AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(3): 6512–6545.
DOI: 10.3934/math.2025298
Received: 16 December 2024
Revised: 17 March 2025
Accepted: 19 March 2025
Published: 24 March 2025

Research article

A discrete logistic model with conditional Hyers–Ulam stability

Douglas R. Anderson[1] **and Masakazu Onitsuka**[2,*]

[1] Department of Mathematics, Concordia College, Moorhead, MN 56562, USA
[2] Department of Applied Mathematics, Okayama University of Science, Okayama 700-0005, Japan

* **Correspondence:** Email: onitsuka@ous.ac.jp.

Abstract: This study investigates the conditional Hyers–Ulam stability of a first-order nonlinear h-difference equation, specifically a discrete logistic model. Identifying bounds on both the relative size of the perturbation and the initial population size is an important issue for nonlinear Hyers–Ulam stability analysis. Utilizing a novel approach, we derive explicit expressions for the optimal lower bound of the initial value region and the upper bound of the perturbation amplitude, surpassing the precision of previous research. Furthermore, we obtain a sharper Hyers–Ulam stability constant, which quantifies the error between true and approximate solutions, thereby demonstrating enhanced stability. The Hyers–Ulam stability constant is proven to be in terms of the step-size h and the growth rate but independent of the carrying capacity. Detailed examples are provided illustrating the applicability and sharpness of our results on conditional stability. In addition, a sensitivity analysis of the parameters appearing in the model is also performed.

Keywords: h-difference equation; Hyers–Ulam stability; Hyers–Ulam stability constant; perturbation; conditional stability; logistic growth
Mathematics Subject Classification: 39A30, 39A60

1. Introduction

Hyers–Ulam stability is concerned with ascertaining whether, given a solution of a perturbed equation, a solution to the unperturbed equation exists that remains close to the given solution of the perturbed equation. The recent monographs from Brzdęk et al. [1] and Tripathy [2] provide excellent overviews of the area. Initially, much of the Hyers–Ulam stability analysis for differential and difference equations was concerned with linear equations. For example, Baias et al. [3] investigated the Hyers–Ulam stability of first-order linear difference equations. Similarly, Bora and Shankar [4], Chen and Si [5], and Kerekes et al. [6] explored the Hyers–Ulam stability of second-order linear difference equations. Additionally, Novac et al. [7] and Shen and Li [8] examined the Hyers–Ulam stability of higher order linear difference equations. Furthermore, Buşe et al. [9] analyzed the stability of first-order matrix two-dimensional differential and difference systems. However, there is a growing interest in the analysis of Hyers–Ulam stability for nonlinear equations, which may be conditional stability. It is often the case in nonlinear analysis that the perturbation must be bounded above and the initial condition must be bounded above or below for Hyers–Ulam stability to be possible. Popa et al. [10] explore approximate solutions of the logistic equation and Hyers–Ulam stability, followed by Onitsuka [11, 12] investigating conditional Hyers–Ulam stability and its application to the logistic model and approximate solutions of the generalized logistic equation, respectively. Also in the continuous case, Backes et al. establish conditional Lipschitz shadowing for ordinary differential equations in [13]. In the discrete case, Jung and Nam [14] analyze the Hyers–Ulam stability of the Pielou logistic difference equation, while Nam [15–17] studies the Hyers–Ulam stability of elliptic, hyperbolic, and loxodromic Möbius difference equations, respectively. Models in population ecology can be continuous or discrete. One of the advantages of the discrete case is modeling seasonal reproduction rather than continuous reproduction. See, for example, [18]. Models in economics can also be continuous or discrete. One can model logistically the relationship between advertising and sales of a product as a series of discrete expenditures, of step-size h, with diminishing impact on sales over time, see for example [19].

Motivated by these works and the relative sparsity of results related to conditional Hyers–Ulam stability and its application to nonlinear difference equations, in this study, we address the logistic growth h-difference equation for step-size h. As h converges to zero, it will be demonstrated that our conditional stability results are consistent with those derived for the continuous case. In contrast, setting $h = 1$ leads to significant advancements over previous research. We successfully identify the optimal lower bound of the initial value and the upper bound of the perturbation amplitude, which are essential for ensuring stability in nonlinear systems. Most importantly, we demonstrate a substantial improvement in the Hyers–Ulam stability constant, a measure of stability, compared to prior work.

This study will proceed as follows. In Section 2, we introduce the discrete logistic equation model, define conditional Hyers–Ulam stability, and derive important inequalities related to solutions of the logistic model and solutions of perturbations on the model. We explain that the perturbations must be bounded above in size, while the initial population size must be bounded away from zero for stability to occur. In Section 3 there are three technical lemmas based on the relative smallness of the perturbation and the relative largeness of the initial condition. In Section 4 we present the main result, proving the conditions under which the discrete logistic model is Hyers–Ulam stable and giving a Hyers–Ulam stability constant. In Section 5 we provide detailed examples with both analytical and numerical evidence that illustrate our results and the conditional nature of the Hyers–Ulam stability. In Section 6, we conduct a sensitivity analysis on each parameter of the logistic model, emphasizing its relevance to ecological applications. In the final section, we present the conclusions drawn from this research.

2. Discrete logistic equation model

The form of the discrete logistic equation model that we study in this work is based on the discussion found in [20, Section 2.4]. Given $h > 0$, set

$$\mathbb{T} := \{0, h, 2h, 3h, \ldots\},$$

and define

$$\Delta_h P(t) := \frac{P(t+h) - P(t)}{h}.$$

We consider the logistic growth h-difference equation

$$\Delta_h P(t) = \frac{rP(t)\,(K - P(t))}{K + hrP(t)}, \tag{2.1}$$

where P is the population size at time t of some species, $r > 0$ is a growth-rate coefficient, $h > 0$ is the step size, and $K > 0$ is the carrying capacity. When $h = 1$, this equation is called the Beverton–Holt equation (see [21, 22]). Let $\varepsilon > 0$ be arbitrarily given. Then the following equations

$$\Delta_h \beta(t) = \frac{r\beta(t)\,(K - \beta(t))}{K + hr\beta(t)} + q(t), \quad |q(t)| \le \varepsilon, \tag{2.2}$$

$$\Delta_h \ell(t) = \frac{r\ell(t)\,(K - \ell(t))}{K + hr\ell(t)} - \varepsilon, \tag{2.3}$$

and

$$\Delta_h u(t) = \frac{ru(t)\,(K - u(t))}{K + hru(t)} + \varepsilon \tag{2.4}$$

for $t \ge 0$, where q: $\mathbb{T} \to \mathbb{R}$, are perturbations of (2.1) that will play a key role in the analysis that follows below. Throughout this paper, we assume the initial conditions

$$P(0) = \beta(0) = \ell(0) = u(0) = P_0. \tag{2.5}$$

We can see that the right-hand side of (2.1)–(2.4), respectively, is well defined with respect to $P > 0$, $\beta > 0$, $\ell > 0$, and $u > 0$. That is, the right-hand sides of these equations are continuously differentiable with respect to the positive dependent variable. Consequently, if a positive initial condition (2.5) is given, then the local existence and uniqueness of the solutions are guaranteed in the positive domain (for more details, see [23, Section 8.2]). However, we must pay attention to the global existence of the solutions. By limiting the initial values and the relative size of the allowed perturbations, the existence of global solutions is guaranteed (see Proposition 2).

Definition 1. *Let*

$$[0, T_P)_h := [0, T_P) \cap \mathbb{T}$$

be the maximal interval of existence for a function P. Let D be a nonempty subset of the real numbers. Define the class of functions C_D as

$$C_D := \{P : [0, T_P)_h \to \mathbb{R} : P(0) \in D \subseteq \mathbb{R}, T_P > 0 \text{ with } T_P = \infty \text{ or } |P(t)| \text{ undefined at } t = T_P\}.$$

Let

$$\mathcal{S} \subseteq (0, \infty).$$

The nonlinear h-difference equation

$$\Delta_h P(t) = F(P(t)) \tag{2.6}$$

is conditionally Hyers–Ulam stable in class C_D *on* $\left[0,\ \min\left\{T_P, T_\phi\right\}\right)_h$*, with* $\mathcal{S}$ *if there exists a constant* $\mathcal{H} > 0$ *such that for every* $\varepsilon \in \mathcal{S}$ *and every approximate solution* $\phi \in C_D$ *that satisfies*

$$|\Delta_h \phi(t) - F(\phi(t))| \le \varepsilon \quad \text{for} \quad 0 \le t < T_\phi, \tag{2.7}$$

there exists a solution $P \in C_D$ *of* (2.6) *such that*

$$|\phi(t) - P(t)| \le \mathcal{H}\varepsilon \quad \text{for} \quad 0 \le t < \min\{T_P, T_\phi\}.$$

Such a constant $\mathcal{H}$ *is known as a Hyers–Ulam stability constant for* (2.6) *on* $\left[0,\ \min\left\{T_P, T_\phi\right\}\right)_h$.

Note that if

$$\mathcal{S} = (0, \infty) \quad \text{and} \quad D = \mathbb{R},$$

then this definition is precisely the canonical definition of Hyers–Ulam stability. In addition, note that Definition 1 does not require the uniqueness of a solution to (2.6) or (2.7).

Proposition 2. *Let*

$$P : [0, T_P)_h \to \mathbb{R}, \quad \beta : [0, T_\beta)_h \to \mathbb{R}, \quad \ell : [0, T_\ell)_h \to \mathbb{R},$$

and

$$u : [0, T_u)_h \to \mathbb{R}$$

be the solutions of (2.1)–(2.4) *with initial condition* (2.5)*, respectively. If*

$$0 < \varepsilon \le \frac{K\left(\sqrt{1+hr} - 1\right)^2}{h^2 r} \quad \text{and} \quad P_0 \ge \frac{K\left(\sqrt{1+hr} - 1\right)}{hr},$$

then

$$T_P = T_\beta = T_\ell = T_u = \infty,$$

and

$$\frac{K\left(\sqrt{1+hr} - 1\right)}{hr} \le \ell(t) \le \beta(t) \le u(t) \quad \text{and} \quad \ell(t) < P(t) < u(t)$$

hold for all $t \in (0, \infty)_h$.

Proof. Assume that

$$0 < \varepsilon \le \frac{K\left(\sqrt{1+hr} - 1\right)^2}{h^2 r}, \qquad P_0 \ge \frac{K\left(\sqrt{1+hr} - 1\right)}{hr}.$$

This proof is divided into four steps.

