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Research article

A new constant in Banach spaces based on the Zbăganu constant $C_Z(B)$

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Abstract: In Banach spaces, first we present a new geometric constant, $C_Z^{(q)}(t,B)$, which is closely related to the Zbăganu constant. We prove that $\frac{(t+2)^q}{2^{q-1}(2^{q-1}+t^q)}$ and $\frac{4}{3}$ are respectively the lower and upper bounds for $C_Z^{(q)}(t,B)$. Furthermore, we also derive that $C_Z^{(q)}(t,B) = C_Z^{(q)}(t,\widetilde{B})$, where $\widetilde{B}$ denotes the ultrapower spaces of B. Then, we can establish certain sufficient conditions that ensure a Banach spaces possesses a normal structure, which involve different constants such as the Zbăganu constant and Domínguez–Benavides coefficient.

Keywords: geometric constants; Banach space; Zbăganu constant; ultrapower space; normal structure
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1. Introduction and preliminaries

In recent years, geometric constants have been extensively cited and extensively studied in academic research in functional analysis. These constants serve as quantitative measures of the geometric attributes of spaces, thereby facilitating a more streamlined approach to tackling challenges within Banach spaces. The elegance of these mathematical constructs is undeniable, and they exhibit an intricate web of interrelationships. For scholars who are enthused by geometric constants of study, we suggest consulting the references listed in [1–3] along with any additional citations mentioned within. Furthermore, it is important to recognize that geometric constants have a significant impact on addressing a range of mathematical challenges, including investigations into the Banach–Stone theorem, the Bishop–Phelps–Bollobas theorem, and Tingley's problem. These represent significant avenues of investigation within the field of functional analysis, and if the reader wishes to go into more depth, they may refer to the literature in [4, 5].

Brodskii and Milman initially introduced the notion of normal structure in [6]. Subsequently, Kirk showed in [7] that a single-valued nonexpansive mapping in a Banach space with normal structure has the fixed–point property. As a result, the examination of normal structure constitutes the most crucial aspect of research in fixed point theory [8–11].

The configuration of the unit sphere in a Banach space dictates its normal structure; however, it does not adequately capture the spatial geometric structure's features. Hence, in recent years, numerous scholars have introduced a variety of geometric constants to characterize the geometric structure of Banach spaces.

The James, Jordan-von Neumann, and Zbǎganu constants are widely studied classical constant in [12–14]. In this article, for Banach spaces B, $B^{'}$ denotes the dual spaces of B. Let U_B and V_B represent the unit sphere and the unit ball of the spaces B, respectively:

$$J(B) = sup\left\{min\{\|a+b\|, \|a-b\|\} : a, b \in V_B\right\},$$

$$C_{NJ}(B) = sup\left\{\frac{\|a+b\|^2 + \|a-b\|^2}{2(\|a\|^2 + \|b\|^2)} : a, b \in B, \|a\| + \|b\| > 0\right\},$$

$$C_Z(B) := sup\left\{\frac{\|a+b\|\|a-b\|}{\|a\|^2 + \|b\|^2} : a, b \in B, (a, b) \neq (0_B, 0_B)\right\}.$$

In order to generalize the results of geometric constants, first Dhompongsa introduced two constants, $J(t, B)$ and $C_{NJ}(t, B)$ (see [1, 15]), where $t \geqslant 0$,

$$J(t, B) = \sup\left\{\min\{\|a+b\|, \|a-b\|\} : a, b, c \in U_B\right\},$$

$$C_{NJ}(t, B) = \sup\left\{\frac{\|a+b\|^2 + \|a-c\|^2}{2\|a\|^2 + \|b\|^2 + \|c\|^2} : a, b, c \in B\right\},$$

when a, b, c are not all zero and $\|b - c\| \leqslant t\|a\|$. The constants have the following properties:

(1) $C_{NJ}(0, B) = C_{NJ}(B)$, $J(0, B) = J(B)$;

(2) $C_{NJ}(t, B)$ and $J(t, B)$ are increasing continuous functions;

(3) When X is Banach spaces, $1 + \frac{4t}{4+t^2} \leqslant C_{NJ}(t, B) \leqslant 2$ for all $t \geqslant 0$. For Hilbert spaces, we have $C_{NJ}(t, B) = 1 + \frac{4t}{4+t^2}$ for all $t \in [0, 2]$.

