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Research article

Characterizations of the product of asymmetric dual truncated Toeplitz operators

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Abstract: The asymmetric dual truncated Toeplitz operator (ADTTO) is a compression multiplication operator acting on the orthogonal complement of two different model spaces. In this paper, we present an operator equation characterization of an ADTTO using the compressed shift operator. As an application, the product of two ADTTOs with certain symbols being another ADTTO is obtained.

Keywords: asymmetric dual truncated Toeplitz operator; Hardy space; model space; product problem
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1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$, its boundary the unit circle $\mathbb{T} = \{z : |z| = 1\}$. L^2 is the space of square-integrable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure dσ. It is known that L^2 is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\mathbb{T}} f\bar{g}d\sigma.$$

Let $e_n(z) = z^n, z \in \mathbb{T}, n \in \mathbb{Z}$; then $\{e_n : n \in \mathbb{Z}\}$ forms a standard orthogonal basis for L^2. For each $h \in L^2$, it is well known that

$$h = \Sigma_{n\in\mathbb{Z}}\langle h, e_n\rangle e_n$$

and

$$||h||^2 = \sum_{n\in\mathbb{Z}} |\langle h, e_n\rangle|^2.$$

The classical Hardy space H^2 is a closed subspace of L^2 consisting of h with $h = \sum_{n\geq 0}\langle h, e_n\rangle e_n$. So $L^2 = H^2 \oplus \overline{zH^2}$, where $\overline{zH^2} = \{\overline{zf} : f \in H^2\}$. Since the evaluation at each point in $\mathbb{D}$ is continuous, then

H^2 becomes a reproducing Hilbert space with the reproducing kernel given by

$$k_w(z) = \frac{1}{1 - \overline{w}z}, \quad w \in \mathbb{D}, z \in \mathbb{T}.$$

Let P be the orthogonal projection from L^2 onto H^2; then

$$Pf(z) = \langle f, k_z \rangle, \quad f \in L^2.$$

Denote L^∞ and H^∞ as the algebras of bounded functions in L^2 and H^2, respectively. Define the Toeplitz operator T_φ on the Hardy space H^2 with symbol $\varphi \in L^\infty$ by

$$T_\varphi f = P[\varphi f], \quad f \in H^2.$$

It is obvious that T_φ is a bounded linear operator on H^2.

If $\theta \in H^\infty$ has $|\theta| = 1$ almost everywhere on the unit circle $\mathbb{T}$, then θ is called an inner function, and the corresponding model space K_θ is the orthogonal complement of θH^2 in H^2, i.e., $K_\theta = H^2 \ominus \theta H^2$. It is known that for inner functions u and v, $K_{uv} = K_u \oplus uK_v$. The shift operator S is defined by $Sf(z) = zf(z)$; its adjoint operator is called the backward unilateral shift operator, which is $S^*f(z) = \frac{f(z)-f(0)}{z}$. The model space is an invariant subspace of the backward unilateral shift operator S^*, and also a reproducing kernel Hilbert space whose reproducing kernel is

$$k_w^\theta(z) = \frac{1 - \overline{\theta(w)}\theta(z)}{1 - \overline{w}z}, \quad w \in \mathbb{D}, \ z \in \mathbb{T}.$$

Since k_w^θ is bounded, the set $K_\theta^\infty = K_\theta \cap H^\infty$ is dense in K_θ.

For $\varphi \in L^\infty$, the truncated Toeplitz operator (TTO) A_φ^θ systematically studied by Sarason in [7] is defined on the model space K_θ by

$$A_\varphi^\theta f = P_\theta[\varphi f], \quad f \in K_\theta,$$

where P_θ is the orthogonal projection from L^2 onto K_θ. As is known to all, TTO is a natural generalization of Toeplitz matrices that appear in many contexts, such as in the study of finite-interval convolution equations, signal processing, control theory, probability, and diffraction problems [4,5,7]. Actually, A_φ^θ is the compression of T_φ on the model space K_θ, i.e., $A_\varphi^\theta = P_\theta T_\varphi|_{K_\theta}$. Sedlock [8] has ever defined the Sedlock class to study the product problem of truncated Toeplitz operators. For more information about model spaces and their operators, one is referred to [4].

Notice that

$$K_\theta^\perp = L^2 \ominus K_\theta = \overline{zH^2} \oplus \theta H^2.$$

It is easy to see that $\{e_{-n} : n \geq 1\}$ and $\{\theta e_n : n \geq 0\}$ are standard orthonormal bases for $\overline{zH^2}$ and θH^2, respectively, and so

$$\{e_{-n} : n \geq 1\} \cup \{\theta e_n : n \geq 0\}$$

forms a standard orthonormal basis for $K_\theta^\perp$.

Let $Q_\theta = P_\theta^\perp = I - P_\theta$ be the orthogonal projection from L^2 onto $K_\theta^\perp$. So $Q_\theta f = Qf + \theta P(\overline{\theta} f)$, where $Q = I - P$. In 2018, Ding and Sang [3] introduced the dual truncated Toeplitz operator (DTTO), which is defined on $K_\theta^\perp$ by

$$D_\varphi^\theta f = Q_\theta[\varphi f], \quad f \in K_\theta^\perp,$$

or written as

$$D_\varphi^\theta f = Q[\varphi f] + \theta P[\overline{\theta}\varphi f], \quad f \in K_\theta^\perp.$$

It is clear that $(D_\varphi^\theta)^* = D_{\overline{\varphi}}^\theta$. Furthermore, for any complex constant λ, we have $D_\lambda^\theta = \lambda I$.

Câmara [2] discussed the asymmetric dual truncated Toeplitz operator (ADTTO), which is defined on $K_\theta^\perp$ by

$$D_\varphi^{\theta,\alpha} f = P_\alpha^\perp[\varphi f] = Q[\varphi f] + \alpha P[\overline{\alpha}\varphi f], \quad f \in K_\theta^\perp.$$

Also, $(D_\varphi^{\theta,\alpha})^* = D_{\overline{\varphi}}^{\alpha,\theta}$. DTTO and ADTTO, acting on these spaces, have realizations, for example, in long–distance communication links with several regenerators along the path that cancel low–frequency noise using high–pass filters, or in the description of wave propagation in the presence of finite–length obstacles.

In 1964, Brown and Halmos [1] proved that a bounded operator A on H^2 is a Toeplitz operator if and only if $A - S^*AS = 0$. This operator equation plays a significant role in the study of Toeplitz operators and related topics. In 2007, Sarason proved a result on truncated Toeplitz operators in [7] that is similar to Brown and Halmos: Let $S_\theta = A_z^\theta$, the compressed shift operator on K_θ, then a bounded operator A on K_θ is a truncated Toeplitz operator if and only if $A - S_\theta^* A S_\theta$ is at most a rank-2 operator, more precisely,

$$A - S_\theta^* A S_\theta = \psi \otimes \widetilde{k_0^\theta} + \widetilde{k_0^\theta} \otimes \chi$$

for some $\psi, \chi \in K_\theta$, where

$$\widetilde{k_0^\theta} = \overline{z}[\theta(z) - \theta(0)].$$

In 2021, Gu [6] studied the dual truncated Toeplitz operators, and obtained that a bounded operator A on $K_\theta^\perp$ is a dual truncated Toeplitz operator if and only if $A - D_{\overline{z}}^\theta A D_z^\theta$ is at most a rank-2 operator and

$$A\theta = D_\varphi^\theta \theta, \quad A^*\theta = (D_\varphi^\theta)^*\theta$$

for some $\varphi \in L^\infty$, where D_z^θ is the compressed shift on $K_\theta^\perp$. This result is similar to Sarason's.