Step 1. Define

$$F(P) := \frac{rP\,(K - P)}{K + hrP}$$

for $P \geq 0$ and $h, r, K > 0$. Note that F is the function that appears on the right-hand side of (2.1). Clearly, the two equilibrium points of (2.1) are $P \equiv 0$ and K determined by

$$F(0) = F(K) = 0,$$

where $P \equiv K$ is attracting and $P \equiv 0$ is repelling. Now, we examine the shape of the function F. Define

$$P^* := \frac{K\left(\sqrt{1+hr}-1\right)}{hr}. \tag{2.8}$$

Since

$$F'(P) = \frac{r(K^2 - 2KP - hrP^2)}{(K+hrP)^2},$$

we see that $F'(P) > 0$ for $0 < P < P^*$, $F'(P^*) = 0$, and $F'(P) < 0$ for $P > P^*$. This implies that the function $F(P)$ takes the maximum value

$$F_{\max} := \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} \tag{2.9}$$

when

$$P = P^*.$$

Moreover, we see that $F(P) > 0$ on $(0, K)$ and $F(P) < 0$ on (K, ∞).

Step 2. At the outset, we prove

$$\ell(t) \geq P^* = \frac{K\left(\sqrt{1+hr}-1\right)}{hr}$$

for all $t \in [0, T_\ell)_h$ with $T_\ell = \infty$, where ℓ is determined by (2.3). Now, we consider the function $F(\ell) - \varepsilon$, which is the function that appears on the right-hand side of (2.3).

Case (a). First, we consider the case

$$\varepsilon = F_{\max},$$

where $F_{\max}$ is defined by (2.9). Then,

$$F(P^*) - \varepsilon = F(P^*) - F_{\max} = 0,$$

so that $P = P^*$ is the unique equilibrium point of (2.3). Thus,

$$\ell(t) \equiv P^*$$

is the unique global solution of (2.3) with

$$\ell(0) = P^*.$$

By the uniqueness of the solutions, $\ell(0) > P^*$ implies $\ell(t) > P^*$ for all $t \in [0, T_\ell)_h$. In addition, since

$$F(\ell) - F_{\max} < 0$$

holds for $\ell > P^*$, we have $\Delta_h \ell < 0$ for $\ell > P^*$. This implies that

$$T_\ell = \infty$$

and $\ell(t) > P^*$ for all $t \in [0, \infty)_h$.

Case (b). Next, we will consider the case

$$0 < \varepsilon < F_{\max},$$

where $F_{\max}$ is given by (2.9). In this case, we have

$$F(P^*) - \varepsilon > 0,$$

where P^* is given by (2.8). This indicates that (2.3) has two positive equilibria Q_1 and Q_2 that satisfy

$$F(Q_1) - \varepsilon = F(Q_2) - \varepsilon = 0$$

and

$$0 < Q_1 < P^* < Q_2.$$

Now

$$\ell(t) \equiv Q_2$$

is a globally unique solution of (2.3).

As

$$F(\ell) - \varepsilon > 0$$

for

$$P^* \leq \ell < Q_2,$$

we have $\Delta_h \ell > 0$ for

$$P^* \leq \ell < Q_2.$$

Consequently, summing this inequality yields

$$\ell(t) \geq \ell(0) \geq P^*, \quad t \in [0, T_\ell)_h.$$

Because of this and the uniqueness of the solutions, we have

$$P^* \leq \ell(0) < Q_2$$

means

$$Q_2 > \ell(t) \geq \ell(0) \geq P^*, \quad t \in [0, T_\ell)_h.$$

Therefore, if

$$P^* \leq \ell(0) < Q_2,$$

then

$$T_\ell = \infty.$$

On the other hand, because

$$F(\ell) - \varepsilon < 0$$

holds for $\ell > Q_2$, we have $\Delta_h \ell < 0$ for $\ell > Q_2$. Consequently, if $\ell(0) > Q_2$, then

$$Q_2 < \ell(t) \leq \ell(0) < \infty, \quad t \in [0, T_\ell)_h,$$

and so if $\ell(0) > Q_2$, then

$$T_\ell = \infty.$$

Considering Cases (a) and (b) together, we can conclude that

$$\ell(0) \geq P^* = \frac{K\left(\sqrt{1+hr}-1\right)}{hr}$$

implies the global existence of the solution $\ell(t)$ of (2.3) and $\ell(t) \geq P^*$ for all $t \in [0, \infty)_h$.

Step 3. We prove

$$\ell(t) \leq \beta(t) \leq u(t)$$

for $t \in [0, \infty)_h$. Let

$$\delta(t) := \beta(t) - u(t)$$

for

$$t \in [0, T_\beta)_h \cap [0, T_u)_h.$$

Our goal here is to show that $\delta(t) \leq 0$ for all

$$t \in [0, T_\beta)_h \cap [0, T_u)_h.$$

By way of contradiction, let

$$t_1 \in [0, T_\beta)_h \cap [0, T_u)_h$$

be the first input value such that $\delta(t_1) > 0$. Since

$$\delta(0) = 0$$

by initial condition (2.5), $\delta(t) \leq 0$ for all $t \in [0, t_1 - h]_h$. Then, we have

$$\begin{aligned}
\Delta_h \delta(t) &= \Delta_h \beta(t) - \Delta_h u(t) \\
&= \frac{r\beta(t)\,(K-\beta(t))}{K+hr\beta(t)} + q(t) - \frac{ru(t)\,(K-u(t))}{K+hru(t)} - \varepsilon \\
&\leq \left(\frac{r(K^2 - hr\beta(t)u(t) - K(\beta(t)+u(t)))}{(K+hr\beta(t))(K+hru(t))}\right)(\beta(t)-u(t)) + |q(t)| - \varepsilon \\
&\leq \left(\frac{r(K^2 - hr\beta(t)u(t) - K(\beta(t)+u(t)))}{(K+hr\beta(t))(K+hru(t))}\right)\delta(t)
\end{aligned}$$

for $t \in [0, t_1 - h]_h$. Let

$$\alpha(t) := \frac{r(K^2 - hr\beta(t)u(t) - K(\beta(t)+u(t)))}{(K+hr\beta(t))(K+hru(t))}.$$

Then

$$\Delta_h \delta(t) \le \alpha(t)\delta(t)$$

from the inequality above, and checking the regressivity condition for α, we have

$$1 + h\left(\frac{r(K^2 - hr\beta(t)u(t) - K(\beta(t) + u(t)))}{(K + hr\beta(t))(K + hru(t))}\right) = \frac{K^2(1 + hr)}{(K + hr\beta(t))(K + hru(t))} > 0,$$

so α is a positively regressive function. Considering this coefficient function α, let

$$e_\alpha(t, 0) := \prod_{j=0}^{\frac{t}{h}-1}(1 + h\alpha(jh)) > 0.$$

It follows from the quotient rule on time scales that

$$\Delta_h\left(\frac{\delta(t)}{e_\alpha(t, 0)}\right) = \frac{e_\alpha(t, 0)\Delta_h\delta(t) - \delta(t)\alpha(t)e_\alpha(t, 0)}{e_\alpha(t + h, 0)e_\alpha(t, 0)} = \frac{\Delta_h\delta(t) - \alpha(t)\delta(t)}{e_\alpha(t + h, 0)}.$$

Since α is a positively regressive function, we have

$$e_\alpha(t + h, 0) > 0.$$

This yields

$$\Delta_h\left(\frac{\delta(t)}{e_\alpha(t, 0)}\right) \le 0,$$

since

$$\Delta_h\delta(t) \le \alpha(t)\delta(t).$$

Summing this inequality and using $\delta(0) = 0$, we have

$$\sum_{j=0}^{\frac{t_1-h}{h}} h\Delta_h\left(\frac{\delta(jh)}{e_\alpha(jh, 0)}\right) = \frac{\delta(t_1)}{e_\alpha(t_1, 0)} \le 0,$$

a contradiction of the assumption $\delta(t_1) > 0$. Thus, we have

$$\beta(t) \le u(t)$$

for all

$$t \in [0, T_\beta)_h \cap [0, T_u)_h.$$

In a similar manner, we have that

$$\ell(t) \le \beta(t)$$

for all

$$t \in [0, \infty)_h \cap [0, T_\beta)_h.$$

Hence

$$0 < P^* \le \ell(t) \le \beta(t) \le u(t), \quad t \in [0, \infty)_h \cap [0, T_\beta)_h \cap [0, T_u)_h. \tag{2.10}$$

Next we show that

$$T_\beta = T_u = \infty.$$

We consider the function $F(u)+\varepsilon$, which is the function that appears on the right-hand side of (2.4). In this case, there is $Q_3 > K$ such that

$$F(Q_3)+\varepsilon = 0;$$

that is, $u \equiv Q_3$ is the unique positive equilibrium point of (2.4). Since

$$F(Q_3)+\varepsilon < 0$$

for $u > Q_3$, we have $\Delta_h u < 0$ for $u > Q_3$. Hence, if $u(0) > Q_3$ implies that

$$u(0) \geq u(t) > Q_3$$

for all $t \in [0, T_u)_h$, and thus $T_u = \infty$ when $u(0) > Q_3$. On the other hand, if

$$0 \leq u(0) < Q_3,$$

then we can obtain $T_u = \infty$. Actually, from (2.10) and the uniqueness of the solution, we have

$$0 < P^* \leq \ell(t) \leq u(t) < Q_3, \quad t \in [0, T_u)_h.$$

This means that $T_u = \infty$ when

$$0 \leq u(0) < Q_3.$$

Therefore, we have

$$T_u = \infty$$

for any case. Furthermore, since (2.4) holds, $\beta(t)$ is always sandwiched between $\ell(t)$ and $u(t)$, so

$$T_\beta = \infty$$

also holds.

