AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(3): 6469–6479.
DOI: 10.3934/math.2025295
Received: 16 December 2024
Revised: 21 January 2025
Accepted: 13 March 2025
Published: 24 March 2025

Research article

GBD and $\mathcal{L}$-positive semidefinite elements in C^*-algebras

Kezheng Zuo[1], **Dragana S. Cvetković-Ilić**[2,*] **and Jiale Gao**[3]

[1] School of Mathematics and Statistics, Hubei Normal University, Huangshi, China
[2] College of Science, Kuwait University, Kuwait
[3] Department of Mathematics, Shanghai University, Shanghai, China

* **Correspondence:** Email: dragana.ilic@ku.edu.kw.

Abstract: This paper focused on the generalized Bott-Duffin (GBD) inverse and the GBD elements in Banach algebra with involution and C^*-algebra, as well as on the property of the p-positive semidefinite elements that are a generalization of the $\mathcal{L}$-positive semidefinite matrices closely related to the GBD inverse. Also, using matrix equalities, inclusion relations of subspaces, and projectors, we established various characterizations of the GBD property in the matrix sets, especially on the set of $\mathcal{L}$-positive semidefinite matrices. Additionally, we compared the methods and tools that we have at our disposal in the matrix set on one side and in Banach and C^*-algebras on the other. Using the GBD inverse as an example, we would like to compare the results and their proofs in both sets and explain steps to quite easily skip from one set to the other, as well as situations in which we must pay additional attention in order to avoid mistakes.

Keywords: generalized Bott-Duffin inverse; p-positive semidefinite elements; GBD elements; GBD matrices
Mathematics Subject Classification: 15A09, 46L05

1. Introduction

Bott and Duffin [1] introduced an important tool called the "constrained inverse" of a square matrix. This inverse was later called the Bott-Duffin inverse, and it is widely used in linear statistical estimation, two-dimensional interpolation, optimization, etc. Since for many classes of linear systems, the Bott-Duffin inverse is not a powerful enough tool, Chen [2] introduced the generalized Bott-Duffin (GBD) inverse that provides a presentation of an automatic analytical method for a system of simultaneous linear equations with the subsidiary condition of unknowns and has many applications in static and dynamic contact analyses.

In [3], we can find some characterizations of the GBD inverse in terms of range, nullspace, and certain matrix equations, as well as some relationships between the GBD inverse and two classes of nonsingular bordered matrices. Some properties and expressions for the GBD inverse of an operator on a Hilbert space were studied in [4, 5]. More interesting results on the GBD inverse can be found in [6–10].

Throughout the paper we use $*$-Banach algebra to mean a Banach algebra with an involution $*$ [11–13]. Let $\mathcal{A}$ be a complex $*$-Banach algebra with unity 1. For an element $a \in \mathcal{A}$, if there exists some $x \in \mathcal{A}$ such that

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa,$$

then a is called Moore-Penrose invertible (MP-invertible) and x, denoted by $a^\dagger$, is called the Moore-Penrose inverse of a [14–16]. Let $a\{i, j, \cdots, k\}$ denote the set of all x that satisfy the equations $(i), (j), \cdots, (k)$ from the above Eqs (1)–(4). In this case $x \in a\{i, j, \cdots, k\}$ is a $\{i, j, \cdots, k\}$-inverse of a and is denoted by $a^{(i,j,\cdots,k)}$. Also, a is called regular (in the sense of von Neumann) if it has an inner inverse, that is, there exists $x \in \mathcal{A}$ such that $axa = a$.
Let $a \in \mathcal{A}$ and let $p \in \mathcal{A}$ be idempotent ($p = p^2$). Then, we write

$$a = pap + pa(1-p) + (1-p)ap + (1-p)a(1-p),$$

and use the notations

$$a_1 = pap, \quad a_2 = pa(1-p), \quad a_3 = (1-p)ap, \quad a_4 = (1-p)a(1-p).$$

Thus, every idempotent $p \in \mathcal{A}$ induces a representation of an arbitrary element $a \in \mathcal{A}$ given by the following matrix:

$$a = \begin{bmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}_p = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p. \tag{1.1}$$

For a projector p (self-adjoint and idempotent) by $p^\perp$ we denote $1-p$.
Using the representation of an arbitrary $a \in \mathcal{A}$ with respect to the projector p we can introduce the following definition of the GBD inverse in the $*$-Banach algebra case.

Definition 1.1. *Let $a \in \mathcal{A}$ and let $p \in \mathcal{A}$ be a projector such that $ap + p^\perp$ is MP-invertible. Then, we call*

$$a_p^{(+)} = p(ap + p^\perp)^\dagger,$$

the GBD inverse of a with respect to p.

Koliha [17] studied an element a in a C^*-algebra to commute with its Moore-Penrose inverse, $aa^\dagger = a^\dagger a$, which is later called the EP element [18].
Inspired by the above work, we naturally put forward the question of characterizations of an element in $*$-Banach algebra to commute with its GBD inverse.