Cui [16] and Dinarvand [17] respectively introduced the constants $C_{NJ}^{(q)}(B)$ and $C_{NJ}^{(q)}(t, B)$ and give sufficient conditions for its normal structure when $t \geqslant 0$, $1 \leqslant q < \infty$,

$$C_{NJ}^{(q)}(B) = \sup\left\{\frac{\|a+b\|^q + \|a-b\|^q}{2^{q-1}\left(\|a\|^q + \|b\|^q\right)} : a, b \in B, (a, b) \neq (0, 0)\right\},$$

$$C_{NJ}^{(q)}(t, B) = \sup\left\{\frac{\|a+b\|^q + \|a-c\|^q}{2^{q-1}\|a\|^q + 2^{q-2}\left(\|b\|^q + \|c\|^q\right)} : a, b, c \in B\right\},$$

when a, b, c are not all zero and $\|b - c\| \leqslant t\|a\|$.To delve deeper into the geometric characteristics of Banach spaces, including properties like normal structure and uniform non-squareness, certain researchers have brought forth the concepts of the constants (see [2, 18, 19]). $C_{-\infty}(B), C_{-\infty}(t, B)$ and $C_{-\infty}^{(q)}(B)$, when $t \geqslant 0$ and $1 \leqslant q < \infty$. For further insights into these geometric constants:

$$C_{-\infty}(B) = \sup\left\{\frac{\min\left\{\|a+b\|^2, \|a-b\|^2\right\}}{\|a\|^2 + \|b\|^2} : a, b \in B, \|a\| + \|b\| > 0\right\},$$

$$C_{-\infty}^{(q)}(B) = \sup\left\{\frac{\min\left\{\|a+b\|^q, \|a-b\|^q\right\}}{2^{q-2}\left(\|a\|^q + \|b\|^q\right)} : a, b \in B, (a, b) \neq (0, 0)\right\},$$

$$C_{-\infty}(t, B) = \sup\left\{\frac{2\min\left\{\|a+b\|^2, \|a-c\|^2\right\}}{2\|a\|^2 + \|b\|^2 + \|c\|^2} : a, b, c \in B\right\},$$

when a, b, c are not all zero and $\|b - c\| \leqslant t\|a\|$.

In the subsequent section, we provide several essential notations and definitions.

Definition 1.1 ([3]). *Let B be Banach spaces, if there exists $\sigma > 0$, we have*

$$\min\{\|a + b\|, \|a - b\|\} < 2(1 - \sigma),$$

for any $a, b \in V_B$. Then Banach spaces B is uniformly non-square.

Definition 1.2 ([6]). *Banach spaces B is said to possess a (weak) normal structure, which implies that for any (weakly compact) closed bounded convex subset G in B that contains more than one point, there exists an element $B_0 \in G$ such that the following condition holds for*

$$\sup\{\|a_0 - b\| : b \in G\} < \sup\{\|a - b\| : a, b \in G\},$$

and then if there exists $c \in (0, 1)$ such that the following condition holds for

$$\sup\{\|a_0 - b\| : b \in G\} < c \sup\{\|a - b\| : a, b \in G\}.$$

Then B is characterized as having a uniform normal structure.

For the purpose of the proofs in the later sections, primarily we present basic information about the ultrapower spaces in [20–22].

When B is Banach spaces and $\mathfrak{E}$ is a filter on $\mathbb{N}$. Relative to $\mathfrak{E}$, a sequence $\{a_n\}$ in B converges on A, denoted $\lim_{\mathfrak{E}} a_n = A$, then for each neighborhood V of A, we have $\left\{y \in: A_y \in V\right\} \in \mathfrak{E}$. A filter $\mathfrak{V}$ on $\mathbb{N}$ is said to be an ultrafilter if it is the greatest element under the subset relation. An ultrafilter is said to be trivial if it is of the form $\{Z \subset: y_0 \in Z\}$ for a fixed subset $y_0 \in \mathbb{N}$, if not, then it is said to be nontrivial. When $L_\infty(B)$ denote the subspace of the product spaces $\prod_{n\in\mathbb{N}} B$, then we have $\|(a_n)\|_\infty = \sup_{n\in\mathbb{N}} \|a_n\| < \infty$.

If $\mathfrak{V}$ is an ultrafilter on $\mathbb{N}$, then we have

$$N_{\mathfrak{V}}(B) = \left\{A = (a_n) \in L_\infty(B) : \lim_{\mathfrak{V}} \|a_n\| = 0\right\}.$$

The ultrapower spaces $\widetilde{B}$ of B is the quotient space $\frac{L_\infty(B)}{N_{\mathfrak{V}}(B)}$ which is equipped with the quotient norm. Let $(a_n)_{\mathfrak{V}}$ denote the elements of the ultrapower. It follows from the definition of the commercial specification that $\|(a_n)_{\mathfrak{V}}\| = \lim_{\mathfrak{V}} \|a_n\|$. So we can know that if $\mathfrak{V}$ is nontrivial, then a can isometrically be mapped into $\widetilde{B}$. Moreover, an important result on ultrapower spaces is that if B is super-reflexive, i.e., $\widetilde{B'} = (\widetilde{B})'$, then $\widetilde{B}$ has normal structure if and only if B has uniform normal structure (see [21]).