Step 4. We show that

$$\ell(t) < P(t) < u(t)$$

for $t \in (0, \infty)_h$. Note that we have already shown in Step 3 that

$$T_\beta = T_\ell = T_u = \infty.$$

Clearly, $T_P = \infty$ is true. Let

$$\mathscr{D}(t) := u(t) - P(t)$$

for $t \in [0, \infty)_h$. From the above inequality with $q(t) \equiv 0$, in other words, P replaces β above if $q(t) \equiv 0$, we see that

$$\mathscr{D}(t) \geq 0$$

for $t \in [0, \infty)_h$. By $\mathscr{D}(0) = 0$, that is

$$P_0 = P(0) = u(0),$$

we have

$$\Delta_h \mathscr{D}(0) = \frac{ru(0)\,(K-u(0))}{K+hru(0)} + \varepsilon + \frac{rP(0)\,(K-P(0))}{K+hrP(0)} = \varepsilon > 0.$$

This implies that

$$\mathscr{D}(h) = h\varepsilon > 0.$$

By way of contradiction, we suppose that there exists $t_2 > 0$ such that $\mathscr{D}(t_2) \leq 0$ and $\mathscr{D}(t) > 0$ for $t \in [h, t_2 - h]_h$. Then, we have

$$\Delta_h \mathscr{D}(t) > \left(\frac{r(K^2 - hru(t)P(t) - K(u(t) + P(t)))}{(K + hru(t))(K + hrP(t))} \right) \mathscr{D}(t).$$

Let

$$\varphi(t) := \frac{r(K^2 - hru(t)P(t) - K(u(t) + P(t)))}{(K + hru(t))(K + hrP(t))}.$$

Then

$$\Delta_h \mathscr{D}(t) > \varphi(t)\mathscr{D}(t)$$

from the inequality above and checking the regressivity condition for φ, we have

$$1 + h\left(\frac{r(K^2 - hru(t)P(t) - K(u(t) + P(t)))}{(K + hru(t))(K + hrP(t))} \right) = \frac{K^2(1 + hr)}{(K + hru(t))(K + hrP(t))} > 0,$$

so φ is a positively regressive function. Considering this coefficient function φ, let

$$e_\varphi(t, 0) := \prod_{j=0}^{\frac{t}{h}-1} (1 + h\varphi(jh)) > 0.$$

It follows that

$$\Delta_h \left(\frac{\mathscr{D}(t)}{e_\varphi(t, 0)} \right) = \frac{e_\varphi(t, 0)\Delta_h \mathscr{D}(t) - \mathscr{D}(t)\varphi(t)e_\varphi(t, 0)}{e_\varphi(t + h, 0)e_\varphi(t, 0)} = \frac{\Delta_h \mathscr{D}(t) - \varphi(t)\mathscr{D}(t)}{e_\varphi(t + h, 0)}.$$

Since φ is a positively regressive function, we have

$$e_\varphi(t + h, 0) > 0.$$

This yields

$$\Delta_h \left(\frac{\mathscr{D}(t)}{e_\varphi(t, 0)} \right) > 0,$$

since

$$\Delta_h \mathscr{D}(t) > \varphi(t)\mathscr{D}(t).$$

Summing this inequality and using $\mathscr{D}(0) = 0$, we have

$$\sum_{j=0}^{\frac{t_2-h}{h}} h\Delta_h \left(\frac{\mathscr{D}(jh)}{e_\varphi(jh, 0)} \right) = \frac{\mathscr{D}(t_2)}{e_\varphi(t_2, 0)} > 0,$$

a contradiction of the assumption $\mathscr{D}(t_2) \leq 0$. Thus, we have $P(t) < u(t)$ for all $t \in (0, \infty)_h$. In a similar manner, we have that $\ell(t) < P(t)$ for all $t \in (0, \infty)_h$. □

To illustrate the proposition, the following example is provided.

Example 3. *Consider* (2.1)–(2.4) *with*

$$h = r = K = 1.$$

According to Proposition 2, if

$$0 < \varepsilon \leq \left(\sqrt{2} - 1\right)^2 \quad \text{and} \quad P_0 \geq \sqrt{2} - 1$$

hold, then the solutions

$$P : [0, T_P)_1 \to \mathbb{R}, \quad \beta : [0, T_\beta)_1 \to \mathbb{R}, \quad \ell : [0, T_\ell)_1 \to \mathbb{R},$$

and

$$u : [0, T_u)_1 \to \mathbb{R}$$

of (2.1)–(2.4), *respectively, with initial condition* (2.5) *satisfy*

$$T_P = T_\beta = T_\ell = T_u = \infty$$

and

$$\sqrt{2} - 1 \leq \ell(t) \leq \beta(t) \leq u(t) \quad \text{and} \quad \ell(t) < P(t) < u(t)$$

for all $t \in [0, \infty)_1$.

Figure 1 illustrates the solution orbits of (2.2) *(red) with* $h = r = K = 1$ *and* $q(t) = 0.01(-1)^t$, (2.3) *(black), and* (2.4) *(blue), given the initial condition*

$$\beta(0) = \ell(0) = u(0) = 0.5$$

and $\varepsilon = 0.01$. *Notice that the solution orbit of* (2.2) *(red) is bounded between the others.*

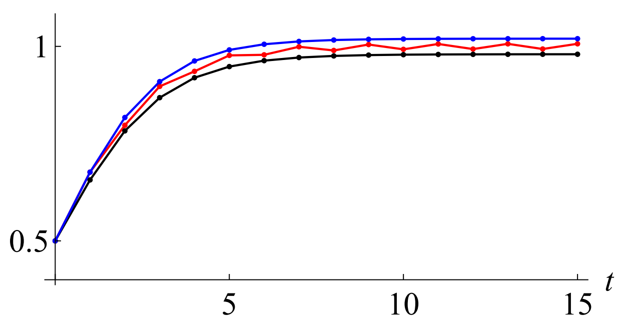

Figure 1. The solution orbits of (2.2) (red), (2.3) (black), and (2.4) (blue) with $\beta(0) = \ell(0) = u(0) = 0.5$.

Remark 4. *Let*

$$\varepsilon > \frac{K\left(\sqrt{1 + hr} - 1\right)^2}{h^2 r} \quad \text{and} \quad P_0 \geq \frac{K\left(\sqrt{1 + hr} - 1\right)}{hr}.$$

Define

$$F(P) := \frac{rP\,(K-P)}{K+hrP}$$

for

$$P > -\frac{K}{hr}$$

and $h, r, K > 0$. *By the proof of Proposition 2, we see that*

$$\Delta_h \ell(t) = \frac{r\ell(t)\,(K-\ell(t))}{K+hr\ell(t)} - \varepsilon \le F_{\max} - \varepsilon < 0$$

for $t \ge 0$ *and* $\ell(t) \in \left(-\frac{K}{hr}, \infty\right)$. *Summing this inequality, we have*

$$\ell(t) - \ell(0) = \sum_{j=0}^{\frac{t-h}{h}} h\Delta_h \ell(jh) \le (F_{\max} - \varepsilon)t$$

for $t \ge 0$ *and* $\ell(t) \in \left(-\frac{K}{hr}, \infty\right)$. *This inequality implies that for any* $h > 0$ *there exists* $t_h \ge 0$ *such that*

$$\ell(t) \le (F_{\max} - \varepsilon)t + \ell(0) \le -\frac{K}{2hr}$$

for $t \ge t_h$ *and* $\ell(t) \in \left(-\frac{K}{hr}, \infty\right)$. *This shows that any solution* $\ell(t)$ *of* (2.3) *with* $\ell(0) \in \mathbb{R}$ *diverges to* $-\infty$ *as* $h \to 0^+$.

On the other hand, any solution $P(t)$ *of Eq* (2.1) *satisfying*

$$P(0) = P_0 \ge \frac{K\left(\sqrt{1+hr}-1\right)}{hr}$$

exists globally in time and is greater than or equal to

$$\frac{K\left(\sqrt{1+hr}-1\right)}{hr}.$$

In fact, (2.1) *has two equilibria* $P = 0, K$; $F(P) > 0$ *for* $0 < P < K$; *and*

$$0 < \frac{K\left(\sqrt{1+hr}-1\right)}{hr} < K$$

holds. As $F(P) > 0$ *for*

$$\frac{K\left(\sqrt{1+hr}-1\right)}{hr} \le P < K,$$

we have $\Delta_h P > 0$ *for*

$$\frac{K\left(\sqrt{1+hr}-1\right)}{hr} \le P < K.$$

Consequently, summing this inequality yields

$$P(t) \ge P(0) \ge \frac{K\left(\sqrt{1+hr}-1\right)}{hr}, \quad t \ge 0.$$

Because of this and the uniqueness of the solutions, we have

$$P(0) \in \left[\frac{K\left(\sqrt{1+hr}-1\right)}{hr}, \infty\right)$$

means

$$P(t) \geq P(0) \geq \frac{K\left(\sqrt{1+hr}-1\right)}{hr}$$

for all $t \geq 0$. *This implies that* $|P(t)-\ell(t)|$ *diverges to* ∞ *as* $h \to 0^+$; *that is, Eq* (2.1) *is not conditionally Hyers–Ulam stable in class* C_D *on* $[0,\infty)_h$ *when* $h \to 0^+$, *where*

$$D = \left[\frac{K\left(\sqrt{1+hr}-1\right)}{hr}, \infty\right).$$

Therefore, we can see that

$$\varepsilon = \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r}$$

is the threshold value.

Remark 5. *Let*

$$\varepsilon = \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r}$$

and

$$0 < P_0 < \frac{K\left(\sqrt{1+hr}-1\right)}{hr}.$$

Define

$$F(P) := \frac{rP\,(K-P)}{K+hrP}$$

for

$$P > -\frac{K}{hr}$$

and $h, r, K > 0$. *By the proof of Proposition 2, we see that*

$$F'(P) = \frac{r(K^2 - 2KP - hrP^2)}{(K+hrP)^2} > 0$$

for

$$-\frac{K}{hr} < P < \frac{K\left(\sqrt{1+hr}-1\right)}{hr};$$

and

$$\ell = \frac{K\left(\sqrt{1+hr}-1\right)}{hr}$$

is the unique equilibrium point of (2.3)*; and*

$$F(\ell) - \varepsilon \le F(P_0) - \varepsilon < F\left(\frac{K\left(\sqrt{1+hr}-1\right)}{hr}\right) - \varepsilon = 0$$

for

$$-\frac{K}{hr} < \ell < \frac{K\left(\sqrt{1+hr}-1\right)}{hr}.$$

Let $\ell(0) = P_0$*. Then, we have*

$$\Delta_h \ell(t) \le F(P_0) - \varepsilon < 0$$

for $t \ge 0$ *and*

$$\ell(t) \in \left(-\frac{K}{hr}, \frac{K\left(\sqrt{1+hr}-1\right)}{hr}\right).$$

Summing this inequality, we have

$$\ell(t) - P_0 = \sum_{j=0}^{\frac{t-h}{h}} h\Delta_h \ell(jh) \le (F(P_0) - \varepsilon)t$$

for $t \ge 0$ *and*

$$\ell(t) \in \left(-\frac{K}{hr}, \frac{K\left(\sqrt{1+hr}-1\right)}{hr}\right).$$

This inequality implies that for any $h > 0$*, there exists* $t_h \ge 0$ *such that*

$$\ell(t) \le (F(P_0) - \varepsilon)t + P_0 \le -\frac{K}{2hr}$$

for $t \ge t_h$ *and*

$$\ell(t) \in \left(-\frac{K}{hr}, \frac{K\left(\sqrt{1+hr}-1\right)}{hr}\right).$$

In a similar manner as Remark 4, we see that Eq (2.1) *is not conditionally Hyers–Ulam stable on* $[0,\infty)_h$ *when* $h \to 0^+$*. Therefore, we can conclude that*

$$P_0 = \frac{K\left(\sqrt{1+hr}-1\right)}{hr}$$

is the threshold value.