For the product problem of when two Toeplitz operators are another Toeplitz operator, Brown and Halmos in [1] established a necessary and sufficient condition based on the above operator equation characterization of the Toeplitz operator. In 2011, N. Sedlock did the same thing for truncated Toeplitz operators in [8].

Inspired by the above work, in this paper, we will establish an operator equation to obtain an equivalent characterization of the ADTTO, which is similar to Sarason's; see Theorem 3.1. Based on this, we follow a method taken in Sedlock's paper [8] and study the product problem of when the product of two ADTTOs with symbols in model spaces is another ADTTO; see Theorem 4.1.

2. preliminaries

In what follows, for $g \in L^2(\mathbb{T})$, we write g_n ($n \in \mathbb{Z}$) as the n-th Fourier coefficient $\langle h, e_n\rangle$ of g, unless otherwise stated.

The following lemmas come from [6].

Lemma 2.1. *For any function $h \in K_\theta^\perp$, we have*

$$D_z^\theta h = zh + \langle h, e_{-1}\rangle(\overline{\theta_0}\theta - e_0);$$
$$D_{\overline{z}}^\theta h = \overline{z}h + \langle h, \theta\rangle(\theta_0 - \theta)e_{-1}.$$

Lemma 2.2. *On $K_\theta^\perp$, we have*

$$I-(D_z^\theta)^*D_z^\theta=(1-|\theta_0|^2)e_{-1}\otimes e_{-1};$$
$$I-D_z^\theta(D_z^\theta)^*=(1-|\theta_0|^2)\theta\otimes\theta.$$

We also need the following result, which says that an ADTTO satisfies an operator equation.

Lemma 2.3. *The operator $D_\varphi^{\theta,\alpha}$ satisfies the following equation.*

$$D_\varphi^{\theta,\alpha}-D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}D_z^\theta=e_{-1}\otimes(\beta_\varphi^{\alpha,\theta}-\langle\beta_\varphi^{\alpha,\theta},\bar z\rangle e_{-1})+(\beta_{\bar\varphi}^{\theta,\alpha}+\delta e_{-1})\otimes e_{-1},$$

where

$$\beta_\varphi^{\alpha,\theta}=P_\theta^\perp[\overline{\varphi z}(1-\overline{\alpha_0}\alpha)],\quad \delta=\overline{\theta_0}\langle\varphi\theta\overline{k_0^\alpha},e_0\rangle.$$

Proof. For $h\in K_\theta^\perp\ominus\operatorname{span}\{e_{-1}\}$, using Lemma 2.1, direct calculations show that

$$\begin{aligned}
D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}D_z^\theta[h]&=D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}[zh]\\
&=\bar zD_\varphi^{\theta,\alpha}[zh]+\langle D_\varphi^{\theta,\alpha}[zh],\alpha\rangle(\alpha_0-\alpha)\bar z\\
&=\bar zQ[\varphi zh]+\bar z\alpha P[\bar\alpha\varphi zh]+\langle\varphi zh,\alpha\rangle(\alpha_0-\alpha)\bar z\\
&=Q[\varphi h]-\bar z(\varphi h)_{-1}+\alpha P[\bar\alpha\varphi h]+\bar z\alpha(\bar\alpha\varphi h)_{-1}+(\bar\alpha\varphi h)_{-1}(\alpha_0-\alpha)\bar z\\
&=D_\varphi^{\theta,\alpha}[h]-\langle\varphi h-\alpha_0(\bar\alpha\varphi h),\bar z\rangle\bar z\\
&=D_\varphi^{\theta,\alpha}[h]-\langle h,\ P_\theta^\perp[\overline{\varphi z}(1-\overline{\alpha_0}\alpha)]\rangle\bar z\\
&=D_\varphi^{\theta,\alpha}[h]-e_{-1}\otimes\beta_\varphi^{\alpha,\theta}[h].
\end{aligned}$$

Set $r=(D_\varphi^{\theta,\alpha}-D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}D_z^\theta)[\bar z]$, then for $h+\overline{zc}$, where $h\in K_\theta^\perp\ominus\operatorname{span}\{e_{-1}\}$ and c is a constant, we have

$$\begin{aligned}
&D_\varphi^{\theta,\alpha}[h+\overline{zc}]-(D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}D_z^\theta)[h+\overline{zc}]\\
&=D_\varphi^{\theta,\alpha}[h]-(D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}D_z^\theta)[h]+(D_\varphi^{\theta,\alpha}-D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}D_z^\theta)[\overline{zc}]\\
&=\langle h,\beta_\varphi^{\alpha,\theta}\rangle e_{-1}+\bar cr=\langle h+\overline{zc},\beta_\varphi^{\alpha,\theta}\rangle e_{-1}+\bar cr-\ \bar c\langle\bar z,\beta_\varphi^{\alpha,\theta}\rangle e_{-1}\\
&=(e_{-1}\otimes(\beta_\varphi^{\alpha,\theta}-\langle\beta_\varphi^{\alpha,\theta},\bar z\rangle e_{-1}))[h+\overline{zc}]+r\otimes e_{-1}[h+\overline{zc}].
\end{aligned}\tag{2.1}$$

Now we calculate $r=(D_\varphi^{\theta,\alpha}-D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}D_z^\theta)[\bar z]$. Since

$$\begin{aligned}
(D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}D_z^\theta)[\bar z]&=D_{\bar z}^\alpha[Q[\varphi\theta\overline{\theta_0}]+\alpha P[\bar\alpha\varphi\theta\overline{\theta_0}]]\\
&=\bar zQ[\varphi\theta\overline{\theta_0}]+\bar z\alpha P[\bar\alpha\varphi\theta\overline{\theta_0}]+\langle\varphi\bar\alpha\theta\overline{\theta_0},e_0\rangle(\alpha_0-\alpha)\bar z\\
&=P_\alpha^\perp[\varphi\theta\overline{\theta_0}\bar z]-(\varphi\theta)_0\overline{\theta_0}e_{-1}+(\varphi\bar\alpha\theta)_0\overline{\theta_0}\alpha_0e_{-1},
\end{aligned}$$

thus, we obtain

$$\begin{aligned}
r&=(D_\varphi^{\theta,\alpha}-D_{\bar z}^\alpha D_\varphi^{\theta,\alpha}D_z^\theta)[\bar z]\\
&=P_\alpha^\perp[\varphi\bar z(1-\overline{\theta_0}\theta)]+\langle\varphi\theta\overline{\theta_0}-\varphi\bar\alpha\theta\overline{\theta_0}\alpha_0,e_0\rangle e_{-1}\\
&=\beta_{\bar\varphi}^{\theta,\alpha}+\overline{\theta_0}\langle\varphi\theta\overline{k_0^\alpha},e_0\rangle e_{-1}=\beta_{\bar\varphi}^{\theta,\alpha}+\delta e_{-1}.
\end{aligned}$$

Substituting the above into (2.1) proves the result. □

When $\theta = \alpha$, the above result becomes the following one, which was obtained by Gu [6].