2. New constant $C_Z^{(q)}(t, B)$

First, the new constant is defined as $C_Z^{(q)}(t, B)$, which takes the form

$$C_Z^{(q)}(t, B) = \sup\left\{\frac{\|a + b\|^{\frac{q}{2}}\|a - c\|^{\frac{q}{2}}}{2^{q-2}(\|a\|^q + \|b\|^q + \|c\|^q)} : a, b, c \in B, (a, b, c) \neq (0, 0, 0)\right\},$$

for all $1 \leq q < \infty$ and $t \geq 0$, where $\|b - c\| \leq t\|a\|$. Next, we are going to introduce some properties of this new constant.

Remark 2.1. *When B is Banach spaces and $1 \leq q < \infty$, the following statements are all valid.*

(1) For $t \geq 0$, we can know that $C_Z^{(q)}(t, B)$ is a continuous and increasing function;

(2) B is uniformly non-square if and only if $C_Z^{(q)}(t, B) < \frac{4}{3}$,then $C_Z(B) < \frac{4}{3}$;

(3) For any $t \geq 0$, we can get $C_Z^{(q)}(t, B) \leq C_Z(B) \leq \frac{4}{3}$.

Theorem 2.1. *For the Banach spaces X, when $1 \leq q < \infty$, we can obtain*

$$\frac{(t+2)^q}{2^{q-1}\left(2^{q-1}+t^q\right)} \leq C_Z^{(q)}(t, B) \leq \frac{4}{3}.$$

Proof. According to Remark 2.1, for all $t \geq 0$, we have $C_Z^{(q)}(t, B) \leqslant \frac{4}{3}$. When $a \in V_B, b = -c = \frac{t}{2}a$ such that $b - c = ta$. Moreover,

$$\|a+b\| = \left\|a + \frac{t}{2}a\right\| = \frac{t+2}{2},$$
$$\|a-c\| = \left\|a + \frac{t}{2}a\right\| = \frac{t+2}{2}.$$

Then

$$C_Z^{(q)}(t, B) \geqslant \frac{\|a+b\|^{\frac{q}{2}}\|a-c\|^{\frac{q}{2}}}{2^{q-2}(\|a\|^q + \|b\|^q + \|c\|^q)} = \frac{\left(\frac{t+2}{2}\right)^q}{2^{q-2}(1+\frac{t^q}{2^{q-1}})} = \frac{(t+2)^q}{2^{q-1}\left(2^{q-1}+t^q\right)}.$$

□

Example 2.1. *Define B to be $\mathbb{R}^2$ with the metric induced by the $l_1 - l_\infty$ norm*

$$\|(a_1, a_2)\| = \begin{cases} \|(a_1, a_2)\|_1, & a_1a_2 \geqslant 0, \\ \|(a_1, a_2)\|_\infty, & a_1a_2 \leqslant 0. \end{cases}$$

Then $C_Z^{(q)}(1, B) = \frac{4}{3}$.

Proof. From Theorem 2.1, we have $C_Z^{(q)}(1, B) \leqslant \frac{4}{3}$. If $a = (1, -1), b = (1, 0), c = (0, 1)$, we can obtain

$$b - c = (1, -1) = a, \quad \|a+b\| = \|(2, -1)\|_\infty = 2, \quad \|a-c\| = \|(1, -2)\|_\infty = 2.$$

From the definition of the $C_Z^{(q)}(t, B)$, we can obtain

$$C_Z^{(q)}(1, B) \geqslant \frac{\|a+b\|^{\frac{q}{2}}\|a-c\|^{\frac{q}{2}}}{2^{q-2}(\|a\|^q + \|b\|^q + \|c\|^q)} = \frac{2^q}{3 \cdot 2^{q-2}} = \frac{4}{3}.$$

Hence, $C_Z^{(q)}(1, B) = \frac{4}{3}$. □

Example 2.2. *Define B to be $\mathbb{R}^2$ with the metric induced by the $l_1 - l_2$ norm*

$$\|(a_1, a_2)\| = \begin{cases} \|(a_1, a_2)\|_1, & a_1a_2 \geqslant 0, \\ \|(a_1, a_2)\|_2, & a_1a_2 \leqslant 0. \end{cases}$$

Then $C_Z^{(q)}(2, B) = \frac{4}{3}$.

