varrho\left(\frac{\int_{X_0^c} p(1-q)\,d\mu}{\int_{X_0^c}(1-q)\,d\mu}\right)\int_{X_0^c}(1-q)\,d\mu\\
&= \mathbb{Z}(\varrho, p, q, X_0, \mu),
\end{aligned}
$$

and the first inequality in (3.1) holds.

The argument in the proof and the discrete Jensen's inequality show that the equality in the first inequality holds if and only if either ϱ is affine on $[\min A, \max A]$ or A has a single point.

The idea in the proof of the second inequality is to apply Jensen's inequality four times in an appropriate way. We have

$$
\begin{aligned}
\mathbb{Z}(\varrho, p, q, X_0, \mu) &= \varrho\left(\frac{\int_{X_0} pq\,d\mu}{\int_{X_0} q\,d\mu}\right)\int_{X_0} q\,d\mu + \varrho\left(\frac{\int_{X_0} p(1-q)\,d\mu}{\int_{X_0}(1-q)\,d\mu}\right)\int_{X_0}(1-q)\,d\mu\\
&\quad + \varrho\left(\frac{\int_{X_0^c} pq\,d\mu}{\int_{X_0^c} q\,d\mu}\right)\int_{X_0^c} q\,d\mu + \varrho\left(\frac{\int_{X_0^c} p(1-q)\,d\mu}{\int_{X_0^c}(1-q)\,d\mu}\right)\int_{X_0^c}(1-q)\,d\mu\\
&\le \frac{\int_{X_0}(\varrho\circ p)\,q\,d\mu}{\int_{X_0} q\,d\mu}\int_{X_0} q\,d\mu + \frac{\int_{X_0}(\varrho\circ p)\,(1-q)\,d\mu}{\int_{X_0}(1-q)\,d\mu}\int_{X_0}(1-q)\,d\mu\\
&\quad + \frac{\int_{X_0^c}(\varrho\circ p)\,q\,d\mu}{\int_{X_0^c} q\,d\mu}\int_{X_0^c} q\,d\mu + \frac{\int_{X_0^c}(\varrho\circ p)\,(1-q)\,d\mu}{\int_{X_0^c}(1-q)\,d\mu}\int_{X_0^c}(1-q)\,d\mu\\
&= \int_{X_0}(\varrho\circ p)\,q\,d\mu + \int_{X_0}(\varrho\circ p)\,(1-q)\,d\mu + \int_{X_0^c}(\varrho\circ p)\,q\,d\mu + \int_{X_0^c}(\varrho\circ p)\,(1-q)\,d\mu\\
&= \int_{X_0}(\varrho\circ p)\,d\mu + \int_{X_0^c}(\varrho\circ p)\,d\mu\\
&= \int_X(\varrho\circ p)\,d\mu.
\end{aligned}
$$

The argument in the proof shows that the equality in the second inequality holds if and only if the four following equalities hold:

$$\varrho\left(\frac{\int_{X_0} pq\,d\mu}{\int_{X_0} q\,d\mu}\right) = \frac{\int_{X_0}(\varrho\circ p)\,q\,d\mu}{\int_{X_0} q\,d\mu},$$
$$\varrho\left(\frac{\int_{X_0} p(1-q)\,d\mu}{\int_{X_0}(1-q)\,d\mu}\right) = \frac{\int_{X_0}(\varrho\circ p)\,(1-q)\,d\mu}{\int_{X_0}(1-q)\,d\mu},$$
$$\varrho\left(\frac{\int_{X_0^c} pq\,d\mu}{\int_{X_0^c} q\,d\mu}\right) = \frac{\int_{X_0^c}(\varrho\circ p)\,q\,d\mu}{\int_{X_0^c} q\,d\mu},$$
$$\varrho\left(\frac{\int_{X_0^c} p(1-q)\,d\mu}{\int_{X_0^c}(1-q)\,d\mu}\right) = \frac{\int_{X_0^c}(\varrho\circ p)\,(1-q)\,d\mu}{\int_{X_0^c}(1-q)\,d\mu}.$$