Corollary 2.1. *The operator D^θ_φ satisfies the following equation:*

$$D^\theta_\varphi - D^\theta_{\bar{z}} D^\theta_\varphi D^\theta_z = e_{-1} \otimes (\beta^\theta_\varphi - \langle \beta^\theta_\varphi, \bar{z}\rangle\, e_{-1}) + (\beta^\theta_{\overline{\varphi}} + \delta e_{-1}) \otimes e_{-1},$$

where

$$\beta^\theta_\varphi = P^\perp_\theta[\overline{\varphi z}(1 - \overline{\theta_0}\theta)], \quad \delta = \overline{\theta_0}\langle \varphi(\theta - \theta_0, e_0\rangle.$$

It is remarked passing that Câmara [2] also obtained that

$$D^{\theta,\alpha}_\varphi - D^\alpha_{\bar{z}} D^{\theta,\alpha}_\varphi D^\theta_z = e_{-1} \otimes \beta^{\alpha,\theta}_\varphi + \beta^{\theta,\alpha}_{\overline{\varphi}} \otimes e_{-1}.$$

It is not true. The following is a counterexample.

Example 2.1. *When $\theta = \alpha,\ \theta_0\theta_1 \neq 0$ and $\varphi = \bar{z}$, then using Lemma 2.1 we obtain*

$$(D^\theta_\varphi - D^\theta_{\bar{z}} D^\theta_{\bar{z}} D^\theta_z)[\bar{z}] = (1 - |\theta_0|^2)\bar{z}^2$$

and

$$(e_{-1} \otimes \beta^\theta_\varphi + \beta^\theta_{\overline{\varphi}} \otimes e_{-1})[\bar{z}] = (1 - |\theta_0|^2)(\bar{z}^2) - \overline{\theta_0}\theta_1\bar{z},$$

so it is clear that

$$(D^\theta_\varphi - D^\alpha_{\bar{z}} D^\theta_{\bar{z}} D^\theta_z)[\bar{z}] \neq (e_{-1} \otimes \beta^\theta_\varphi + \beta^\theta_{\overline{\varphi}} \otimes e_{-1})[\bar{z}].$$

Using a proof similar to Lemma 2.3, we can show that the operator $D^{\theta,\alpha}_\varphi$ satisfies the following equations:

$$D^{\theta,\alpha}_\varphi D^\theta_z - D^\alpha_z D^{\theta,\alpha}_\varphi = \alpha \otimes P^\perp_\theta[\overline{\varphi z}(\alpha - \alpha_0)] - P^\perp_\alpha[\varphi(1 - \overline{\theta_0}\theta)] \otimes e_{-1}$$

and

$$D^{\theta,\alpha}_\varphi - D^\alpha_z D^{\theta,\alpha}_\varphi D^\theta_{\bar{z}} = \alpha \otimes [R^{\alpha,\theta}_{\bar{z}\varphi} - \langle R^{\alpha,\theta}_{\bar{z}\varphi}, \theta\rangle\theta] + [R^{\theta,\alpha}_{\overline{z\varphi}} + \theta_0\langle \varphi(\overline{\alpha} - \overline{\alpha_0}), e_0\rangle\alpha] \otimes \theta,$$

where $R^{\alpha,\theta}_\varphi = P^\perp_\theta[\overline{\varphi z}(\alpha - \alpha_0)]$. The first equation was also obtained by Câmara using a different method in [2].

3. Characterization when an operator being an ADTTO

We denote $\mathcal{B}(K^\perp_\theta, K^\perp_\alpha)$ as the set of all bounded linear operators from $K^\perp_\theta$ to $K^\perp_\alpha$. For $A \in \mathcal{B}(K^\perp_\theta, K^\perp_\alpha)$, suppose $A = D^\alpha_{\bar{z}} A D^\theta_z$. Note that

$$D^\theta_z e_{-n} = e_{-(n-1)},$$

then we have

$$Ae_{-n} = (D^\alpha_{\bar{z}})^n A (D^\theta_z)^n [e_{-n}] = (D^\alpha_{\bar{z}})^n A D^\theta_z [e_{-1}], \quad n > 1, \tag{3.1}$$

and for each $n \geq 0$, $m \geq 1$,

$$\begin{aligned} \langle A(\theta e_n),\ e_{-m}\rangle &= \langle (D^\alpha_{\bar{z}})^m A[\theta e_{n+m}],\ e_{-m}\rangle \\ &= \langle A[\theta e_{n+m}],\ (D^\alpha_z)^m e_{-m}\rangle = \langle A[\theta e_{n+m}],\ D^\alpha_z e_{-1}\rangle. \end{aligned} \tag{3.2}$$

The above observations will be used frequently in the proof of the following result.

Proposition 3.1. *Let $A \in \mathcal{B}(K_\theta^\perp, K_\alpha^\perp)$ and $A - D_{\bar{z}}^\alpha A D_z^\theta = 0$.*

(a) *If $\alpha_0 = 0$, then there exist $\psi \in H^\infty$ and $\omega \in L^\infty$ such that $A = F_\omega + G_\psi$, where F_ω and G_ψ are defined by*

$$F_\omega h = \alpha P[\omega\bar{\theta}Ph], \quad G_\psi h = \overline{\theta_0}\alpha P[\psi Qh]$$

for $h \in K_\theta^\perp$ and $\psi = P\omega$.

(b) *If $\alpha_0 \neq 0$, then there exist $\psi, \omega \in H^\infty$ such that $A = F_\omega + G_\psi$, where F_ω, G_ψ are defined as follows: for $h \in K_\theta^\perp$,*

$$F_\omega h = \overline{\theta_0}(\overline{z\omega})Qh + Q[\overline{z\omega\theta}Ph] + \frac{\alpha}{\alpha_0}P[\overline{z\omega\theta}Ph],$$
$$G_\psi h = \overline{\theta_0}\alpha_0 Q[\psi Qh] + \overline{\theta_0}\alpha P[\psi Qh] + \alpha\psi\bar{\theta}Ph.$$

In particular, $A\theta = \overline{z\omega} + \alpha\psi$.