Proof. From Theorem 2.1, we have $C_Z^{(q)}(2, B) \leqslant \frac{4}{3}$. Let $a = \left(\frac{1}{2}, \frac{1}{2}\right), b = (0, 1), c = (-1, 0)$. We have

$$b - c = (1, 1) = 2\left(\frac{1}{2}, \frac{1}{2}\right) = 2a, \quad \|a + b\| = \left\|\left(\frac{1}{2}, \frac{3}{2}\right)\right\|_1 = 2, \quad \|a - c\| = \left\|\left(\frac{3}{2}, \frac{1}{2}\right)\right\|_1 = 2.$$

From the definition of the $C_Z^{(q)}(t, B)$, we have

$$C_Z^{(q)}(1, B) \geqslant \frac{\|a + b\|^{\frac{q}{2}}\|a - c\|^{\frac{q}{2}}}{2^{q-2}(\|a\|^q + \|b\|^q + \|c\|^q)} = \frac{2^q}{3 \cdot 2^{q-2}} = \frac{4}{3}.$$

Then $C_Z^{(q)}(2, B) = \frac{4}{3}$. □

According to the definitions of the constants $J(t, B)$ and $C_Z^{(q)}(t, B)$, we can effortlessly deduce the subsequent lemma.

Lemma 2.1. *If B is a Banach space and $1 < q < \infty$. Then for all $t \geqslant 0$, we have*

$$J(t, B) \leqslant \left(\sqrt[q]{3 \cdot 2^{q-2}}\right)\left(\sqrt[q]{C_Z^{(q)}(t, B)}\right).$$

Proof. For $1 < q < \infty$ and $t \geqslant 0$, we have

$$\begin{aligned} C_Z^{(q)}(t, B) &\geq \frac{\|a + b\|^{\frac{q}{2}}\|a - c\|^{\frac{q}{2}}}{3 \cdot 2^{q-2}} \\ &\geq \frac{[\min\{\|a + b\|, \|a - c\|\}]^q}{3 \cdot 2^{q-2}} \\ &\geq \frac{[J(t, B)]^q}{3 \cdot 2^{q-2}}. \end{aligned}$$

So we know that

$$\begin{aligned} [J(t, B)]^q &\leq 3 \cdot 2^{q-2} C_Z^{(q)}(t, B), \\ J(t, B) &\leq \left(\sqrt[q]{3 \cdot 2^{q-2}}\right)\left(\sqrt[q]{C_Z^{(q)}(t, B)}\right). \end{aligned}$$

Then we can obtain the result. □

Lemma 2.2. *For Banach spaces B, when $1 \leqslant q < \infty$. We have*

$$C_Z^{(q)}(t, B) = C_Z^{(q)}(t, \widetilde{B}).$$

Proof. For $1 \leqslant q < \infty$, we can obtain

$$C_Z^{(q)}(t, B) \leqslant C_Z^{(q)}(t, \widetilde{B}).$$

Next, we prove that $C_Z^{(q)}(t, B) \geqslant C_Z^{(q)}(t, \tilde{B})$. If $\sigma > 0, \beta \in [0, t]$. Assume that $\widetilde{a}, \widetilde{b}, \widetilde{z} \in \widetilde{B}$ satisfy $\|\widetilde{b} - \widetilde{c}\| = \beta\|\widetilde{a}\|$. For $\widetilde{a} = 0$, we have

$$\frac{\|\widetilde{a} + \widetilde{b}\|^{\frac{q}{2}}\|\widetilde{a} - \widetilde{c}\|^{\frac{q}{2}}}{2^{q-2}(\|\widetilde{a}\|^q + \|\widetilde{b}\|^q + \|\widetilde{c}\|^q)} = \frac{\|\widetilde{b}\|^{\frac{q}{2}}\|\widetilde{c}\|^{\frac{q}{2}}}{2^{q-2}(\|\widetilde{b}\|^q + \|\widetilde{c}\|^q)} = 2^{1-q},$$

thus $2^{1-q} \leqslant C_Z^{(q)}(t,B)$. When $\widetilde{a} \neq 0$, then for all $\phi > 0$ such that $\phi < \sigma\|\widetilde{a}\|$. We can get $\|\widetilde{a}\| = \lim_{\mathcal{U}} \|a_n\|$ and

$$k = \frac{\|\widetilde{a}+\widetilde{b}\|^{\frac{q}{2}}\|\widetilde{a}-\widetilde{c}\|^{\frac{q}{2}}}{2^{q-2}(\|\widetilde{a}\|^q+\|\widetilde{b}\|^q+\|\widetilde{c}\|^q)} = \lim_{\mathcal{U}} \frac{\|a_n+b_n\|^{\frac{q}{2}}\|a_n-c_n\|^{\frac{q}{2}}}{2^{q-2}(\|a_n\|^q+\|b_n\|^q+\|c_n\|^q)} = \lim_{\mathcal{U}} k_n.$$