Definition 1.2. *Let $a \in \mathcal{A}$ and let $p \in \mathcal{A}$ be a projector such that $ap + p^\perp$ is MP-invertible. If $aa_p^{(+)} = a_p^{(+)}a$, then a is called a* GBD *element. In particular, if $\mathcal{A}$ is the set of all $n \times n$ complex matrices, then a* GBD *element is also called a* GBD *matrix.*

We emphasize the following main contributions of this paper. Initially, we give a representation of the GBD inverse in the C^*-algebra case (and in general in the $*$-Banach algebra case), as well as a representation of the GBD inverse for the GBD element together with the necessary and sufficient conditions for an element to be GBD. Then, we present various characterizations of GBD matrices related to different matrix equalities, inclusion relations of subspaces, and projectors with a special emphasis on $\mathcal{L}$-positive semidefinite matrices. Meanwhile, we compare the methods of the proof and the results in the matrix set on one side and in $*$-Banach algebra and C^*-algebra on the other, as well as point out difficulties and problems that occur in both cases.

2. Preliminaries

In this section, we introduce some necessary notations, definitions, and results.

Let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices. We denote the identity matrix in $\mathbb{C}^{n\times n}$ by I_n, and the null matrix of the appropriate order by 0. Symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^* denote the range space, null space and conjugate transpose of $A \in \mathbb{C}^{m\times n}$, respectively. For two subspaces $\mathcal{T}, \mathcal{S} \subseteq \mathbb{C}^n$, $P_{\mathcal{T},\mathcal{S}}$ stands for an idempotent matrix on $\mathcal{T}$ along $\mathcal{S}$. In particular, $P_{\mathcal{T}} = P_{\mathcal{T},\mathcal{T}^{\perp}}$ denotes the projector (self-adjoint and idempotent) onto $\mathcal{T}$, where $\mathcal{T}^{\perp}$ is the orthogonal complement of $\mathcal{T}$. For $A \in \mathbb{C}^{m\times n}$ and two subspaces $\mathcal{T} \subseteq \mathbb{C}^n$ and $\mathcal{S} \subseteq \mathbb{C}^m$, if $X \in \mathbb{C}^{n\times m}$ satisfies $XAX = X$, $\mathcal{R}(X) = \mathcal{T}$, and $\mathcal{N}(X) = \mathcal{S}$, then X is unique and is denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$.

Definition 2.1. *[2] Let $A \in \mathbb{C}^{n\times n}$ and let $\mathcal{L}$ be a subspace of $\mathbb{C}^n$. Then,*

$$A^{(+)}_{(\mathcal{L})} = P_{\mathcal{L}}(AP_{\mathcal{L}} + P_{\mathcal{L}^{\perp}})^{\dagger},$$

is the GBD inverse of A with respect to $\mathcal{L}$.

Definition 2.2. *[2] Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix and let $\mathcal{L}$ be a subspace of $\mathbb{C}^n$. If*

(1) $x^*Ax \geq 0$ *for all* $x \in \mathcal{L}$,

(2) $x^*Ax = 0$ *for* $x \in \mathcal{L}$, *implies that* $Ax = 0$,

then A is called an $\mathcal{L}$-positive semidefinite ($\mathcal{L}$-p.s.d.) matrix.

If $\mathcal{L} \subseteq \mathbb{C}^n$ is a given subspace and $A \in \mathbb{C}^{n\times n}$ is an $\mathcal{L}$-p.s.d. matrix, then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $P_{\mathcal{L}}$ and A can be represented as

$$P_{\mathcal{L}} = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*, \quad A = U\begin{bmatrix} A_1 & A_2 \\ A_2^* & A_4 \end{bmatrix}U^*, \tag{2.1}$$

where $A_1 \in \mathbb{C}^{r\times r}$ is positive semidefinite, $A_4 \in \mathbb{C}^{(n-r)\times(n-r)}$ is Hermitian, and $A_2 \in \mathbb{C}^{r\times(n-r)}$ is such that $A_2^* = TA_1$ for some $T \in \mathbb{C}^{(n-r)\times r}$, where $r = \dim(\mathcal{L})$. This decomposition implies many characterizations of $\mathcal{L}$-p.s.d. matrices and is the motivation for the next definition of a p-positive semidefinite element in $*$-Banach algebra.

Definition 2.3. *Let $a \in \mathcal{A}$ be a Hermitian element, and let $p \in \mathcal{A}$ be a projector. If*

$$a = \begin{bmatrix} a_1 & a_2 \\ a_2^* & a_4 \end{bmatrix}_p,$$

where a_1 is positive semidefinite, a_4 is Hermitian, and $a_2^* = ta_1$ for some $t \in \mathcal{A}$, then a is called a p-positive semidefinite (p-p.s.d.) element.

Lemma 2.4. *[2] Let $A \in \mathbb{C}^{n\times n}$ be $\mathcal{L}$-p.s.d., and let $\mathcal{S} = \mathcal{R}(P_{\mathcal{L}}A)$ and $\mathcal{T} = \mathcal{R}(AP_{\mathcal{L}})$. Then,*

$$A^{(+)}_{(\mathcal{L})} = A^{(2)}_{\mathcal{S},\mathcal{S}^{\perp}} = (P_{\mathcal{L}}AP_{\mathcal{L}})^{\dagger}, \tag{2.2}$$

$$AA^{(+)}_{(\mathcal{L})} = P_{\mathcal{T},\mathcal{S}^{\perp}},\ A^{(+)}_{(\mathcal{L})}A = P_{\mathcal{S},\mathcal{T}^{\perp}}, \tag{2.3}$$

$$A^{(+)}_{(\mathcal{L})}AP_{\mathcal{L}} = P_{\mathcal{L}}AA^{(+)}_{(\mathcal{L})} = P_{\mathcal{S}}. \tag{2.4}$$

3. Results

We will start this section with the following $*$-Banach algebra result (and later with a comment for the C^*-algebra case) that gives us a representation of the GBD inverse. This result gives us a deeper understanding of the structure of the GBD inverse of a C^*-algebra element, and its application in the matrix set as a result will have very simple proofs of most of the results that avoid long computations. So, let us suppose that in the next theorem $\mathcal{A}$ is a $*$-Banach algebra. We say that p is a projector, if p is self-adjoint ($p = p^*$) and idempotent ($p = p^2$).