Proof. (a) $\alpha_0 = 0$. For this case, $D_z^\alpha e_{-1} = 0$. There are two cases to be discussed:

(a1) $\theta_0 = 0$. In this case, $D_z^\theta e_{-1} = 0$ and $(D_z^\theta)^*\theta = 0$. Then by (3.1) we have

$$Ae_{-n} = (D_{\bar{z}}^\alpha)^n A D_z^\theta[e_{-1}] = 0, \quad n \geq 2,$$

so $A|_{\overline{zH^2}} = 0$. Furthermore, for $n \geq 0,\ m \geq 1$, by (3.2) we obtain

$$\langle A\theta e_n,\ e_{-m}\rangle = \langle A[\theta e_{n+m}],\ D_z^\alpha e_{-1}\rangle = 0. \tag{3.3}$$

So we can regard A as a bounded operator from θH^2 to αH^2. For $n \geq 0$, there exists a function sequence $\{h_n\} \subset H^2$, such that

$$A\theta e_n = \alpha h_n.$$

Define a linear operator B on H^2 by $Be_n = h_n$. Since θ and α are inner functions, B is bounded, and $\|B\| = \|A\|$. For A,

$$\begin{aligned}\alpha h_n = A\theta e_n &= D_{\bar{z}}^\alpha A D_z^\theta[\theta e_n] = D_{\bar{z}}^\alpha A[\theta e_{n+1}]\\ &= D_{\bar{z}}^\alpha[\alpha h_{n+1}] = \alpha P[\overline{\alpha z}\alpha h_{n+1}] = \alpha P[\bar{z}h_{n+1}].\end{aligned}$$

So $h_n = P[\bar{z}h_{n+1}]$, and therefore

$$Be_n = h_n = P[\bar{z}h_{n+1}] = T_z^* Be_{n+1} = T_z^* BT_z e_n,\ n \geq 0.$$

It implies that $B = T_z^* BT_z$. By the Brown-Halmos theorem [1], it tells that B is a Toeplitz operator, so there exists $\omega \in L^\infty$ such that $B = T_\omega$. Hence $Ah = \alpha P[\omega\bar{\theta}Ph]$.

(a2) $\theta_0 \neq 0$. For this case, when $n \geq 0$ and $m \geq 1$, by (3.3) we may regard A as a bounded operator from $K_\theta^\perp$ to αH^2, so there exists $\psi \in H^2$ such that $A\theta = \alpha\psi$. Note that $D_z^\theta e_{-1} = \overline{\theta_0}\theta$, then for $n \geq 1$, by (3.1) again we have

$$Ae_{-n} = (D_{\bar{z}}^\alpha)^n A D_z^\theta[e_{-1}] = \overline{\theta_0}(D_{\bar{z}}^\alpha)^n A\theta = \overline{\theta_0}(D_{\bar{z}}^\alpha)^n(\alpha\psi). \tag{3.4}$$

Write $\psi = \sum_{k\geq 0} a_k e_k$. Notice that

$$D_{\bar{z}}^\alpha[\alpha] = \alpha e_{-1} + \langle\alpha, \alpha\rangle(\alpha_0 - \alpha)e_{-1} = 0$$

and

$$D_{\bar{z}}^{\alpha}[\alpha a_k e_k] = \alpha a_k e_k e_{-1} + \langle \alpha a_k e_k, \alpha\rangle(\alpha_0 - \alpha)e_{-1} = \alpha a_k e_{k-1}$$

for $k \geq 1$, combining with (3.4), we obtain

$$\begin{aligned} Ae_{-n} &= \overline{\theta_0}(D_{\bar{z}}^{\alpha})^n\Big[\alpha \sum_{k\geq 0} a_k e_k\Big] \\ &= \overline{\theta_0}\Big(\alpha \sum_{k\geq n} a_k e_{k-n}\Big) = \overline{\theta_0}\alpha P[e_{-n}\psi]. \end{aligned}$$

Therefore, for $h \in \overline{zH^2}$, we have $Ah = \overline{\theta_0}\alpha P[\psi h]$.

When $h \in \theta H^2$, similar to (a1), it can be shown that there exists $\omega \in L^\infty$ such that $Ah = \alpha P[\omega\bar{\theta}Ph]$. Hence, we obtain that

$$Ah = \alpha P[\omega\bar{\theta}Ph] + \overline{\theta_0}\alpha P[\psi Qh] = F_\omega[h] + G_\psi[h], \quad h \in K_\theta^\perp.$$

The above gives $A\theta = \alpha P\omega$, by $A\theta = \alpha\psi$ obtained before, we get $\psi = P\omega$.

(b) We first suppose $\alpha_0 \neq 0$ and $\theta_0 \neq 0$. In this case, by Lemma 2.1, we see that $D_{\bar{z}}^{\alpha} e_{-n} = e_{-(n+1)}$ for $n \geq 1$; $D_{\bar{z}}^{\alpha}\alpha = \alpha_0 e_{-1}$; $D_{\bar{z}}^{\alpha}(\alpha e_{m+1}) = \alpha e_m$ for $m \geq 0$. So $D_{\bar{z}}^{\alpha}$ is invertible, and

$$(D_{\bar{z}}^{\alpha})^{-1}e_{-(n+1)} = e_{-n}, \quad (D_{\bar{z}}^{\alpha})^{-1}e_{-1} = \alpha/\alpha_0, \quad n \geq 1,$$

$$(D_{\bar{z}}^{\alpha})^{-1}(\alpha e_m) = \alpha e_{m+1}, \quad m \geq 0.$$

Let

$$A\theta = \sum_{m\geq 1} b_m e_{-m} + \alpha \sum_{k\geq 0} a_k e_k = \overline{z\omega} + \alpha\psi,$$

where $\omega,\ \psi \in H^2$.

Assume that $A\theta = \overline{z\omega}$, $\omega \in H^\infty$. For $n \geq 1$, like (3.4), it has

$$Ae_{-n} = \overline{\theta_0}(D_{\bar{z}}^{\alpha})^n A\theta = \overline{\theta_0}(D_{\bar{z}}^{\alpha})^n[\overline{z\omega}] = \overline{\theta_0}\,\overline{z\omega}e_{-n}.$$

Therefore, for $g \in H^2$, we have

$$A[\overline{zg}] = \overline{\theta_0}\overline{z\omega}\overline{zg}.$$

Because

$$A\theta = (D_{\bar{z}}^{\alpha})^n A(D_z^{\theta})^n\theta = (D_{\bar{z}}^{\alpha})^n A(\theta e_n),$$

we have

$$A(\theta e_n) = (D_{\bar{z}}^{\alpha})^{-n}A\theta = (D_{\bar{z}}^{\alpha})^{-n}[\overline{z\omega}].$$

So

$$\begin{aligned} A(\theta e_n) &= (D_{\bar{z}}^{\alpha})^{-n}[\overline{z\omega}] = (D_{\bar{z}}^{\alpha})^{-n}\Big[\sum_{m\geq 1} b_m e_{-m}\Big] \\ &= \sum_{k=1}^{n} b_k \frac{\alpha}{\alpha_0} e_{n-k} + \sum_{m\geq n+1} b_m e_{-m+n} \\ &= \frac{\alpha}{\alpha_0}P[\overline{z\omega}e_n] + Q[\overline{z\omega}e_n]. \end{aligned}$$

Therefore,

$$A(\theta f) = \frac{\alpha}{\alpha_0} P[\overline{z\omega}\bar{\theta}\theta f] + Q[\overline{z\omega}\bar{\theta}\theta f]$$

for $f \in H^2$.