Then, Jensen's inequality show that the equality in the second inequality holds if and only if the four following statements hold:

(1) ϱ is affine on $p(X_0)$ or p is constant $q\mu$-a.e. on X_0,
(2) ϱ is affine on $p(X_0)$ or p is constant $(1-q)\mu$-a.e. on X_0,
(3) ϱ is affine on $p(X_0^c)$ or p is constant $q\mu$-a.e. on X_0^c,
(4) ϱ is affine on $p(X_0^c)$ or p is constant $(1-q)\mu$-a.e. on X_0^c.

Finally, this is equivalent to the two following statements:

(a) ϱ is affine on $p(X_0)$ or p is constant μ-a.e. on X_0,
(b) ϱ is affine on $p(X_0^c)$ or p is constant μ-a.e. on X_0^c.

If we replace ϱ by $-\varrho$, we obtain reverse of inequalities if ϱ is a concave function. □

Example 5. *Let μ be the Lebesgue measure on $X = [0,1]$, and $X_0 = [0,\frac{1}{2}]$. Consider also $p(x) = x^2, q(x) = x$ and $\varrho(x) = e^x$. Then,*

$$\varrho\left(\int_X p\,d\mu\right) = \varrho\left(\int_0^1 x^2\,dx\right) = \varrho\left(\left[\frac{x^3}{3}\right]_0^1\right) = e^{\frac{1}{3}},$$

$$\int_X(\varrho\circ p)\,d\mu = \int_0^1 e^{x^2}\,dx = \sum_{n=0}^{\infty}\frac{1}{n!(2n+1)}.$$

Also,

$$\varrho\left(\frac{\int_{X_0} pq\,d\mu}{\int_{X_0} q\,d\mu}\right)\int_{X_0} q\,d\mu = \varrho\left(\frac{\int_0^{\frac{1}{2}} x^3\,dx}{\int_0^{\frac{1}{2}} x\,dx}\right)\int_0^{\frac{1}{2}} x\,dx = \varrho\left(\frac{\left[\frac{x^4}{4}\right]_0^{\frac{1}{2}}}{\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}}}\right)\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} = \frac{1}{8}e^{1/8},$$
$$\varrho\left(\frac{\int_{X_0^c} pq\,d\mu}{\int_{X_0^c} q\,d\mu}\right)\int_{X_0^c} q\,d\mu = \varrho\left(\frac{\int_{\frac{1}{2}}^1 x^3\,dx}{\int_{\frac{1}{2}}^1 x\,dx}\right)\int_{\frac{1}{2}}^1 x\,dx = \varrho\left(\frac{\left[\frac{x^4}{4}\right]_{\frac{1}{2}}^1}{\left[\frac{x^2}{2}\right]_{\frac{1}{2}}^1}\right)\left[\frac{x^2}{2}\right]_{\frac{1}{2}}^1 = \frac{3}{8}e^{5/8},$$

$$\varrho\left(\frac{\int_{X_0} p(1-q)\,d\mu}{\int_{X_0}(1-q)\,d\mu}\right)\int_{X_0}(1-q)\,d\mu=\varrho\left(\frac{\int_0^{\frac{1}{2}}(x^2-x^3)\,dx}{\int_0^{\frac{1}{2}}(1-x)\,dx}\right)\int_0^{\frac{1}{2}}(1-x)\,dx$$
$$=\varrho\left(\frac{\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^{1/2}}{\left[x-\frac{x^2}{2}\right]_0^{1/2}}\right)\left[x-\frac{x^2}{2}\right]_0^{1/2}=\frac{3}{8}e^{5/72},$$
$$\varrho\left(\frac{\int_{X_0^c} p(1-q)\,d\mu}{\int_{X_0^c}(1-q)\,d\mu}\right)\int_{X_0^c}(1-q)\,d\mu=\varrho\left(\frac{\int_{\frac{1}{2}}^1(x^2-x^3)\,dx}{\int_{\frac{1}{2}}^1(1-x)\,dx}\right)\int_{\frac{1}{2}}^1(1-x)\,dx$$
$$=\varrho\left(\frac{\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_{1/2}^{1}}{\left[x-\frac{x^2}{2}\right]_{1/2}^{1}}\right)\left[x-\frac{x^2}{2}\right]_{1/2}^{1}=\frac{1}{8}e^{11/24}.$$