3. Technical lemmas

Before presenting the main theorem and its proof in the next section, we give some important lemmas.

Lemma 6. *Suppose that*

$$0 < \varepsilon \leq \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} \quad \textit{and} \quad P_0 \geq \frac{K\left(\sqrt{1+hr}-1\right)}{hr}.$$

Let

$$P : [0, T_P)_h \to \mathbb{R}, \ \ \ell : [0, T_\ell)_h \to \mathbb{R}$$

and

$$u : [0, T_u)_h \to \mathbb{R}$$

be the solutions of (2.1)*,* (2.3)*, and* (2.4) *with initial condition* (2.5)*, respectively. Then,*

$$T_P = T_\ell = T_u = \infty$$

and

$$\frac{r(K^2 - hr\ell(t)P(t) - K(\ell(t) + P(t)))}{(K + hr\ell(t))(K + hrP(t))} < -\frac{\sqrt{1+hr}-1}{h\sqrt{1+hr}} \frac{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} - (1+hr)^{-\frac{t}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t}{2h}}}$$

and

$$\frac{r(K^2 - hru(t)P(t) - K(u(t) + P(t)))}{(K + hru(t))(K + hrP(t))} < -\frac{\sqrt{1+hr}-1}{h\sqrt{1+hr}} \frac{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} - (1+hr)^{-\frac{t}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t}{2h}}}$$

hold for all $t \in (0, \infty)_h$*.*

Proof. By Proposition 2, we have

$$T_P = T_\ell = T_u = \infty$$

and

$$\frac{K\left(\sqrt{1+hr}-1\right)}{hr} \leq \ell(t) < P(t) < u(t)$$

for all $t \in (0, \infty)_h$. As the proofs of the two inequalities in the statement above are the same, the second one is omitted. For convenience, we write

$$F(t) := \frac{r(K^2 - hr\ell(t)P(t) - K(\ell(t) + P(t)))}{(K + hr\ell(t))(K + hrP(t))}$$

for $t \in (0, \infty)_h$. Since (2.1) can be solved directly for P, we have

$$P(t, P_0) = \frac{P_0 K (1+hr)^{\frac{t}{h}}}{P_0 (1+hr)^{\frac{t}{h}} + K - P_0}, \qquad P_0 := P(0).$$

Notice that $P(t, P_0)$ is increasing in P_0. For fixed $t \geq 0$, it thus follows that

$$P(t, P_0) > P\left(t, \ \frac{K\left(\sqrt{1+hr}-1\right)}{hr}\right) = \frac{K}{1 + (1+hr)^{\frac{1}{2}-\frac{t}{h}}}$$

for $t \in (0, \infty)_h$. Also notice that for fixed t the function F is decreasing in $P > 0$ and $\ell > 0$, since

$$\frac{\partial F}{\partial P} = -\frac{K^2 r(1+hr)}{(K + h\ell r)(K + hPr)^2} < 0,$$

$$\frac{\partial F}{\partial \ell} = -\frac{K^2 r(1+hr)}{(K+h\ell r)^2(K+hPr)} < 0.$$

This yields

$$\begin{aligned}
F(t) &= \frac{r(K^2 - hr\ell(t)P(t) - K(\ell(t)+P(t)))}{(K+hr\ell(t))(K+hrP(t))} \\
&< \frac{r\left(K^2 - hr\frac{K(\sqrt{1+hr}-1)}{hr}\frac{K}{1+(1+hr)^{\frac{1}{2}-\frac{t}{h}}} - K\left(\frac{K(\sqrt{1+hr}-1)}{hr} + \frac{K}{1+(1+hr)^{\frac{1}{2}-\frac{t}{h}}}\right)\right)}{\left(K + hr\frac{K(\sqrt{1+hr}-1)}{hr}\right)\left(K + hr\frac{K}{1+(1+hr)^{\frac{1}{2}-\frac{t}{h}}}\right)} \\
&= \frac{\sqrt{1+hr} - 1 - \frac{hr}{1+(1+hr)^{\frac{1}{2}-\frac{t}{h}}}}{h\left(1 + \frac{hr}{1+(1+hr)^{\frac{1}{2}-\frac{t}{h}}}\right)} = \frac{(\sqrt{1+hr}-1)\left(1+(1+hr)^{\frac{1}{2}-\frac{t}{h}}\right) - hr}{h\sqrt{1+hr}\left(\sqrt{1+hr} + (1+hr)^{-\frac{t}{h}}\right)} \\
&= -\frac{(\sqrt{1+hr}-1)\left(\sqrt{1+hr} - (1+hr)^{-\frac{t}{h}+\frac{1}{2}}\right)}{h\sqrt{1+hr}\left(\sqrt{1+hr} + (1+hr)^{-\frac{t}{h}}\right)} \\
&= -\frac{\sqrt{1+hr}-1}{h\sqrt{1+hr}}\frac{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} - (1+hr)^{-\frac{t}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t}{2h}}}
\end{aligned}$$

for $t \in (0,\infty)_h$. Thus, we obtain the first inequality in the statement of this lemma. □

Lemma 7. *Let* $\varepsilon > 0$*, and let*

$$\mathscr{F}(t) := -\frac{\sqrt{1+hr}-1}{h\sqrt{1+hr}}\frac{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} - (1+hr)^{-\frac{t}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t}{2h}}}. \tag{3.1}$$

Then the function

$$\Omega(t) := \varepsilon h\sqrt{1+hr}\left(e_{\mathscr{F}}(t,0) + \frac{1}{\sqrt{1+hr}-1}\frac{(1+hr)^{\frac{t-h}{2h}+\frac{1}{2}} - (1+hr)^{-\frac{t-h}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t-h}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t-h}{2h}}}\right) \tag{3.2}$$

solves the linear h-difference equation

$$\Delta_h\Omega(t) = \mathscr{F}(t)\Omega(t) + \varepsilon\sqrt{1+hr}\frac{(1+hr)^{\frac{t-h}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t-h}{2h}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t}{2h}}} \tag{3.3}$$

with the initial condition $\Omega(0) = 0$.

Proof. For simplicity, let $\alpha = 1 + hr$,

$$f(t) = \alpha^{\frac{t}{2h}+\frac{1}{2}} - \alpha^{-\frac{t}{2h}+\frac{1}{2}}, \quad g(t) = \alpha^{\frac{t}{2h}+\frac{1}{2}} + \alpha^{-\frac{t}{2h}}, \quad \text{and} \quad \Omega_p(t) = \frac{\varepsilon h\sqrt{\alpha}}{\sqrt{\alpha}-1}\frac{f(t-h)}{g(t-h)}.$$

First, we show that the function $\Omega_p(t)$ is a particular solution of Eq (3.3). This fact can be confirmed by direct calculation, but to aid in the calculation, we calculate some difference operators in advance. In fact, by

$$\Delta_h\alpha^{\frac{t}{2h}} = \frac{\alpha^{\frac{1}{2}}-1}{h}\alpha^{\frac{t}{2h}} \quad \text{and} \quad \Delta_h\alpha^{-\frac{t}{2h}} = \frac{\alpha^{-\frac{1}{2}}-1}{h}\alpha^{-\frac{t}{2h}},$$

we have

$$\Delta_h f(t) = \frac{\alpha^{\frac{1}{2}} - 1}{h} g(t) > 0 \quad \text{and} \quad \Delta_h g(t) = \frac{\alpha^{-\frac{1}{2}} - 1}{h\alpha^{\frac{1}{2}}} f(t+h) > 0.$$

It follows from the quotient rule on time scales that

$$\begin{aligned}
\Delta_h \Omega_p(t) &= \frac{\varepsilon h \sqrt{\alpha}}{\sqrt{\alpha} - 1} \Delta_h \left(\frac{f(t-h)}{g(t-h)} \right) \\
&= \frac{\varepsilon h \sqrt{\alpha}}{\sqrt{\alpha} - 1} \frac{g(t-h)\Delta_h f(t-h) - f(t-h)\Delta_h g(t-h)}{g(t-h)g(t)} \\
&= \frac{\varepsilon h \sqrt{\alpha}}{\sqrt{\alpha} - 1} \frac{\frac{\alpha^{\frac{1}{2}}-1}{h} g(t-h)^2 - \frac{\alpha^{-\frac{1}{2}}-1}{h\alpha^{\frac{1}{2}}} f(t-h)f(t)}{g(t-h)g(t)} \\
&= \varepsilon \sqrt{\alpha} \frac{g(t-h)}{g(t)} + \mathscr{F}(t)\Omega(t).
\end{aligned}$$

Thus, $\Omega_p(t)$ is a particular solution of Eq (3.3).

Next we consider the function $e_{\mathscr{F}}(t,0)$. Since

$$1 + h\mathscr{F}(t) = \frac{\alpha^{\frac{t}{2h}} + \alpha^{-\frac{t}{2h}+\frac{1}{2}}}{\alpha^{\frac{t}{2h}+\frac{1}{2}} + \alpha^{-\frac{t}{2h}}} > 0$$

holds, $\mathscr{F}$ is a positively regressive function. Hence $e_{\mathscr{F}}(t,0)$ solves the linear h-difference equation

$$\Delta_h e_{\mathscr{F}}(t,0) = \mathscr{F}(t) e_{\mathscr{F}}(t,0),$$

and is positive for $t \in [0,\infty)_h$. From the superposition principle, we see that

$$\Omega(t) = \varepsilon h \sqrt{\alpha} e_{\mathscr{F}}(t,0) + \Omega_p(t)$$

is a solution to Eq (3.3). Moreover, we have

$$\Omega(0) = \varepsilon h \sqrt{\alpha} \left(1 + \frac{1}{\sqrt{\alpha} - 1} \frac{1 - \alpha}{1 + \alpha^{\frac{1}{2}}} \right) = 0.$$