For $h \in K_\theta^\perp$, $h = \overline{zg} + \theta f = Qh + Ph$, we get

$$Ah = \overline{\theta_0}\,\overline{z\omega}Qh + Q[\overline{z\omega}\bar{\theta}Ph] + \frac{\alpha}{\alpha_0} P[\overline{z\omega}\bar{\theta}Ph],$$

denote the right side of the above equation as $F_\omega h$.

Now, assuming $A\theta = \alpha\psi$, where $\psi \in H^\infty$, and for $n \geq 1$, by (3.4) we have

$$Ae_{-n} = \overline{\theta_0}(D^{\alpha}_{\bar{z}})^n[\alpha\psi] = \overline{\theta_0}\alpha_0 Q[\psi e_{-n}] + \overline{\theta_0}\alpha P[\psi e_{-n}].$$

Therefore,

$$A[\overline{zg}] = \overline{\theta_0}\alpha_0 Q[\psi\overline{zg}] + \overline{\theta_0}\alpha P[\psi\overline{zg}]$$

for $g \in H^2$. On the other hand, for $n \geq 0$, we obtain that

$$\begin{aligned} A(\theta e_n) &= (D^{\alpha}_{\bar{z}})^{-n} A\theta = (D^{\alpha}_{\bar{z}})^{-n}[\alpha\psi] \\ &= (D^{\alpha}_{\bar{z}})^{-n}\Big[\alpha\sum_{k\geq 0} a_k e_k\Big] = e_n\Big[\alpha\sum_{k\geq 0} a_k e_k\Big] = e_n\alpha\psi. \end{aligned}$$

Hence, $A[\theta f] = \alpha\psi\bar{\theta}\theta f$ for $f \in H^2$.

For $h \in K_\theta^\perp$, $h = \overline{zg} + \theta f = Qh + Ph$, it induces that

$$Ah = \overline{\theta_0}\alpha_0 Q[\psi Qh] + \overline{\theta_0}\alpha P[\psi Qh] + \alpha\psi\bar{\theta}Ph,$$

denote the right side of the above equation as $G_\psi h$.

Therefore, it follows that $A = F_\omega + G_\psi$ and $A\theta = \overline{z\omega} + \alpha\psi$. The proof is similar for the case of $\alpha_0 \neq 0$ and $\theta_0 = 0$. ◻

By Proposition 3.1, we have the following result.

Corollary 3.1. *Let $A \in \mathcal{B}(K_\theta^\perp, K_\alpha^\perp)$.*

(*a*) *If $\alpha_0 = 0$, then $A = 0$ if and only if $A - D^{\alpha}_{\bar{z}} A D^{\theta}_{z} = 0$, $A\theta = 0$, and $A^*\alpha = 0$;*

(*b*) *If $\alpha_0 \neq 0$, then $A = 0$ if and only if $A - D^{\alpha}_{\bar{z}} A D^{\theta}_{z} = 0$ and $A\theta = 0$.*

Proof. It only needs to show the sufficiency.

(a) $\alpha_0 = 0$. The proof is divided into the following two cases.

(a1) Suppose $\theta_0 = 0$. By Proposition 3.1, we have

$$Ah = \alpha P[\omega\bar{\theta}Ph] = PD^{\theta,\alpha}_{\omega\alpha\bar{\theta}}Ph,$$

where $h \in K_\theta^\perp$. Therefore, $A = PD^{\theta,\alpha}_{\omega\alpha\bar{\theta}}P$ and $A^* = PD^{\alpha,\theta}_{\overline{\omega\alpha}\theta}P$. Since $A\theta = \alpha P[\omega] = 0$ and $A^*\alpha = \theta P[\overline{\omega}] = 0$, we have $\omega = 0$. Thus, $A = 0$.

(a2) Suppose $\theta_0 \neq 0$. In this case, by Proposition 3.1 we have

$$Ah = \alpha P[\omega\bar{\theta}Ph] + \overline{\theta_0}\alpha P[\psi Qh], \quad h \in K_\theta^\perp,$$

and $A\theta = \alpha\psi$. So by $A\theta = 0$, we have $\psi = 0$. Then, $Ah = \alpha P[\omega\overline{\theta}Ph]$ for $h \in K_\theta^\perp$. Similar to (a1), we can obtain $\omega = 0$. Hence, $A = 0$.

(b) $\alpha_0 \neq 0$. By Proposition 3.1, $A\theta = \overline{z\omega} + \alpha\psi = 0$, thus we have $\omega = 0$ and $\psi = 0$, which induces $A = 0$. □

Let

$$A - D_{\overline{z}}^{\alpha} A D_z^{\theta} = e_{-1} \otimes (\beta_\varphi^{\alpha,\theta} - \langle \beta_\varphi^{\alpha,\theta}, \overline{z}\rangle e_{-1}) + (\beta_{\overline{\varphi}}^{\theta,\alpha} + \delta_\varphi e_{-1}) \otimes e_{-1}, \tag{3.5}$$

where

$$\varphi \in L^\infty, \quad \beta_\varphi^{\alpha,\theta} = P_\theta^\perp[\overline{\varphi z}(1 - \overline{\alpha_0}\alpha)], \quad \delta_\varphi = \overline{\theta_0}\langle \varphi\theta\overline{k_0^\alpha},\ e_0\rangle.$$

It follows from Lemma 2.3 that $A = D_\varphi^{\theta,\alpha}$ satisfies the above equation, then by Proposition 3.1, we can easily obtain the following theorem.

Theorem 3.1. *Let $A \in \mathcal{B}(K_\theta^\perp, K_\alpha^\perp)$ and $\varphi \in L^\infty$.*
(*a*) *If $\alpha_0 = 0$, then $A = D_\varphi^{\theta,\alpha}$ if and only if A satisfies* (3.5)*, $A\theta = D_\varphi^{\theta,\alpha}\theta$ and $A^*\alpha = (D_\varphi^{\theta,\alpha})^*\alpha$;*
(*b*) *If $\alpha_0 \neq 0$, then $A = D_\varphi^{\theta,\alpha}$ if and only if A satisfies* (3.5) *and $A\theta = D_\varphi^{\theta,\alpha}\theta$.*

It is remarked that in [2], the authors also obtained the characterization for ADTTO with different presentations.