Therefore,

$$e^{1/3}<\frac{1}{8}\left(e^{1/8}+3e^{5/8}+3e^{5/72}+e^{11/24}\right)<\sum_{n=0}^{\infty}\frac{1}{n!(2n+1)}\,.$$

By approximately calculating these numbers, we confirm the inequalities, i.e.,

$$1.3956<1.4419<1.4627.$$

Theorem 4 enables us to obtain the following remark for Theorem 1.

Remark 6. *Under the hypotheses in Theorem 1, let us define*

$$B=\left\{\frac{\sum_{i\in J}w_iv_ix_i}{\sum_{i\in J}w_iv_i},\frac{\sum_{i\in J}w_i(1-v_i)x_i}{\sum_{i\in J}w_i(1-v_i)},\frac{\sum_{i\in J^c}w_iv_ix_i}{\sum_{i\in J^c}w_iv_i},\frac{\sum_{i\in J^c}w_i(1-v_i)x_i}{\sum_{i\in J^c}w_i(1-v_i)}\right\}.$$

The equality in the first inequality in Theorem 1 holds if and only if either ϱ is affine on $[\min B,\max B]$ *or B has a single point.*

The equality in the second inequality in Theorem 1 holds if and only if the two following statements hold:

(*a*) *ϱ is affine on $\cup_{i\in J}\{x_i\}$ or $x_i=x_j$ for every $i,j\in J$,*

(*b*) *ϱ is affine on $\cup_{i\in J^c}\{x_i\}$ or $x_i=x_j$ for every $i,j\in J^c$.*

Now, we prove the following refinements of Theorems 2 and 3.

Theorem 7. *Let μ be a probability measure on space X, let $X_0\subset X$ be a measurable subset, and let $I\subset\mathbb{R}\setminus\{0\}$ be an interval. Also, let $p:X\to I$ be a measurable function such that $\frac{1}{p}\in L^1(X,\mu)$, and let q be a (X_0,μ)-compatible function. If $\varrho:I\to\mathbb{R}$ is a harmonic convex function, then $\frac{q}{p}$ and $\frac{1-q}{p}$ are μ-integrable functions and*

$$\varrho\left(\frac{1}{\int_X\frac{d\mu}{p}}\right)\leq\mathbb{Z}_h(\varrho,p,q,X_0,\mu)\leq\int_X(\varrho\circ p)\,d\mu,\tag{3.2}$$

where

$$\mathbb{Z}_h(\varrho,p,q,X_0,\mu)=\varrho\left(\frac{\int_{X_0} q\,d\mu}{\int_{X_0}\frac{q}{p}\,d\mu}\right)\int_{X_0} q\,d\mu+\varrho\left(\frac{\int_{X_0}(1-q)\,d\mu}{\int_{X_0}\frac{1-q}{p}\,d\mu}\right)\int_{X_0}(1-q)\,d\mu$$
$$+\varrho\left(\frac{\int_{X_0^c} q\,d\mu}{\int_{X_0^c}\frac{q}{p}\,d\mu}\right)\int_{X_0^c} q\,d\mu+\varrho\left(\frac{\int_{X_0^c}(1-q)\,d\mu}{\int_{X_0^c}\frac{1-q}{p}\,d\mu}\right)\int_{X_0^c}(1-q)\,d\mu.$$