Then, the set

$$Q = \{n \in \mathbb{N} : |k_n - k| < \sigma, \|b_n - c_n\| \leqslant \beta\|a_n\| + \phi < (\beta+\sigma)\|a_n\|\}$$

in the ultrafilter about $\widetilde{B}$. Specifically, for all $n \in Q$, notice that $a_n \neq 0$, there exists n such that

$$k < \frac{\|\widetilde{a}+\widetilde{b}\|^{\frac{q}{2}}\|\widetilde{a}-\widetilde{c}\|^{\frac{q}{2}}}{2^{q-2}(\|\widetilde{a}\|^q+\|\widetilde{b}\|^q+\|\widetilde{c}\|^q)} + \sigma \leqslant C_Z^{(q)}(t+\sigma, B) + \sigma.$$

Hence, according to the continuity of $C_Z^{(q)}(\cdot, B)$ and the arbitrariness of σ, we have the inequality

$$C_Z^{(q)}(t, \widetilde{B}) \leqslant C_Z^{(q)}(t, B).$$

□

Lemma 2.3 ([23]). *If super-reflexive Banach spaces B have no normal structure, for $d \in (0,1]$, then we have $\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3} \in V_{\widetilde{B}}, \widetilde{g_1}, \widetilde{g_2}, \widetilde{g_3} \in V_{(\tilde{B})^*}$ such that*

(1) $\left\|\widetilde{a_i} - \widetilde{a_j}\right\| = 1$ and $\widetilde{g_i}\left(\widetilde{a_j}\right) = 0$ for all $i \neq j, (i,j = (1,2,3))$;

(2) $\widetilde{g_i}(\widetilde{a_i}) = 1$ when $i = 1,2,3$;

(3) $\|\widetilde{a_3} - (\widetilde{a_2} + d\widetilde{a_1})\| \geqslant \|\widetilde{a_2} + d\widetilde{a_1}\|$.

Theorem 2.2. *For Banach spaces X, when $t \in [0,1]$, then we have*

$$C_Z^{(q)}(t,B) < \frac{1}{3\cdot 2^{q-2}}\left(1+t+\frac{1-t^2}{J(t,B)+2t}\right)^q,$$

where $1 < q < \infty$, then we can know that B has a normal structure.

Proof. According to Lemma 2.1, we can obtain $(J(t,B))^q \leqslant 3\cdot 2^{q-2}\cdot C_Z^{(q)}(t,B)$. So B is a uniform non-square from the inequality in [1], therefore, B is also super-reflexive. If B has a normal structure and for any $t \in [0,1]$, the inequality holds. Then, we have $\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3} \in V_{\widetilde{B}}, \widetilde{g_1}, \widetilde{g_2}, \widetilde{g_3} \in V_{(\widetilde{B})^*}$ conforming all the specifications in Lemma 2.3.

Let

$$\begin{cases} \widetilde{a} = \widetilde{a_3} - \widetilde{a_1}, \\ \widetilde{b} = t\widetilde{a_3} + (1-t)\frac{\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|}, \\ \tilde{c} = t\widetilde{a_1} + (1-t)\frac{\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|}. \end{cases}$$

Then $\|\widetilde{a}\| = 1, \|\widetilde{b}\| \leqslant 1, \|\widetilde{c}\| \leqslant 1, \widetilde{b} - \widetilde{c} = t\widetilde{a}$. Then, we can obtain

$$\begin{aligned} \|\widetilde{a}+\widetilde{b}\| &= \left\|\widetilde{a_3} - \widetilde{a_1} + t\widetilde{a_3} + (1-t)\frac{\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|}\right\| \\ &\geqslant \widetilde{g_3}\left(\widetilde{a_3} - \widetilde{a_1} + t\widetilde{a_3} + (1-t)\frac{\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|}\right) \\ &= 1+t+\frac{1-t}{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|}, \end{aligned}$$

$$\begin{aligned}
\|\widetilde{a}-\widetilde{c}\| &= \left\|\widetilde{a_3}-\widetilde{a_1}-t\widetilde{a_1}-(1-t)\frac{\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|}\right\| \\
&\geqslant (-\widetilde{g_1})\left(\widetilde{a_3}-\widetilde{a_1}+t\widetilde{a_1}+(1-t)\frac{\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|}\right) \\
&= 1+t+\frac{1-t}{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|}.
\end{aligned}$$