Lemma 3.1. *Let $a, p \in \mathcal{A}$ be such that p is a projector and $ap + p^{\perp}$ is MP-invertible. If a is given by* (1.1)*, then*

$$a^{(+)}_{p} = \begin{bmatrix} a_1^{(1,2,3)} & y \\ 0 & 0 \end{bmatrix}_p,$$

where $a_1^{(1,2,3)}$ and y satisfy that

$$a_1y = 0,\ y(1 + a_3a_3^*) = a_3^* - a_1^{(1,2,3)}a_1a_3^* \quad \text{and} \quad a_1^{(1,2,3)}a_1 + ya_3 \text{ is Hermitian.} \tag{3.1}$$

Proof. Since $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_p$ and $p^{\perp} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_p$, we have that $ap + p^{\perp} = \begin{bmatrix} a_1 & 0 \\ a_3 & 1 \end{bmatrix}_p$. Let us suppose that $(ap + p^{\perp})^{\dagger} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}_p$, for some $x, y, z, w \in \mathcal{A}$. Then, the first MP-equation is equivalent with

$$a_1y = 0,\ x \in a_1\{1\},\ a_3xa_1 + za_1 = 0,\ w = 1 - a_3y. \tag{3.2}$$

Now, having in mind (3.2), we have that the third MP-equation is equivalent with

$$z = -a_3x,\ x \in a_1\{3\}. \tag{3.3}$$

By (3.2) and (3.3), we get that the second MP-equation is equivalent with $x \in a_1\{2\}$. Finally, the forth MP-equation is equivalent with the fact that a_3y and $xa_1 + ya_3$ are Hermitian and $a_3(1-(xa_1+ya_3)) = y^*$. Hence,

$$(ap + p^{\perp})^{\dagger} = \begin{bmatrix} a_1^{(1,2,3)} & y \\ -a_3x & 1 - a_3y \end{bmatrix}_p \text{ and } a^{(+)}_p = p(ap + p^{\perp})^{\dagger} = \begin{bmatrix} a_1^{(1,2,3)} & y \\ 0 & 0 \end{bmatrix}_p,$$

for y and $a_1^{(1,2,3)}$ that satisfy

$$a_1y = 0,\ a_3 - a_3(a_1^{(1,2,3)}a_1 + ya_3) = y^* \quad \text{and} \quad a_1^{(1,2,3)}a_1 + ya_3, a_3y \text{ are Hermitian.} \tag{3.4}$$

Now, using the fact that $a_1^{(1,2,3)}a_1 + ya_3$ is Hermitian, the second equality from (3.4) is equivalent with

$$y = a_3^* - (a_1^{(1,2,3)}a_1 + ya_3)a_3^*. \tag{3.5}$$

Now, by (3.5), it is clear that the fact that $a_1^{(1,2,3)}a_1 + ya_3$ is Hermitian implies that a_3y is Hermitian. Thus, (3.4) is equivalent with (3.1).

Notice that a ring $\mathcal{R}$ with involution has the Gelfand-Naimark property (GN-property) if $1 + a^*a$ is invertible for any $a \in \mathcal{R}$ and that it is well known that C^*-algebras possess the GN-property. The element a in a ring $\mathcal{R}$ with the Gelfand-Naimark property is MP-invertible if and only if it is regular, that is, there exists $b \in \mathcal{R}$ such that $aba = a$. Thus, in a C^*-algebra, from the above proof we have that $y = (1 - a_1^{(1,2,3)}a_1)a_3^*(1 + a_3a_3^*)^{-1}$, which implies that $a_1y = 0$. So, in the following theorem we directly obtain the representation of the GBD inverse in the C^*-algebra case.

Lemma 3.2. *Let $a, p \in \mathcal{A}$ be such that p is a projector and $ap + p^\perp$ is MP-invertible. If a is given by* (1.1)*, then*

$$a_p^{(+)} = \begin{bmatrix} a_1^{(1,2,3)} & y \\ 0 & 0 \end{bmatrix}_p, \tag{3.6}$$

where $a_1^{(1,2,3)}$ and $y = (1 - a_1^{(1,2,3)}a_1)a_3^*(1 + a_3a_3^*)^{-1}$ satisfy that

$$a_1^{(1,2,3)}a_1 + ya_3 \textit{ is Hermitian}. \tag{3.7}$$

In the next theorem, we will give a complete characterization of the class of GBD elements and a representation of its GBD inverse in the case when $\mathcal{A}$ is a C^*-algebra. Later, we will give a comment on our reason to consider the C^*-algebra case and how we can get the analogous result in the case of $*$-Banach algebra.