Proof. Since $\frac{1}{p}\in L^1(X,\mu)$ and $0\le q\le 1$, then $\frac{q}{p}$ and $\frac{1-q}{p}$ are μ-integrable functions. Since p takes values on I and q is a (X_0,μ)-compatible function, one can check that

$$\frac{\int_{X_0} q\,d\mu}{\int_{X_0}\frac{q}{p}\,d\mu},\ \frac{\int_{X_0}(1-q)\,d\mu}{\int_{X_0}\frac{1-q}{p}\,d\mu},\ \frac{\int_{X_0^c} q\,d\mu}{\int_{X_0^c}\frac{q}{p}\,d\mu},\ \frac{\int_{X_0^c}(1-q)\,d\mu}{\int_{X_0^c}\frac{1-q}{p}\,d\mu}\in I.$$

Since q is a (X_0,μ)-compatible function, Theorem 2 (the discrete Jensen's inequality for harmonic convex functions) implies

$$\begin{aligned}
\varrho\left(\frac{1}{\int_X\frac{d\mu}{p}}\right)&=\varrho\left(\frac{1}{\frac{\int_{X_0} q\,d\mu}{\left(\frac{\int_{X_0} q\,d\mu}{\int_{X_0}\frac{q}{p}\,d\mu}\right)}+\frac{\int_{X_0}(1-q)\,d\mu}{\left(\frac{\int_{X_0}(1-q)\,d\mu}{\int_{X_0}\frac{1-q}{p}\,d\mu}\right)}+\frac{\int_{X_0^c} q\,d\mu}{\left(\frac{\int_{X_0^c} q\,d\mu}{\int_{X_0^c}\frac{q}{p}\,d\mu}\right)}+\frac{\int_{X_0^c}(1-q)\,d\mu}{\left(\frac{\int_{X_0^c}(1-q)\,d\mu}{\int_{X_0^c}\frac{1-q}{p}\,d\mu}\right)}}\right)\\
&\le\varrho\left(\frac{\int_{X_0} q\,d\mu}{\int_{X_0}\frac{q}{p}\,d\mu}\right)\int_{X_0} q\,d\mu+\varrho\left(\frac{\int_{X_0}(1-q)\,d\mu}{\int_{X_0}\frac{1-q}{p}\,d\mu}\right)\int_{X_0}(1-q)\,d\mu\\
&\quad+\varrho\left(\frac{\int_{X_0^c} q\,d\mu}{\int_{X_0^c}\frac{q}{p}\,d\mu}\right)\int_{X_0^c} q\,d\mu+\varrho\left(\frac{\int_{X_0^c}(1-q)\,d\mu}{\int_{X_0^c}\frac{1-q}{p}\,d\mu}\right)\int_{X_0^c}(1-q)\,d\mu\\
&=\mathbb{Z}_h(\varrho,p,q,X_0,\mu),
\end{aligned}$$

and the first inequality in (3.2) holds.

We also have

$$\begin{aligned}
\mathbb{Z}_h(\varrho,p,q,X_0,\mu)&=\varrho\left(\frac{\int_{X_0} q\,d\mu}{\int_{X_0}\frac{q}{p}\,d\mu}\right)\int_{X_0} q\,d\mu+\varrho\left(\frac{\int_{X_0}(1-q)\,d\mu}{\int_{X_0}\frac{1-q}{p}\,d\mu}\right)\int_{X_0}(1-q)\,d\mu\\
&\quad+\varrho\left(\frac{\int_{X_0^c} q\,d\mu}{\int_{X_0^c}\frac{q}{p}\,d\mu}\right)\int_{X_0^c} q\,d\mu+\varrho\left(\frac{\int_{X_0^c}(1-q)\,d\mu}{\int_{X_0^c}\frac{1-q}{p}\,d\mu}\right)\int_{X_0^c}(1-q)\,d\mu\\
&=\varrho\left(\frac{1}{\frac{\int_{X_0}\frac{q}{p}\,d\mu}{\int_{X_0} q\,d\mu}}\right)\int_{X_0} q\,d\mu+\varrho\left(\frac{1}{\frac{\int_{X_0}\frac{1-q}{p}\,d\mu}{\int_{X_0}(1-q)\,d\mu}}\right)\int_{X_0}(1-q)\,d\mu
\end{aligned}$$