From Lemma 2.2, we can obtain

$$\begin{aligned}
C_Z^{(q)}(t,B) &= C_Z^{(q)}(t,\widetilde{B}) \\
&\geqslant \frac{\|\widetilde{a}+\widetilde{b}\|^{\frac{q}{2}}\|\widetilde{a}-\widetilde{c}\|^{\frac{q}{2}}}{2^{q-2}(\|\widetilde{a}\|^q+\|\widetilde{b}\|^q+\|\widetilde{c}\|^q)} \\
&\geqslant \frac{1}{3\cdot 2^{q-2}}\left(1+t+\frac{1-t}{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|}\right)^q.
\end{aligned}$$

By taking

$$\begin{cases}
\widetilde{a}=\widetilde{a_2}-\widetilde{a_1}, \\
\widetilde{b}=t\widetilde{a_2}+(1-t)\frac{-\widetilde{a_3}+\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_3}-\widetilde{a_2}-\widetilde{a_1}\|}, \\
\widetilde{c}=t\widetilde{a_1}+(1-t)\frac{-\widetilde{a_3}+\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_3}-\widetilde{a_2}-\widetilde{a_1}\|}.
\end{cases}$$

Then $\|\widetilde{a}\|=1, \|\widetilde{b}\|\leqslant 1, \|\widetilde{c}\|\leqslant 1, \widetilde{b}-\widetilde{c}=t\widetilde{a}$. Similarly, we have

$$\begin{aligned}
C_Z^{(q)}(t,B) &= C_Z^{(q)}(t,\widetilde{B}) \\
&\geqslant \frac{\|\widetilde{a}+\widetilde{b}\|^{\frac{q}{2}}\|\widetilde{a}-\widetilde{c}\|^{\frac{q}{2}}}{2^{q-2}(\|\widetilde{a}\|^q+\|\widetilde{b}\|^q+\|\widetilde{c}\|^q)} \\
&\geqslant \frac{1}{3\cdot 2^{q-2}}\left(1+t+\frac{1-t}{\|\widetilde{a_3}-\widetilde{a_2}-\widetilde{a_1}\|}\right)^q.
\end{aligned}$$

By taking

$$\begin{cases}
\widetilde{a}=\widetilde{a_3}-\widetilde{a_2}, \\
\widetilde{b}=t\widetilde{a_3}+(1-t)\widetilde{a_1}, \\
\widetilde{c}=t\widetilde{a_2}+(1-t)\widetilde{a_1}.
\end{cases}$$

When $\|\widetilde{a}\|=1, \|\widetilde{b}\|\leqslant 1, \|\widetilde{c}\|\leqslant 1, \widetilde{b}-\widetilde{c}=t\widetilde{a}$. Hence,

$$\begin{gathered}
\|\widetilde{a}+\widetilde{b}\| = \|\widetilde{a_3}-\widetilde{a_2}+t\widetilde{a_3}+(1-t)\widetilde{a_1}\| \geqslant (1+t)\,\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|-t\,\|\widetilde{a_2}-2\widetilde{a_1}\|, \\
\|\widetilde{a}-\widetilde{c}\| = \|\widetilde{a_3}-(1+t)\tilde{a}_2-(1-t)\widetilde{a_1}\| \geqslant (1+t)\,\|\widetilde{a_3}-\widetilde{a_2}-\widetilde{a_1}\|-t\,\|\widetilde{a_3}-2\widetilde{a_1}\|, \\
\|\widetilde{a_2}-2\widetilde{a_1}\| \leqslant \|\widetilde{a_2}-\widetilde{a_1}\|+\|\widetilde{a_1}\|=2, \|\widetilde{a_3}-2\widetilde{a_1}\|\leqslant 2.
\end{gathered}$$

From the definition of $J(t,B)$, we can obtain

$$\begin{aligned}
J(t,B) &= J(t,\widetilde{B}) \\
&\geqslant \min\{\|\widetilde{a}+\widetilde{b}\|, \|\widetilde{a}-\widetilde{c}\|\} \\
&\geqslant (1+t)\min\{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|, \|\widetilde{a_3}-\widetilde{a_2}-\widetilde{a_1}\|\}-2t.
\end{aligned}$$