Theorem 3.3. *Let $a, p \in \mathcal{A}$ be such that p is a projector and $ap + p^\perp$ is MP-invertible, and let a be given by* (1.1)*. Then, a is a* GBD *element if and only if $a_3 = 0$, $a_1a_1^\dagger = a_1^\dagger a_1$ and $a_1a_2 = 0$, in which case $a_p^{(+)}$ is given by*

$$a_p^{(+)} = \begin{bmatrix} a_1^\dagger & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Proof. By Lemma 3.2, we have that $a_p^{(+)}$ is given by (3.6) for $a_1^{(1,2,3)}$ that satisfies (3.7) for $y = (1 - a_1^{(1,2,3)}a_1)a_3^*(1 + a_3a_3^*)^{-1}$. Now, $aa_p^{(+)} = a_p^{(+)}a$ is equivalent with

$$a_1a_1^{(1,2,3)} = a_1^{(1,2,3)}a_1 + ya_3, \tag{3.8}$$
$$a_1y = a_1^{(1,2,3)}a_2 + ya_4, \tag{3.9}$$
$$a_3a_1^{(1,2,3)} = 0, \tag{3.10}$$
$$a_3y = 0. \tag{3.11}$$

Now, by (3.10) and (3.11) and the condition that $a_3 - a_3(a_1^{(1,2,3)}a_1 + ya_3) = y^*$, we get $y^* = a_3$. Thus, (3.11) implies $a_3 = 0$, that is, $y = 0$.
Now, evidently $aa_p^{(+)} = a_p^{(+)}a$ if and only if $a_1a_1^{(1,2,3)} = a_1^{(1,2,3)}a_1$, $a_1^{(1,2,3)}a_2 = 0$, $a_3 = 0$. Now, since y and $a_1^{(1,2,3)}$ satisfy (3.7), we get that $a_1^{(1,2,3)}a_1$ is Hermitian, that is, $a_1^{(1,2,3)} = a_1^\dagger$. Hence, $aa_p^{(+)} = a_p^{(+)}a$ if and only if $a_1a_1^\dagger = a_1^\dagger a_1$, $a_1^\dagger a_2 = 0$ and $a_3 = 0$, which is further equivalent with $a_1a_1^\dagger = a_1^\dagger a_1$, $a_1a_2 = 0$ and $a_3 = 0$.

Remark 3.4. (1) *From the above proof, we can easily get an analogous result in the* $*$*-Banach algebra case, where in general we do not have the following implication*

$$a_3a_3^* = 0 \Rightarrow a_3 = 0,$$

which is the fact that we used in the above proof and is valid in C^**-algebras or *-reducing algebras. So, instead of the conclusion that* $a_3 = 0$ *that we get in the* C^**-algebra case, in the* $*$*-Banach algebra case we will get that* $a_3a_3^* = 0$*.*
(2) *One more fact that we must comment on here is that we used* $a_1^\dagger$ *without assuming that* a_1 *is MP-invertible. Indeed, the MP-invertibility of* $ap + p^\perp$ *implies that* a_1 *is regular, which is in the* C^**-algebra case equivalent with the MP-invertibility of* a_1*. The equivalence between regularity and MP-invertibility is also valid in a ring with involution that satisfies GN-property (see [19]). On the other hand, such equivalence is not valid in a* $*$*-Banach algebra case that is demonstrated by the following example:*

Example 3.5. *Let us consider the Banach algebra* $\mathcal{B}(\mathbb{C}^2, \|\cdot\|)$ *of all bounded and linear maps defined on* $(\mathbb{C}^2, \|\cdot\|)$*, where*

$$\|(x, y)\| = |x| + |y|, \ (x, y) \in \mathbb{C}^2,$$

and $|x|$ *is the module of* x*. Let* $S(x, y) = (x - y, 0)$*,* $(x, y) \in \mathbb{C}^2$*. Then,* S *is an idempotent (regular), but* S *does not have a Moore-Penrose inverse in* $\mathcal{B}(\mathbb{C}^2, \|\cdot\|)$*.*

It is interesting to mention that a necessary and sufficient condition for a Banach space of dimension greater than 3 to be a Hilbert space is that the set of all regular operators coincides with the one of all MP-invertible (see [20]).

In some recent literature, we can find many long and mainly computational proofs with a lot of unnecessary steps that make the reader feels that she/he has spent much time over it without getting the essence of the argument. If we go deeper into the structures using a number of well-known decompositions in either set (such as the partial singular value decomposition, core-nilpotent decomposition, etc.), we will see that the transition from one set to the other will go rather smoothly and that many proofs can be made much easier. For instance, using our Lemma 3.2, we can give a very simple proof of Lemma 2.4 in the C^*-algebra case. Indeed, we will prove (2.2) and give a representation of $a_p^{(+)}$ that will directly imply properties (2.3) and (2.4).