$$+\varrho\left(\frac{1}{\frac{\int_{X_0^c}\frac{q}{p}\,d\mu}{\int_{X_0^c}q\,d\mu}}\right)\int_{X_0^c}q\,d\mu+\varrho\left(\frac{1}{\frac{\int_{X_0^c}\frac{1-q}{p}\,d\mu}{\int_{X_0^c}(1-q)\,d\mu}}\right)\int_{X_0^c}(1-q)\,d\mu$$
$$\leq\int_{X_0}(\varrho\circ p)\,q\,d\mu+\int_{X_0}(\varrho\circ p)\,(1-q)\,d\mu+\int_{X_0^c}(\varrho\circ p)\,q\,d\mu+\int_{X_0^c}(\varrho\circ p)\,(1-q)\,d\mu$$
$$=\int_{X_0}(\varrho\circ p)\,d\mu+\int_{X_0^c}(\varrho\circ p)\,d\mu$$
$$=\int_X(\varrho\circ p)\,d\mu.$$

□

Theorem 7 directly implies the following result.

Corollary 8. *Consider an interval $I\subset(0,\infty)$, $x_i\in I$, $0<v_i<1$ for $i=1,2,\ldots,n$, and w_i positive numbers such that $\sum_{i=1}^n w_i=1$. Also, let J be a proper subset of $\{1,2,\ldots,n\}$ and $J^c=\{1,2,\ldots,n\}\setminus J$. If $\varrho:I\to\mathbb{R}$ is a harmonic convex function, then*

$$\varrho\left(\frac{1}{\sum_{i=1}^n\frac{w_i}{x_i}}\right)\leq\mathbb{Z}_h(\varrho,\boldsymbol{w},\boldsymbol{v},\boldsymbol{x},J)\leq\sum_{i=1}^n\varrho(x_i)w_i,\tag{3.3}$$

where

$$\mathbb{Z}_h(\varrho,\boldsymbol{w},\boldsymbol{v},\boldsymbol{x},J)=\varrho\left(\frac{\sum_{i\in J}v_iw_i}{\sum_{i\in J}\frac{v_iw_i}{x_i}}\right)\sum_{i\in J}v_iw_i+\varrho\left(\frac{\sum_{i\in J}(1-v_i)w_i}{\sum_{i\in J}\frac{(1-v_i)w_i}{x_i}}\right)\sum_{i\in J}(1-v_i)w_i$$
$$+\varrho\left(\frac{\sum_{i\in J^c}v_iw_i}{\sum_{i\in J^c}\frac{v_iw_i}{x_i}}\right)\sum_{i\in J^c}v_iw_i+\varrho\left(\frac{\sum_{i\in J^c}(1-v_i)w_i}{\sum_{i\in J^c}\frac{(1-v_i)w_i}{x_i}}\right)\sum_{i\in J^c}(1-v_i)w_i.$$

Proof. Let $X=\{1,\ldots,n\}$ and $X_0=J$. Consider the measure μ on X given by $\mu(\{i\})=w_i$ for $i=1,\ldots,n$. Theorem 7, with the functions $p:X\to I$ and $q:X\to\mathbb{R}^+$, defined by $p(i)=x_i$ and $q(i)=v_i$ for $i=1,\ldots,n$, gives the desired inequalities. □

4. Generalized Caputo derivative

In [9], Michele Caputo introduced a new fractional derivative. This definition has an important property associated with the resolution of differential equations, since it is not necessary to define the initial conditions of fractional order. For multiple applications of the so-called Caputo differential operator, the interested reader is encouraged to consult [10].