Consequently,

$$\frac{1}{\min\{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|,\|\widetilde{a_3}-\widetilde{a_2}-\widetilde{a_1}\|\}} \geqslant \frac{1+t}{J(t,B)+2t}.$$

As a result, we obtain

$$\begin{aligned} C_Z^{(q)}(t,B) &\geqslant \frac{1}{3\cdot 2^{q-2}}(1+t+\frac{1-t}{\min\{\|\widetilde{a_3}-\widetilde{a_2}+\widetilde{a_1}\|,\|\widetilde{a_3}-\widetilde{a_2}-\widetilde{a_1}\|\}})^q \\ &\geqslant \frac{1}{3\cdot 2^{q-2}}\left(1+t+\frac{1-t}{J(t,B)+2t}\right)^q. \end{aligned}$$

This contradicts the hypothesis. Therefore, we know that Banach spaces B have normal structure. □

3. Domínguez-Benavides coefficient

Domínguez-Benavides introduce the constant $D(t,B)$ in 1996 in [24], for $t \geqslant 0$,

$$D(t,B) = \sup\left\{\liminf_{n\to\infty}\{\|a_n + a\|\}\right\},$$

when the supremum satisfies all $a \in B$ with $\|a\| \leqslant t$. Then all weakly null sequences $\{a_n\}$ in U_B satisfy

$$D[(a_n)] = \limsup_{n\to\infty}\limsup_{m\to\infty}\|a_n - a_m\| \leqslant 1.$$

Particularly, when $t = 1$, we have $1 \leqslant D(1,B) \leqslant 2$.

Lemma 3.1 ([23]). *A super-reflexive Banach space B has no normal structure; then we have* $\widetilde{a_1},\widetilde{a_2} \in V_{\widetilde{B}}, \widetilde{g_1},\widetilde{g_2} \in V_{(\tilde{B})^*}$ *such that*

(1) $\|\widetilde{a_1}-\widetilde{a_2}\| = 1$ and $\widetilde{g_i}\left(\widetilde{a_j}\right) = 0$ for $i \neq j, (i,j = (1,2))$;

(2) $\widetilde{g_i}(\widetilde{a_i}) = 1$ when $i = 1,2$;

(3) $\|\widetilde{a_2}+\widetilde{a_1}\| \leqslant D(1,B)$.

Theorem 3.1. *Banach spaces B, where there exists* $t \in [0,1]$*, if*

$$C_Z^{(q)}(t,B) < \frac{1}{3\cdot 2^{q-2}}\left(1+t+\frac{1-t}{D(1,B)}\right)^q,$$

when $1 < q < \infty$*, then B has a normal structure.*

Proof. Suppose that $1 < q < \infty$; if $t = 1$, we can know that B has a normal structure. When $0 \leqslant t < 1$ and $D(1,B) \geqslant 1$, we have $C_Z^{(q)}(t,B) < \frac{4}{3}$, then we can obtain B is a uniform non–square; therefore, B is super–reflexive.

Assume that B has no normal structure. According to Lemma 3.1, we have $\widetilde{a_1},\widetilde{a_2} \in V_{\widetilde{B}}, \widetilde{g_1},\widetilde{g_2} \in V_{(\widetilde{B})^*}$ conforming to all the conditions delineated in Lemma 3.1. By taking

$$\begin{cases} \widetilde{a} = \widetilde{a_2} - \widetilde{a_1}, \\ \widetilde{b} = t\widetilde{a_2} + (1-t)\frac{\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_2}+\widetilde{a_1}\|}, \\ \widetilde{c} = t\widetilde{a_1} + (1-t)\frac{\widetilde{a_2}+\widetilde{a_1}}{\|\widetilde{a_2}+\widetilde{a_1}\|}. \end{cases}$$

Then $\|\widetilde{a}\| = 1, \|\widetilde{b}\| \leqslant 1, \|\widetilde{c}\| \leqslant 1$, and $\|\widetilde{a_2} + \widetilde{a_1}\| \geqslant \widetilde{g_1}\,(\widetilde{a_2} + \widetilde{a_1}) = 1$. We can obtain