4. The product problem of two ADTTOs for certain symbols

For the inner function θ, we define a class of conjugation linear operators $C_\theta : L^2 \to L^2$ by

$$(C_\theta f)(z) = \theta\overline{zf}, \quad f \in L^2,$$

which satisfies that $\langle C_\theta f, C_\theta g\rangle = \langle g, f\rangle$, and $(C_\theta)^2 = I$. According to the definition of C_θ, we have

$$C_\theta e_{-n} = \theta e_{n-1}, \;\; C_\theta(\theta e_{n-1}) = e_{-n}, \quad n \geq 1.$$

Hence, it is clear $C_\theta K_\theta = K_\theta$, $C_\theta(\theta H^2) = \overline{zH^2}$ and $C_\theta(\overline{zH^2}) = \theta H^2$.

Before we present the result of the product problem of when the product of two ADTTOs is another ADTTO for certain symbols, we first give the following lemma.

Lemma 4.1. *Suppose $\varphi, \psi \in H^\infty$, $h \in L^\infty$, and $K_\alpha \subseteq K_\gamma \subseteq K_\theta$. If*

$$D_\varphi^{\gamma,\alpha} D_\psi^{\theta,\gamma}[\theta] = D_h^{\theta,\alpha}[\theta], \quad D_\varphi^{\gamma,\alpha} D_\psi^{\theta,\gamma}[\overline{z}] = D_h^{\theta,\alpha}[\overline{z}],$$

then $h = \varphi\psi$.

Proof. First note that $K_\alpha \subseteq K_\gamma \subseteq K_\theta$ means that γ/α, θ/α, and θ/γ all are inner functions.

Notice that $D_\varphi^{\gamma,\alpha} D_\psi^{\theta,\gamma}[\theta] = \varphi\psi\theta$ and $D_h^{\theta,\alpha}[\theta] = Q[h\theta] + \alpha P[\overline{\alpha}h\theta]$, so it is obvious that $h\theta \in H^2$ and $P[\overline{\alpha}h\theta] = \overline{\alpha}\varphi\psi\theta$.

Let $\overline{\alpha}h\theta = \overline{\alpha}\varphi\psi\theta + \overline{zf} + \overline{zg}$, where $f \in K_\alpha$, $g \in \alpha H^2$, then

$$h\theta = \varphi\psi\theta + C_\alpha f + C_\alpha g.$$

Because $h\theta \in H^2$, the above means that $C_\alpha g \in H^2$, which gives $g = 0$ since $C_\alpha g \in \overline{zH^2}$. Thus

$$h = \varphi\psi + \overline{\theta}C_\alpha f. \tag{4.1}$$

Also, it is noted that

$$D_\varphi^{\gamma,\alpha} D_\psi^{\theta,\gamma}[\overline{z}] = \varphi_0\psi_0\overline{z} + \alpha P[\overline{\alpha}\varphi\gamma P[\overline{\gamma z}\psi]],$$
$$D_h^{\theta,\alpha}[\overline{z}] = Q[h\overline{z}] + \alpha P[\overline{\alpha}h\overline{z}].$$

So $Q[h\overline{z}] = \varphi_0\psi_0\overline{z}$, and we see that $h \in H^2$ and $h_0 = \varphi_0\psi_0$. Now by (4.1),

$$h = \varphi\psi + \overline{\theta}C_\alpha f = \varphi\psi + \overline{z}\overline{\theta/\alpha f}.$$

Since $\overline{z}\overline{\theta/\alpha f} \in \overline{zH^2}$ and $h \in H^2$, it has $f = 0$ and hence $h = \varphi\psi$, so we obtain the desired conclusion. □

It is worth noting that we can use the result of Ding [3, Theorem 4.7] and Lemma 4.1 to obtain the following characterization of the product problem for DTTOs.

Corollary 4.1. *For $\varphi, \psi \in H^\infty$, $D_\varphi^\theta D_\psi^\theta = D_h^\theta$ if and only if there exists $\lambda \in \mathbb{C}$ such that $\overline{\varphi}(\theta - \lambda), \overline{\psi}(\theta - \lambda), \overline{\varphi\psi}(\theta - \lambda) \in H^2$ or one of φ and ψ is a constant, in which case $h = \varphi\psi$.*

We are ready to solve the product problem of two ADTTOs with certain analytic symbols.

Theorem 4.1. *Let $\varphi, \psi \in K_\alpha \subseteq K_\gamma \subseteq K_\theta$ and $\varphi, \psi \in H^\infty$, $q^{\theta,\alpha} = \alpha P[\overline{\alpha}\theta\overline{z}]$.*
(a) If $\alpha_0 = 0$, then $D_\varphi^{\gamma,\alpha} D_\psi^{\theta,\gamma} = D_h^{\theta,\alpha}$ for some $h \in L^\infty$ if and only if $h = \varphi\psi \in K_{z\alpha}$.
(b) If $\alpha_0 \neq 0$, then $D_\varphi^{\gamma,\alpha} D_\psi^{\theta,\gamma} = D_h^{\theta,\alpha}$ for some $h \in L^\infty$ if and only if $h = \varphi\psi \in K_{z\alpha}$ and

$$\begin{aligned}\theta P[\overline{\theta z}\varphi\psi] = \overline{\theta_0}\psi_0\{&C_\theta[\overline{\varphi - \varphi_0}] - (\theta\overline{\gamma})_0 C_\gamma[\overline{\varphi - \varphi_0}] \\ &+ \varphi_0 q^{\theta,\alpha} - \varphi q^{\theta,\gamma} - \varphi_0(\theta\overline{\gamma})_0 q^{\gamma,\alpha}\}.\end{aligned} \tag{4.2}$$

Proof. First, suppose that $D_\varphi^{\gamma,\alpha} D_\psi^{\theta,\gamma} = D_h^{\theta,\alpha}$. We notice that when $\varphi, \psi \in K_\alpha \subseteq K_\gamma \subseteq K_\theta$,

$$\beta_\varphi^{\alpha,\theta} = \overline{\varphi z} - \overline{\alpha_0}P_\theta^\perp[C_\alpha\varphi] = \overline{\varphi z},$$

$$\beta_{\overline{\varphi}}^{\theta,\alpha} = \varphi_0\overline{z} - \overline{\theta_0}P_\alpha^\perp[C_\theta\overline{\varphi}] = (1 - |\theta_0|^2)\varphi_0\overline{z} - \overline{\theta_0}\varphi_0 q^{\theta,\alpha} - \overline{\theta_0}C_\theta[\overline{\varphi - \varphi_0}], \tag{4.3}$$

$$\delta_\varphi = \overline{\theta_0}\langle\varphi\theta - \alpha_0\varphi\theta\overline{\alpha}, 1\rangle = 0,$$

and

$$\langle\beta_\varphi^{\alpha,\theta}, \overline{z}\rangle = \overline{\varphi_0}, \quad \langle\beta_{\overline{\varphi}}^{\theta,\alpha}, \overline{z}\rangle = (1 - |\theta_0|^2)\varphi_0.$$