The *Caputo derivative* of a differentiable function g of order $0<\alpha<1$ is defined as

$${}^{C}D_a^\alpha g(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{g'(s)}{(t-s)^\alpha}\,ds.\tag{4.1}$$

An extension of ${}^{C}D_a^\alpha$ is the so-called *Caputo-Fabricio derivative* (see [4, 11]), given by:

$${}^{CF}D_a^\alpha g(t)=\frac{N(\alpha)}{1-\alpha}\int_a^t g'(s)e^{-\frac{\alpha(t-s)}{1-\alpha}}\,ds,\tag{4.2}$$

such that $N(0) = N(1) = 1$ and $N(\alpha)$ is a normalization function.

A more recent extension is the *Atangana-Baleanu derivative*, defined in [3] by

$$ {}^{ABC}D_a^\alpha g(t) = \frac{N(\alpha)}{1-\alpha} \int_a^t g'(s) E_\alpha\Big(-\frac{\alpha(t-s)^\alpha}{1-\alpha}\Big)\, ds, \tag{4.3} $$

where,

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} $$

is the *Mittag-Leffler function*. Note that $E_1(z) = e^z$.

Definition 9. *We say that K is an* admissible kernel *for the interval $[a, b]$ if $K : [0, b-a] \times (0, 1) \to [0, \infty)$ is a non-negative continuous function such that*

$$ \mathbb{K}(\alpha) = \int_0^{b-a} \frac{ds}{K(s, \alpha)} < \infty $$

for each $\alpha \in (0, 1)$. K is an admissible kernel *for $[a, \infty)$ if it is admissible for $[a, b]$ for every $b > a$.*

Next, we introduce the definition of the generalized Caputo derivative as provided in [5].

Definition 10. *Let $\alpha \in (0, 1)$, K be an admissible kernel for $[a, b]$, $g : [a, b] \to \mathbb{R}$ be a differentiable function and $t \in [a, b]$. The* generalized Caputo derivative *of order α of the function g at the point t is*

$$ {}^{C}D_{K,a}^\alpha g(t) = \int_a^t \frac{g'(s)}{K(t-s, \alpha)}\, ds. \tag{4.4} $$

Remark 11. *If $K(x, \alpha) = \Gamma(1-\alpha)\, x^\alpha$, then we obtain the classical Caputo derivative. Similarly, we can obtain the kernels for Caputo-Fabricio and Atangana-Baleanu extensions. Hence, each result for ${}^{C}D_{K,a}^\alpha$ has as a consequence the same result for the classical Caputo derivative, Caputo-Fabricio, and Atangana-Baleanu extensions. This shows that the generalized Caputo derivative is quite general and unifying.*

The following integral operator is associated with the generalized Caputo derivative in a natural way.

Definition 12. *Let $\alpha \in (0, 1)$, let K be an admissible kernel for $[a, b]$ and let $g : [a, b] \to \mathbb{R}$ be a measurable function such that*

$$ \frac{g(s)}{K(t-s, \alpha)} \in L^1([a, b], ds). $$

The generalized Caputo integral operator *of order α of the function g at the point $t \in [a, b]$ is*

$$ {}^{C}J_{K,a}^\alpha g(t) = \int_a^t \frac{g(s)}{K(t-s, \alpha)}\, ds. \tag{4.5} $$

Hence, if g is differentiable, we have

$$ {}^{C}D_{K,a}^\alpha g(t) = {}^{C}J_{K,a}^\alpha g'(t). $$

We also use the functional defined by

$$^{C}J_{K}^{\alpha}(g) = {}^{C}J_{K,a}^{\alpha}g(b) = \int_{a}^{b} \frac{g(s)}{K(b-s,\alpha)}\,ds.$$

It is clear that the above operators are non-local linear operators.

Certain properties of the generalized Caputo derivative and its associated integral operator can be found in [5].