Therefore, the statement in the lemma is true. □

Lemma 8. *Let* $\varepsilon > 0$*, and let* $\omega(t)$ *satisfy* $\omega(0) = 0$ *and the linear h-difference inequality*

$$\Delta_h \omega(t) \le \mathscr{F}(t)\omega(t) + \varepsilon \tag{3.4}$$

for $t \in [0,\infty)_h$*, where* $\mathscr{F}(t)$ *is given by* (3.1)*. Let* $\Omega(t)$ *be given by* (3.2)*. Then*

$$\Omega(t) \ge \omega(t)$$

for all $t \in [0,\infty)_h$*.*

Proof. Define

$$\delta(t) := \Omega(t) - \omega(t).$$

Note that by Lemma 7, $\Omega(t)$ is the solution of Eq (3.3) with $\Omega(0) = 0$. Then $\delta(0) = 0$. By way of contradiction, we suppose that there exists $t_1 > 0$ such that $\delta(t_1) < 0$ and $\delta(t) \geq 0$ for $t \in [0, t_1 - h]_h$. Then, we have

$$\begin{aligned}\Delta_h\delta(t) &\geq \mathscr{F}(t)\delta(t) + \varepsilon\left(\sqrt{1+hr}\frac{(1+hr)^{\frac{t-h}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t-h}{2h}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t}{2h}}} - 1\right)\\ &= \mathscr{F}(t)\delta(t) + \frac{\varepsilon hr(1+hr)^{-\frac{t}{2h}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t}{2h}}}\\ &> \mathscr{F}(t)\delta(t)\end{aligned}$$

for $t \in [0, t_1 - h]_h$. As shown in the proof of the previous lemma, $\mathscr{F}$ is a positively regressive function, and

$$e_{\mathscr{F}}(t,0) := \prod_{j=0}^{\frac{t}{h}-1}(1 + h\mathscr{F}(jh)) > 0$$

holds. Consequently,

$$\Delta_h\left(\frac{\delta(t)}{e_{\mathscr{F}}(t,0)}\right) = \frac{e_{\mathscr{F}}(t,0)\Delta_h\delta(t) - \delta(t)\mathscr{F}(t)e_{\mathscr{F}}(t,0)}{e_{\mathscr{F}}(t+h,0)e_{\mathscr{F}}(t,0)} = \frac{\Delta_h\delta(t) - \mathscr{F}(t)\delta(t)}{e_{\mathscr{F}}(t+h,0)}.$$

Since $\mathscr{F}$ is a positively regressive function, we have

$$e_{\mathscr{F}}(t+h,0) > 0.$$

This yields

$$\Delta_h\left(\frac{\delta(t)}{e_{\mathscr{F}}(t,0)}\right) > 0,$$

since

$$\Delta_h\delta(t) > \mathscr{F}(t)\delta(t).$$

Summing this inequality and using $\delta(0) = 0$, we have

$$\sum_{j=0}^{\frac{t_1-h}{h}} h\Delta_h\left(\frac{\delta(jh)}{e_{\mathscr{F}}(jh,0)}\right) = \frac{\delta(t_1)}{e_{\mathscr{F}}(t_1,0)} > 0,$$

a contradiction of the assumption $\delta(t_1) < 0$. Thus, we have

$$\Omega(t) \geq \omega(t)$$

for all $t \in [0, \infty)_h$. □

4. Conditional Hyers–Ulam stability

The following theorem is the main result obtained in this study.

Theorem 9. *Suppose that*

$$0 < \varepsilon \le \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} \quad \textit{and} \quad P_0 \ge \frac{K\left(\sqrt{1+hr}-1\right)}{hr}.$$

Let

$$P : [0, T_P)_h \to \mathbb{R}$$

and

$$\beta : [0, T_\beta)_h \to \mathbb{R}$$

be the solutions of (2.1) *and* (2.2) *with initial condition* (2.5)*, respectively. Then,*

$$T_P = T_\beta = \infty,$$

and

$$|\beta(t) - P(t)| \le \frac{h(1+hr)}{\sqrt{1+hr}-1}\varepsilon$$

holds for all $t \in [0, \infty)_h$. That is, Eq (2.1) *is conditionally Hyers–Ulam stable with Hyers–Ulam stability constant*

$$\mathcal{H} = \frac{h(1+hr)}{\sqrt{1+hr}-1}.$$

Proof. Assume that

$$0 < \varepsilon \le \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} \quad \text{and} \quad P_0 \ge \frac{K\left(\sqrt{1+hr}-1\right)}{hr}.$$

Let

$$P : [0, T_P)_h \to \mathbb{R}, \;\; \beta : [0, T_\beta)_h \to \mathbb{R}, \;\; \ell : [0, T_\ell)_h \to \mathbb{R},$$

and

$$u : [0, T_u)_h \to \mathbb{R}$$

be the solutions of (2.1)–(2.4) with (2.5), respectively. It follows from Proposition 2 that

$$T_P = T_\beta = T_\ell = T_u = \infty,$$

and

$$\frac{K\left(\sqrt{1+hr}-1\right)}{hr} \le \ell(t) \le \beta(t) \le u(t) \quad \text{and} \quad \ell(t) < P(t) < u(t)$$

hold for all $t \in (0, \infty)_h$. As a result,

$$|\beta(t) - P(t)| \le \max\{u(t) - P(t), P(t) - \ell(t)\}.$$

Let

$$\mathscr{D}(t) := u(t) - P(t) > 0$$

for $t \in (0,\infty)_h$. Then, we have

$$\Delta_h \mathscr{D}(t) = \left(\frac{r(K^2 - hru(t)P(t) - K(u(t) + P(t)))}{(K + hru(t))(K + hrP(t))} \right) \mathscr{D}(t) + \varepsilon$$

for $t \in (0,\infty)_h$. Using Lemma 6,

$$\Delta_h \mathscr{D}(t) < -\frac{\sqrt{1+hr}-1}{h\sqrt{1+hr}} \frac{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} - (1+hr)^{-\frac{t}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t}{2h}}} \mathscr{D}(t) + \varepsilon = \mathscr{F}(t)\mathscr{D}(t) + \varepsilon$$

holds for all $t \in (0,\infty)_h$, where $\mathscr{F}(t)$ is given by (3.1). Note that by (2.5), we have

$$\mathscr{D}(0) = u(0) - P(0) = 0,$$

and

$$\Delta_h \mathscr{D}(0) = \varepsilon,$$

and so that

$$\Delta_h \mathscr{D}(t) \le \mathscr{F}(t)\mathscr{D}(t) + \varepsilon$$

for all $t \in [0,\infty)_h$. Now we consider the function $\Omega(t)$ defined by (3.2). Then, by Lemma 8,

$$u(t) - P(t) = \mathscr{D}(t) \le \Omega(t)$$

for all $t \in [0,\infty)_h$. In a similar manner, we have that

$$P(t) - \ell(t) \le \Omega(t)$$

for all $t \in [0,\infty)_h$. Hence we obtain

$$|\beta(t) - P(t)| \le \Omega(t) = \varepsilon h\sqrt{1+hr} \left(e_{\mathscr{F}}(t,0) + \frac{1}{\sqrt{1+hr}-1} \frac{(1+hr)^{\frac{t-h}{2h}+\frac{1}{2}} - (1+hr)^{-\frac{t-h}{2h}+\frac{1}{2}}}{(1+hr)^{\frac{t-h}{2h}+\frac{1}{2}} + (1+hr)^{-\frac{t-h}{2h}}} \right)$$

for all $t \in [0,\infty)_h$. As shown in the proof of Lemma 7, $\mathscr{F}$ is a positively regressive function, and $e_{\mathscr{F}}(t,0) > 0$ for $t \in [0,\infty)_h$. In addition, by (3.1), $\mathscr{F}$ is non-positive for $t \in [0,\infty)_h$. That is, $\Delta_h e_{\mathscr{F}}(t,0) \le 0$ for $t \in [0,\infty)_h$. Hence, we see that

$$0 < e_{\mathscr{F}}(t,0) \le 1$$

for $t \in [0,\infty)_h$. Using this, we obtain

$$|\beta(t) - P(t)| \le \varepsilon h\sqrt{1+hr} \left(1 + \frac{1}{\sqrt{1+hr}-1} \right) = \frac{h(1+hr)}{\sqrt{1+hr}-1} \varepsilon$$

for all $t \in [0,\infty)_h$. □

Remark 10. *Theorem 9 implies the following fact: the Eq* (2.1) *is conditionally Hyers–Ulam stable in class* C_D *on* $[0,\infty)_h$*, with*

$$\mathcal{S} = \left(0, \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r}\right],$$

and with a Hyers–Ulam stability constant

$$\mathcal{H} = \frac{h(1+hr)}{\sqrt{1+hr}-1},$$

where

$$D = \left[\frac{K\left(\sqrt{1+hr}-1\right)}{hr}, \infty\right).$$

For the three key constants given here, we note that as the step-size $h > 0$ *tends to zero, we have*

$$\lim_{h\to 0^+} \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} = \frac{rK}{4}, \quad \lim_{h\to 0^+} \frac{K\left(\sqrt{1+hr}-1\right)}{hr} = \frac{K}{2}, \quad \text{and} \quad \lim_{h\to 0^+} \frac{h(1+hr)}{\sqrt{1+hr}-1} = \frac{2}{r}.$$

These limiting values match the values found for the continuous logistic model [11, Example 3.2].

Remark 11. *If* $h = 1$*, then* (2.1) *can be rewritten as the iteration equation*

$$P(t+1) = \frac{\sqrt{1+r}P(t)}{\frac{r}{K\sqrt{1+r}}P(t) + \frac{1}{\sqrt{1+r}}}. \tag{4.1}$$

Letting

$$a = \sqrt{1+r}, \quad b = 0, \quad c = \frac{r}{K\sqrt{1+r}}, \quad d = \frac{1}{\sqrt{1+r}},$$

we see that

$$P(t+1) = \frac{aP(t)+b}{cP(t)+d} \quad \text{with} \quad ad - bc = 1 \quad \text{and} \quad a + d > 2.$$

This is an example of a loxodromic Möbius difference equation. For more on Hyers–Ulam stability of loxodromic Möbius difference equations, see Nam [17].