So by (3.5) we obtain

$$\begin{aligned}D_\varphi^{\theta,\alpha} - D_{\overline{z}}^\alpha D_\varphi^{\theta,\alpha} D_z^\theta &= e_{-1} \otimes (\overline{\varphi z - \varphi_0 z}) + (\varphi_0\overline{z} - \overline{\theta_0}P_\alpha^\perp[C_\theta\overline{\varphi}]) \otimes e_{-1} \\ &= e_{-1} \otimes (\overline{\varphi z}) - (\overline{\theta_0}P_\alpha^\perp[C_\theta\overline{\varphi}]) \otimes e_{-1}.\end{aligned} \tag{4.4}$$

By Lemma 2.2, we have

$$
\begin{aligned}
D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi} - D^{\alpha}_{\bar{z}}D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi}D^{\theta}_{z} &= D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi} - D^{\alpha}_{\bar{z}}D^{\gamma,\alpha}_{\varphi}(D^{\gamma}_{z}D^{\gamma}_{\bar{z}} + (1-|\gamma_0|^2)\gamma\otimes\gamma)D^{\theta,\gamma}_{\psi}D^{\theta}_{z} \\
&= D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi} - D^{\alpha}_{\bar{z}}D^{\gamma,\alpha}_{\varphi}D^{\gamma}_{z}D^{\gamma}_{\bar{z}}D^{\theta,\gamma}_{\psi}D^{\theta}_{z} \\
&\quad - (1-|\gamma_0|^2)[D^{\alpha}_{\bar{z}}D^{\gamma,\alpha}_{\varphi}\gamma]\otimes[D^{\theta}_{\bar{z}}D^{\gamma,\theta}_{\bar{\psi}}\gamma].
\end{aligned}
\tag{4.5}
$$

Making use of (4.3) and (4.4), it follows that

$$
\begin{aligned}
D^{\alpha}_{\bar{z}}D^{\gamma,\alpha}_{\varphi}D^{\gamma}_{z}D^{\gamma}_{\bar{z}}D^{\theta,\gamma}_{\psi}D^{\theta}_{z} &= [D^{\gamma,\alpha}_{\varphi} - e_{-1}\otimes(\overline{\varphi z}) + (\overline{\gamma_0}P^{\perp}_{\alpha}[C_\gamma\overline{\varphi}])\otimes e_{-1}] \\
&\quad \times [D^{\theta,\gamma}_{\psi} - e_{-1}\otimes(\overline{\psi z}) + (\overline{\theta_0}P^{\perp}_{\gamma}[C_\theta\overline{\psi}])\otimes e_{-1}] \\
&= D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi} + (\overline{\theta_0}D^{\gamma,\alpha}_{\varphi}P^{\perp}_{\gamma}[C_\theta\overline{\psi}])\otimes e_{-1} \\
&\quad - e_{-1}\otimes(\overline{\varphi\psi z}) - e_{-1}\otimes(\langle\overline{\varphi z},(\overline{\theta_0}P^{\perp}_{\gamma}[C_\theta\overline{\psi}])\rangle e_{-1}) \\
&\quad + (\overline{\gamma_0}P^{\perp}_{\alpha}[C_\gamma\overline{\varphi}])\otimes(\langle e_{-1},\overline{\theta_0}P^{\perp}_{\gamma}[C_\theta\overline{\psi}]\rangle e_{-1}) \\
&= D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi} + (D^{\gamma,\alpha}_{\varphi}(|\theta_0|^2\psi_0\bar{z} + \overline{\theta_0}\psi_0 q^{\theta,\gamma} + \overline{\theta_0}C_\theta[\overline{\psi-\psi_0}]))\otimes e_{-1} \\
&\quad - e_{-1}\otimes(\overline{\varphi\psi z}) - e_{-1}\otimes(\langle\overline{\varphi z},(|\theta_0|^2\psi_0\bar{z} + \overline{\theta_0}\psi_0 q^{\theta,\gamma} \\
&\quad + \overline{\theta_0}C_\theta[\overline{\psi-\psi_0}])\rangle e_{-1}) + (|\gamma_0|^2\varphi_0\bar{z} + \overline{\gamma_0}\varphi_0 q^{\gamma,\alpha} + \overline{\gamma_0}C_\gamma[\overline{\varphi-\varphi_0}]) \\
&\quad \otimes(\langle e_{-1},(|\theta_0|^2\psi_0\bar{z} + \overline{\theta_0}\psi_0 q^{\theta,\gamma} + \overline{\theta_0}C_\theta[\overline{\psi-\psi_0}])\rangle e_{-1}) \\
&= D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi} + (\varphi_0\psi_0|\theta_0|^2 e_{-1} + \overline{\theta_0}\psi_0\varphi q^{\theta,\gamma} + \varphi\overline{\theta_0}C_\theta[\overline{\psi-\psi_0}])\otimes e_{-1} \\
&\quad - e_{-1}\otimes(\overline{\varphi\psi z} - \overline{(\varphi\psi)_0}\bar{z}) - e_{-1}\otimes\overline{(\varphi\psi)_0}e_{-1} - \varphi_0\psi_0|\theta_0|^2 e_{-1}\otimes e_{-1}) \\
&\quad + |\theta_0|^2\psi_0(|\gamma_0|^2\varphi_0\bar{z} + \overline{\gamma_0}\varphi_0 q^{\gamma,\alpha} + \overline{\gamma_0}C_\gamma[\overline{\varphi-\varphi_0}])\otimes e_{-1} \\
&= D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi} - e_{-1}\otimes[\overline{\varphi\psi z} - (\overline{\varphi\psi})_0\bar{z}] \\
&\quad + (\varphi_0\psi_0|\theta_0|^2|\gamma_0|^2 - \varphi_0\psi_0)e_{-1}\otimes e_{-1} + \phi\otimes e_{-1},
\end{aligned}
\tag{4.6}
$$

where ϕ denotes

$$
\overline{\theta_0}\psi_0\varphi q^{\theta,\gamma} + \overline{\theta_0}\varphi C_\theta[\overline{\psi-\psi_0}] + \varphi_0\psi_0|\theta_0|^2\overline{\gamma_0}q^{\gamma,\alpha} + \psi_0|\theta_0|^2\overline{\gamma_0}C_\gamma[\overline{\varphi-\varphi_0}],
$$

which satisfies $\phi\perp e_{-1}$. Also,

$$
[D^{\alpha}_{\bar{z}}D^{\gamma,\alpha}_{\varphi}\gamma]\otimes[D^{\theta}_{\bar{z}}D^{\gamma,\theta}_{\bar{\psi}}\gamma] = \psi_0\overline{\theta_0}(\theta\overline{\gamma})_0[C_\gamma[\overline{\varphi-\varphi_0}] + \varphi_0 q^{\gamma,\alpha}]\otimes e_{-1} + (\varphi_0\psi_0|\theta_0|^2)e_{-1}\otimes e_{-1}.
$$