Theorems 4 and 7 have the following implications for the general fractional integrals of Caputo-type.

Proposition 13. *Assume that K is an admissible kernel for the interval $[a,b]$ with*

$$\mathbb{K}(\alpha) = \int_{a}^{b} \frac{1}{K(b-s,\alpha)}\,ds = \int_{0}^{b-a} \frac{ds}{K(s,\alpha)} < \infty,$$

and consider the measure $d\mu(s) = \frac{ds}{\mathbb{K}(\alpha)K(b-s,\alpha)}$ on $[a,b]$. Also, let $I \subset \mathbb{R}$ be an interval, let $p : [a,b] \to I$ be a μ-integrable function, let $X_0 \subset [a,b]$ be a nonempty subinterval, and let q be a (X_0,μ)-compatible function. If $\varrho : I \to \mathbb{R}$ is a convex function, then $pq, p(1-q) \in L^1([a,b],\mu)$ and

$$\varrho\left(\frac{1}{\mathbb{K}(\alpha)} \int_{a}^{b} \frac{p(s)}{K(b-s,\alpha)}\,ds\right) \le \mathbb{Z}(\varrho,p,q,X_0,\mu) \le \frac{1}{\mathbb{K}(\alpha)} \int_{a}^{b} \frac{\varrho(p(s))}{K(b-s,\alpha)}\,ds.$$

Proposition 14. *Assume that K is an admissible kernel for the interval $[a,b]$ with*

$$\mathbb{K}(\alpha) = \int_{a}^{b} \frac{1}{K(b-s,\alpha)}\,ds = \int_{0}^{b-a} \frac{ds}{K(s,\alpha)} < \infty,$$

and consider the measure $d\mu(s) = \frac{ds}{\mathbb{K}(\alpha)K(b-s,\alpha)}$ on $[a,b]$. Also, let $I \subset \mathbb{R} \setminus \{0\}$ be an interval, let $p : [a,b] \to I$ be a measurable function such that $\frac{1}{p} \in L^1([a,b],\mu)$, let $X_0 \subset [a,b]$ be a nonempty subinterval, and let q be a (X_0,μ)-compatible function. If $\varrho : I \to \mathbb{R}$ is a harmonic convex function, then $\frac{q}{p}, \frac{1-q}{p} \in L^1([a,b],\mu)$ and

$$\varrho\left(\frac{\mathbb{K}(\alpha)}{\int_{a}^{b} \frac{ds}{p(s)K(b-s,\alpha)}}\right) \le \mathbb{Z}_h(\varrho,p,q,X_0,\mu) \le \frac{1}{\mathbb{K}(\alpha)} \int_{a}^{b} \frac{\varrho(p(s))}{K(b-s,\alpha)}\,ds.$$

5. Generalized local fractional derivative

The definition of the generalized local fractional derivative, as stated in [2,6,14,15], is as follows.

Definition 15. *Given an interval $I \subseteq \mathbb{R}$, $p : I \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and a positive continuous function $H(t,\alpha)$ on $I \times (0,\infty)$, the* derivative *$G_H^{\alpha}p$ of p of order α at the point $t \in I$ is defined by*

$$G_H^{\alpha}p(t) = \lim_{h\to 0} \frac{1}{h^{\lceil\alpha\rceil}} \sum_{k=0}^{\lceil\alpha\rceil} (-1)^k \binom{\lceil\alpha\rceil}{k} p(t-khH(t,\alpha)). \tag{5.1}$$

If $a = \inf\{t \in I\}$ (respectively, $b = \sup\{t \in I\}$), then $G_H^{\alpha} p(a)$ (respectively, $G_H^{\alpha} p(b)$) is defined with $h \to 0^-$ (respectively, $h \to 0^+$) instead of $h \to 0$ in the limit.

The reader is referred to [1, 17, 19] for further information on conformable fractional derivatives.