In 2017, Jung and Nam [14, Example 4.1] gave an example of the conditional Hyers–Ulam stability of the iteration equation

$$P(t+1) = \frac{AP(t)}{CP(t)+1},$$

which is equivalent to (4.1)*, where*

$$A = 1 + r \quad \text{and} \quad C = \frac{r}{K}.$$

Their result, expressed in the terms of our paper, is as follows: The Eq (4.1) *(resp.,* (2.1)*) is conditionally Hyers–Ulam stable in class* C_{D^*} *on*

$$[0,\infty)_1 = \mathbb{N}_0,$$

with

$$\mathcal{S}^* = \left(0, \frac{A\sqrt{A} - 2A + \sqrt{A}}{(A - \sqrt{A} + 1)C}\right),$$

and with a Hyers–Ulam stability constant

$$\mathcal{H}^* := \frac{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^2}{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^2 - 1},$$

where

$$D^* = \left(-\infty, -\frac{A - \sqrt{A} + 2}{C}\right) \cup \left(\frac{A - \sqrt{A}}{C}, \infty\right).$$

We note here that the term "conditional Hyers–Ulam stability" is not used in [14], and their original result shows that if $\beta(0)$ *is in* D^**, then there exists* $P(t)$ *which satisfies* (4.1) *and*

$$|\beta(t) - P(t)| \leq \frac{|\beta(0) - P(0)|}{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{2t}} + \sum_{j=0}^{t-1} \frac{\varepsilon}{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{2j}}$$

for all $t \in \mathbb{N}_0$*, where* $\beta(t)$ *is a solution of* (2.2)*. In our paper settings,* $\beta(0) = P(0)$ *(see* (2.5)*), so the first term on the right-hand side is* 0*. The second term can be evaluated as follows:*

$$\sup_{t\in\mathbb{N}_0} \sum_{j=0}^{t-1} \frac{\varepsilon}{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{2j}} = \sup_{t\in\mathbb{N}_0} \frac{1 - \left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{-2t}}{1 - \left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^{-2}}\varepsilon = \frac{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^2}{\left(\sqrt{A} + \frac{1}{\sqrt{A}} - 1\right)^2 - 1}\varepsilon.$$

Thus, we have the Hyers–Ulam stability constant $\mathcal{H}^*$*.*

We now compare the three important constants obtained in Theorem 9 with those appearing in the above mentioned $\mathcal{S}^*$*,* D^**, and* $\mathcal{H}^*$*, but note that the negative region of* D^* *is omitted since it is not of interest in our paper. First, we compare our result with theirs for the upper bound of* ε*. Using*

$$A = 1 + r \quad \text{and} \quad C = \frac{r}{K},$$

we have

$$\frac{A\sqrt{A} - 2A + \sqrt{A}}{(A - \sqrt{A} + 1)C} = \frac{K\left(\sqrt{1+r} - 1\right)^2}{r} \times \frac{\sqrt{1+r}}{2 + r - \sqrt{1+r}} < \frac{K\left(\sqrt{1+r} - 1\right)^2}{r}.$$

From this inequality, we can claim that our result (Theorem 9) is sharper than theirs because a smaller ε *is more stable. Next, we compare the infimum values of the initial value. Since*

$$\frac{A - \sqrt{A}}{C} = \frac{K\left(\sqrt{1+r} - 1\right)}{r}\sqrt{1+r} > \frac{K\left(\sqrt{1+r} - 1\right)}{r}$$

holds, we can claim that our result (Theorem 9) is sharper than theirs for this point. Simply from the qualitative aspect of ensuring Hyers–Ulam stability, we can conclude that our result that guarantees Hyers–Ulam stability for larger ε *and smaller initial value is sharp.*

Finally, we compare the Hyers–Ulam stability constants. Define

$$H(r) := \mathcal{H} = \frac{1+r}{\sqrt{1+r}-1} \quad \textit{and} \quad H^*(r) := \mathcal{H}^* = \frac{\left(\sqrt{1+r}+\frac{1}{\sqrt{1+r}}-1\right)^2}{\left(\sqrt{1+r}+\frac{1}{\sqrt{1+r}}-1\right)^2-1} \tag{4.2}$$

for $r > 0$*. The graphs of functions* H *and* H^* *are shown in Figure 2. The red curve shows the graph for* H*, and the blue curve shows the graph for* H^**.*

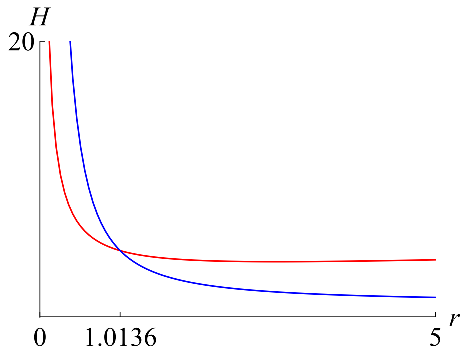

Figure 2. The graphs of $H(r)$ (red curve) and $H^*(r)$ (blue curve).

Note that $r \approx 1.013624$ *solves*

$$H(r) = H^*(r).$$

Thus, if $0 < r < 1.013624$*, then our Hyers–Ulam stability constant*

$$H(r) = \mathcal{H}$$

is better than theirs. However, this statement may be reversed if $r > 1.013624$*. But, the next section gives an example where this conjecture is not necessarily true (see Example 13).*

There is a reason why the Hyers–Ulam stability constants diverge as r *approaches* 0*. If* $h = 1$ *and* $r = 0$*, then* (4.1) *(resp.,* (2.1)*) and* (2.2) *become*

$$\Delta P(t) = 0$$

and

$$\Delta\beta(t) = q(t)$$

with

$$|q(t)| \le \varepsilon$$

for all $t \in \mathbb{N}_0$*. We put*

$$q(t) \equiv \varepsilon.$$

Then we have a solution

$$\beta(t) = \varepsilon t.$$

Since

$$P(t) \equiv C$$

is any solution of the equation $\Delta P(t) = 0$*, where* C *is an arbitrary constant, we see that*

$$\lim_{t\to\infty} |\beta(t) - P(t)| = \lim_{t\to\infty} |\varepsilon t - C| = \infty.$$

This means that (4.1) *is not Hyers–Ulam stable on* $\mathbb{N}_0$*. Therefore, it is a natural consequence that*

$$\lim_{r\to 0^+} H(r) = \lim_{r\to 0^+} H^*(r) = \infty.$$

In addition, we have

$$\lim_{r\to 0^+} (H^*(r) - H(r)) = \infty.$$

That is, $H^*(r)$ *is much larger near* $r = 0$ *than* $H(r)$*.*

5. Examples

In this section we present detailed examples with specific parameter values that illustrate our main conditional stability results.

Example 12. *In* (2.1)*,* (2.3)*, and* (2.4) *take* $h = 1$*,* $r = \frac{1}{3}$*,* $K = 9$*,* $\varepsilon = \frac{3}{5}$*, and*

$$P_0 = 9\left(-3 + 2\sqrt{3}\right).$$

According to Theorem 9, since

$$0 < \varepsilon \le \frac{K\left(\sqrt{1+hr} - 1\right)^2}{h^2 r} = 63 - 36\sqrt{3} \approx 0.646171$$

and

$$P_0 \ge \frac{K\left(\sqrt{1+hr} - 1\right)}{hr} = 9\left(-3 + 2\sqrt{3}\right),$$

then solutions

$$P : [0, T_P)_h \to \mathbb{R}, \quad \ell : [0, T_\ell)_h \to \mathbb{R},$$

and

$$u : [0, T_u)_h \to \mathbb{R}$$

of (2.1)*,* (2.3)*, and* (2.4)*, respectively, with initial condition* (2.5) *satisfy*

$$T_P = T_\ell = T_u = \infty$$

and

$$|\ell(t) - P(t)|, |u(t) - P(t)| \le \frac{h(1+hr)}{\sqrt{1+hr} - 1}\varepsilon = \frac{4}{5}\left(3 + 2\sqrt{3}\right) \approx 5.17128 \tag{5.1}$$

for all $t \in [0, \infty)_h$*. Note that in this specific instance we have*

$$P(t) = \frac{1}{\frac{1}{9} + 2^{1-2t} 3^{-\frac{5}{2}+t}}, \quad \ell(t) = 3 + \frac{12}{5 + 3^{\frac{3}{2}-3t} 25^t},$$

and

$$u(t)=\frac{9\left(53-\sqrt{109}\right)^{t}\left(21-16\sqrt{3}+\rho\right)+9\left(53+\sqrt{109}\right)^{t}\left(16\sqrt{3}-21+\rho\right)}{\left(53-\sqrt{109}\right)^{t}\left(53-30\sqrt{3}+\sqrt{109}\right)+\left(53+\sqrt{109}\right)^{t}\left(-53+30\sqrt{3}+\sqrt{109}\right)},$$

where

$$\rho=\sqrt{327\left(7-4\sqrt{3}\right)},$$

so that we have the numerical comparison for $t=0,\ldots,10$ *given in Table 1.*

Table 1. Solutions and errors with $h=1$, $r=\frac{1}{3}$, $K=9$, $\varepsilon=\frac{3}{5}$, and $P(0)=\ell(0)=u(0)=P_0=9\left(-3+2\sqrt{3}\right)$ for Eqs (2.1), (2.3), and (2.4), respectively.

t	$P(t)$	$\ell(t)$	$u(t)$	$P(t)-\ell(t)$	$u(t)-P(t)$
0	4.17691	4.17691	4.17691	0.0	0.0
1	4.82309	4.22309	5.42309	0.6	0.6
2	5.45614	4.26919	6.62136	1.18695	1.16522
3	6.05189	4.31509	7.68981	1.7368	1.63792
4	6.59169	4.36065	8.58024	2.23105	1.98855
5	7.06428	4.40574	9.28147	2.65853	2.21719
6	7.46571	4.45025	9.80946	3.01546	2.34375
7	7.79806	4.49404	10.1937	3.30401	2.39569
8	8.0674	4.53702	10.4666	3.53039	2.39917
9	8.28195	4.57908	10.6569	3.70287	2.37492
10	8.4505	4.62013	10.788	3.83038	2.33748

In the two right-most columns of Table 1, we see that inequality (5.1) *holds and the conditional Hyers–Ulam stability is guaranteed.*

If we keep all the parameter values the same but take $\varepsilon=\frac{4}{5}$ *instead of* $\varepsilon=\frac{3}{5}$*, then*

$$\varepsilon>\frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2r}=63-36\sqrt{3}\approx 0.646171$$

and the right-hand side of (5.1) *becomes*

$$\frac{h(1+hr)}{\sqrt{1+hr}-1}\varepsilon=\frac{16}{15}\left(3+2\sqrt{3}\right)\approx 6.89504, \tag{5.2}$$

so one of the hypotheses of Theorem 9 is not met. Indeed, below we compare the values for solutions

$$P:[0,T_P)_h\to\mathbb{R}$$

and

$$\ell:[0,T_\ell)_h\to\mathbb{R}$$

of (2.1) *and* (2.3)*, respectively, with initial condition* (2.5)*. We have the numerical comparison for* $t=0,\ldots,20$ *given in Table 2.*

We see that inequality (5.1) *does not hold eventually in the right-most column of Table 2 since*

$$|\ell(t)-P(t)|\nleq\frac{h(1+hr)}{\sqrt{1+hr}-1}\varepsilon=\frac{16}{15}\left(3+2\sqrt{3}\right)\approx 6.89504$$

using (5.2)*, making the equation unstable. This shows the impact of the value of the perturbation* ε *being too large, as noted in Remark 4, and highlights the conditional nature of the Hyers–Ulam stability result in Theorem 9.*

Table 2. Solutions and errors with $h = 1$, $r = \frac{1}{3}$, $K = 9$, $\varepsilon = \frac{4}{5}$, and $P(0) = \ell(0) = P_0 = 9\left(-3 + 2\sqrt{3}\right)$ for Eqs (2.1) and (2.3), respectively.