By (4.6) and the above equation, (4.5) becomes that

$$
\begin{aligned}
&D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi} - D^{\alpha}_{\bar{z}}D^{\gamma,\alpha}_{\varphi}D^{\theta,\gamma}_{\psi}D^{\theta}_{z} \\
&= e_{-1}\otimes[\overline{\varphi\psi z} - (\overline{\varphi\psi})_0\bar{z}] - \Phi\otimes e_{-1} + [(1-|\theta_0|^2)\varphi_0\psi_0]e_{-1}\otimes e_{-1},
\end{aligned}
\tag{4.7}
$$

where Φ denotes

$$
\overline{\theta_0}\psi_0\varphi q^{\theta,\gamma} + \overline{\theta_0}\varphi C_\theta[\overline{\psi-\psi_0}] + \varphi_0\psi_0\overline{\theta_0}(\theta\overline{\gamma})_0 q^{\gamma,\alpha} + \psi_0\overline{\theta_0}(\theta\overline{\gamma})_0 C_\gamma[\overline{\varphi-\varphi_0}].
$$

By Lemma 2.4, we see that $h = \varphi\psi$. In this case, we have

$$
\beta^{\alpha,\theta}_{h} = \overline{\varphi\psi z} - \overline{\alpha_0}P^{\perp}_{\theta}[C_\alpha[\overline{\varphi\psi}]],
$$

$$\beta_{\overline{h}}^{\theta,\alpha} = [(1-|\theta_0|^2)\varphi_0\psi_0]\overline{z} + \theta P[\overline{\theta z}\varphi\psi] - \overline{\theta_0}C_\theta[\overline{\varphi\psi - (\varphi\psi)_0}] - \overline{\theta_0}(\varphi\psi)_0 q^{\theta,\alpha},$$
$$\delta_h = 0,$$

and

$$\langle \beta_h^{\alpha,\theta}, \overline{z}\rangle = \overline{(\varphi\psi)_0} - \overline{\alpha_0}\langle C_\alpha[\varphi\psi], \overline{z}\rangle, \quad \langle \beta_{\overline{h}}^{\theta,\alpha}, \overline{z}\rangle = (1-|\theta_0|^2)\varphi_0\psi_0.$$

So

$$\begin{aligned} D_h^{\theta,\alpha} &- D_{\overline{z}}^{\alpha} D_h^{\theta,\alpha} D_z^{\theta} \\ &= e_{-1}\otimes[\overline{\varphi\psi z} - \overline{(\varphi\psi)_0}\overline{z} - \overline{\alpha_0}(P_\theta^{\perp}[C_\alpha[\varphi\psi]] - \langle C_\alpha[\varphi\psi], \overline{z}\rangle\overline{z})] \\ &\quad + [\theta P[\overline{\theta z}\varphi\psi] - \overline{\theta_0}C_\theta[\overline{\varphi\psi - (\varphi\psi)_0}] - \overline{\theta_0}(\varphi\psi)_0 q^{\theta,\alpha}]\otimes e_{-1} \\ &\quad + (1-|\theta_0|^2)\varphi_0\psi_0 e_{-1}\otimes e_{-1}. \end{aligned} \tag{4.8}$$

Now, by comparing the equalities (4.7) and (4.8), we obtain

$$\overline{\alpha_0}P_\theta^{\perp}[C_\alpha[\varphi\psi]] = \overline{\alpha_0}\langle C_\alpha[\varphi\psi], \overline{z}\rangle\overline{z} \tag{4.9}$$

and

$$\begin{aligned} \theta P[\overline{\theta z}\varphi\psi] = \overline{\theta_0}\psi_0\{C_\theta[\overline{\varphi-\varphi_0}] - (\theta\overline{\gamma})_0 C_\gamma[\overline{\varphi-\varphi_0}] \\ + \varphi_0 q^{\theta,\alpha} - \varphi q^{\theta,\gamma} - \varphi_0(\theta\overline{\gamma})_0 q^{\gamma,\alpha}\}. \end{aligned} \tag{4.10}$$

The above is (4.2).

If $\alpha_0 \neq 0$, then (4.9) gives that

$$P_\theta^{\perp}[C_\alpha(\varphi\psi)] = \langle C_\alpha[\varphi\psi], \overline{z}\rangle\overline{z}.$$

Simple computation shows that it is

$$\alpha\overline{z\varphi\psi} - P(\alpha\overline{z\varphi\psi}) = \langle\alpha\overline{z\varphi\psi}, \overline{z}\rangle\overline{z},$$

or

$$\alpha\overline{\varphi\psi} - zP(\alpha\overline{z\varphi\psi}) = \langle\alpha\overline{z\varphi\psi}, \overline{z}\rangle,$$

which is equivalent to that $\alpha\overline{\varphi\psi} \in H^2$. Thus $\varphi\psi \in \alpha\overline{H^2} = zK_\alpha \oplus \overline{H^2}$, which implies that $\varphi\psi \in K_{z\alpha}$. Hence, by Theorem 3.1, we see (b) holds.

If $\alpha_0 = 0$, then also $\theta_0 = 0$ since θ/α is an inner function. In this case, the equality (4.9) holds naturally and (4.10) yields that $P[\overline{\theta z}\varphi\psi] = 0$, to obtain $\theta\overline{\varphi\psi} \in H^2$. On the other hand, it is easily seen that $(D_\varphi^{\gamma,\alpha}D_\psi^{\theta,\gamma})^*\alpha = (D_{\varphi\psi}^{\theta,\alpha})^*\alpha$ is

$$(\bar{\theta}\alpha)_0(\overline{\varphi\psi})_0\theta = Q(\overline{\varphi\psi}\alpha) + (\bar{\theta}\alpha)_0(\overline{\varphi\psi})_0\theta,$$

or $Q(\overline{\varphi\psi}\alpha) = 0$, that is, $\alpha\overline{\varphi\psi} \in H^2$. Also by Theorem 3.1, we have (a).

It is easy to see the converse holds. We finish the proof. □

We notice that when $\theta = \gamma = \alpha$, it has $q^{\theta,\alpha} = q^{\theta,\gamma} = q^{\gamma,\alpha} = 0$, so we can derive quickly a result: Let $\varphi, \psi \in K_\theta$, then $D_\varphi^\theta D_\psi^\theta = D_h^\theta$ if and only if $h = \varphi\psi \in K_{z\theta}$.

Obviously, it is a special case of Corollary 4.1 for the dual truncated Toeplitz operators.

The following corollary is also obvious.

Corollary 4.2. *Let $\varphi, \psi \in K_\alpha \subseteq K_\gamma \subseteq K_\theta$ and $\varphi, \psi \in H^\infty$, then $D_\varphi^{\gamma,\alpha}D_\psi^{\theta,\gamma} = 0$ if and only if $\varphi = 0$ or $\psi = 0$.*

5. Conclusions

In this paper, we studied ADTTO acting on the orthogonal complement of two different model spaces. More precisely, we characterized when a given operator is an ADTTO with an operator equation. As applications of this result, we solved the product problem of two ADTTOs with certain analytic symbols. In future work, we will investigate the product problem of two ADTTOs with general symbols.

Author contributions

Zhenhui Zhu: Writing-original draft; Qi Wu: Writing-review and editing; Yong Chen: Supervision, Writing-review and editing, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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