Theorem 3.6. *Let* $a, p \in \mathcal{A}$ *be such that* p *is a projector,* $ap + p^\perp$ *is MP-invertible, and* a *is* p*-p.s.d. If* a *is given by* (1.1)*, then*

$$a_p^{(+)} = \begin{bmatrix} a_1^\dagger & 0 \\ 0 & 0 \end{bmatrix}_p. \tag{3.12}$$

Proof. By Lemma 3.2, we have that $a_p^{(+)}$ is given by (3.6) where $a_1^{(1,2,3)}$ satisfies (3.7) for $y = (1 - a_1^{(1,2,3)}a_1)a_3^*(1 + a_3a_3^*)^{-1}$. Evidently, $a_1y = 0$ and $y^* = a_3 - a_3(a_1^{(1,2,3)}a_1 + ya_3)$. Since a is p-p.s.d., then $a_3 = a_2^* = ta_1$ for some $t \in \mathcal{A}$. Thus,

$$y^* = a_3 - a_3(a_1^{(1,2,3)}a_1 + ya_3) = ta_1 - ta_1a_1^{(1,2,3)}a_1 - ta_1ya_3 = 0,$$

that is, $y = 0$. Now, (3.7) gives that $a_1^{(1,2,3)}a_1$ is Hermitian, that is, $a_1^{(1,2,3)} = a_1^\dagger$. Hence, $a_p^{(+)}$ is given by (3.12).

Theorem 3.7. *Let $a, p \in \mathcal{A}$ be such that p is a projector, $ap + p^{\perp}$ is MP-invertible, and a is p-p.s.d. The following are equivalent:*

(a) *a is a* GBD *element,*
(b) $pa(1 - p) = 0$,
(c) $ap = pa$.

If a is MP-invertible, then any of items (a)–(c) *is equivalent with any of the following:*

(d) $a^{\dagger} p = pa^{\dagger}$,
(e) $(1 - p)a^{\dagger} p = 0$.

Proof. Let us suppose that a is given by (1.1). By Theorem 3.6, we have that $a_p^{(+)}$ is given by (3.12), which implies that (a) is equivalent with

$$a_3 a_1^{\dagger} = 0, \ a_1^{\dagger} a_2 = 0. \tag{3.13}$$

Since a is p-p.s.d, we have that $a_2 = a_1 t$ for some $t \in \mathcal{A}$, so we get that (3.13) is equivalent with $a_2 = 0$, that is, $pa(1 - p) = 0$. Hence, (a) and (b) are equivalent. Now, using that p is a projector and a is Hermitian, we can eaisly verify that (c) is equivalent with (b). If we suppose that a is MP-invertible, then (e) together with the fact that a is Hermitian (so $a^{\dagger}$ is also Hermitian) implies that

$$a^{\dagger} = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}_p,$$

for some $b_1, b_4 \in \mathcal{A}$. So, it is easy to check that both b_1 and b_4 are MP-invertible and

$$a = (a^{\dagger})^{\dagger} = \begin{bmatrix} b_1^{\dagger} & 0 \\ 0 & b_4^{\dagger} \end{bmatrix}_p.$$

Evidently, $pa(1 - p) = 0$. Hence, (e) implies (b). That (b) implies (e) follows analogously. Also, (d) and (e) are equivalent.
Using a completely computational method we can give different characterizations of the GBD property on the set of $\mathcal{L}$-p.s.d. matrices; however, all of them are just another way to write that $P_{\mathcal{L}} A P_{\mathcal{L}^{\perp}} = 0$ which is by Theorem 3.7, a necessary and sufficient condition for A to be a GBD matrix. The next theorem will show one good side of the computational method which is sometimes very short and clear.

Theorem 3.8. *Let $A \in \mathbb{C}^{n\times n}$ be* $\mathcal{L}$-p.s.d. *Then, the following statements are equivalent:*

(1) *A is a* GBD *matrix*, (2) $A_{(\mathcal{L})}^{(+)} = (A_{(\mathcal{L})}^{(+)})^2 A$, (3) $P_{\mathcal{L}} A A_{(\mathcal{L})}^{(+)} = A_{(\mathcal{L})}^{(+)} A$,

(4) $P_{\mathcal{L}} A A_{(\mathcal{L})}^{(+)} = A_{(\mathcal{L})}^{(+)} A^2 A_{(\mathcal{L})}^{(+)}$, (5) $A A_{(\mathcal{L})}^{(+)} = A (A_{(\mathcal{L})}^{(+)})^2 A^2 A_{(\mathcal{L})}^{(+)}$, (6) $A_{(\mathcal{L})}^{(+)} = (A_{(\mathcal{L})}^{(+)})^2 A^2 A_{(\mathcal{L})}^{(+)}$.

Proof. (1) $\Rightarrow$ (2) : Multiplying $AA_{(\mathcal{L})}^{(+)} = A_{(\mathcal{L})}^{(+)} A$ from the left by $A_{(\mathcal{L})}^{(+)}$ and using the fact that $A_{(\mathcal{L})}^{(+)}$ is an outer inverse of A, we get (2).
(2) $\Rightarrow$ (3) : Multiplying (2) with $P_{\mathcal{L}} A$ from the left side and using (2.4) and (2.3), we have the following

$$P_{\mathcal{L}} A A_{(\mathcal{L})}^{(+)} = P_{\mathcal{L}} A (A_{(\mathcal{L})}^{(+)})^2 A = P_{\mathcal{S}} A_{(\mathcal{L})}^{(+)} A = A_{(\mathcal{L})}^{(+)} A.$$