t	$P(t)$	$\ell(t)$	$P(t) - \ell(t)$
0	4.17691	4.17691	0.0
1	4.82309	4.02309	0.8
2	5.45614	3.86849	1.58764
3	6.05189	3.71158	2.3403
4	6.59169	3.5507	3.04099
5	7.06428	3.38404	3.68024
6	7.46571	3.20952	4.25619
7	7.79806	3.02471	4.77334
8	8.0674	2.82667	5.24074
9	8.28195	2.61171	5.67024
10	8.4505	2.37515	6.07535
11	8.58149	2.11081	6.47068
12	8.68243	1.81034	6.87209
13	8.7597	1.46211	7.29759
14	8.81856	1.04934	7.76923
15	8.86323	0.546773	8.31646
16	8.89703	−0.08544	8.98247
17	8.92255	−0.914282	9.83684
18	8.94179	−2.06177	11.0036
19	8.95627	−3.7763	12.7326
20	8.96716	−6.6538	15.621

Example 13. *In* (2.1)*,* (2.3)*, and* (2.4)*, take $h = K = 1$, $r = 3$, and $\varepsilon = P_0 = \frac{1}{3}$. According to Theorem 9, since*

$$0 < \varepsilon \leq \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} = \frac{1}{3} \quad \text{and} \quad P_0 \geq \frac{K\left(\sqrt{1+hr}-1\right)}{hr} = \frac{1}{3},$$

then solutions

$$P : [0, T_P)_h \to \mathbb{R}, \quad \ell : [0, T_\ell)_h \to \mathbb{R},$$

and

$$u : [0, T_u)_h \to \mathbb{R}$$

of (2.1)*,* (2.3)*, and* (2.4)*, respectively, with initial condition* (2.5)*, satisfy*

$$T_P = T_\ell = T_u = \infty$$

and

$$|\ell(t) - P(t)|, |u(t) - P(t)| \leq \frac{h(1+hr)}{\sqrt{1+hr}-1}\varepsilon = \frac{4}{3} \tag{5.3}$$

for all $t \in [0, \infty)_h$. Note that in this specific instance we have

$$P(t) = \frac{4^t}{2+4^t}, \quad \ell(t) \equiv \frac{1}{3},$$

and

$$u(t) = \frac{\left(\frac{3}{4}\right)^t\left(-3+\sqrt{5}\right)^{t+1}+\left(-\frac{3}{4}\right)^t\left(3+\sqrt{5}\right)^{t+1}}{3\left(\left(\frac{3}{4}\left(-3+\sqrt{5}\right)\right)^t\left(1+\sqrt{5}\right)+\left(-1+\sqrt{5}\right)\left(-\frac{3}{4}\left(3+\sqrt{5}\right)\right)^t\right)},$$

so that we have the numerical comparison for $t = 0, \ldots, 10$ *given in Table 3.*

Table 3. Solutions and errors with $h = K = 1$, $r = 3$, $\varepsilon = \frac{1}{3}$, and $P(0) = \ell(0) = u(0) = P_0 = \frac{1}{3}$ for Eqs (2.1), (2.3), and (2.4), respectively.

t	$P(t)$	$\ell(t)$	$u(t)$	$P(t) - \ell(t)$	$u(t) - P(t)$
0	0.333333	0.333333	0.33333	0.0	0.0
1	0.666667	0.333333	1.0	0.333333	0.333333
2	0.888889	0.333333	1.33333	0.555556	0.444444
3	0.969697	0.333333	1.4	0.636364	0.430303
4	0.992248	0.333333	1.41026	0.658915	0.418008
5	0.998051	0.333333	1.41176	0.664717	0.413714
6	0.999512	0.333333	1.41199	0.666179	0.412473
7	0.999878	0.333333	1.41202	0.666545	0.412139
8	0.999969	0.333333	1.41202	0.666636	0.412052
9	0.999992	0.333333	1.41202	0.666659	0.41203
10	0.999998	0.333333	1.41202	0.666665	0.412025

In the two right-most columns of Table 3, we see that inequality (5.3) *holds and the conditional Hyers–Ulam stability is guaranteed.*

Note that we cannot use the results of Jung and Nam [14] for this example because ε *and* P_0 *are the critical values given in Theorem 9 (see also Remark 11). Let* $H^*(r)$ *be given in* (4.2). *That is,* $H^*(r)$ *is a Hyers–Ulam stability constant derived in [14]. In this example,* $r = 3$, *so we can see from Figure 2 that* $H^*(3)$ *is smaller than our constant* $H(3)$. *Actually, we have*

$$H^*(3) = \frac{9}{5} < 4 = H(3).$$

But, in the two right-most columns of Table 3, we see that the inequality

$$|\ell(t) - P(t)| \leq H^*(3)\varepsilon = \frac{3}{5} = 0.6$$

does not hold for all t. From this fact, it becomes clear that their result shows that the Hyers–Ulam stability constant can be smaller when $r > 1.013624$ *because* ε *and* P_0 *are more limited than ours.*

If we keep all the parameter values the same but take $\varepsilon = \frac{2}{5}$ *instead of* $\varepsilon = \frac{1}{3}$, *then*

$$\varepsilon > \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} = \frac{1}{3}$$

and the right-hand side of (5.3) *becomes*

$$\frac{h(1+hr)}{\sqrt{1+hr}-1}\varepsilon = \frac{8}{5} = 1.6, \tag{5.4}$$

so one of the hypotheses of Theorem 9 is not met. Indeed, if we compare the values for solutions

$$P : [0, T_P)_h \to \mathbb{R}$$

and

$$\ell : [0, T_\ell)_h \to \mathbb{R}$$

of (2.1) *and* (2.3)*, respectively, with initial condition* (2.5)*, we have the numerical comparison for* $t = 0, \ldots, 20$ *given in Table 4.*

Table 4. Solutions and errors with $h = K = 1$, $r = 3$, $\varepsilon = \frac{2}{5}$, and $P(0) = \ell(0) = P_0 = \frac{1}{3}$ for Eqs (2.1) and (2.3), respectively.

t	$P(t)$	$\ell(t)$	$\lvert\ell(t) - P(t)\rvert$
0	0.333333	0.333333	0.0
1	0.666667	0.266667	0.4
2	0.888889	0.192593	0.696296
3	0.969697	0.0882629	0.881434
4	0.992248	−0.120861	1.11311
5	0.998051	−1.15844	2.15649
6	0.999512	1.47198	0.47247
7	0.999878	0.687147	0.312731
8	0.999969	0.497808	0.502161
9	0.999992	0.398594	0.601399
10	0.999998	0.326108	0.67389
11	1.0	0.259362	0.740637
12	1.0	0.183464	0.816536
13	1.0	0.0733356	0.926664
14	1.0	−0.159557	1.15956
15	1.0	−1.62423	2.62423
16	1.0	1.27762	0.277625
17	1.0	0.657445	0.342555
18	1.0	0.484752	0.515248
19	1.0	0.39006	0.60994
20	1.0	0.318945	0.681055

We see that inequality (5.3) *does not hold for all t in the right-most column of Table 4 since*

$$|\ell(t) - P(t)| \nleq \frac{h(1+hr)}{\sqrt{1+hr}-1}\varepsilon = \frac{8}{5} = 1.6$$

using (5.4)*, for some* $t \in [0, \infty)_h$*, making the equation unstable. This shows the impact of the value of the perturbation* ε *being too large, as noted in Remark 4, and highlights the conditional nature of the Hyers–Ulam stability result in Theorem 9.*

Finally, again take $h = K = 1$*,* $r = 3$*, and* $\varepsilon = \frac{1}{3}$*, but let*

$$P(0) = \ell(0) = P_0 = \frac{1}{4}.$$

In this case, note that

$$0 < \varepsilon \leq \frac{K\left(\sqrt{1+hr}-1\right)^2}{h^2 r} = \frac{1}{3} \quad \textit{but} \quad P_0 = \frac{1}{4} < \frac{K\left(\sqrt{1+hr}-1\right)}{hr} = \frac{1}{3},$$

so again one of the hypotheses of Theorem 9 is not met. Note that in this specific instance we have

$$P(t) = \frac{4^t}{3+4^t} \quad \textit{and} \quad \ell(t) = \frac{t-6}{3(t-8)}.$$

Indeed, if we compare the values for solutions

$$P : [0, T_P)_h \to \mathbb{R}$$

and

$$\ell : [0, T_\ell)_h \to \mathbb{R}$$

of (2.1) *and* (2.3)*, respectively, with initial condition* (2.5)*, we have the numerical comparison for* $t = 0, \ldots, 8$ *given in Table 5.*

Table 5. Solutions and errors with $h = K = 1$, $r = 3$, $\varepsilon = \frac{1}{3}$, and $P(0) = \ell(0) = P_0 = \frac{1}{4}$ for Eqs (2.1) and (2.3), respectively.

t	$P(t)$	$\ell(t)$	$\lvert\ell(t) - P(t)\rvert$
0	0.25	0.25	0.0
1	0.571429	0.238095	0.333333
2	0.842105	0.222222	0.619883
3	0.955224	0.2	0.755224
4	0.988417	0.166667	0.82175
5	0.997079	0.111111	0.885968
6	0.999268	0.0	0.999268
7	0.999817	−0.33333	1.33315
8	0.999954	∞	∞

Since

$$P_0 = \frac{1}{4} < \frac{1}{3}$$

and

$$T_\ell = 8,$$

the equation is unstable. This shows the impact of the value of P_0 *being too small, as noted in Remark 5, and highlights the conditional nature of the Hyers–Ulam stability result in Theorem 9.*

6. Sensitivity analysis

We now proceed to investigate the local sensitivity analysis related to the parameters of Eq (2.1). It is important to note that the population dynamics represented by the logistic model are biologically meaningful within the range $0 < P < K$. Therefore, we perform the local sensitivity analysis under the assumption that $0 < P < K$.

Since Eq (2.1) can be rewritten as

$$P(t + h) = \frac{K(1 + hr)P(t)}{K + hrP(t)},$$

if we further define

$$n := \frac{t}{h}$$

and

$$x(n) := P(hn),$$

we obtain the following difference equation:

$$x(n+1) = \frac{K(1+hr)x(n)}{K+hrx(n)}. \tag{6.1}$$

First, we perform a sensitivity analysis of the parameter K, which represents the carrying capacity. Differentiating (6.1) with respect to K, we obtain

$$\frac{\partial x(n+1)}{\partial K} = \frac{hr(1+hr)}{\left(\frac{K}{x(n)} + hr\right)^2}.$$

Therefore, the sensitivity coefficient for the parameter K is dependent on the population size, $x(n)$. Given that

$$0 < x(n) = P(ht) < K,$$

we observe that the sensitivity is low when the population is small (when $x(n)$ approaches 0), and the sensitivity is high when the population is large (when $x(n)$ approaches K).