$$\begin{aligned}
\|\widetilde{a} + \widetilde{b}\| &= \left\| \widetilde{a_2} - \widetilde{a_1} + t\widetilde{a_2} + (1-t)\frac{\widetilde{a_2} + \widetilde{a_1}}{\|\widetilde{a_2} + \widetilde{a_1}\|} \right\| \\
&\geqslant \widetilde{g_2}\left(\widetilde{a_2} - \widetilde{a_1} + t\widetilde{a_2} + (1-t)\frac{\widetilde{a_2} + \widetilde{a_1}}{\|\widetilde{a_2} + \widetilde{a_1}\|} \right) \\
&= 1 + t + \frac{1-t}{\|\widetilde{a_2} + \widetilde{a_1}\|}, \\
\|\widetilde{a} - \widetilde{c}\| &= \left\| \widetilde{a_2} - \widetilde{a_1} - t\widetilde{a_1} - (1-t)\frac{\widetilde{a_2} + \widetilde{a_1}}{\|\widetilde{a_2} + \widetilde{a_1}\|} \right\| \\
&\geqslant (-\widetilde{g_1})\left(\widetilde{a_2} - \widetilde{a_1} - t\widetilde{a_1} - (1-t)\frac{\widetilde{a_2} + \widetilde{a_1}}{\|\widetilde{a_2} + \widetilde{a_1}\|} \right) \\
&= 1 + t + \frac{1-t}{\|\widetilde{a_2} + \widetilde{a_1}\|}.
\end{aligned}$$

From the definition of the $C_Z^{(q)}(t, B)$, we can obtain

$$\begin{aligned}
C_Z^{(q)}(t, B) &= C_Z^{(q)}(t, \widetilde{B}) \\
&\geqslant \frac{\|\widetilde{a} + \widetilde{b}\|^{\frac{q}{2}} \|\widetilde{a} - \widetilde{c}\|^{\frac{q}{2}}}{2^{q-2}(\|\widetilde{a}\|^q + \|\widetilde{b}\|^q + \|\widetilde{c}\|^q)} \\
&\geqslant \frac{1}{3 \cdot 2^{q-2}} \left(1 + t + \frac{1-t}{\|\widetilde{a_2} + \widetilde{a_1}\|} \right)^q \\
&\geqslant \frac{1}{3 \cdot 2^{q-2}} \left(1 + t + \frac{1-t}{D(1, B)} \right)^q .
\end{aligned}$$

This contradicts the hypothesis. Thus, this completes the proof. □

If $t = 0$, we obtain the following corollary.

Corollary 3.1. *For Banach spaces X, when*

$$C_Z^{(q)}(B) < \frac{1}{3 \cdot 2^{q-2}} \left(1 + \frac{1}{D(1, B)} \right)^q ,$$

where $1 < q < \infty$, *we can get that B has a normal structure.*

We know that if

$$C_Z^{(2)}(t, B) < \frac{2}{3}\left(1 + \left(\frac{(1+t)}{\min\{D(1, B) + t, 2\}}\right)^2 \right),$$

then B has a normal structure [18]. According to Theorem 3.1, it has

$$C_Z^{(2)}(t, B) < \frac{1}{3}\left(1 + t + \frac{1-t}{D(1, B)} \right)^2$$

when $q = 2$. Then, for $t \in [0, 1], 1 \leqslant D(t, B) \leqslant 2$, we have

$$\frac{1}{3}(1 + t + \frac{1-t}{D(1, B)})^2 \in [\frac{3}{4}, \frac{4}{3}], \quad \frac{2}{3}\left(1 + (\frac{(1+t)}{\min\{D(1, B) + t, 2\}})^2 \right) \in [\frac{5}{6}, \frac{4}{3}].$$

In particular, when $q = 2$, according to Corollary 3.1, we have

$$C_Z^{(2)}(t, B) < \frac{1}{3}(1 + \frac{1}{D(1, B)})^2.$$

Since

$$\frac{1}{3}(1 + \frac{1}{D(1, B)})^2 < \frac{2}{3}(1 + \frac{1}{D(1, B)^2})$$

holds, then we have improved the results.

4. Conclusions

In this paper, we present a novel geometric constant $C_Z^{(q)}(t, B)$ in Banach spaces, which bears a significant relationship to the Zbăganu constant. At first, we introduce the definition and some properties of this new constant, and the focus of this paper is on the relationship between the normal structure and the constant $C_Z^{(q)}(t, B)$. Using ultrapower techniques, we establish the relationship between the Zbăganu constant (Domínguez-Benavides coefficient) and $C_Z^{(q)}(t, B)$, which gives us a sufficient condition for the normal structure. Thus, we can improve on the results obtained in [18, 23, 25].

Authors contributions

Qian Li: Writing-original draft; Zhouping Yin: Supervision, Writing-review; Yuxin Wang: Writing-review; Qi Liu: Writing-review; Hongmei Zhang: Methodology. All authors have reviewed and consented to the publication of the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors state that there are no conflicts of interest.

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