(3) ⇒ (4) : Multiplying (3) with $AA^{(+)}_{(\mathcal{L})}$ from the right side and using the fact that $AA^{(+)}_{(\mathcal{L})}$ is an idempotent, we directly obtain (4).
(4) ⇒ (1) : Obviously, by (2.2), we have that

$$(A^{(+)}_{(\mathcal{L})})^* = A^{(+)}_{(\mathcal{L})}. \tag{3.14}$$

Moreover, by (2.4) it follows that

$$\begin{aligned} A^{(+)}_{(\mathcal{L})}A^2A^{(+)}_{(\mathcal{L})} - P_{\mathcal{L}}AA^{(+)}_{(\mathcal{L})} &= A^{(+)}_{(\mathcal{L})}A^2A^{(+)}_{(\mathcal{L})} - A^{(+)}_{(\mathcal{L})}AP_{(\mathcal{L})}P_{(\mathcal{L})}P_{(\mathcal{L})}AA^{(+)}_{(\mathcal{L})} \\ &= A^{(+)}_{(\mathcal{L})}A(I_n - P_{\mathcal{L}})AA^{(+)}_{(\mathcal{L})} = A^{(+)}_{(\mathcal{L})}AP_{\mathcal{L}^\perp}AA^{(+)}_{(\mathcal{L})}. \end{aligned} \tag{3.15}$$

Thus, by (4) we have that $A^{(+)}_{(\mathcal{L})}AP_{\mathcal{L}^\perp}AA^{(+)}_{(\mathcal{L})} = 0$. Now, since $A^{(+)}_{(\mathcal{L})}AP_{\mathcal{L}^\perp}AA^{(+)}_{(\mathcal{L})} = A^{(+)}_{(\mathcal{L})}AP_{\mathcal{L}^\perp}(A^{(+)}_{(\mathcal{L})}AP_{\mathcal{L}^\perp})^*$, we get that $A^{(+)}_{(\mathcal{L})}AP_{\mathcal{L}^\perp} = 0$, that is $P_{\mathcal{L}^\perp}AA^{(+)}_{(\mathcal{L})} = 0$. By (4) we directly get $P_{\mathcal{L}^\perp}A^{(+)}_{(\mathcal{L})}A^2A^{(+)}_{(\mathcal{L})} = 0$ which together with $P_{\mathcal{L}^\perp}AA^{(+)}_{(\mathcal{L})} = 0$ gives that

$$P_{\mathcal{L}^\perp}(I_n - A^{(+)}_{(\mathcal{L})}A)AA^{(+)}_{(\mathcal{L})} = 0. \tag{3.16}$$

Using that (4) is equivalent with

$$P_{\mathcal{L}}(I_n - A^{(+)}_{(\mathcal{L})}A)AA^{(+)}_{(\mathcal{L})} = 0,$$

and by (3.16), we get that $(I_n - A^{(+)}_{(\mathcal{L})}A)AA^{(+)}_{(\mathcal{L})} = 0$, i.e. $AA^{(+)}_{(\mathcal{L})} = A^{(+)}_{(\mathcal{L})}A^2A^{(+)}_{(\mathcal{L})}$. From the last equality, since $A^{(+)}_{(\mathcal{L})}A^2A^{(+)}_{(\mathcal{L})}$ is Hermitian, we have that $AA^{(+)}_{(\mathcal{L})}$ is Hermitian which is equivalent with (1).
(4) ⇔ (5) : Applying (2.3) and (2.4), we get that (4) is equivalent with

$$P_{\mathcal{S}} = P_{\mathcal{S},\mathcal{T}^\perp}P_{\mathcal{T},\mathcal{S}^\perp},$$

while (5) is equivalent with

$$P_{\mathcal{T},\mathcal{S}^\perp} = P_{\mathcal{T},\mathcal{S}^\perp}P_{\mathcal{S},\mathcal{T}^\perp}P_{\mathcal{T},\mathcal{S}^\perp}.$$

Now, the equivalence between (4) and (5) is evident.
(5) ⇔ (6) : Multiplying (5) by $A^{(+)}_{(\mathcal{L})}$ from the left side, we get (6). Similarly, multiplying (6) by A from the left side, we get (5).

In the next theorem, we will give a characterization of the GBD property on the set of $\mathcal{L}$-p.s.d. matrices in terms of idempotents and projectors. We will omit the proof because the proofs of all items can be given using computational techniques that are similar to the ones used in the previous theorem.

Theorem 3.9. *Let $A \in \mathbb{C}^{n\times n}$ be $\mathcal{L}$-p.s.d., and let $\mathcal{S} = \mathcal{R}(P_{\mathcal{L}}A)$. Let us define a set $N = \{AA^{(+)}_{(\mathcal{L})}, A^{(+)}_{(\mathcal{L})}A, A(A^{(+)}_{(\mathcal{L})})^2A, A^{(+)}_{(\mathcal{L})}A^2A^{(+)}_{(\mathcal{L})}, AA^{(+)}_{(\mathcal{L})} - A^{(+)}_{(\mathcal{L})}A\}$. The following statements are equivalent:*

(1) *A is a* GBD *matrix,*
(2) *At least one element of the set N is a projector,*
(3) *All elements of the set N are projectors,*
(4) *$AA^{(+)}_{(\mathcal{L})} - A^{(+)}_{(\mathcal{L})}A$ is idempotent.*

Furthermore, if any of the conditions (1)–(4) *holds, then*

$$AA^{(+)}_{(\mathcal{L})} = A^{(+)}_{(\mathcal{L})}A = A(A^{(+)}_{(\mathcal{L})})^2A = A^{(+)}_{(\mathcal{L})}A^2A^{(+)}_{(\mathcal{L})} = P_{\mathcal{S}}. \tag{3.17}$$

Now, we will consider a connection between the GBD property on the set of $\mathcal{L}$-p.s.d. matrices and the appropriate equality, as well as the inclusion of certain subspaces. The next result is a good example of the evident difference between finite and infinite dimensional spaces. Indeed, using the fact that for given subspaces $\mathcal{T}$ and $\mathcal{S}$ the following implication

$$\dim(\mathcal{T}) = \dim(\mathcal{S}) \text{ and } \mathcal{T} \subseteq \mathcal{S} \Rightarrow \mathcal{T} = \mathcal{S},$$

holds only in the finite-dimensional case, we will get some equivalences that are generally not valid out of the matrix set.