Next, we perform a sensitivity analysis of the parameter r, which represents the growth rate. Differentiating (6.1) with respect to r, we obtain

$$\frac{\partial x(n+1)}{\partial r} = \frac{hKx(n)(K-x(n))}{(K+hrx(n))^2}.$$

Define the function

$$S(x) := \frac{hKx(K-x)}{(K+hrx)^2}$$

for $0 < x < K$. Then

$$S'(x) = \frac{hK^2(K-2x)}{(K+hrx)^2}.$$

This demonstrates that the sensitivity is low when the population is small or large (when $x(n)$ approaches 0 or K), and the sensitivity is high when the population is at an intermediate level (when $x(n)$ approaches $\frac{K}{2}$).

From the form of (6.1), we arrive at the same conclusion regarding the sensitivity with respect to h, as h plays a role analogous to r.

Let us recall that the Hyers–Ulam stability constant in Theorem 9 is

$$\mathcal{H} = \frac{h(1+hr)}{\sqrt{1+hr}-1}.$$

The parameters K, r, and h all influence the initial value and ε, which represents the margin of error in (6.1) and its perturbed equation. However, the Hyers–Ulam stability constant can be chosen independently of K. In other words, the carrying capacity is unrelated to the error between the approximate solution and the true solution of (6.1). Therefore, we can conclude that the carrying capacity K is sensitive when the population is large, but even if some perturbation is added to the equation, it does not affect the error between the approximate solution and the true solution, so it is a parameter that does not need to be treated very delicately. On the other hand, r and h exhibit sensitivity when the population is at an intermediate level, and they also influence the error between the approximate solution and the true solution. In many cases, h is fixed in advance, and from a biological perspective, it is important to investigate how the population changes from the intermediate stage. Therefore, the parameter to which we should truly pay attention is r, which represents the growth rate.

Example 14. *In* (2.1) *and* (2.2) *take*

$$h = K = 1.$$

According to Theorem 9, if

$$0 < \varepsilon \leq \frac{\left(\sqrt{1+r}-1\right)^2}{r} \quad \textit{and} \quad P_0 \geq \frac{\sqrt{1+r}-1}{r}$$

hold, then solutions

$$P : [0, T_P)_1 \to \mathbb{R}$$

and

$$\beta : [0, T_\beta)_1 \to \mathbb{R}$$

of (2.1) *and* (2.2)*, respectively, with initial condition* (2.5) *satisfy*

$$T_P = T_\beta = \infty$$

and

$$|\beta(t) - P(t)| \leq \frac{1+r}{\sqrt{1+r}-1}\varepsilon$$

for all $t \in [0, \infty)_1$. *Table 6 shows the upper bounds of* ε*, the lower bounds of* P_0*, and the Hyers–Ulam stability constants, all of which depend on* r*.*

Table 6. Upper bounds of ε, lower bounds of P_0, and HUS constants, all dependent on r.

r	$\frac{\left(\sqrt{1+r}-1\right)^2}{r}$	$\frac{\sqrt{1+r}-1}{r}$	$\frac{1+r}{\sqrt{1+r}-1}$
0.1	0.023823	0.488088	22.5369
0.2	0.0455488	0.477226	12.5727
0.3	0.0654972	0.467251	9.27409
0.4	0.0839202	0.45804	7.64126
0.5	0.101021	0.44949	6.67423
0.6	0.116963	0.441518	6.03976
0.7	0.131884	0.434058	5.59504
0.8	0.145898	0.427051	5.26869

Now, we consider (2.2) *with*

$$h = K = 1$$

and

$$q(t) = 0.01(-1)^t.$$

Figure 3 shows the solution orbits for the equation with initial condition

$$\beta(0) = P_0 = 0.5$$

when $r = 0.2$ *(red), 0.5 (black), and 0.8 (blue). In addition, the dashed curves show the* $\frac{1+r}{\sqrt{1+r}-1}\varepsilon$*-neighborhoods around the solution orbits, where* $\varepsilon = 0.01$*. As mentioned above, the sensitivity for* r *is high near the solution of the equation to* $\frac{1}{2}$*, and as a result, the three solution orbits are significantly different.*

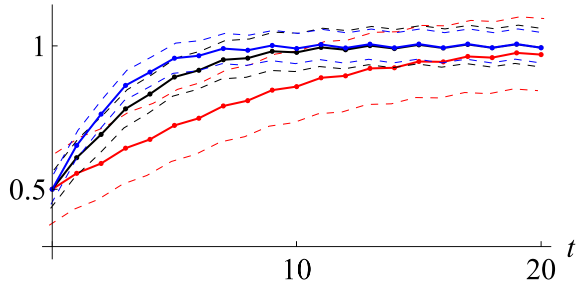

Figure 3. The solution orbits with $r = 0.2$ (red), 0.5 (black), and 0.8 (blue).

7. Conclusions

We establish robust conditional Hyers–Ulam stability results for the logistic h-difference equation, also known as the Beverton–Holt equation if $h = 1$, for any constant step-size $h > 0$. As h tends to zero, our results recover known results for the conditional stability of the continuous logistic-growth model. Additionally, departing from the methodology employed by Jung and Nam [14] in case $h = 1$, we introduce a novel approach to derive sharper results. Specifically, we explicitly determine the optimal lower bound for the initial value region and the upper bound for the perturbation amplitude, demonstrating an improvement over their findings. Furthermore, our analysis yields a sharper Hyers–Ulam constant, which quantifies the error between the true and approximate solutions. Given that a smaller Hyers–Ulam constant indicates greater stability and is desirable for practical applications, our results offer a substantial advancement in precision. The sharpness of our derived bounds and constants is substantiated through illustrative examples.

Author contributions

Douglas R. Anderson: conceptualization, methodology, writing—original draft; writing—review and editing, visualization; Masakazu Onitsuka: conceptualization, methodology, writing—original draft, writing—review and editing, conducted the numerical simulations. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The second author is supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI (Grant number: JP20K03668).

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. Brzdęk, D. Popa, I. Raşa, B. Xu, *Ulam stability of operators*, Academic Press, 2018.
2. A. K. Tripathy, *Hyers–Ulam stability of ordinary differential equations*, Chapman and Hall/CRC, 2021. https://doi.org/10.1201/9781003120179
3. A. R. Baias, F. Blaga, D. Popa, On the best Ulam constant of a first order linear difference equation in Banach spaces, *Acta Math. Hungar.*, **163** (2021), 563–575. https://doi.org/10.1007/s10474-020-01098-3
4. S. N. Bora, M. Shankar, Ulam–Hyers stability of second-order convergent finite difference scheme for first- and second-order nonhomogeneous linear differential equations with constant coefficients, *Results Math.*, **78** (2023), 17. https://doi.org/10.1007/s00025-022-01791-5
5. K. Chen, Y. Si, Ulam type stability for the second-order linear Hahn difference equations, *Appl. Math. Lett.*, **160** (2025), 109355. https://doi.org/10.1016/j.aml.2024.109355
6. D. M. Kerekes, B. Moşneguţu, D. Popa, On Ulam stability of a second order linear difference equation, *AIMS Math.*, **8** (2023), 20254–20268. https://doi.org/10.3934/math.20231032
7. A. Novac, D. Otrocol, D. Popa, Ulam stability of a linear difference equation in locally convex spaces, *Results Math.*, **76** (2021), 33. https://doi.org/10.1007/s00025-021-01344-2
8. Y. Shen, Y. Li, The z-transform method for the Ulam stability of linear difference equations with constant coefficients, *Adv. Differ. Equations*, **2018** (2018), 396. https://doi.org/10.1186/s13662-018-1843-0
9. C. Buşe, V. Lupulescu, D. O'Regan, Hyers–Ulam stability for equations with differences and differential equations with time-dependent and periodic coefficients, *Proc. R. Soc. Edinburgh Sect. A*, **150** (2020), 2175–2188. https://doi.org/10.1017/prm.2019.12
10. D. Popa, I. Raşa, A. Viorel, Approximate solutions of the logistic equation and Ulam stability, *Appl. Math. Lett.*, **85** (2018), 64–69. https://doi.org/10.1016/j.aml.2018.05.018
11. M. Onitsuka, Conditional Ulam stability and its application to the logistic model, *Appl. Math. Lett.*, **122** (2021), 107565. https://doi.org/10.1016/j.aml.2021.107565
12. M. Onitsuka, Approximate solutions of generalized logistic equation, *Discrete Contin. Dyn. Syst.*, **29** (2024), 4505–4526. https://doi.org/10.3934/dcdsb.2024053
13. L. Backes, D. Dragičević, M. Onitsuka, M. Pituk, Conditional Lipschitz shadowing for ordinary differential equations, *J. Dyn. Differ. Equations*, **36** (2024), 3535–3552. https://doi.org/10.1007/s10884-023-10246-6
14. S. M. Jung, Y. W. Nam, Hyers–Ulam stability of Pielou logistic difference equation, *J. Nonlinear Sci. Appl.*, **10** (2017), 3115–3122. https://doi.org/10.22436/jnsa.010.06.26
15. Y. W. Nam, Hyers–Ulam stability of elliptic Möbius difference equation, *Cogent Math. Stat.*, **5** (2018), 1492338. https://doi.org/10.1080/25742558.2018.1492338
16. Y. W. Nam, Hyers–Ulam stability of hyperbolic Möbius difference equation, *Filomat*, **32** (2018), 4555–4575. https://doi.org/10.2298/fil1813555n

17. Y. W. Nam, Hyers–Ulam stability of loxodromic Möbius difference equation, *Appl. Math. Comput.*, **356** (2019), 119–136. https://doi.org/10.1016/j.amc.2019.03.033

18. A. S. Ackleh, Y. M. Dib, S. R. J. Jang, A three-stage discrete-time population model: continuous versus seasonal reproduction, *J. Biol. Dyn.*, **1** (2007), 291–304. https://doi.org/10.1080/17513750701605440

19. L. L. Albu, Non-linear models: applications in economics, *SSRN Electron. J.*, 2006. https://doi.org/10.2139/ssrn.1565345

20. M. Bohner, A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, 2003. https://doi.org/10.1007/978-0-8176-8230-9

21. J. Cushing, S. Henson, A periodically forced Beverton–Holt equation, *J. Differ. Equations Appl.*, **8** (2002), 1119–1120. https://doi.org/10.1080/1023619031000081159

22. S. Elaydi, R. J. Sacker, Global stability of periodic orbits of non-autonomous difference equations and population biology, *J. Differ. Equations*, **208** (2005), 258–273. https://doi.org/10.1016/j.jde.2003.10.024

23. M. Bohner, A. Peterson, *Dynamic equations on time scales: an introduction with applications*, Birkhäuser, 2001. https://doi.org/10.1007/978-1-4612-0201-1

AIMS Press