Let I be an interval $I \subseteq \mathbb{R}$, $a, t \in I$ and $\alpha \in \mathbb{R}$. The generalized local fractional integral $J_{H,a}^{\alpha}$ is defined as

$$J_{H,a}^{\alpha}(p)(t) = \int_a^t \frac{p(s)}{H(s,\alpha)}\, ds, \tag{5.2}$$

for every locally integrable function p on I.

The following results in [2, 6, 14, 15, 20] contain some basic properties of this integral operator. Theorems 4 and 7 have the following implications for the operator $J_{H,a}^{\alpha}$.

Proposition 16. *Let $I \subset \mathbb{R}$ be an interval and $d\mu(s) = \dfrac{ds}{\mathbb{H}(\alpha)H(s,\alpha)}$ on $[a,b]$ with*

$$\mathbb{H}(\alpha) = \int_a^b \frac{1}{H(s,\alpha)}\, ds < \infty.$$

Also, let $p : [a,b] \to I$ be a μ-integrable function, let $X_0 \subset [a,b]$ be a nonempty subinterval, and let q be a (X_0,μ)-compatible function. If $\varrho : I \to \mathbb{R}$ is a convex function, then $pq, p(1-q) \in L^1([a,b],\mu)$ and

$$\varrho\left(\frac{1}{\mathbb{H}(\alpha)} \int_a^b \frac{p(s)}{H(s,\alpha)}\, ds\right) \le \mathbb{Z}(\varrho, p, q, X_0, \mu) \le \frac{1}{\mathbb{H}(\alpha)} \int_a^b \frac{\varrho(p(s))}{H(s,\alpha)}\, ds.$$

Proposition 17. *Let $I \subset \mathbb{R} \setminus \{0\}$ be an interval and $d\mu(s) = \dfrac{ds}{\mathbb{H}(\alpha)H(s,\alpha)}$ on $[a,b]$ with*

$$\mathbb{H}(\alpha) = \int_a^b \frac{1}{H(s,\alpha)}\, ds < \infty.$$

Also, let $p : [a,b] \to I$ be a measurable function such that $\frac{1}{p} \in L^1([a,b],\mu)$, let $X_0 \subset [a,b]$ be a nonempty subinterval, and let q be a (X_0,μ)-compatible function. If $\varrho : I \to \mathbb{R}$ is a harmonic convex function, then $\frac{q}{p}, \frac{1-q}{p} \in L^1([a,b],\mu)$ and

$$\varrho\left(\frac{\mathbb{H}(\alpha)}{\int_a^b \frac{ds}{p(s)H(s,\alpha)}}\right) \le \mathbb{Z}_h(\varrho, p, q, X_0, \mu) \le \frac{1}{\mathbb{H}(\alpha)} \int_a^b \frac{\varrho(p(s))}{H(s,\alpha)}\, ds.$$

6. Conclusions

Our contributions to this research focus on two interrelated domains: the theory of inequalities and the theory of fractional operators.

On the theoretical side, we establish novel refinements of Jensen-type inequalities. Notably, while several advancements in Jensen's inequality already exist, such as the discrete Jensen's inequality with a double inequality discussed in Theorem 1 of [22], our primary results go further by:

(1) Extending these inequalities in the best possible way: It holds on any general measurable space with any probability measure, with a detailed characterization of the equality conditions (see Theorem 4).

(2) Developing a specialized version of these inequalities for harmonic convex functions (see Theorem 7).

On the applied side, we expand the applicability of these inequalities in the realm of fractional integral operators. Specifically, we adapt and extend the results to two classes of fractional operators:

(1) Fractional integral operators associated with the generalized Caputo derivative.
(2) Fractional integral operators tied to the generalized local fractional derivative.

Through these extensions, our findings not only broaden the theoretical scope of Jensen-type inequalities but also provide deeper insights and tools for fractional calculus.

Author contributions

All authors have accepted responsibility for the entire content of the manuscript and approved its submission.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors state no conflicts of interest.

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