Theorem 3.10. *Let $A \in \mathbb{C}^{n\times n}$ be $\mathcal{L}$-p.s.d., and let $\mathcal{S} = \mathcal{R}(P_{\mathcal{L}}A)$ and $\mathcal{T} = \mathcal{R}(AP_{\mathcal{L}})$. Then, the following statements are equivalent:*

$$(1)\ A \text{ is a GBD matrix}, \quad (2)\ \mathcal{S} = \mathcal{T}, \quad (3)\ \mathcal{S} \subseteq \mathcal{T}, \quad (4)\ \mathcal{T} \subseteq \mathcal{S}, \quad (5)\ \mathcal{T} \subseteq \mathcal{L}.$$

Proof. Equivalences between (1)–(4) are evident. Indeed, by

$$AA^{(+)}_{(\mathcal{L})} = A^{(+)}_{(\mathcal{L})}A \Leftrightarrow P_{\mathcal{T},\mathcal{S}^{\perp}} = P_{\mathcal{S},\mathcal{T}^{\perp}} \Leftrightarrow \mathcal{S} = \mathcal{T},$$

we have that (1) is equivalent with (2). Then, since A is Hermitian, we have that $\dim(\mathcal{T}) = \dim(\mathcal{S})$, so any of items (3)–(4) is equivalent with (2).
(5) $\Leftrightarrow$ (1) : From (2.3), (3.14), and (3.15), we have

$$\begin{aligned}\mathcal{T} \subseteq \mathcal{L} \Leftrightarrow P_{\mathcal{L}^{\perp}}P_{\mathcal{T},\mathcal{S}^{\perp}} = 0 \Leftrightarrow P_{\mathcal{L}^{\perp}}AA^{(+)}_{(\mathcal{L})} = 0 \Leftrightarrow (P_{\mathcal{L}^{\perp}}AA^{(+)}_{(\mathcal{L})})^{*}P_{\mathcal{L}^{\perp}}AA^{(+)}_{(\mathcal{L})} = 0\\ \Leftrightarrow A^{(+)}_{(\mathcal{L})}AP_{\mathcal{L}^{\perp}}AA^{(+)}_{(\mathcal{L})} = 0 \Leftrightarrow A^{(+)}_{(\mathcal{L})}A^{2}A^{(+)}_{(\mathcal{L})} = P_{\mathcal{L}}AA^{(+)}_{(\mathcal{L})}.\end{aligned}$$

Hence, $\mathcal{T} \subseteq \mathcal{L}$ is equivalent with $A^{(+)}_{(\mathcal{L})}A^{2}A^{(+)}_{(\mathcal{L})} = P_{\mathcal{L}}AA^{(+)}_{(\mathcal{L})}$ which is, by Theorem 3.8, equivalent with the fact that A is a GBD matrix.

Because the first advantage of the matrix sets is the fact that all the elements are inner and MP-invertible, many situations in the matrix sets are much easier. So, when we skip from matrix sets to any other (infinite dimensional operator case, $*$-Banach algebras, etc.), we must be very wary and always have in mind that before using any elemet's inverses, we must first check their existence. For instance, some of the equivalences from Theorem 3.10 will not be valid in some other sets—in other words, for an element a in a C^{*}-algebra, a projector p, $\mathcal{S} = pa\mathcal{A}$ and $\mathcal{T} = ap\mathcal{A}$, we will have that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) and (1) $\Rightarrow$ (3), but (3) does not imply (1). But unfortunately, as a result of this fact, there are many papers in the operator case obtained just using the copy-paste method of the appropriate matrix case results, and in that case, these results are valid only under the restrictive assumption of the existence of certain inverses. In most of situations, this is not the only option and the problem must and can be considered in the general case without assuming some restrictive conditions. So, our advice is to be careful but not limited.

4. Conclusions

We derive a representation of the GBD inverse in C^{*}-algebras (or $*$-Banach algebras) and several equivalent conditions for an element (or a p-p.s.d. element) to be GBD. Using different methods, we consider how an $\mathcal{L}$-p.s.d. matrix can become a GBD matrix. In addition, by comparing results and proofs given in the paper, we point out difficulties and problems when we skip from matrix sets to $*$-Banach algebras, C^*-algebras, etc., and suggested several ways to overcome them. It is worth noting that Deng and Chen [6] introduced a generalization of the GBD inverse in the matrix set, that is, $A^{(+)}_{\mathcal{T},\mathcal{S}} = P_{\mathcal{T},\mathcal{S}}(AP_{\mathcal{T},\mathcal{S}} + P_{\mathcal{S},\mathcal{T}})^{\dagger}$, where $A \in \mathbb{C}^{n\times n}$ and $\mathcal{T}, \mathcal{S} \subseteq \mathbb{C}^n$ are two subspaces. So, it is a potential research topic to extend $A^{(+)}_{\mathcal{T},\mathcal{S}}$ from the matrix set to a $*$-Banach algebra $\mathcal{A}$, namely, $a^{(+)}_q = q(aq+1-q)^{\dagger}$, where $a \in \mathcal{A}$ and q is an idempotent element such that $aq + 1 - q$ is MP-invertible. Naturally, further consideration will include certain characterisations of an element $a \in \mathcal{A}$ such that $aa^{(+)}_q = a^{(+)}_q a$.

Author contributions

All authors made substantial contributions, participated in writing and approved the manuscript. All authors have read and approved the final version of the manuscript for publication. The contribution of the corresponding author is 70%.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflict of interest.

References

1. R. Bott, R. J. Duffin, On the algebra of networks, *T. Am. Math. Soc.*, **74** (1953), 99–109. https://doi.org/10.2307/1990850
2. Y. Chen, The generalized Bott-Duffin inverse and its applications, *Linear Algebra Appl.*, **134** (1990), 71–91. https://doi.org/10.1016/0024-3795(90)90007-Y
3. J. Gao, K. Zuo, H. Zou, Z. Fu, New characterizations and representations of the generalized Bott-Duffin inverse and its applications, *Linear Multilinear A.*, **71** (2023), 1026–1043. https://doi.org/10.1080/03081087.2022.2050170
4. C. Deng, H. Du, Further results on generalized Bott-Duffin inverses, *Oper. Matrices*, **7** (2013), 713–722. https://doi.org/10.7153/oam-07-39
5. M. Kostadinov, Operator matrix representations of the generalised Bott-Duffin inverse, *B. Iran. Math. Soc.*, **47** (2021), 523–533. https://doi.org/10.1007/s41980-020-00396-4
6. B. Deng, G. Chen, A note on the generalized Bott-Duffin inverse, *Appl. Math. Lett.*, **20** (2007), 746–750. https://doi.org/10.1016/j.aml.2006.06.018
7. G. Chen, G. Liu, Y. Xue, Perturbation theory for the generalized Bott-Duffin inverse and its applications, *Appl. Math. Comput.*, **129** (2002), 145–155. https://doi.org/10.1016/S0096-3003(01)00041-8

8. G. Chen, G. Liu, Y. Xue, Perturbation analysis of the generalized Bott-Duffin inverse of $\mathcal{L}$-zero matrices, *Linear Multilinear A.*, **51** (2003), 11–20. https://doi.org/10.1080/0308108031000053602
9. J. Gao, K. Zuo, Y. Chen, J. Wu, The $\mathcal{L}$-positive semidefinite matrices A such that $AA^{(\dagger)}_{(\mathcal{L})} - A^{(\dagger)}_{(\mathcal{L})}A$ are nonsingular, *Linear Multilinear A.*, **72** (2023), 764–786. https://doi.org/10.1080/03081087.2022.2161461
10. Y. Xue, G. Chen, The expression of the generalized Bott-Duffin inverse and its perturbation theory, *Appl. Math. Comput.*, **132** (2002), 437–444. https://doi.org/10.1016/S0096-3003(01)00204-1
11. M. Karaev, M. Gürdal, S. Saltan, Some applications of Banach algebra techniques, *Math. Nachr.*, **284** (2011), 1678–1689. https://doi.org/10.1002/mana.200910129
12. M. Gurdal, S. Saltan, U. Yamancı, Generators of certain function Banach algebras and related question, *Appl. Math. Inform. Sci.*, **9** (2015), 85–88. http://dx.doi.org/10.12785/amis/090112
13. M. Altıntaş, M. Gürdal, R. Tapdigoglu, Duhamel Banach algebra structure of some space and related topics, *Publ. I. Math.*, **112** (2022), 83–93. https://doi.org/10.2298/PIM2226083A
14. R. Penrose, A generalized inverse for matrices, *Math. Proc. Cambridge*, **51** (1955), 406–413. https://doi.org/10.1017/S0305004100030401
15. R. Harte, M. Mbekhta, On generalized inverses in C^*-algebras, *Stud. Math.*, **103** (1992), 71–77. Available from: `http://eudml.org/doc/215936`.
16. J. J. Koliha, The Drazin and Moore-Penrose inverse in C^*-algebras, *Math. Proc. Roy. Irish Acad.*, **99A** (1999), 17–27. Available from: `http://www.jstor.org/stable/20459739`.
17. J. J. Koliha, Elements of C^*-algebras commuting with their Moore-Penrose inverse, *Stud. Math.*, **139** (2000), 81–90.
18. J. Benítez, Moore-Penrose inverses and commuting elements of C^*-algebras, *J. Math. Anal. Appl.*, **345** (2008), 766–770. https://doi.org/10.1016/j.jmaa.2008.04.062
19. J. J. Koliha, D. S. Djordjević, D. S. C. Ilić, Moore-Penrose inverse in rings with involution, *Linear Algebra Appl.*, **426** (2007), 371–381. https://doi.org/10.1016/j.laa.2007.05.012
20. M. Mbekhta, Partial isometries and generalized inverses, *Acta Sci. Math.(Szeged)*, **70** (2004), 767–781.

AIMS Press